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Energy conditions in $f(\mathcal{G},T)$ gravity

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Abstract The aim of this paper is to introduce a new modified gravity theory named $f(\mathcal{G},T)$ gravity ($\mathcal{G}$ and T are the Gauss–Bonnet invariant and trace of the energy-momentum tensor, respectively) and investigate energy conditions for two reconstructed models in the context of FRW universe. We formulate general field equations, divergence of energy-momentum tensor, equation of motion for test particles as well as corresponding energy conditions. The massive test particles follow non-geodesic lines of geometry due to the presence of an extra force. We express the energy conditions in terms of cosmological parameters like the deceleration, jerk, and snap parameters. The reconstruction technique is applied to this theory using de Sitter and power-law cosmological solutions. We analyze the energy bounds and obtain feasible constraints on the free parameters.

1 Introduction

Current cosmic accelerated expansion has been affirmed from a diverse set of observational data coming from several pieces of astronomical evidence, including supernova type Ia, large scale structure, cosmic microwave background radiation etc. [1–4]. This expanding paradigm is considered as a consequence of mysterious force dubbed dark energy (DE), which possesses a large negative pressure. Modified theories of gravity are considered as the favorite candidates to unveil the enigmatic nature of this energy. These modified theories are usually developed by including scalar invariants and their corresponding generic functions in the Einstein–Hilbert action.

A remarkably interesting gravity theory is the modified Gauss–Bonnet (GB) theory. A linear combination of the form

$$\mathcal{G}=R_{\alpha\beta\xi\eta}R^{\alpha\beta\xi\eta}-4R_{\alpha\beta}R^{\alpha\beta}+R^{2},$$

where $R_{\alpha\beta\xi\eta}$, $R_{\alpha\beta}$ and R represent the Riemann tensor, the Ricci tensor, and the Ricci scalar, respectively, is called a Gauss–Bonnet invariant ($\mathcal{G}$). It is a second order Lovelock scalar invariant and thus free from spin-2 ghosts instabilities [5–7]. The Gauss–Bonnet combination is a four-dimensional topological invariant which does not involve the field equations. However, it provides interesting results in the same dimensions when either coupled with a scalar field or when an arbitrary function $f(\mathcal{G})$ is added to the Einstein–Hilbert action [8–10]. The latter approach is introduced by Nojiri and Odintsov; it is known as the $f(\mathcal{G})$ theory of gravity [11]. Like other modified theories, this theory is an alternative to study DE and is consistent with solar system constraints [12]. In this context, there is a possibility to discuss a transition from decelerated to accelerated as well as from non-phantom to phantom phases and also to explain the unification of early and late times accelerated expansion of the universe [13,14].

The fascinating problem of cosmic accelerated expansion has successfully been discussed by taking into account modified theories of gravity with curvature–matter coupling. The motion of test particles is studied in $f(R)$ and $f(\mathcal{G})$ gravity theories non-minimally coupled with the matter Lagrangian density ($\mathcal{L}_m$). Consequently, the extra force experienced by test particles is found to be orthogonal to their four velocities and the motion becomes non-geodesic [15–17]. It is found that, for certain choices of $\mathcal{L}_m$, the presence of the extra force vanishes in a non-minimal $f(R)$ model, while it remains preserved in a non-minimal $f(\mathcal{G})$ model. The geodesic deviation is weaker in $f(\mathcal{G})$ gravity for small curvatures as compared to non-minimal $f(R)$ gravity. Nojiri et al. [18] studied the non-minimally coupling of $f(R)$ and $f(\mathcal{G})$ theories with $\mathcal{L}_m$ and found that such a coupling naturally unifies the inflationary era with current cosmic accelerated expansion.

In order to describe some realistic matter distribution, certain conditions must be imposed on the energy-momentum tensor ($T_{\alpha\beta}$) known as energy conditions. These conditions originate from the Raychaudhuri equations with the requirement that not only gravity is attractive but also the energy

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density is positive. The null (NEC), weak (WEC), dominant (DEC), and strong (SEC) energy conditions are the four fundamental conditions. They play a key role to study the theorems related to singularity and black hole thermodynamics. The null energy condition is important to discuss the second law of black hole thermodynamics while its violation leads to a Big-Rip singularity of the universe [19]. The proof of the positive mass theorem is based on DEC [20], while SEC is useful to study the Hawking–Penrose singularity theorem [21].

The energy conditions have been investigated in different modified theories of gravity like $f(R)$ gravity, Brans–Dicke theory, $f(\mathcal{G})$ gravity, and generalized teleparallel theory [22–25]. Banijamali et al. [26] investigated the energy conditions for non-minimally coupling $f(\mathcal{G})$ theory with $\mathcal{L}_m$ and found that the WEC is satisfied for specific viable $f(\mathcal{G})$ models. Sharif and Waheed [27] explored the energy bounds in the context of generalized second order scalar-tensor gravity with the help of a power-law ansatz for the scalar field. Sharif and Zubair [28] derived these conditions in $f(R, T, R_{\alpha\beta}T^{\alpha\beta})$ theory of gravity for two specific models and also examined the Dolgov–Kowasaki instability for particular models of $f(R, T)$ gravity.

In this paper, we introduced a new modified theory of gravity named $f(\mathcal{G}, T)$ gravity, in which the gravitational Lagrangian is obtained by adding a generic function $f(\mathcal{G}, T)$ in the Einstein–Hilbert action. We study the energy conditions for the reconstructed $f(\mathcal{G}, T)$ models using an isotropic homogeneous universe model. The paper has the following format. In Sect. 2, we formulate the field equations of this gravity and discuss the equation of motion for test particles, while general expressions for the energy conditions as well as formulations in terms of cosmological parameters are discussed in Sect. 3. The reconstruction of models and their energy bounds is analyzed in Sect. 4. In the last section, we summarize our results.

2 Field equations of $f(\mathcal{G}, T)$ gravity

In this section, we formulate the field equations for $f(\mathcal{G}, T)$ gravity. For this purpose, we assume an action of the following form:

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[R + f(\mathcal{G}, T)] + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{1}$$

where g and κ represent the determinant of the metric tensor ($g_{\alpha\beta}$) and the coupling constant, respectively. The energy-momentum tensor is defined as [29]

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\alpha\beta}}. \tag{2}$$

Assuming that the matter distribution depends on the components of $g_{\alpha\beta}$ but has no dependence on its derivatives, we obtain

$$T_{\alpha\beta} = g_{\alpha\beta}\mathcal{L}_m - 2\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}}. \tag{3}$$

The variation in the action (1) gives

$$0 = \delta\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x[(R + f(\mathcal{G}, T))\delta\sqrt{-g} + \sqrt{-g}(\delta R + f_{\mathcal{G}} \times(\mathcal{G}, T)\delta\mathcal{G} + f_T(\mathcal{G}, T)\delta T)] + \int d^4x\delta(\sqrt{-g}\mathcal{L}_m), \tag{4}$$

where $f_{\mathcal{G}}(\mathcal{G}, T) = \frac{\partial f(\mathcal{G},T)}{\partial\mathcal{G}}$ and $f_T(\mathcal{G}, T) = \frac{\partial f(\mathcal{G},T)}{\partial T}$. The variations of $\sqrt{-g}$, $R^{\xi}_{\alpha\beta\eta}$, $R_{\alpha\eta}$, and R provide the following expressions:

$$\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta},\\
\delta R^{\xi}_{\alpha\beta\eta} &= \nabla_\beta(\delta\Gamma^{\xi}_{\eta\alpha}) - \nabla_\eta(\delta\Gamma^{\xi}_{\beta\alpha}),\\
&= (g_{\alpha\lambda}\nabla_{[\eta}\nabla_{\beta]} + g_{\lambda[\beta}\nabla_{\eta]}\nabla_\alpha)\delta g^{\xi\lambda} + \nabla_{[\eta}\nabla^{\xi}\delta g_{\beta]\alpha},\\
\delta R_{\alpha\eta} &= \delta R^{\xi}_{\alpha\xi\eta}, \quad \delta R = (R_{\alpha\beta} + g_{\alpha\beta}\nabla^2 - \nabla_\alpha\nabla_\beta)\delta g^{\alpha\beta},
\end{aligned} \tag{5}$$

where $\Gamma^{\xi}_{\alpha\beta}$ and ∇_α represent the Christoffel symbol and covariant derivative, respectively. The variations of $\mathcal{G}$ and T yield

$$\begin{aligned}
&\delta\mathcal{G} = 2R\delta R - 4\delta(R_{\alpha\beta}R^{\alpha\beta}) + \delta(R_{\alpha\beta\xi\eta}R^{\alpha\beta\xi\eta}),\\
&\delta T = (T_{\alpha\beta} + \Theta_{\alpha\beta})\delta g^{\alpha\beta}, \quad \Theta_{\alpha\beta} = g^{\xi\eta}\frac{\delta T_{\xi\eta}}{\delta g_{\alpha\beta}}.
\end{aligned} \tag{6}$$

