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# ENERGY DECAY IN A TIMOSHENKO-TYPE SYSTEM WITH HISTORY IN THERMOELASTICITY OF TYPE III

Salim A. Messaoudi
Department of Mathematics and Statistics, KFUPM
Dhahran 31261, Saudi Arabia

Belkacem Said-Houari
Institut de Mathématiques de Toulouse, MIP
Université Paul Sabatier
118, route de Narbonne, 31062 Toulouse Cedex 09, France

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**Abstract.** In this paper we consider a one-dimensional linear thermoelastic system of Timoshenko type with past history acting only in one equation. We consider the model where the heat conduction is given by Green and Naghdi's theories and prove exponential and polynomial stability results for the equal and nonequal wave-speed propagation. Our results are established under conditions on the relaxation function weaker than those in [9].

## 1. Introduction

In 1921, Timoshenko [35] gave, as model for a thick beam, the following system of coupled hyperbolic equations:

$$\begin{aligned} &\rho u_{tt} = (K(u_x - \varphi))_x, \quad \text{in } (0,L)\times(0,+\infty) \\ &I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x-\varphi), \ \text{ in } \ (0,L)\times(0,+\infty), \end{aligned} \tag{1.1}$$

where $t$ denotes the time variable and $x$ is the space variable along the beam of length $L$, in its equilibrium configuration, $u$ is the transverse displacement of the beam and $\varphi$ is the rotation angle of the filament of the beam. The coefficients $\rho, I_\rho, E, I$ and $K$ are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

An important issue of research is to look for a minimum dissipation by which solutions of system (1.1) decay uniformly to the stable state as time goes to infinity. In this regards, several types of dissipative mechanisms have

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been introduced. For instance, Raposo *et al.* [31] used two linear frictional dampings acting on both equations to stabilize the system uniformly. An exponential decay result has been proved. Kim and Renardy [15] considered (1.1) together with two boundary controls of the form

$$K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) = \alpha\frac{\partial u}{\partial t}(L,t), \quad \forall t \geq 0$$
$$EI\frac{\partial \varphi}{\partial x}(L,t) = -\beta\frac{\partial \varphi}{\partial t}(L,t), \quad \forall t \geq 0$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng *et al.* [8], where the stabilization of vibrations in a Timoshenko system was studied. Yan [36] generalized the result of [15] by considering boundary conditions of the form

$$K(\varphi(L,t) - \frac{\partial u}{\partial x}(L,t)) = f_1(\frac{\partial u}{\partial t}(L,t)), \quad \forall t \geq 0$$
$$-EI\frac{\partial \varphi}{\partial x}(L,t) = f_2(\frac{\partial \varphi}{\partial t}(L,t)), \quad \forall t \geq 0,$$

where $f_1, f_2$ are functions with polynomial growth near the origin. The boundary stabilization of the nonuniform Timoshenko beam has also been studied by Ammar-Khodja *et al.* [3]. They considered

$$\begin{aligned}
&\alpha u_{tt} = (\beta(u_x + \varphi))_x, \quad \text{in } (0,L) \times (0,+\infty) \\
&\gamma\varphi_{tt} = (\delta\varphi_x)_x - K(u_t + \varphi), \ \text{ in } (0,L) \times (0,+\infty), \\
&u(0,t) = u(L,t) = 0, \ \varphi_x(0,t) = c\varphi_t(0,t), \ \varphi_x(L,t) = -d\varphi_t(L,t), \quad t > 0,
\end{aligned} \tag{1.2}$$

for positive $C^1$-functions $\alpha(x), \beta(x), \gamma(x), \delta(x)$, and proved that the uniform stability of (1.2) holds if and only if the wave speeds are equal ($\frac{\delta}{\gamma} = \frac{\beta}{\alpha}$ on $(0,L)$); otherwise only the asymptotic stability has been proved. See also recent work by Messaoudi and Mustafa [19], where the decay rate has been discussed for several systems and without imposing any growth condition on the damping functions. Stabilization by one damping has been considered by many authors. Soufyane and Wehbe [34] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. They considered

