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Energy conditions for an imperfect fluid

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Abstract. The weak, dominant and strong energy conditions are investigated for various kinds of imperfect fluids. In this context, attention has been given to the model of a collapsing or expanding sphere of shear-free fluid which conducts heat and radiates energy to infinity.

1. Introduction

In a spacetime with metric $g_{\alpha\beta}$ (signature $-, +, +, +$) the energy-momentum tensor of a viscous fluid with heat flow can be written in the form

$$T_{\alpha\beta} = (\mu + p - \zeta\theta)u_\alpha u_\beta + (p - \zeta\theta)g_{\alpha\beta} - 2n\sigma_{\alpha\beta} + u_\alpha q_\beta + u_\beta q_\alpha \tag{1.1}$$

where u^α is the 4-velocity of the fluid, μ is its rest energy density and p is the isotropic pressure. The contribution of viscosity to $T_{\alpha\beta}$ is described by the term

$$-\zeta\theta(u_\alpha u_\beta + g_{\alpha\beta}) - 2n\sigma_{\alpha\beta} \tag{1.2}$$

where $n \geqslant 0$ is the coefficient of dynamic viscosity and $\zeta \geqslant 0$ is the coefficient of bulk viscosity. The quantities

$$\theta = u^\alpha{}_{;\alpha} \tag{1.3}$$

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} + \dot{u}_{(\alpha}u_{\beta)} - \tfrac{1}{3}\theta(g_{\alpha\beta} + u_\alpha u_\beta) \tag{1.4}$$

(where the round brackets on the indices denote symmetrisation, the semicolon denotes covariant differentiation and the dot denotes differentiation in the direction of u^α) are the expansion and shear velocity of the fluid which, according to (1.4), satisfies the condition $u^\alpha\sigma_{\alpha\beta} = 0$. The heat conduction is described by the heat flux vector q^α defined as follows:

$$q_\alpha u^\alpha = 0 \tag{1.5a}$$

$$q_\alpha n^\alpha = (\text{heat per unit time crossing unit surface perpendicular to } n^\alpha) \tag{1.5b}$$

where n^α is a unit spacelike vector. We note that, because of (1.5a), q^α is spacelike.

In order that this fluid is physically reasonable it must obey the following energy conditions [1, 2].

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(*a*) *The weak energy condition.* For any timelike future pointing vector w^α and at each event of the spacetime, $T_{\alpha\beta}w^\alpha w^\beta \geqslant 0$.

(*b*) *The dominant energy condition.* For any timelike future pointing vector w^α and at each event of the spacetime the 4-momentum density vector, $-T_{\alpha\beta}w^\beta$, must be future pointing and timelike or null.

(*c*) *The strong energy condition.* For any timelike future pointing unit vector w^α and at each event of the spacetime the stresses of the matter are restricted according to the inequality $2T_{\alpha\beta}w^\alpha w^\beta + T \geqslant 0$, where T is the trace of $T_{\alpha\beta}$.

The weak and dominant energy conditions are satisfied by continuity and when the vector w^α is null. The weak energy condition is equivalent to saying that the energy density of the fluid as measured by any observer is non-negative. The dominant energy condition can be interpreted as saying that the speed of energy flow of matter is less than the speed of light for any observer. The strong energy condition can be violated only if the total energy density $T_{\alpha\beta}w^\alpha w^\beta$ is negative or if, for $T_{\alpha\beta}w^\alpha w^\beta > 0$, there exists a large negative principal pressure of $T_{\alpha\beta}$. Note that the three energy conditions are always satisfied by the electromagnetic field. Note also that the dominant energy condition implies the weak energy condition. All known forms of matter obey the above energy conditions. For this reason a cosmological model or a star model based on some fluid which violates these conditions cannot be seriously considered as physically relevant. However, despite this and the fact that in the literature one can find a number of articles where imperfect fluids are considered, as a general rule the discussion of energy conditions is neglected.

An investigation of the energy conditions is, in essence, an algebraic problem. It is closely related to the eigenvalue problem of $T_{\alpha\beta}$ and therefore, on a four-dimensional spacetime manifold, it leads to the search of the roots of a polynomial of degree 4. Because of this, in many situations one is confronted with complicated analytical expressions of the eigenvalues which make the problem intractable and its (non-numerical) discussion practically impossible in full generality. In spite of this we try to include here as many of the most general situations as possible. We consider only energy-momentum tensors of the Segre types [111, 1], [11, 2] and their degeneracies. The types [1, 3] and [11, $z\bar{z}$] are well known to violate even the weak energy condition [3] (see also [1]).