Using these variational relations in Eq. (4), we obtain the field equations of $f(\mathcal{G}, T)$ gravity after simplification as follows:

$$\begin{aligned}
G_{\alpha\beta} = {}& \kappa^2 T_{\alpha\beta} - (T_{\alpha\beta} + \Theta_{\alpha\beta})f_T(\mathcal{G}, T) + \frac{1}{2}g_{\alpha\beta}f(\mathcal{G}, T)\\
& - (2RR_{\alpha\beta} - 4R^{\xi}_{\alpha}R_{\xi\beta} - 4R_{\alpha\xi\beta\eta}R^{\xi\eta}\\
& + 2R^{\xi\eta\delta}_{\alpha}R_{\beta\xi\eta\delta})f_{\mathcal{G}}(\mathcal{G}, T) - (2Rg_{\alpha\beta}\nabla^2\\
& - 2R\nabla_\alpha\nabla_\beta - 4g_{\alpha\beta}R^{\xi\eta}\nabla_\xi\nabla_\eta - 4R_{\alpha\beta}\nabla^2 + 4R^{\xi}_{\alpha}\nabla_\beta\nabla_\xi\\
& + 4R^{\xi}_{\beta}\nabla_\alpha\nabla_\xi + 4R_{\alpha\xi\beta\eta}\nabla^{\xi}\nabla^{\eta})f_{\mathcal{G}}(\mathcal{G}, T),
\end{aligned} \tag{7}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ and $\nabla^2 = \Box = \nabla_\alpha\nabla^\alpha$ denote the Einstein tensor and the d'Alembert operator, respectively. It is worth mentioning here that, for $f(\mathcal{G}, T) = f(\mathcal{G})$, Eq. (7) reduces to the field equations for $f(\mathcal{G})$ gravity, while $\Lambda(T)$ gravity (Λ is the cosmological constant) is obtained in the absence of the quadratic invariant $\mathcal{G}$ [11,30]. Furthermore, the Einstein field equations are recovered when $f(\mathcal{G}, T) = 0$. The trace of Eq. (7) is given by

$$\begin{aligned}
& R + \kappa^2 T - (T + \Theta)f_T(\mathcal{G}, T) + 2f(\mathcal{G}, T)\\
& + 2\mathcal{G}f_{\mathcal{G}}(\mathcal{G}, T) - 2R\nabla^2 f_{\mathcal{G}}(\mathcal{G}, T)\\
& + 4R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{\mathcal{G}}(\mathcal{G}, T) = 0,
\end{aligned}$$

where $\Theta = \Theta^{\alpha}_{\alpha}$. In this theory, the covariant divergence of Eq. (7) is non-zero, given by

$$\nabla^{\alpha} T_{\alpha\beta} = \frac{f_T(\mathcal{G},T)}{\kappa^2 - f_T(\mathcal{G},T)}\Big[(T_{\alpha\beta}+\Theta_{\alpha\beta})\nabla^{\alpha}(\ln f_T(\mathcal{G},T)) + \nabla^{\alpha}\Theta_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\nabla^{\alpha}T\Big]. \quad (8)$$

To obtain a useful expression for $\Theta_{\alpha\beta}$, we differentiate Eq. (3) with respect to the metric tensor

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\xi\eta}} = \frac{\delta g_{\alpha\beta}}{\delta g^{\xi\eta}}\mathcal{L}_m + g_{\alpha\beta}\frac{\partial \mathcal{L}_m}{\partial g^{\xi\eta}} - 2\frac{\partial^2 \mathcal{L}_m}{\partial g^{\xi\eta}\partial g^{\alpha\beta}}. \quad (9)$$

Using the relations

$$\frac{\delta g_{\alpha\beta}}{\delta g^{\xi\eta}} = -g_{\alpha\mu}g_{\beta\nu}\delta^{\mu\nu}_{\xi\eta}, \quad \delta^{\mu\nu}_{\xi\eta} = \frac{\delta g^{\mu\nu}}{\delta g^{\xi\eta}},$$

where $\delta^{\mu\nu}_{\xi\eta}$ is the generalized Kronecker symbol and putting Eq. (9) into (6), we obtain

$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_m - 2g^{\xi\eta}\frac{\partial^2 \mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\xi\eta}}. \quad (10)$$

This shows that once the value of $\mathcal{L}_m$ is determined, we can find the expression for the tensor $\Theta_{\alpha\beta}$.

We consider the matter distribution as a perfect fluid given by

$$T_{\alpha\beta} = (\rho + P)V_{\alpha}V_{\beta} - Pg_{\alpha\beta}, \quad (11)$$

where ρ, P and V_{α} are the density, pressure, and four velocity of the fluid, respectively. The four velocity satisfies the relation $V_{\alpha}V^{\alpha} = 1$ and the corresponding Lagrangian density can be taken as $\mathcal{L}_m = -P$ [31]. Thus Eq. (10) yields

$$\Theta_{\alpha\beta} = -2T_{\alpha\beta} - Pg_{\alpha\beta}. \quad (12)$$

Equation (7) can be written in a form identical to the Einstein field equations as

$$G_{\alpha\beta} = \kappa^2 T^{(\mathrm{eff})}_{\alpha\beta} = \kappa^2(T_{\alpha\beta} + T^{\mathcal{G}T}_{\alpha\beta}), \quad (13)$$

where $T^{\mathcal{G}T}_{\alpha\beta}$ is the $f(\mathcal{G},T)$ contribution. For the case of a perfect fluid, the expression for $T^{\mathcal{G}T}_{\alpha\beta}$ is given by

$$\begin{aligned} T^{\mathcal{G}T}_{\alpha\beta} = \frac{1}{\kappa^2}\Big[&(\rho+P)V_{\alpha}V_{\beta}f_T(\mathcal{G},T) + \frac{1}{2}g_{\alpha\beta}f(\mathcal{G},T) \\ &-(2RR_{\alpha\beta} - 4R^{\xi}_{\alpha}R_{\xi\beta} - 4R_{\alpha\xi\beta\eta}R^{\xi\eta} + 2R^{\xi\eta\delta}_{\alpha}R_{\beta\xi\eta\delta}) \\ &\times f_{\mathcal{G}}(\mathcal{G},T) - (2Rg_{\alpha\beta}\nabla^2 - 2R\nabla_{\alpha}\nabla_{\beta} \\ &-4g_{\alpha\beta}R^{\xi\eta}\nabla_{\xi}\nabla_{\eta} - 4R_{\alpha\beta}\nabla^2 + 4R^{\xi}_{\alpha}\nabla_{\beta}\nabla_{\xi} + 4R^{\xi}_{\beta}\nabla_{\alpha}\nabla_{\xi} \\ &+4R_{\alpha\xi\beta\eta}\nabla^{\xi}\nabla^{\eta})f_{\mathcal{G}}(\mathcal{G},T)\Big]. \end{aligned} \quad (14)$$

The line element for FRW universe model is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \quad (15)$$

where $a(t)$ represents the scale factor. The corresponding field equations are

$$3H^2 = \kappa^2\rho_{\mathrm{eff}}, \quad -(2\dot{H} + 3H^2) = \kappa^2 P_{\mathrm{eff}}, \quad (16)$$

where

$$\rho_{\mathrm{eff}} = \rho + \frac{1}{\kappa^2}\Big[(\rho+P)f_T(\mathcal{G},T) + \frac{1}{2}f(\mathcal{G},T) - 12H^2 \times (H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T) + 12H^3\partial_t f_{\mathcal{G}}(\mathcal{G},T)\Big], \quad (17)$$

$$P_{\mathrm{eff}} = P - \frac{1}{\kappa^2}\Big[\frac{1}{2}f(\mathcal{G},T) - 12H^2(H^2+\dot{H})f_{\mathcal{G}}(\mathcal{G},T) + 8H(H^2+\dot{H})\partial_t f_{\mathcal{G}}(\mathcal{G},T) + 4H^2\partial_{tt}f_{\mathcal{G}}(\mathcal{G},T)\Big], \quad (18)$$

$\mathcal{G} = 24H^2(H^2+\dot{H})$, $H = \dot{a}/a$ is the Hubble parameter and a dot represents the time derivative. The divergence of $T_{\alpha\beta}$ takes the form

$$\dot{\rho} + 3H(\rho+P) = \frac{-1}{\kappa^2 + f_T(\mathcal{G},T)} \times \Big[\Big(\dot{P} + \frac{1}{2}\dot{T}\Big)f_T(\mathcal{G},T) + (\rho+P)\partial_t f_T(\mathcal{G},T)\Big]. \quad (19)$$