$$\begin{aligned}
&\rho u_{tt} = (K(u_x - \varphi))_x, \ \text{ in } (0,L) \times (0,+\infty) \\
&I_\rho\varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b(x)\varphi_t, \ \text{ in } (0,L) \times (0,+\infty)
\end{aligned} \tag{1.3}$$

$$u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \quad \forall t > 0,$$

where $b(x)$ is a positive and continuous function satisfying $b(x) \geq b_0 > 0$, for all $x \in [a_0, a_1] \subset [0, L]$. They proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal ($\frac{K}{\rho} = \frac{EI}{I_\rho}$); otherwise only the asymptotic stability holds. This result has been recently improved by Rivera and Racke [23], where an exponential decay of the solution energy of (1.3) has been established for $b$ with indefinite sign. Ammar-Khodja *et al.* [2] considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0 \end{aligned}$$

in $(0, L) \times (0, +\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal ($\frac{K}{\rho_1} = \frac{b}{\rho_2}$) and $g$ decays uniformly. Precisely, they proved an exponential decay if $g$ decays in an exponential rate and a polynomial decay if $g$ decays in a polynomial rate. They also required some extra technical conditions on both $g'$ and $g''$ to obtain their result. Guesmia and Messaoudi [14] obtained the same uniform decay results without imposing those extra technical conditions on $g'$ and $g''$. Recently, Messaoudi and Mustafa [20] improved the results of [2], [14] by allowing more general relaxation functions. They established a more general decay result, from which the exponential and the polynomial decay results are only special cases. The feedback of memory type has also been used by Santos [33]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. This latter result has been pushed to a multi-dimensional problem by Messaoudi and Soufyane [16]. Also, Rivera and Racke [22] treated a nonlinear Timoshenko-type system of the form

$$\begin{aligned} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{aligned} \tag{1.4}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case. Recently, Fernández Sare and Rivera [9], considered Timoshenko type system with past history acting only in one equation. More precisely they looked into the following problem:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(t)\psi_{xx}(t-s,.)ds + K(\varphi_x + \psi) = 0 \end{aligned} \tag{1.5}$$

and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. For more results concerning well posedness and controllability of Timoshenko systems, we refer the reader to Alabau-Boussouira [1], Fernández Sare and Racke [10], Messaoudi *et al.* [17], and Messaoudi and Mustafa [18].

For Timoshenko systems in classical thermoelasticity, Rivera and Racke [21] considered

$$\begin{aligned} &\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \\ &\rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0 \end{aligned} \tag{1.6}$$

in $(0, \infty) \times (0, L)$, where $\varphi, \psi$, and $\theta$ are functions of $(x, t)$ which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions on $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and nonexponential stability result for the case of different wave speeds.

In system (1.6), the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation. That is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this physical paradox, many theories have merged such as thermoelasticity by second sound or thermoelasticity type III. By the end of the last century, Green and Naghdi [11$-$13] introduced three types of thermoelastic theories based on an entropy equality instead of the usual entropy inequality. In each of these theories, the heat flux is given by a different constitutive assumption. As a result, three theories are obtained and were called thermoelasticity type I, type II, and type III respectively. This theory is developed in a rational way in order to obtain a fully consistent explanation, which will incorporate thermal pulse transmission in a very logical manner and elevate the unphysical infinite speed of heat propagation induced by the classical theory of heat conduction. When the theory of type I is linearized the parabolic equation of the heat conduction arises, whereas the theory of type II does not admit dissipation of energy and it is known as thermoelasticity without dissipation. It is a limiting case of thermoelasticity type III. See in this regard $[4-6]$ and [32] for more details. To understand these new theories and their applications, several mathematical and physical contributions have been made; see for example $[4-6], [24-30]$ and [37]. In particular, we must mention the survey paper of Chandrasekharaiah [6], in which the author has focussed attention on the work done during the last two decades. He reviewed the theory of thermoelasticity with thermal relaxation and the temperature rate dependent thermoelasticity. He also described the thermoelasticity without dissipation and clarified its properties. By the end of his paper, he made a brief discussion of the new theories, including what is called dual-phase-lag effects. We recall here the type III thermoelasticity characterized by the following constitutive equations for the heat flux:

$$q = -\kappa^* \tau_x - \widetilde{\kappa} \theta_x,$$

where $\theta$ denotes the temperature, $\tau$ is the thermal displacement which satisfies $\tau_t = \theta$, and $\kappa^*, \widetilde{\kappa}$ are positive constants.

Zhang and Zuazua [37] discussed the long time behavior of the solution of the system

$$\begin{aligned} &u_{tt} - \mu \triangle u - (\mu + \lambda)\nabla(divu) + \beta \nabla \theta = 0 \\ &\theta_{tt} - \Delta \theta + divu_{tt} - \Delta \theta_t = 0 \end{aligned} \tag{1.7}$$

in $\Omega \times (0, \infty)$, subject to initial data and boundary conditions of Dirichlet-Dirichlet type. They concluded the following: "For most domains, the energy of the system does not decay uniformly. But under suitable conditions on the domain, which might be described in terms of geometric optics, the energy of the system decays exponentially. For most domains in two space dimensions, the energy of smooth solutions decays in a polynomial rate."

In [29], Quintanilla and Racke considered a system similar to (1.7) and used the spectral analysis method and the energy method to obtain the exponential stability in one dimension for different boundary conditions (Dirichlet- Dirichlet or Dirichlet- Neuman). They also proved a decay of energy result for the radially symmetric situations in the multi-dimensional case ($n = 2, 3$). We also recall the contribution of Quintanilla [28], in which he proved that solutions of thermoelasticity of type III converge to solutions of the classical thermoelasticity as well as to the solution of thermoelasticity without energy dissipation and Quintanilla [26], in which he established a structural stability result on the coupling coefficients and continuous dependence on the external data in thermoelasticity type III.

In the present work we study the following system:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(x, t-s)ds + K(\varphi_x + \psi) + \beta\theta_x = 0 \\ &\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 \end{aligned} \tag{1.8}$$

in $(0, 1) \times (0, \infty)$, subject to the initial and boundary conditions

$$\begin{aligned} &\varphi(.,0) = \varphi_0,\ \varphi_t(.,0) = \varphi_1,\ \psi(t.,0) = \psi_0,\ \psi_1(.,0) = \psi_1, \\ &\theta(.,0) = \theta_0,\ \theta_t(.,0) = \theta_1 \end{aligned} \tag{1.9}$$

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 \tag{1.10}$$

and prove uniform decay results. Precisely, we will show that, for $\frac{\rho_1}{K} = \frac{\rho_2}{b}$, the first energy decays exponentially (respectively polynomially) if $g$ decays exponentially (respectively polynomially). In the case of different wave speeds, we show that the decay is of polynomial type. This system models the transverse vibration of a thick beam, taking into account the heat conduction given by Green and Naghdi' s theory. Following the idea of Dafermos [7], we introduce

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s),\ s \geq 0; \tag{1.11}$$

consequently we obtain the following initial and boundary conditions

$$\eta^t(x,0) = 0,\ \forall t \geq 0 \tag{1.12}$$

$$\eta^t(0,s) = \eta^t(1,s) = 0,\ \forall s,t \geq 0 \tag{1.13}$$

$$\eta^0(x,s) = \eta_0(s),\ \forall s \geq 0. \tag{1.14}$$

Clearly, (1.11) gives

$$\eta_t^t(x,s) + \eta_s^t(x,s) = \psi_t(x,t). \tag{1.15}$$

We also assume that $g$ is a differentiable function satisfying

$$g(t) > 0, \qquad \widehat{b} = b - \int_0^\infty g(s)ds > 0, \quad g'(t) \leq -k_0 g^p(t) \tag{1.16}$$

for a positive constant $k_0$ and $1 \leq p < 3/2$.