The easiest way to write down the energy conditions is to calculate the eigenvalues of the energy-momentum tensor which, for the types [111, 1] and [11, 2], must be real. For the type [111, 1] and if λ_0 denotes the eigenvalue corresponding to the timelike eigenvector, the energy conditions are equivalent to the following simple relations between the eigenvalues [2, § 9.2, p 219]:

$$-\lambda_0 \geqslant 0 \qquad -\lambda_0 + \lambda_i \geqslant 0 \qquad (i = 1, 2, 3) \tag{1.6}$$

for the weak energy condition,

$$-\lambda_0 \geqslant 0 \qquad \lambda_0 \leqslant \lambda_i \leqslant -\lambda_0 \qquad (i = 1, 2, 3) \tag{1.7}$$

for the dominant energy condition, and

$$-\lambda_0 + \sum_{i=1}^{i=3} \lambda_i \geqslant 0 \qquad -\lambda_0 + \lambda_i \geqslant 0 \qquad (i = 1, 2, 3) \tag{1.8}$$

for the strong energy condition. For the type [11, 2] let us denote by λ_1 the eigenvalue of multiplicity 2 corresponding to the unique null eigenvector l^α, and by ν the quantity

$$\nu = T_{\alpha\beta}k^\alpha k^\beta \tag{1.9}$$

where k^α is a null vector non-collinear with l^α, orthogonal to the eigenvectors corresponding to the other two eigenvalues λ_2 and λ_3 and normalised so that $k^\alpha l_\alpha = 1$. Then the energy conditions are equivalent to the relations†

$$-\lambda_1 \geqslant 0 \qquad \nu > 0 \qquad \lambda_i \geqslant \lambda_1 \qquad (i = 2, 3) \tag{1.10}$$

for the weak energy condition,

$$-\lambda_1 \geqslant 0 \qquad \nu > 0 \qquad \lambda_1 \leqslant \lambda_i \leqslant -\lambda_1 \qquad (i = 2, 3) \tag{1.11}$$

for the dominant energy condition, and

$$\nu > 0 \qquad \lambda_2 + \lambda_3 \geqslant 0 \qquad \lambda_i \geqslant \lambda_1 \qquad (i = 2, 3) \tag{1.12}$$

for the strong energy condition.

In the next section the eigenvalues are calculated in a locally Minkowskian coordinate system. However, we took care that the energy conditions are finally written in a coordinate-free manner so that they are of easy use in any coordinate system. Each energy condition is studied separately after substitution of the eigenvalues into the corresponding inequalities. In some cases these inequalities are not all independent. When this happens we always give a minimal set of independent conditions. The results are presented in the form of theorems so that the reader can use them quickly without entering into the details of our investigations. In this respect, we believe that this paper can save time for anyone who investigates imperfect fluids and would like to test if the energy conditions are fulfilled. A point worthwhile noting is that any energy-momentum tensor can be written in the general form [4] of equation (1.1). The dominant energy condition for a non-viscous fluid with heat flow is given by Hall [5] and Hall and Negm [4]. Also, in these papers the eigenvalues for some simple cases of combinations of energy-momentum tensors (e.g. a perfect fluid with pure radiation field, a perfect fluid with non-null Maxwell field, a combination of two pure radiation fields and a combination of a non-null Maxwell field with a pure radiation field) are given explicitly.

In § 3, we consider a well known collapsing model in spherical symmetry with emphasis on the energy conditions. The fluid we consider there is non-viscous with heat flow. From the results of this section we observe the important role played by the junction conditions which, together with some physically reasonable assumptions about the gradients of the energy density, pressure and heat flow, are sufficient to guarantee the validity of the energy conditions.