To obtain a standard conservation equation,

$$\dot{\rho} + 3H(\rho+P) = 0, \quad (20)$$

we need an additional constraint by taking the right side of Eq. (19) equal to zero:

$$\Big(\dot{P} + \frac{1}{2}\dot{T}\Big)f_T(\mathcal{G},T) + (\rho+P)\partial_t f_T(\mathcal{G},T) = 0. \quad (21)$$

Now, we briefly discuss the motion of test particles in $f(\mathcal{G},T)$ gravity. For this purpose, using Eqs. (11) and (12) in (8), the divergence of the energy-momentum tensor for perfect fluid is given by

$$\begin{aligned} &\nabla_{\beta}(\rho+P)V^{\alpha}V^{\beta} + (\rho+P)[V^{\beta}\nabla_{\beta}V^{\alpha} + V^{\alpha}\nabla_{\beta}V^{\beta}] - g^{\alpha\beta}\nabla_{\beta}P \\ &= \frac{-2}{2\kappa^2 + 3f_T(\mathcal{G},T)}\left[T^{\alpha\beta}\nabla_{\beta}f_T(\mathcal{G},T) + g^{\alpha\beta}\nabla_{\beta}(Pf_T(\mathcal{G},T))\right]. \end{aligned}$$

The contraction of the above equation with the projection operator ($h_{\alpha\xi} = g_{\alpha\xi} - V_{\alpha\xi}$) gives the following expression:

$$g_{\alpha\xi}V^{\beta}\nabla_{\beta}V^{\alpha} = \frac{(2\kappa^2 + f_T(\mathcal{G},T))\nabla_{\beta}}{(\rho+P)(2\kappa^2 + 3f_T(\mathcal{G},T))}h^{\beta}_{\xi}, \quad (22)$$

where we have used the relations $V^{\alpha}\nabla_{\beta}V_{\alpha} = 0$, $h_{\alpha\xi}V^{\alpha} = 0$, and $h_{\alpha\xi}T^{\alpha\beta} = -Ph^{\beta}_{\xi}$. Multiplying Eq. (22) with $g^{\mu\xi}$ and using the following identity [31]:

$$V^{\beta}\nabla_{\beta}V^{\alpha} = \frac{\mathrm{d}^2x^{\alpha}}{\mathrm{d}s^2} + \Gamma^{\alpha}_{\beta\xi}V^{\beta}V^{\xi},$$

we obtain the equation of motion for massive test particles in this model of gravity as

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\xi} V^\beta V^\xi = \zeta^\alpha, \quad (23)$$

where

$$\zeta^\alpha = \frac{(2\kappa^2 + f_T(\mathcal{G}, T))}{(\rho + P)(2\kappa^2 + 3 f_T(\mathcal{G}, T))}(g^{\alpha\beta} - V^\alpha V^\beta)\nabla_\beta P \quad (24)$$

represents the extra force acting on the test particles and is perpendicular to the four velocity of the fluid ($\zeta^\alpha V_\alpha = 0$). For a pressureless fluid, Eq. (24) gives $\zeta^\alpha = 0$ and hence the dust particles follow the geodesic trajectories both in general relativity as well as in $f(\mathcal{G}, T)$ gravity. The equation of motion for a perfect fluid in general relativity is recovered in the absence of coupling between matter and geometry [32].

3 Energy conditions

The energy conditions are the coordinate invariant which incorporate the common characteristics shared by almost every matter field. The concept of energy conditions came from the Raychaudhuri equations which play a key role in any discussion of the congruence of null and timelike geodesics with the requirement that not only the gravity is attractive but also the energy density is positive. These equations describe the temporal evolution of the expansion scalar (θ) as follows [33]:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 + \omega_{\alpha\beta}\omega^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\alpha\beta}u^\alpha u^\beta, \quad (25)$$

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 + \omega_{\alpha\beta}\omega^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta, \quad (26)$$

where $\omega_{\alpha\beta}$, $\sigma_{\alpha\beta}$, u^α and k^α represent the rotation, shear tensor, timelike, and null tangent vectors in the congruences, respectively. For non-geodesic congruences, the temporal evolution of θ is affected by the presence of an acceleration term which arises due to a non-gravitational force like pressure gradient as [34,35]

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 + \omega_{\alpha\beta}\omega^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \nabla_\alpha(V^\beta\nabla_\beta V^\alpha) - R_{\alpha\beta}V^\alpha V^\beta. \quad (27)$$

Neglecting the quadratic terms due to rotation-free as well as small distortions described by $\sigma_{\alpha\beta}$, Eqs. (25) and (26) yield

$$\theta = -\tau R_{\alpha\beta}u^\alpha u^\beta, \quad \theta = -\tau R_{\alpha\beta}k^\alpha k^\beta.$$

Using the condition for gravity to be attractive, i.e., $\theta < 0$, we obtain $R_{\alpha\beta}u^\alpha u^\beta \geq 0$ and $R_{\alpha\beta}k^\alpha k^\beta \geq 0$. The equivalent form of these inequalities can be obtained by the inversion of the Einstein field equations as

$$\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)u^\alpha u^\beta \geq 0, \quad \left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)k^\alpha k^\beta \geq 0.$$

For a perfect fluid matter distribution, these inequalities provide the energy constraints defined by:

- NEC: $\rho + P \geq 0$,
- WEC: $\rho + P \geq 0$, $\rho \geq 0$,
- SEC: $\rho + P \geq 0$, $\rho + 3P \geq 0$,
- DEC: $\rho \pm P \geq 0$, $\rho \geq 0$.

These conditions show that the violation of the NEC leads to the violation of all other conditions. Due to the purely geometric nature of the Raychaudhuri equations, the concept of energy bounds in modified theories of gravity can be extended with the assumption that the total cosmic matter distribution acts like a perfect fluid. The energy conditions can be formulated by replacing ρ and P with ρ_{eff} and P_{eff}, respectively. The geodesic lines of geometry are followed by dust particles in $f(\mathcal{G}, T)$ gravity, therefore we consider a pressureless fluid to discuss the energy conditions. These conditions take the following form:

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G}, T) + 4H \times (H^2 - 2\dot{H})\partial_t f_\mathcal{G}(\mathcal{G}, T) - 4H^2\partial_{tt} f_\mathcal{G}(\mathcal{G}, T)\Big] \geq 0, \quad (28)$$

$$\textbf{WEC}: \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\Big[2\rho f_T(\mathcal{G}, T) + f(\mathcal{G}, T) - 24H^2 \times (H^2 + \dot{H}) f_\mathcal{G}(\mathcal{G}, T) + 24H^3\partial_t f_\mathcal{G}(\mathcal{G}, T)\Big] \geq 0, \quad (29)$$

$$\textbf{SEC}: \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho - \frac{1}{\kappa^2} \times \Big[f(\mathcal{G}, T) - \rho f_T(\mathcal{G}, T) - 24H^2(H^2 + \dot{H}) \times f_\mathcal{G}(\mathcal{G}, T) + 12H(H^2 + 2\dot{H})\partial_t f_\mathcal{G}(\mathcal{G}, T) + 12H^2\partial_{tt} f_\mathcal{G}(\mathcal{G}, T)\Big] \geq 0, \quad (30)$$

$$\textbf{DEC}: \quad \rho_{\text{eff}} - P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[\rho f_T(\mathcal{G}, T) + f(\mathcal{G}, T) - 24H^2(H^2 + \dot{H}) \times f_\mathcal{G}(\mathcal{G}, T) + 4H(5H^2 + 2\dot{H})\partial_t f_\mathcal{G}(\mathcal{G}, T) + 4H^2\partial_{tt} f_\mathcal{G}(\mathcal{G}, T)\Big] \geq 0. \quad (31)$$

The Hubble parameter, the Ricci scalar, the GB invariant, and their derivatives can be written in terms of cosmic parameters as

$$\dot{H} = -H^2(1 + q), \quad \ddot{H} = H^3(j + 3q + 2),$$
$$\dddot{H} = H^4(s - 4j - 3q^2 - 12q - 6), \quad (32)$$
$$R = -6H^2(1 - q), \quad \dot{R} = -6H^3(j - q - 2),$$

$$\ddot{R} = -6H^4(s + 8q + q^2 + 6), \tag{33}$$

$$\mathcal{G} = -24qH^4, \quad \dot{\mathcal{G}} = 24H^5(j + 3q + 2q^2),$$
$$\ddot{\mathcal{G}} = 24H^6(s - 6j - 6qj - 12q - 15q^2 - 2q^3), \tag{34}$$

where q, j, and s denote the deceleration, jerk, and snap parameters, respectively, and are defined as [36,37]

$$q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3}\frac{\dddot{a}}{a}, \quad s = \frac{1}{H^4}\frac{\ddddot{a}}{a}. \tag{35}$$