**Remark 1.1**. Under condition (1.16), it is easy to verify that

$$G_0 = \int_0^\infty g^{1/2}(s)ds < \infty, \quad G_p = \int_0^\infty g^{2-p}(s)ds < \infty, \quad 1 \leq p < 3/2.$$

## 2. Uniform decay $\frac{\rho_1}{K} = \frac{\rho_2}{b}$

In this section, we state and prove our main decay result. In order to exhibit the dissipative nature of system (1.8), we introduce the new variables $\phi = \varphi_t$, $\Psi = \psi_t$, and $\widehat{\eta}^t = \eta_t^t$. Thus, (1.8)-(1.15) yield

$$\begin{aligned}
&\rho_1\phi_{tt} - K(\phi_x + \Psi)_x = 0,\\
&\rho_2\Psi_{tt} - \widehat{b}\Psi_{xx} - \int_0^\infty g(s)\widehat{\eta}^t_{xx}(x,s)ds + K(\phi_x + \Psi) + \beta\theta_{tx} = 0\\
&\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\Psi_{tx} - k\theta_{txx} = 0\\
&\widehat{\eta}^t_t + \widehat{\eta}^t_s - \Psi_t = 0,
\end{aligned} \tag{2.1}$$

where $x \in (0,1)$, $t \geq 0$ and $s \geq 0$. We also obtain the following boundary and initial conditions:

$$\begin{aligned}
\phi(.,0) &= \phi_0,\ \phi_t(.,0) = \phi_1,\ \Psi(t.,0) = \Psi_0,\ \Psi_1(.,0) = \Psi_1\\
\theta(.,0) &= \theta_0,\ \theta_t(.,0) = \theta_1
\end{aligned} \tag{2.2}$$

$$\phi(0,t) = \phi(1,t) = \Psi(0,t) = \Psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 \tag{2.3}$$

$$\begin{aligned}
\widehat{\eta}^t(x,0) &= 0,\ \forall t \geq 0\\
\widehat{\eta}^t(0,s) &= \widehat{\eta}^t(1,s) = 0,\ \forall s,t \geq 0\\
\widehat{\eta}^0(x,s) &= \widehat{\eta}_0(s),\ \forall s \geq 0.
\end{aligned} \tag{2.4}$$

In order to use the Poincaré inequality for $\theta$, we introduce

$$\overline{\theta} = \theta(x,t) - t\int_0^1 \theta_1(x)dx - \int_0^1 \theta_0(x)dx.$$

Then by $(2.1)_3$ we easily verify that

$$\int_0^1 \overline{\theta}(x,t)dx = 0$$

for all $t \geq 0$; in this case the Poincaré inequality is applicable for $\overline{\theta}$. On the other hand $(\phi, \Psi, \overline{\theta}, \widehat{\eta}^t)$ satisfies the same partial differential equations (2.1) and boundary conditions (2.2) $-$ (2.4). In the sequel we shall work with $\overline{\theta}$ but we write $\theta$ for simplicity. Then the associated first-order energy is

$$E(t) = E_1(\phi, \Psi, \theta, \widehat{\eta}^t) = \frac{\gamma}{2} \int_0^1 (\rho_1 \phi_t^2 + \rho_2 \Psi_t^2 + K \left|\phi_x + \Psi\right|^2 + \widehat{b} \Psi_x^2) dx$$
$$+ \frac{\beta}{2} \int_0^1 (\rho_3 \theta_t^2 + \delta \theta_x^2) dx + \frac{\gamma}{2} \int_0^1 \int_0^\infty g(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx. \tag{2.5}$$

**Theorem 2.1.** *Suppose that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b} \tag{2.6}$$

*and let* $\phi_0, \Psi_0,\ \theta_0 \in H_0^1(0,1)$, $\widehat{\eta}_0^t \in L_g^2(\mathbb{R}^+, H_0^1(0,1))$, $\phi_1, \Psi_1,\ \theta_1 \in L^2(0,1)$. *Then there exist two positive constants* $C$ *and* $\xi$, *such that*

$$E(t) \leq Ce^{-\xi t}, \qquad p = 1 \tag{2.7}$$
$$E(t) \leq \frac{C}{(t+1)^{1/(p-1)}} \qquad p > 1. \tag{2.8}$$

The proof of our result will be established through several lemmas.