2. Energy conditions for an imperfect fluid

The eigenvalues λ of the energy-momentum tensor (1.1) are the roots of the equation

$$|T_{\alpha\beta} - \lambda g_{\alpha\beta}| = 0. \tag{2.1}$$

Because λ is a scalar, we can simplify our calculations if, in the event of the spacetime under consideration, we use a locally Minkowskian coordinate system. Furthermore,

† The dominant energy condition for the type [11, 2] is given in [3].

by making in this event Lorentz transformations of the coordinate axis, we can achieve the following:

$$u^\alpha = \delta_0^\alpha \tag{2.2}$$

$$q^\alpha = q\delta_1^\alpha \tag{2.3}$$

$$\sigma_{12} = 0 \tag{2.4}$$

where in (2.3) q is the magnitude of q^α:

$$q = (q^\alpha q_\alpha)^{1/2}. \tag{2.5}$$

The coordinate x^0 is timelike and x^1, x^2, x^3 are spacelike. The heat flow holds in the direction of the coordinate x^1. With (2.2)-(2.4) the coordinate system is completely specified. Now, (2.1) can be written as

$$\begin{vmatrix} \mu+\lambda & -q & 0 & 0 \\ -q & \tilde{p}-\lambda-2n\sigma_{11} & 0 & -2n\sigma_{13} \\ 0 & 0 & \tilde{p}-\lambda-2n\sigma_{22} & -2n\sigma_{23} \\ 0 & -2n\sigma_{13} & -2n\sigma_{23} & \tilde{p}-\lambda-2n\sigma_{33} \end{vmatrix} = 0 \tag{2.6}$$

where

$$\tilde{p} = p - \zeta\theta \tag{2.7}$$

is the effective pressure. It turns out that the eigenvalues are determined by μ, $\tilde{p}$, q and the four invariants α, σ, β, γ constructed from q^α and $\sigma_{\alpha\beta}$ as follows:

$$\alpha q^2 = q^\alpha q^\beta \sigma_{\alpha\beta} \tag{2.8}$$

$$\sigma^2 = 2\sigma_{\alpha\beta}\sigma^{\alpha\beta} \tag{2.9}$$

$$\beta = 4q^\alpha q^\beta \sigma_\alpha^\gamma \sigma_{\gamma\beta} \tag{2.10}$$

$$\gamma = \tfrac{8}{3}\sigma_\beta^\alpha \sigma_\gamma^\beta \sigma_\alpha^\gamma. \tag{2.11}$$

By setting

$$x = \tilde{p} - \lambda \tag{2.12}$$

we can easily prove that the characteristic polynomial assumes the form

$$x^4 - (\mu+\tilde{p})x^3 + (q^2 - n^2\sigma^2)x^2 + [2n\alpha q^2 + (\mu+\tilde{p})n^2\sigma^2 - \gamma n^3]x + (\mu+\tilde{p})\gamma n^3 + n^2\beta - q^2n^2\sigma^2 = 0. \tag{2.13}$$

The roots of this equation can be explicitly found but they are too complicated to permit us to follow our study in full generality. In order to have simpler expressions we must introduce here some restriction on the coefficients of the characteristic polynomial. We will suppose that, if $n \neq 0$, there exist no shear velocities between neighbourhood surface elements orthogonal to the direction of the heat flux. This means that the heat flux vector is an eigenvector of $\sigma_{\alpha\beta}$:

$$n(q^\alpha \sigma_{\alpha\beta} - \alpha q_\beta) = 0. \tag{2.14}$$

When $n \neq 0$, we can easily prove [4] that (2.14) is satisfied if and only if the heat flux vector and fluid 4-velocity vector span a timelike invariant 2-space of $T_{\alpha\beta}$. We also observe that (2.14) is equivalent to saying that the heat flux vector is an eigenvector of the tensor (1.2) describing the fluid viscosity. Of course, (2.14) is a restrictive assumption but we believe that it is physically relevant as it is expected to be satisfied by every fluid in which the heat is transmitted only by convection. Furthermore, and independent of this, it is identically satisfied if the spacetime is spherically symmetric and the fluid moves radially† (i.e. the 4-velocity vector is orthogonal to the isometry group orbits). Now (2.14) permits a simple factorisation of (2.13). This can be done by starting from (2.13) and by using the equations $\gamma = 8\alpha^3 - 2\alpha\sigma^2$ and $\beta = 4\alpha^2 q^2$ which are valid in this case, but it is better to proceed directly from (2.6). In fact, in the locally Minkowskian coordinate system specified by (2.2)-(2.4), because of (2.14) we have $\sigma_{13} = 0$. The four eigenvalues assume the form