The energy conditions (28)–(31) in the form of the above parameters are

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[\rho f_T + 4H^3 \times(3 + 2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) - 4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0, \tag{36}$$

$$\textbf{WEC}: \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\Big[f + 2\rho f_T + 24qH^4 f_{\mathcal{G}} + 24H^3(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T})\Big] \geq 0, \tag{37}$$

$$\textbf{SEC}: \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Big[-f + \rho f_T - 24qH^4 f_{\mathcal{G}} + 12H^3(1 + 2q) \times(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) - 12H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0, \tag{38}$$

$$\textbf{DEC}: \quad \rho_{\text{eff}} - P_{\text{eff}} = \rho + \frac{1}{\kappa^2}[f + \rho f_T + 24qH^4 f_{\mathcal{G}} + 4H^3(3 - 2q)(f_{\mathcal{G}\mathcal{G}}\dot{\mathcal{G}} + f_{\mathcal{G}T}\dot{T}) + 4H^2(f_{\mathcal{G}\mathcal{G}\mathcal{G}}\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}\ddot{\mathcal{G}} + f_{\mathcal{G}T}\ddot{T})\Big] \geq 0. \tag{39}$$

4 Reconstruction of $f(\mathcal{G}, T)$ models

In this section, we use the reconstruction technique and discuss the energy conditions for de Sitter and power-law universe models.

4.1 de Sitter universe model

This cosmological model explains the exponential expansion of the universe with constant Hubble expansion rate. The scale factor is defined as [38]

$$a(t) = a_0 e^{H_0 t}, \quad H = H_0, \tag{40}$$

where a_0 is constant at t_0. The values of R and the GB invariant are

$$R = -12H_0^2, \quad \mathcal{G} = 24H_0^4. \tag{41}$$

For pressureless fluid, Eq. (20) gives the energy density of the form

$$\rho = \rho_0 e^{-3H_0 t}. \tag{42}$$

The trace of the energy-momentum tensor and its derivatives have the following expressions:

$$T = \rho, \quad \dot{T} = -3H_0 T, \quad \ddot{T} = 9H_0^2 T. \tag{43}$$

Using Eqs. (40)–(43) in Eq. (16), we obtain a partial differential equation

$$\kappa^2 T + \frac{1}{2}f(\mathcal{G}, T) - 12H_0^4 f_{\mathcal{G}}(\mathcal{G}, T) + Tf_T(\mathcal{G}, T) - 36H_0^4 T f_{\mathcal{G}T}(\mathcal{G}, T) - 3H_0^2 = 0, \tag{44}$$

whose solution is given by

$$f(\mathcal{G}, T) = c_1 c_2\big(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}\big) + \gamma_3 T + \gamma_4, \tag{45}$$

where the c_i are integration constants and

$$\gamma_1 = -\frac{1}{2}\left(\frac{1 - 24c_1H_0^4}{1 - 36c_1H_0^4}\right), \quad \gamma_2 = -\frac{1}{2},$$
$$\gamma_3 = -\frac{2}{3}\kappa^2, \quad \gamma_4 = 6H_0^2.$$

The additional constraint (21) becomes

$$c_1c_2\frac{(1 - 24c_1H_0^4)(1 - 30c_1H_0^4)}{(1 - 36c_1H_0^4)^2}e^{c_1\mathcal{G}}T^{\gamma_1} + c_1c_2T^{\gamma_2} + \gamma_3 T = 0.$$

This equation splits Eq. (45) into two $f(\mathcal{G}, T)$ functions with some additional constant relations between the coefficients. The reconstructed model (45) can be written as a combination of those functions. We analyze the energy conditions for the $f(\mathcal{G}, T)$ model given in Eq. (45) instead of analyzing them separately. Using model (45) in the energy conditions (28)–(31), it follows that

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\big[\rho\{c_1c_2(\gamma_1 e^{c_1\mathcal{G}}T^{(\gamma_1-1)} + \gamma_2 T^{(\gamma_2-1)}) + \gamma_3\} + 12c_1^2c_2\gamma_1H_0^4(1 - 3\gamma_1)e^{c_1\mathcal{G}}T^{\gamma_1}\big] \geq 0, \tag{46}$$

$$\textbf{WEC}: \quad \rho_{\text{eff}} = \rho + \frac{1}{2\kappa^2}\big[2\rho\{c_1c_2 \times(e^{c_1\mathcal{G}}\gamma_1T^{(\gamma_1-1)} + \gamma_2T^{(\gamma_2-1)}) + \gamma_3\} + \{c_1c_2(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}) + \gamma_3T + \gamma_4\} - 24c_1^2c_2H_0^4e^{c_1\mathcal{G}}T^{\gamma_1}(1 + 3\gamma_1)\big] \geq 0, \tag{47}$$

$$\textbf{SEC}: \quad \rho_{\text{eff}} + 3P_{\text{eff}} = \rho - \frac{1}{\kappa^2}\big[c_1c_2(e^{c_1\mathcal{G}}T^{\gamma_1} + T^{\gamma_2}) + \gamma_3T + \gamma_4 - \rho \times\big\{c_1c_2\big(\gamma_1e^{c_1\mathcal{G}}T^{(\gamma_1-1)} + \gamma_2T^{(\gamma_2-1)}\big) + \gamma_3\big\} - 12c_1^2c_2e^{c_1\mathcal{G}}H_0^4T^{\gamma_1} \times\{2 + 3\gamma_1 - 9\gamma_1^2\}\big] \geq 0, \tag{48}$$

DEC: $\rho_{\text{eff}} - P_{\text{eff}} = \rho$

$$+ \frac{1}{\kappa^2}[\rho\{c_1 c_2(e^{c_1 \mathcal{G}} \gamma_1 T^{(\gamma_1 - 1)} + \gamma_2 T^{(\gamma_2 - 1)}) + \gamma_3\} + \{c_1 c_2(e^{c_1 \mathcal{G}} T^{\gamma_1} + T^{\gamma_2}) + \gamma_3 T + \gamma_4\} - 12 c_1^2 c_2 H_0^4 e^{c_1 \mathcal{G}} T^{\gamma_1} \times \{2 + \gamma_1(5 - 3\gamma_1)\}] \geq 0. \tag{49}$$

Figures 1 and 2 show the variation of the NEC and WEC for the case $c_1 > 0$ and $c_2 < 0$ with $\kappa = 1$. We use the following values of the cosmological parameters: $H_0 = 0.718$, $q = -0.64$, $j = 1.02$ and $s = -0.39$ [39–41]. In these plots, we fix the constant c_1 for two arbitrarily chosen values, while c_2 varies from $[-10, 0]$. Figure 1 shows the positively increasing behavior of the NEC as well as WEC with respect to time in the considered interval of c_2. Figure 2 shows a similar behavior for $c_1 = 4$. In this case, both conditions are satisfied for all values of c_1 and c_2. The energy conditions for $(c_1, c_2) > 0$ are discussed in Figs. 3 and 4. The left plot of Fig. 3 shows that the NEC is satisfied for $t < 3$, $t < 2.28$ and $t = 2$ for $c_2 = 0.005$, 0.05 and 0.1, respectively. Figure 4 (left) shows a similar decreasing behavior of time as the value of c_2 increases for $c_1 = 0.01$. It is also observed that as the value of c_1 increases, the time interval for a valid NEC decreases, while the positivity of ρ_{eff} is shown in the right panel of both figures. For the case $(c_1, c_2) > 0$, both NEC and WEC are satisfied for small values of c_1 and c_2 in a very small time interval.

Figures 5 and 6 deal with the case $c_1 < 0$ and $c_2 > 0$. For arbitrarily chosen values of c_1, the increasing behavior of the NEC with respect to time is observed in the left panel of both figures for all values of c_2. The right plot of Fig. 5 shows the positivity of ρ_{eff} for $t < 34$, while it remains positive throughout the time interval for $c_1 = -0.001$ as shown in Fig. 6 (right panel). The last possibility, i.e., $c_1 < 0$ and $c_2 < 0$ is examined in Figs. 7 and 8. The left panels of both figures show the decreasing and increasing behavior of the NEC as the time and integration constant c_2 increase, respectively. The effective energy density exhibits a constant behavior for the assumed values of c_1 in the considered interval of c_2.