**Lemma 2.2.** *Let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* (2.1)-(2.4). *Then we have*

$$\frac{dE(t)}{dt} = -\beta k \int_0^1 \theta_{tx}^2 dx + \frac{\gamma}{2} \int_0^1 \int_0^\infty g'(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx \leq 0. \tag{2.9}$$

**Proof.** Multiplying equation $(2.1)_1$ by $\gamma \phi_t$, $(2.1)_2$ by $\gamma \Psi_t$ and $(2.1)_3$ by $\beta \theta_t$, integrating over $(0,1)$, and summing up, using (1.16), we obtain (2.9).

**Lemma 2.3**. *Let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* (2.1)-(2.4). *Then we have, for* $1 < p < 3/2$,

$$\Big( \int_0^1 \int_0^\infty g(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx \Big)^{2p-1} \leq C_0 \int_0^1 \int_0^\infty g^p(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx \tag{2.10}$$

*for a constant* $C_0 > 0$.

**Proof.** Using Hölder's inequality, it is straightforward to see that, for any $r > 1$, we have

$$\int_0^1 \int_0^\infty g(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx$$
$$= \int_0^1 \int_0^\infty g^{\frac{1}{2r}}(s) \left|\widehat{\eta}_x^t(s)\right|^{\frac{2}{r}} g^{\frac{2r-1}{2r}}(s) \left|\widehat{\eta}_x^t(s)\right|^{\frac{2r-2}{r}} ds dx$$

$$\leq \Big(\int_0^1\int_0^\infty g^{\frac{1}{2}}(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big)^{\frac{1}{r}}\Big(\int_0^1\int_0^\infty g^{\frac{2r-1}{2r-2}}(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big)^{\frac{r-1}{r}}.$$

Remark 1.1, (1.11), (2.5), and (2.9) lead to

$$\int_0^1\int_0^\infty g^{1/2}(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx \leq 2E(0)\int_0^\infty g^{1/2}(s)ds = 2G_0E(0).$$

By taking $r=(2p-1)/(2p-2)$, (2.10) follows.

**Lemma 2.4**. *For* $1\leq p\leq 3/2$, *we have*

$$\int_0^1\Big(\int_0^\infty g(s)\widehat{\eta}_x^t(s)ds\Big)^2dx \leq G_p\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx. \qquad (2.11)$$

**Proof.** Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_0^\infty g(s)\widehat{\eta}_x^t(s)ds &= \int_0^\infty g^{1-\frac{p}{2}}(s)g^{\frac{p}{2}}(s)\widehat{\eta}_x^t(s)ds\\ &\leq \Big(\int_0^\infty g^{2-p}(s)\Big)^{1/2}\Big(\int_0^\infty g^p(s)|\widehat{\eta}_x^t(s)|^2ds\Big)^{1/2}.\end{aligned}$$

Therefore, (2.11) follows by Remark 1.1.

As in [21], let

$$I_1 := \int_0^1(\rho_2\Psi_t\Psi+\rho_1\phi_t\omega)dx, \qquad (2.12)$$

where $\omega$ is the solution of

$$-\omega_{xx}=\Psi_x, \quad \omega(0)=\omega(1)=0.$$

**Lemma 2.5** *Let* $(\phi,\Psi,\theta,\widehat{\eta}^t)$ *be a solution of* (2.1)-(2.4). *Then we have, for any* $\varepsilon_1,\lambda_1>0$,

$$\begin{aligned}\frac{dI_1(t)}{dt} \quad &\leq \quad (-\frac{\widehat{b}}{2}+\lambda_1)\int_0^1\Psi_x^2dx+\varepsilon_1\rho_1\int_{0t}^1\phi_t^2dx+(\rho_2+\frac{\rho_1}{4\varepsilon_1})\int_0^1\Psi_t^2dx\\ &+\frac{\beta^2}{2\widehat{b}}\int_0^1\theta_{tx}^2dx+\frac{G_p}{4\lambda_1}\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx. \qquad (2.13)\end{aligned}$$