$$\lambda_0 = \tfrac{1}{2}(\tilde{p} - \mu - 2n\alpha - \Delta) \tag{2.15}$$

$$\lambda_1 = \tfrac{1}{2}(\tilde{p} - \mu - 2n\alpha + \Delta) \tag{2.16}$$

$$\lambda_2 = \tilde{p} + n\alpha - n(\sigma^2 - 3\alpha^2)^{1/2} \tag{2.17}$$

$$\lambda_3 = \tilde{p} + n\alpha + n(\sigma^2 - 3\alpha^2)^{1/2} \tag{2.18}$$

where

$$\Delta = [(\tilde{p} + \mu - 2n\alpha)^2 - 4q^2]^{1/2}. \tag{2.19}$$

As we noticed before, in order that the energy conditions are satisfied, the eigenvalues must be real. Thus, if $n \neq 0$, we must have

$$(\tilde{p} + \mu - 2n\alpha)^2 \geqslant 4q^2 \tag{2.20}$$

$$\sigma^2 \geqslant 3\alpha^2. \tag{2.21}$$

If $\mu + \tilde{p} - 2n\alpha \leqslant 0$, the eigenvector corresponding to the eigenvalue λ_1 is timelike or null. Independent of the Segre type of $T_{\alpha\beta}$ the weak energy condition yields, in this case, $-\lambda_1 + \lambda_0 \geqslant 0$. On the other hand, from (2.15) and (2.16) we have $-\lambda_1 + \lambda_0 \leqslant 0$ and thus $\lambda_0 = \lambda_1$ and $\mu + \tilde{p} - 2n\alpha = -2q$. Now we can easily see that the Segre type of $T_{\alpha\beta}$ is [11, 2] with the unique null eigenvector $(1/q)q^\alpha - u^\alpha$. A direct calculation of the quantity ν defined by (1.9) yields $\nu = -q < 0$ which violates the three energy conditions.

By taking into account (2.20) we must therefore have

$$\mu + \tilde{p} - 2n\alpha \geqslant 2q. \tag{2.22}$$

Because of (2.22) the eigenvector corresponding to the eigenvalue λ_0 is timelike or null. We can easily prove that the null case holds if and only if $\mu + \tilde{p} - 2n\alpha = 2q$ and this relation implies that the Segre type of $T_{\alpha\beta}$ is [11, 2] with $\nu = q > 0$. By substitution of (2.15)-(2.18) into (1.6)-(1.8) for the type [111, 1] and into (1.10)-(1.12) for the type [11, 2] we obtain the restrictions that the energy conditions impose on the quantities μ, $\tilde{p}$, α, q, σ and n. It turns out that the results obtained for the type [11, 2] follow from those of the type [111, 1] if we set $\mu + \tilde{p} - 2n\alpha = 2q$. Finally, for the weak energy condition we obtained

$$\mu - \tilde{p} + 2n\alpha + \Delta \geqslant 0 \tag{2.23}$$

$$\mu + \tilde{p} + 4n\alpha + \Delta - 2n(\sigma^2 - 3\alpha^2)^{1/2} \geqslant 0. \tag{2.24}$$

† This can be proved as follows. Because u^α is radial we can always introduce spherical and comoving coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, so that $u^\alpha \sim \delta^\alpha_0$. In this coordinate system, spherical symmetry and the Einstein field equations yield $T_{02} = T_{03} = T_{12} = T_{13} = 0$. From the equations $T_{02} = T_{03} = \sigma_{02} = \sigma_{03} = 0$ and (1.1) we take $q_2 = q_3 = 0$, i.e. q^α is radial and thus $q^\alpha \sim \delta^\alpha_1$. This and the equations $T_{12} = T_{13} = 0$ yield $\sigma_{12} = \sigma_{13} = 0$. In addition, as $\sigma_{10} = 0$, it follows that the vector $q^\alpha \sigma^\beta_\alpha$ is collinear with q^β.

For the dominant energy condition, in addition to (2.24) these inequalities must be satisfied:

$$\mu-\tilde{p}+2n\alpha\geqslant 0 \tag{2.25}$$

$$\mu-3\tilde{p}+\Delta-2n(\sigma^2-3\alpha^2)^{1/2}\geqslant 0. \tag{2.26}$$

For the strong energy condition, in addition to (2.24) we must have

$$2(\tilde{p}+n\alpha)+\Delta\geqslant 0. \tag{2.27}$$

The above results are recapitulated in the following theorem.