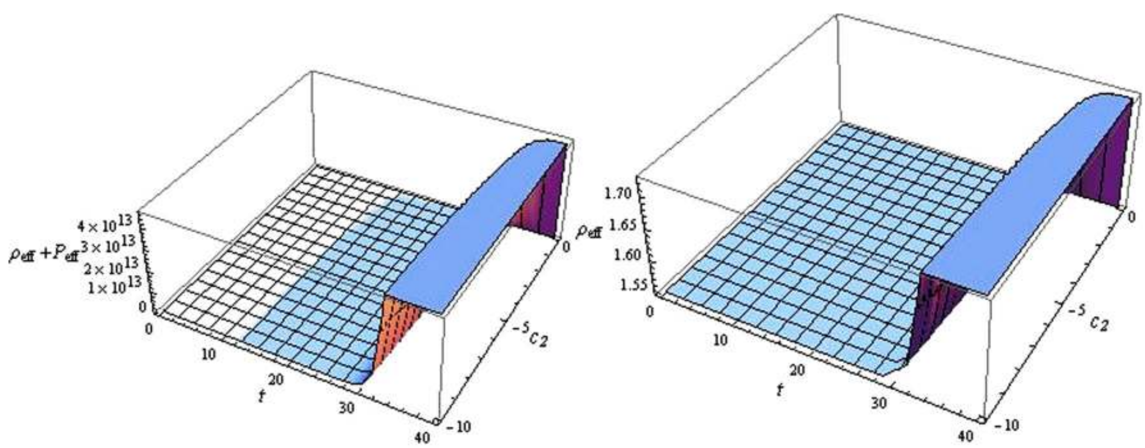

Fig. 1 Energy conditions for $c_1 = 0.001$

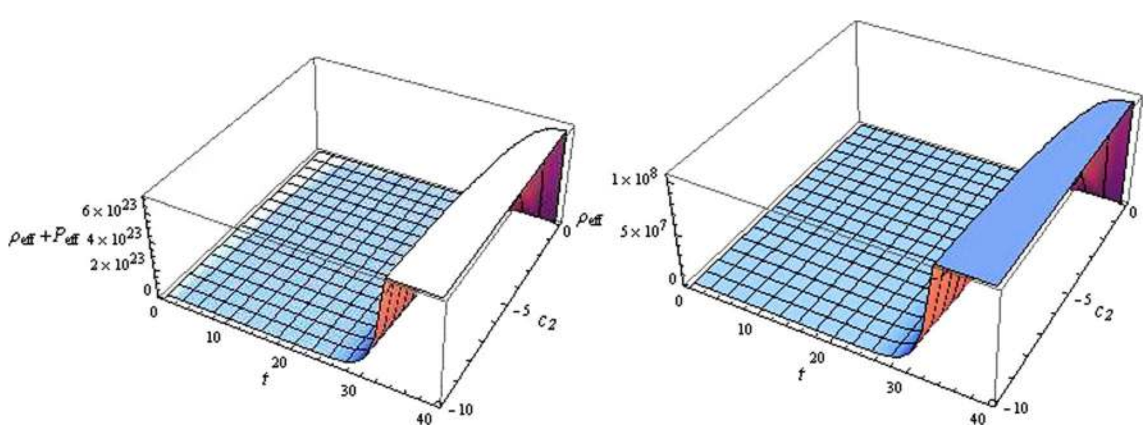

Fig. 2 Energy conditions for $c_1 = 4$

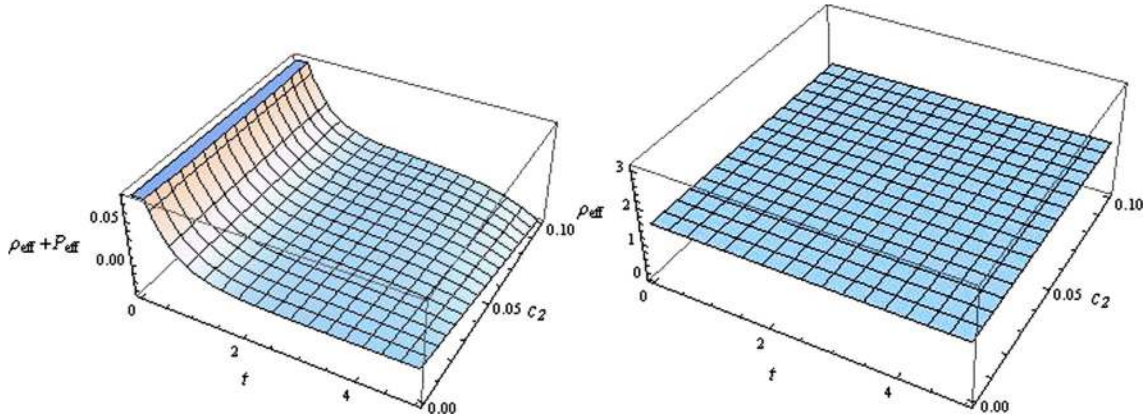

Fig. 3 Energy conditions for $c_1 = 0.001$

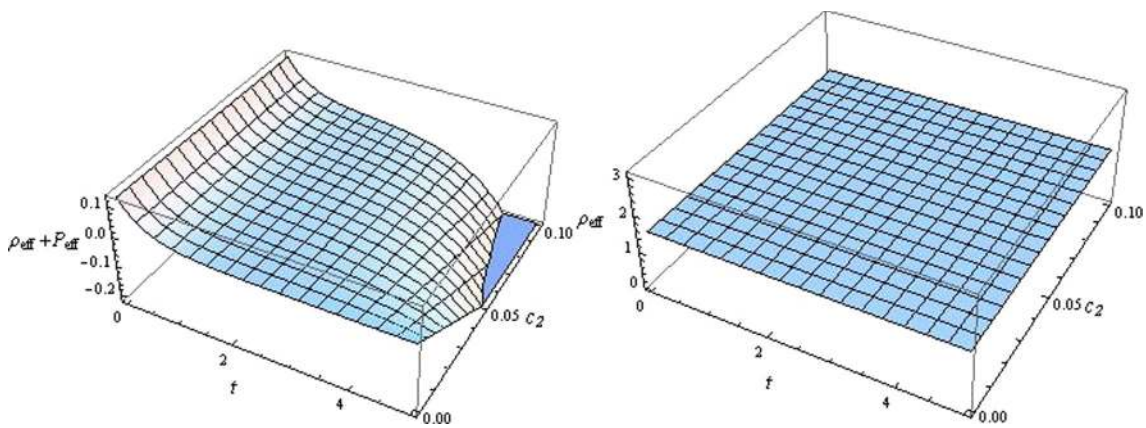

Fig. 4 Energy conditions for $c_1 = 0.01$

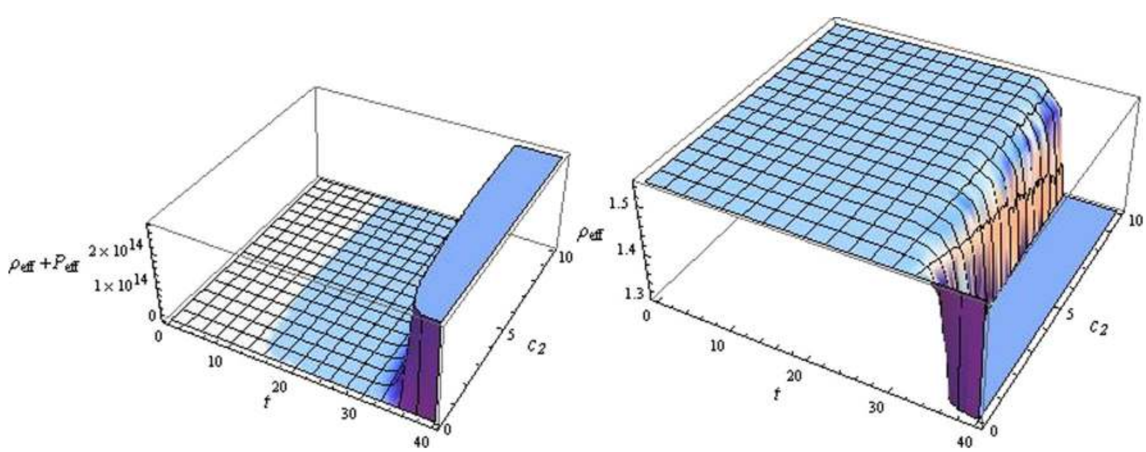

Fig. 5 Energy conditions for $c_1 = -0.01$

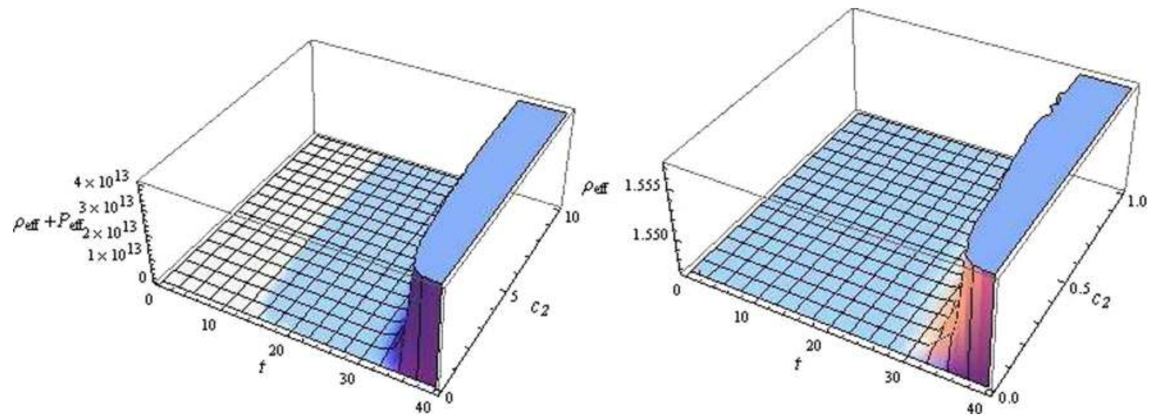

Fig. 6 Energy conditions for $c_1 = -0.001$

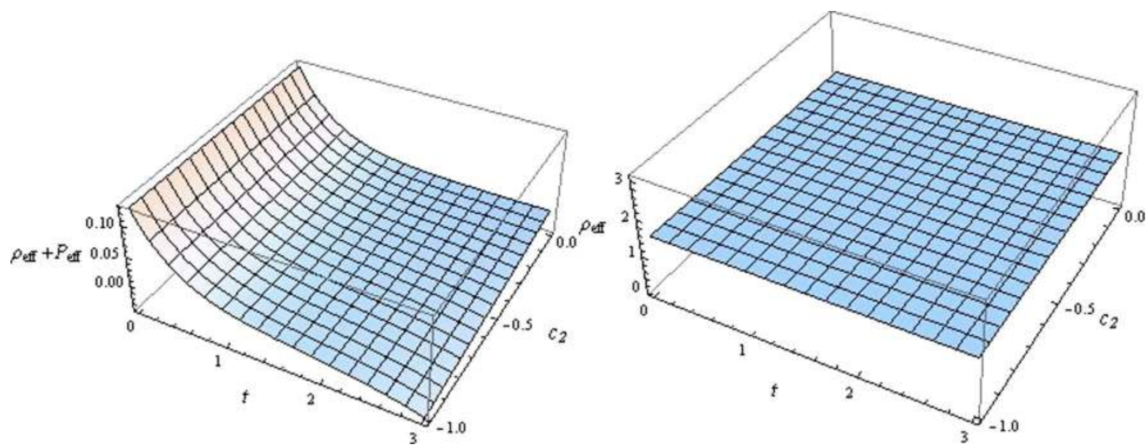

Fig. 7 Energy conditions for $c_1 = -0.001$

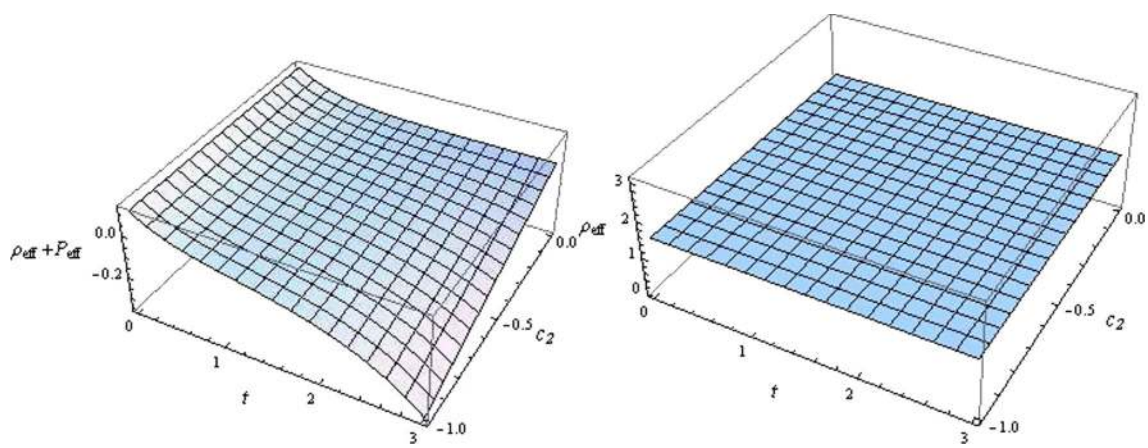

Fig. 8 Energy conditions for $c_1 = -0.01$

4.2 Power-law solution

The power-law solution is of great interest to discuss the cosmic evolution and its scale factor is defined as [38]

$$a(t) = a_0 t^n, \quad H = \frac{n}{t}, \tag{50}$$

where $n > 0$. For $0 < n < 1$, we have a decelerated universe, which leads to a radiation dominated era for $n = \frac{1}{2}$ and a dust dominated era for $n = \frac{2}{3}$, while a cosmic accelerated era is observed for $n > 1$. The Ricci scalar and GB invariant are

$$R = \frac{6n}{t^2}(1 - 2n), \quad \mathcal{G} = \frac{24n^3}{t^4}(n - 1). \tag{51}$$

The energy density for dust fluid is obtained from Eq. (20) as

$$\rho = \rho_0 t^{-3n}. \tag{52}$$

The trace of $T_{\alpha\beta}$ and its time derivatives take the form

$$T = \rho, \quad \dot{T} = -\frac{3n}{t}T, \quad \ddot{T} = \frac{3n}{t^2}(1+3n)T. \tag{53}$$

Inserting Eqs. (50)–(53) in the first field equation (16), we obtain

$$\kappa^2 T + \frac{1}{2}f(\mathcal{G},T) - \frac{1}{2}\mathcal{G}f_{\mathcal{G}}(\mathcal{G},T) + Tf_T(\mathcal{G},T) - \left(\frac{2}{n-1}\right)\mathcal{G}^2 f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T) - \left(\frac{3n}{2(n-1)}\right)\mathcal{G}Tf_{\mathcal{G}T}(\mathcal{G},T) - 3n^2\left(\frac{T}{\rho_0}\right)^{\frac{2}{3n}} = 0, \tag{54}$$

whose solution is given by

$$f(\mathcal{G},T) = d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} + \chi_3 T + d_1d_2T^{\chi_4} + \chi_5 T^{\chi_6}, \tag{55}$$

where d_i are constants of integration and

$$\chi_1 = \frac{1}{2}\left[n^2(1+3d_2(3d_2+2)) + 2d_2(n-16) + 3(2n+3)\right]^{\frac{1}{2}},$$
$$\chi_2 = \frac{1}{2}\left[5-n(1+3d_2)\right], \quad \chi_3 = -\frac{2}{3}\kappa^2, \quad \chi_4 = -\frac{1}{2},$$
$$\chi_5 = \left(\frac{18n^3}{2+3n}\right)\rho_0^{-\frac{2}{3n}}, \quad \chi_6 = \frac{2}{3n}.$$

In this case, Eq. (21) takes the form