**Proof.** By taking a derivative of (2.12) and using equations (2.1) we conclude

$$\begin{aligned}\frac{dI_1(t)}{dt} \quad &= \quad -\widehat{b}\int_0^1\Psi_x^2dx+\rho_2\int_0^1\Psi_t^2dx-K\int_0^1\Psi^2dx-\beta\int_0^1\Psi\theta_{tx}dx\\ &+K\int_0^1\omega_x^2dx+\rho_1\int_0^1\phi_t\omega_tdx-\int_0^1\Psi_x\int_0^\infty g(s)\widehat{\eta}_x^t(s)dsdx.\end{aligned}$$

By using Young' s inequality and

$$\begin{aligned} \int_0^1 \omega_x^2 dx &\leq \int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx \\ \int_0^1 \omega_t^2 dx &\leq \int_0^1 \omega_{tx}^2 dx \leq \int_0^1 \Psi_t^2 dx, \end{aligned}$$

we find that

$$\begin{aligned} \frac{dI_1(t)}{dt} \leq &-\widehat{b} \int_0^1 \Psi_x^2 dx + \varepsilon_1 \rho_1 \int_0^1 \phi_t^2 dx + (\rho_2 + \frac{\rho_1}{4\varepsilon_1}) \int_0^1 \Psi_t^2 dx \\ &+ \frac{\beta^2}{2\widehat{b}} \int_0^1 \theta_{tx}^2 dx + \frac{\widehat{b}}{2} \int_0^1 \Psi_x^2 dx - \int_0^1 \Psi_x \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds dx. \end{aligned} \tag{2.14}$$

Using Young's inequality and (2.11), the last term in the right-hand side of (2.14) can be estimated as follows:

$$\begin{aligned} \int_0^1 \Psi_x \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds dx &\leq \frac{1}{4\lambda_1} \int_0^1 \Big( \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds \Big)^2 dx + \lambda_1 \int_0^1 \Psi_x^2 dx \\ &\leq \frac{G_p}{4\lambda_1} \int_0^1 \int_0^\infty g^p(s) |\widehat{\eta}_x^t(s)|^2 ds dx + \lambda_1 \int_0^1 \Psi_x^2 dx, \qquad \lambda_1 > 0. \end{aligned} \tag{2.15}$$

Inserting (2.15) into (2.14), we obtain the desired result.

Next, we set

$$I_2 := -\rho_2 \int_0^1 \Psi_t(x,t) \int_0^\infty g(s) \widehat{\eta}^t(s) ds dx. \tag{2.16}$$

**Lemma 2.6** *Let $(\phi, \Psi, \theta, \widehat{\eta}^t)$ be a solution of $(2.1) - (2.4)$. Then we have, for any $\varepsilon_2 > 0$,*

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq &-\frac{\rho_2 g_0}{2} \int_0^1 \Psi_t^2 dx + \varepsilon_2 \widehat{b}^2 \int_0^1 \Psi_x^2 dx + \varepsilon_2 K^2 \int_0^1 (\phi_x + \Psi)^2 dx \\ &+ \frac{\beta^2}{2} \int_0^1 \theta_{tx}^2 dx + G_p \Big( 1 + \frac{1}{4\varepsilon_2} + \frac{C^*}{4\varepsilon_2} + \frac{C^*}{2} \Big) \int_0^1 \int_0^\infty g^p(s) \left| \widehat{\eta}_x^t(s) \right|^2 ds dx \\ &- \frac{C^* g(0)}{2\rho_2} \int_0^1 \int_0^\infty g'(s) \left| \widehat{\eta}_x^t(s) \right|^2 ds dx. \end{aligned} \tag{2.17}$$