Theorem. A fluid with non-zero heat flow restricted by (2.14) fulfils the weak energy condition if and only if (2.21)-(2.24) are satisfied. The dominant energy condition is fulfilled if and only if, in addition to (2.21), (2.22) and (2.24) are satisfied and also the inequalities (2.25) and (2.26). Finally, the strong energy condition is satisfied if and only if, in addition to (2.21), (2.22) and (2.24) are satisfied and also the inequality (2.27).

It must be noted that a necessary condition for the fulfillment of the weak energy condition is that the energy density in the fluid rest frame is positive: $\mu\geqslant 0$. In fact, one can easily see that this follows from (2.22) and (2.23). From this theorem by setting $n=0$ or $\sigma_{\mu\nu}=0$ we obtain as a corollary the energy conditions for a fluid with vanishing coefficient of dynamic viscosity or the energy conditions for a fluid with shear-free motion. As we pointed out in § 1, the dominant energy condition in this case is given in a paper by Hall and Negm [4]. However, we believe that it is useful to present this corollary here because it gives separately the three energy conditions and because we will use it in the next section.

Corollary 1. A fluid which undergoes shear-free motion or has a vanishing coefficient of dynamic viscosity satisfies the weak energy condition if and only if

$$\mu+\tilde{p}\geqslant 2q \tag{2.28}$$

$$\mu-\tilde{p}+\Delta>0. \tag{2.29}$$

The dominant energy condition is satisfied if and only if, in addition to (2.28), we have

$$\mu-p\geqslant 0 \tag{2.30}$$

$$\mu-3\tilde{p}+\Delta\geqslant 0. \tag{2.31}$$

Finally, the strong energy condition is satisfied if and only if, in addition to (2.28), we have

$$2\tilde{p}+\Delta\geqslant 0. \tag{2.32}$$

We shall close this section with the investigation of a viscous fluid with $n\neq 0$ and vanishing heat flux, $q=0$. In this case we have the obvious eigenvalue

$$\lambda_0=-\mu. \tag{2.33}$$

To this eigenvalue corresponds the timelike eigenvector u^α and the Segre type of $T_{\alpha\beta}$ is [111, 1]. The other three eigenvalues are determined by the equation

$$x^3-n^2\sigma^2x-n^3\gamma=0. \tag{2.34}$$

The eigenvalues of the symmetric tensor $\sigma_{\alpha\beta}$ in a Euclidean space with signature $(+,+,+)$ are necessarily real. It follows that the roots of the equation (2.34) are real. Thus, the inequality $(\sigma^2/3)^3 \geqslant (\gamma/2)^2$ is identically satisfied. If $\gamma \neq 0$, this inequality permits us to put

$$y = -(\gamma/|\gamma|)(n|\sigma|/\sqrt{3}) \tag{2.35}$$

$$\cos\omega = \tfrac{3}{2}\sqrt{3}|\gamma|/|\sigma|^3 \qquad (0° \leqslant \omega \leqslant 90°) \tag{2.36}$$

and then the remaining eigenvalues are given by

$$\lambda_1 = \tilde{p} + 2y\cos\tfrac{1}{3}\omega \tag{2.37}$$

$$\lambda_2 = \tilde{p} - 2y\cos(60° - \tfrac{1}{3}\omega) \tag{2.38}$$

$$\lambda_3 = \tilde{p} - 2y\cos(60° + \tfrac{1}{3}\omega). \tag{2.39}$$

By substitution of (2.37)-(2.39) into (1.6)-(1.8) we obtain the restrictions that the energy conditions impose on μ, $\tilde{p}$, n and $\sigma_{\mu\nu}$. Thus, we see that the weak energy condition is fulfilled if and only if

$$\mu \geqslant 0 \tag{2.40}$$

$$\mu + \tilde{p} + 2y\cos\tfrac{1}{3}\omega \geqslant 0 \tag{2.41}$$

$$\mu + \tilde{p} - 2y\cos(60° - \tfrac{1}{3}\omega) \geqslant 0. \tag{2.42}$$

The inequality $-\lambda_0 + \lambda_3 \geqslant 0$ is omitted here because it is proved to be a consequence of (2.41) and (2.42). The dominant energy condition is satisfied if and only if, in addition to (2.40)-(2.42), we have