$$d_1d_3T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)}\left[\frac{d_2}{6n}\{3n(2d_2-1)+2(\chi_1+\chi_2)\}\right] + d_2d_3T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} \times \left[\frac{d_2}{6n}\{3n(2d_2-1)-2(\chi_1-\chi_2)\}\right] + \chi_3 T + d_1d_2\chi_4^2T^{\chi_4} + \chi_5\chi_6^2T^{\chi_6} = 0.$$

Solving Eq. (55) with the above equation as in the previous section, we obtain two functions whose combination is equivalent to the reconstructed power-law $f(\mathcal{G},T)$ model.

Inserting the model (55) in the energy conditions (36)–(39), we obtain

$$\textbf{NEC}: \quad \rho_{\text{eff}} + P_{\text{eff}} = \rho + \frac{1}{\kappa^2}\Bigg[4H^3(3+2q)\Bigg(\Bigg[\frac{1}{4}d_1d_3 \times (\chi_1+\chi_2)\left[\frac{1}{4}(\chi_1+\chi_2)-1\right] \times T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2d_3(\chi_1-\chi_2) \times \left[\frac{1}{4}(\chi_1-\chi_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]\dot{\mathcal{G}} + \Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\dot{T}\Bigg)$$
$$-4H^2\Bigg(\Bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2) \times \left[\frac{1}{4}(\chi_1+\chi_2)-1\right]\left[\frac{1}{4}(\chi_1+\chi_2)-2\right]T^{d_2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3} - \frac{1}{4}d_2d_3(\chi_1-\chi_2)\left[\frac{1}{4}(\chi_1-\chi_2)+1\right] \times \left[\frac{1}{4}(\chi_1-\chi_2)+2\right] \times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\Bigg]\dot{\mathcal{G}}^2 + 2\Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2) \times \left[\frac{1}{4}(\chi_1+\chi_2)-1\right]T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2^2d_3(\chi_1-\chi_2)\left[\frac{1}{4}(\chi_1-\chi_2)+1\right] \times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]$$
$$\times \dot{\mathcal{G}}\dot{T} + \Bigg[\frac{1}{4}d_1d_2d_3(d_2-1)(\chi_1+\chi_2)T^{d_2-2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2d_3(d_2-1) \times (\chi_1-\chi_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\dot{T}^2 + \Bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2) \times \left[\frac{1}{4}(\chi_1+\chi_2)-1\right] \times T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2d_3(\chi_1-\chi_2) \times \left[\frac{1}{4}(\chi_1-\chi_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Bigg]$$
$$\times \ddot{\mathcal{G}} + \Bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]\ddot{T}\Bigg) + \rho\big[d_1d_2d_3T^{d_2-1}$$

$$\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}\Big]\Big] \geq 0, \tag{56}$$