$$\mu - \tilde{p} - 2y\cos\tfrac{1}{3}\omega \geqslant 0 \tag{2.43}$$

$$\mu - \tilde{p} + 2y\cos(60° - \tfrac{1}{3}\omega) \geqslant 0. \tag{2.44}$$

Again, the inequality corresponding to λ_3 is a consequence of (2.43) and (2.44) and it is omitted. Finally, the strong energy condition is fulfilled if and only if, in addition to (2.41) and (2.42), we have

$$\mu + 3\tilde{p} \geqslant 0. \tag{2.45}$$

From (2.41) and (2.42) we observe that a necessary condition for the fulfillment of the weak energy condition is

$$\mu + \tilde{p} \geqslant 0 \tag{2.46}$$

where the inequality $-\lambda_0 + \lambda_3 \geqslant 0$ is also used. Similarly, from (2.43) and (2.44) and $-\lambda_0 - \lambda_i \geqslant 0$ we notice that a necessary condition for the fulfillment of the dominant energy condition is

$$\mu \geqslant \tilde{p}. \tag{2.47}$$

It must be noted that, according to the sign of γ, the system of the inequalities (2.41) and (2.42) collapses to only one inequality. So, if $\gamma > 0$, it is equivalent to (2.41) while if $\gamma < 0$ it is equivalent to (2.42). Similarly, the system (2.43) and (2.44) is equivalent to (2.44) if $\gamma > 0$ and to (2.43) if $\gamma < 0$. These results are recapitulated in the following theorem.

Theorem. A fluid with vanishing heat flow and $\gamma \neq 0$ fulfils the weak energy condition if and only if, in addition to (2.40), the inequality (2.41) is satisfied if $\gamma > 0$, or the inequality (2.42) if $\gamma < 0$. The dominant energy condition is fulfilled if and only if, in addition to the above conditions, the inequality (2.44) is satisfied if $\gamma > 0$, or the inequality (2.43) if $\gamma < 0$. Finally, the strong energy condition is fulfilled if and only if, in addition to (2.45), the inequality (2.41) is satisfied if $\gamma > 0$, or the inequality (2.42) if $\gamma < 0$.

Except perhaps for some very particular cases, the investigation of the validity of (2.41)-(2.44) needs the use of numerical calculations. However, the following corollary could be proved very useful in practice. Its proof is very simple and for this reason it is omitted.

Corollary 2. A sufficient condition for a fluid with vanishing heat flow and $\gamma \neq 0$ to fulfil the weak energy condition is that it satisfies (2.40) and

$$\mu + \tilde{p} \geqslant \frac{2}{\sqrt{3}} n|\sigma|. \qquad (2.48)$$

A sufficient condition for the validity of the dominant energy condition is that the fluid satisfies

$$\mu \geqslant \frac{2}{\sqrt{3}} n|\sigma| + |\tilde{p}|. \qquad (2.49)$$

The strong energy condition is fulfilled if (2.48) and (2.45) are satisfied.

To close the investigation of the fluids with vanishing heat flow it remains to consider the case

$$\gamma = 0. \qquad (2.50)$$

Now (2.34) is resolved to yield the eigenvalues

$$\lambda_1 = \tilde{p} \qquad (2.51)$$

$$\lambda_2 = \tilde{p} + n\sigma \qquad (2.52)$$

$$\lambda_3 = \tilde{p} - n\sigma. \qquad (2.53)$$

The energy conditions are easily obtained by substitution into (1.6)-(1.8). The results are recapitulated in the next theorem.