WEC:

$$\begin{aligned}\rho_{\text{eff}} &= \rho + \frac{1}{2\kappa^2}\Big[d_1 d_3 T^{d_2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6} + 2\rho[d_1 d_2 d_3 T^{d_2-1} \\ &\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}] + 24qH^4 \\ &\times \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2)T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big] \\ &+ 24H^3\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3 \times (\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \\ &\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\dot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \times T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3 \\ &\times (\chi_1-\chi_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}\Big)\Big] \geq 0,\end{aligned} \tag{57}$$

SEC:

$$\begin{aligned}\rho_{\text{eff}} + 3P_{\text{eff}} &= \rho + \frac{1}{\kappa^2}\Big[-[d_1 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6}] \\ &+ \rho[d_1 d_2 d_3 T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}] - 24qH^4 \\ &\times \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2)T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big] \\ &+ 12H^3(1+2q)\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \\ &\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\dot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3 \times (\chi_1+\chi_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3(\chi_1-\chi_2) \\ &\times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}\Big) \\ &- 12H^2\Big(\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2)\Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big] \times \Big[\frac{1}{4}(\chi_1+\chi_2)-2\Big]T^{d_2} \\ &\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times \Big[\frac{1}{4}(\chi_1-\chi_2)+2\Big] \\ &\times T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\Big]\dot{\mathcal{G}}^2 + 2\Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2-1} \\ &\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2^2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big] \times T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big] \\ &\times \dot{\mathcal{G}}\dot{T} + \Big[\frac{1}{4}d_1 d_2 d_3(d_2-1)(\chi_1+\chi_2)T^{d_2-2} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3(d_2-1) \\ &\times (\chi_1-\chi_2)T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\dot{T}^2 + \Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2) \times \Big[\frac{1}{4}(\chi_1+\chi_2)-1\Big]T^{d_2} \\ &\times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2} + \frac{1}{4}d_2 d_3(\chi_1-\chi_2)\Big[\frac{1}{4}(\chi_1-\chi_2)+1\Big]T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\Big]\ddot{\mathcal{G}} + \Big[\frac{1}{4}d_1 d_2 d_3(\chi_1+\chi_2) \\ &\times T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2^2 d_3 \times (\chi_1-\chi_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Big]\ddot{T}\Big)\Big] \geq 0,\end{aligned} \tag{58}$$

DEC:

$$\begin{aligned}\rho_{\text{eff}} - P_{\text{eff}} &= \rho + \frac{1}{\kappa^2}\Big[\Big[d_1 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2 d_3 T^{d_2} \times \mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 T + d_1 d_2 T^{\chi_4} + \chi_5 T^{\chi_6}\Big] \\ &+ \rho[d_1 d_2 d_3 T^{d_2-1} \times \mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)} + d_2^2 d_3 T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)} - \chi_3 + d_1 d_2 \chi_4 T^{\chi_4-1} + \chi_5\chi_6 T^{\chi_6-1}\Big] \\ &+ 24qH^4\Big[\frac{1}{4}d_1 d_3(\chi_1+\chi_2)T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1} - \frac{1}{4}d_2 d_3(\chi_1-\chi_2)T^{d_2}\end{aligned}$$

$$\times\,\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\Bigg]+4H^3(3-2q)\Bigg(\Bigg[\bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)$$
$$\times\bigg[\frac{1}{4}(\chi_1+\chi_2)-1\bigg]$$
$$\times\,T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}+\frac{1}{4}d_2d_3(\chi_1-\chi_2)\bigg[\frac{1}{4}(\chi_1-\chi_2)+1\bigg]$$
$$\times\,T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\bigg]$$
$$\times\,\dot{\mathcal{G}}+\bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}$$
$$-\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)T^{d_2-1}$$
$$\times\,\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\bigg]\dot{T}\Bigg)+4H^2\Bigg(\Bigg[\bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)$$
$$\times\bigg[\frac{1}{4}(\chi_1+\chi_2)-1\bigg]$$
$$\times\bigg[\frac{1}{4}(\chi_1+\chi_2)-2\bigg]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-3}-\frac{1}{4}d_2d_3(\chi_1-\chi_2)$$
$$\times\bigg[\frac{1}{4}(\chi_1-\chi_2)+1\bigg]\bigg[\frac{1}{4}(\chi_1-\chi_2)+2\bigg]T^{d_2}$$
$$\times\,\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-3}\bigg]\dot{\mathcal{G}}^2+2\bigg[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)$$
$$\times\bigg[\frac{1}{4}(\chi_1+\chi_2)-1\bigg]T^{d_2-1}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}+\frac{1}{4}d_2^2d_3(\chi_1-\chi_2)$$
$$\times\bigg[\frac{1}{4}(\chi_1-\chi_2)+1\bigg]T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}\bigg]$$
$$\times\,\dot{\mathcal{G}}\dot{T}+\bigg[\frac{1}{4}d_1d_2d_3(d_2-1)(\chi_1+\chi_2)$$
$$\times\,T^{d_2-2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}-\frac{1}{4}d_2^2d_3(d_2-1)(\chi_1-\chi_2)$$
$$\times\,T^{d_2-2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\bigg]\dot{T}^2$$
$$+\bigg[\frac{1}{4}d_1d_3(\chi_1+\chi_2)\bigg[\frac{1}{4}(\chi_1+\chi_2)-1\bigg]T^{d_2}\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-2}$$
$$+\frac{1}{4}d_2d_3(\chi_1-\chi_2)$$

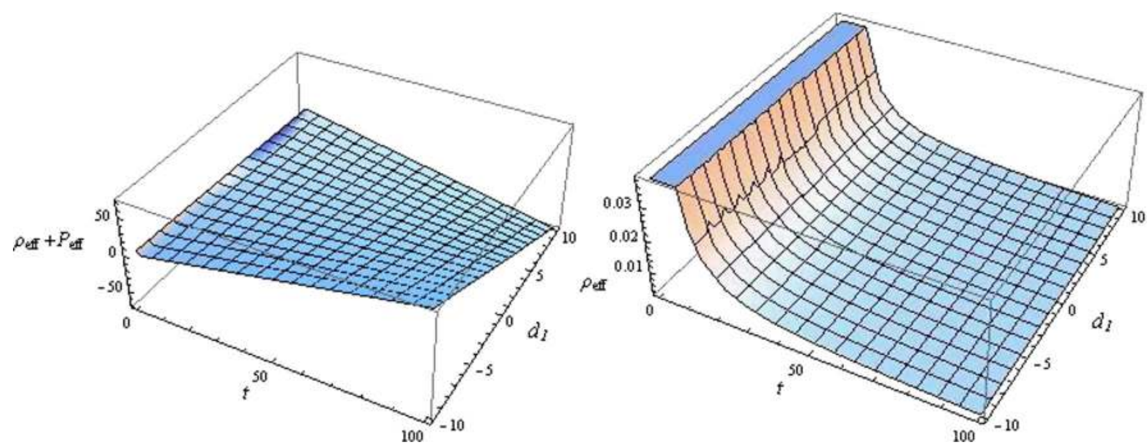

Fig. 9 Energy conditions for $d_2=0.1$ and $d_3=1$

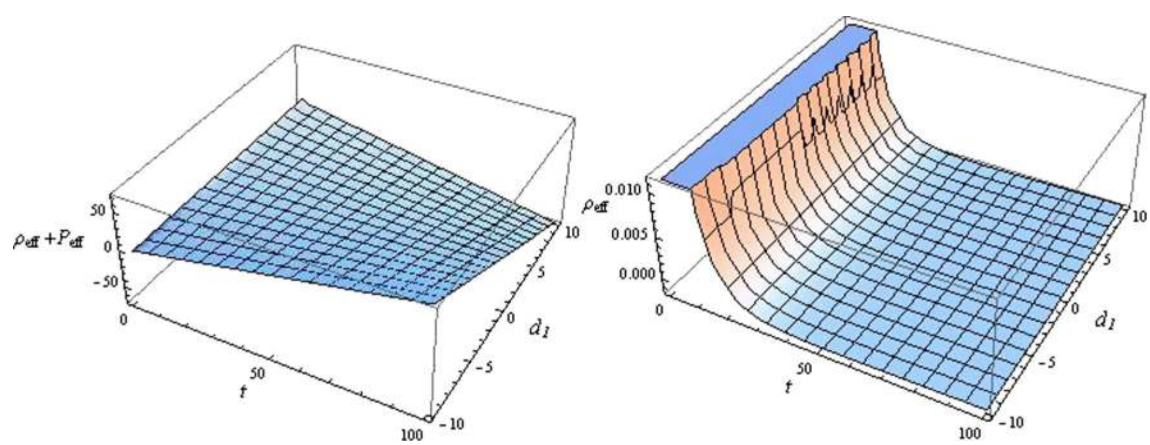

Fig. 10 Energy conditions for $d_2=0.1$ and $d_3=-0.5$

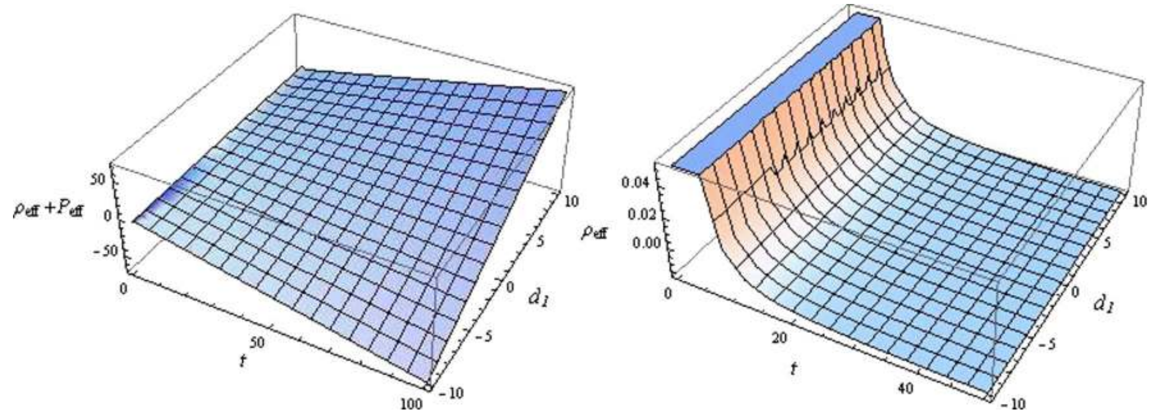

Fig. 11 Energy conditions for $d_2 = -0.1$ and $d_3 = 0.5$

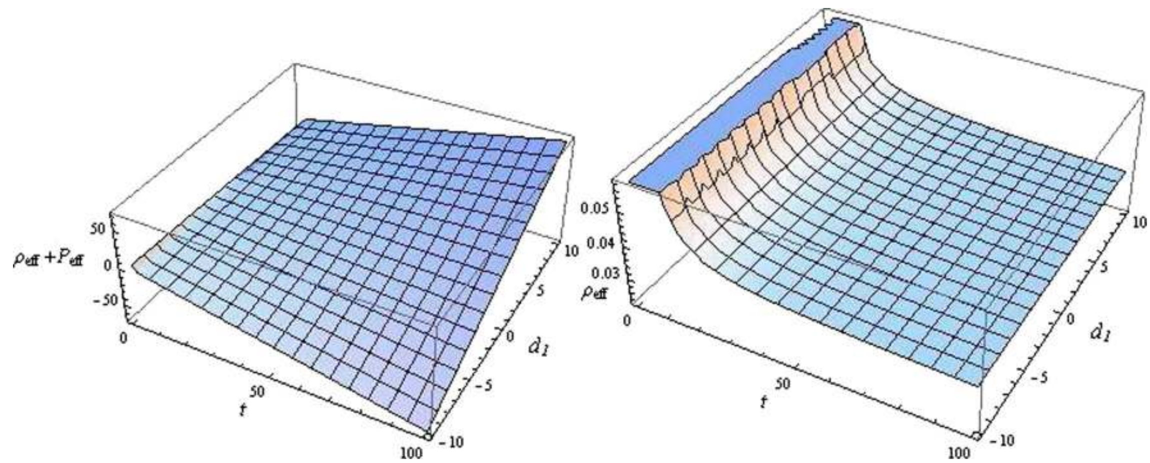

Fig. 12 Energy conditions for $d_2 = -0.1$ and $d_3 = -1$

$$\times\left[\frac{1}{4}(\chi_1-\chi_2)+1\right]T^{d_2}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-2}]\ddot{\mathcal{G}}$$
$$+\left[\frac{1}{4}d_1d_2d_3(\chi_1+\chi_2)T^{d_2-1}\right.$$
$$\times\mathcal{G}^{\frac{1}{4}(\chi_1+\chi_2)-1}-\frac{1}{4}d_2^2d_3$$
$$\left.\left.\left.\times(\chi_1-\chi_2)T^{d_2-1}\mathcal{G}^{-\frac{1}{4}(\chi_1-\chi_2)-1}\right]\ddot{T}\right)\right]\geq 0. \qquad (59)$$

The NEC and WEC depend on four parameters $t,\ d_1,\ d_2$ and d_3. We plot these conditions against t and d_1 for $n=\frac{2}{3}$ with possible signs of d_2 and d_3. The left plot of Fig. 9 shows a positively increasing behavior of NEC for $-10\leq d_1\leq 0$ with respect to time while invalid for $d_1>0$. The effective energy density remains positive for all values of (t, d_1) as shown in Fig. 9 (right). The same behavior of both conditions are obtained for $0<d_2\leq 0.51$ with $d_3>0$ as well as for $d_2>0$ with $d_3=0$. The left plot of Fig. 10 shows a similar behavior of the NEC for $d_2>0$ and $d_3<0$, while $\rho_{\rm eff}$ remains positive for $0<t<23$. Similarly, for $d_3=-1$ and -10, WEC is valid for $0<t<14$ and $0<t<4.5$, respectively, with $d_2=0.1$. The right plots of Figs. 11 and 12 show the validity of NEC for $d_1\geq 0$, while it does not hold for negative values of d_1. The effective energy density remains positive for the time interval $1\leq t\leq 10$ with $d_3=0.5$ as shown in Fig. 11 (right panel), while for $d_3=1$ and 10, the acceptable intervals are $1\leq t\leq 7$ and $1\leq t\leq 3$, respectively. This shows that the validity region of the WEC decreases as the value of integration constant d_3 increases. The right plot of Fig. 12 shows the positivity of $\rho_{\rm eff}$ for $(d_2, d_3)<0$, which confirms the positivity of the WEC with $d_1>0$.

5 Final remarks

In this paper, we have presented a generalized modified theory of gravity with an arbitrary coupling between geometry and matter. The gravitational Lagrangian is obtained by adding an arbitrary function $f(\mathcal{G}, T)$ in the Einstein–Hilbert action. We have formulated the corresponding field equations using the least action principle and calculated the non-zero covariant divergence of $T_{\alpha\beta}$ consistent with $f(R, T)$ theory [31]. Consequently, the test particles follow non-geodesic trajectories due to the presence of an extra force originating from the non-minimal coupling, while they move along geodesics for a pressureless fluid. We have constructed the energy conditions for an FRW universe model filled with dust fluid in terms of the deceleration, jerk, and snap (q, j, s) cosmological parameters. The reconstruction technique has been applied to $f(\mathcal{G}, T)$ gravity using the well-known de Sitter and power-law universe models. The results are summarized as follows.

- In the de Sitter reconstructed model, the energy bounds have dependence on three parameters t, c_1 and c_2. We have plotted NEC and WEC against t and c_2 with four possible signatures of c_1 and c_2 as shown in Figs. 1, 2, 3, 4, 5, 6, 7, and 8. It is found that NEC and WEC are satisfied for $c_1 > 0$ and $c_2 < 0$ throughout the time interval for cases $(c_1, c_2) > 0$ and $(c_1, c_2) < 0$ that the energy conditions are satisfied for small values of the c_i in a very small time interval. It is observed that the NEC shows a positively increasing behavior for all negative values of c_1 with $c_2 > 0$, while the validity ranges of the WEC show dependence on c_1.
- For a power-law reconstructed model, we have explored the behavior of the four parameters t, d_1, d_2, and d_3 with $n = \frac{2}{3}$. In this case, we have plotted the energy conditions against (t, d_1) and analyzed the possible behavior of remaining constants. In Figs. 9, 10, 11, and 12, we have taken $-10 \le d_1 \le 10$ and found the valid regions where the energy conditions are satisfied.

Finally, we conclude that the NEC and WEC are satisfied in both reconstructed $f(\mathcal{G}, T)$ models with a suitable choice of the free parameters.

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