Theorem. A viscous fluid with vanishing heat flow and $\gamma = 0$ fulfils the weak energy condition if and only if

$$\mu \geqslant 0 \qquad (2.54)$$

$$\mu + \tilde{p} \geqslant n|\sigma|. \qquad (2.55)$$

It fulfils the dominant energy condition if and only if

$$\mu \geqslant \tilde{p} + n|\sigma|. \qquad (2.56)$$

Finally, the strong energy condition is fulfilled if and only if, in addition to (2.55), we have

$$\mu + 3\tilde{p} \geqslant 0. \qquad (2.57)$$

3. Energy conditions for a collapsing radiating sphere

Let us consider in the context of general relativity theory the energy conditions for a collapsing (or expanding) sphere consisting of a non-viscous fluid with energy-momentum tensor:

$$T_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + pg_{\alpha\beta} + u_\alpha q_\beta + u_\beta q_\alpha. \tag{3.1}$$

Because of the spherical symmetry the vector q^α points in the radial direction and according to (1.5b) the heat per unit time crossing a unit spherical surface is

$$q_\alpha n^\alpha = (q_\alpha q^\alpha)^{1/2} \tag{3.2}$$

where n^α is a unit radial vector. In order that the collapse (or the expansion) is physically plausible we expect that the energy density in the rest frame of the fluid μ and the pressure p decrease outward:

$$\mu' < 0 \tag{3.3}$$

$$p' < 0 \tag{3.4}$$

where the prime denotes differentiation in the radial direction. It seems that there exist no general physical reasons which imply a definite sign for the gradient of the heat flow. However, as far as the heat is transmitted by convection and because the mobility of the fluid is expected to be greater in the outer regions of the sphere, the heat flow should increase outward. So we will make the hypothesis

$$(q_\alpha q^\alpha)' > 0 \tag{3.5}$$

and will prove that it has the merit to imply, together with (3.3), (3.4) and $\mu \geq 3p$, the validity of the energy conditions. For such a collapsing (or expanding) model, energy must be dissipated by the fluid to the exterior. It follows that the exterior metric matched with the interior on a spherical hypersurface Σ cannot be that of Schwarzschild. If one wishes to make an exact treatment of the field equations it seems that the best choice is to consider as exterior the spherically symmetric outgoing Vaidya metric [6]. Collapsing star models based on the Vaidya metric have been investigated by many authors [7] (these references are far from exhaustive). A model based on the Vaidya metric and on a non-viscous fluid with energy-momentum tensor (3.1) was proposed first by Glass [8] and investigated in more detail by Santos [9]. One interesting consequence of the matching conditions for this model is that the pressure of the fluid interior on Σ is given by

$$p_\Sigma = (q_\alpha q^\alpha)_\Sigma^{1/2} > 0 \tag{3.6}$$

where the suffix Σ means that the quantities are to be calculated on Σ. It must be noted that for an interior solution such that $p_\Sigma < 0$, the matching conditions yield an unphysical outgoing Vaidya metric in which the mass is an increasing function of the retarded time. It must be noted also that the case of a fluid with non-vanishing coefficient of bulk viscosity can be obtained if, in equations (3.1), (3.4) and (3.6), we substitute p by $\tilde{p}$. Nevertheless, we adopt here the case $\zeta = 0$. So, for the investigation of the energy conditions we will apply corollary 1 of the preceding section with the tacit assumption $p = \tilde{p}$.

From (3.4) and (3.6) it follows that $p>0$ and so (2.32) is trivially satisfied. Thus, in this model the strong energy condition is satisfied if the weak energy condition is satisfied. Now, by virtue of (2.30) and (3.3)-(3.6) we have

$$\mu+p>(\mu+p)_\Sigma>2p_\Sigma \tag{3.7}$$

$$p_\Sigma>(q_\alpha q^\alpha)_\Sigma^{1/2}. \tag{3.8}$$

It follows that (2.28) is always fulfilled when (2.30) and (3.3)-(3.5) are satisfied. As far as conditions (2.30) and (2.31) are concerned we notice that both are satisfied if $\mu\geqslant 3p$. If, on the other hand, $\mu<3p$ then the inequality (2.31) is equivalent to

$$2p(\mu-p)\geqslant q_\alpha q^\alpha. \tag{3.9}$$

To recapitulate we state this theorem.

Theorem. We consider a general relativistic star model consisting of a spherically symmetric shear-free fluid whose energy-momentum tensor is given by (3.1) and which radiates energy to infinity in the form of a null dust which is described by the outgoing Vaidya metric. Then the energy conditions are fulfilled if the energy density in the rest frame of the fluid, the pressure and the heat flow satisfy the conditions (3.3)-(3.5) and $\mu\geqslant 3p$. If, however, $\mu<3p$ then, in addition to (3.3)-(3.5), the relations (2.30) and (3.9) must also be satisfied.

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