**Proof.** Using the second and fourth equations of (2.1) we get

$$\frac{dI_2(t)}{dt} = \widehat{b} \int_0^1 \Psi_x \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds dx + \int_0^1 \Big( \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds \Big)^2 dx$$

$$+ K\int_0^1(\phi_x+\Psi)\int_0^\infty g(s)\widehat{\eta}^t(s)dsdx + \beta\int_0^1\theta_{xt}\int_0^\infty g(s)\widehat{\eta}^t(s)dsdx$$
$$-\rho_2 g_0\int_0^1\Psi_t^2dx + \rho_2\int_0^1\Psi_t\int_0^\infty g(s)\widehat{\eta}^t_s(s)dsdx. \tag{2.18}$$

By using (2.11) and Young's inequality, we obtain the following estimates:

$$\begin{aligned}
&\widehat{b}\int_0^1\Psi_x\int_0^\infty g(s)\widehat{\eta}^t_x(s)dsdx\\
&\le \varepsilon_2\widehat{b}^2\int_0^1\Psi_x^2dx + \frac{G_p}{4\varepsilon_2}\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}^t_x(s)\right|^2dsdx\\
&\quad + K\int_0^1(\phi_x+\Psi)\int_0^\infty g(s)\widehat{\eta}^t(s)dsdx\\
&\le \varepsilon_2K^2\int_0^1(\phi_x+\Psi)^2dx + \frac{C^*G_p}{4\varepsilon_2}\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}^t_x(s)\right|^2dsdx,
\end{aligned}$$

$$\beta\int_0^1\theta_{xt}\int_0^\infty g(s)\widehat{\eta}^t(s)dsdx \le \frac{\beta^2}{2}\int_0^1\theta_{xt}^2dx + \frac{C^*G_p}{2}\int_0^1\int_0^\infty g^p(s)|\widehat{\eta}^t_x(s)|^2dsdx,$$

$$\begin{aligned}
&\int_0^1\Psi_t\int_0^\infty g(s)\widehat{\eta}^t_s(s)dsdx = -\int_0^1\Psi_t\int_0^\infty g'(s)\widehat{\eta}^t(s)dsdx\\
&\le \frac{\rho_2 g_0}{2}\int_0^1\Psi_t^2dx - \frac{C^*g(0)}{2\rho_2}\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}^t_x(s)\right|^2dsdx,
\end{aligned}$$

where $C^*$ is the Poincaré constant. By inserting all the above estimates into (2.18), relation (2.17) follows.

Next we introduce the functional

$$\begin{aligned}
J(t)\ \ :\ \ &= \rho_2\int_0^1\Psi_t(\phi_x+\Psi)dx + \frac{\rho_1\widehat{b}}{K}\int_0^1\Psi_x\phi_tdx\\
&+\frac{\rho_1}{K}\int_0^1\phi_t(t)\int_0^\infty g(s)\widehat{\eta}^t_x(s)dsdx.
\end{aligned} \tag{2.19}$$

**Lemma 2.7.** *Let* $(\phi,\Psi,\theta,\widehat{\eta}^t)$ *be a solution of* $(2.1)-(2.4)$. *Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{\widehat{b}+g_0} = \frac{\rho_2}{b}. \tag{2.20}$$

*Then, for* $\varepsilon_3>0$, *we conclude*

$$\frac{dJ(t)}{dt} \le \Big[\phi_x\Big(b\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(x,s)\Big)\Big]_{x=0}^{x=1} - \frac{K}{2}\int_0^1(\phi_x+\Psi)^2dx$$

$$+ \rho_2 \int_0^1 \Psi_t^2 dx + \varepsilon_3 \int_0^1 \phi_t^2 dx + \frac{\beta^2}{2K} \int_0^1 \theta_{tx}^2 dx \tag{2.21}$$
$$- g(0)C(\varepsilon_3) \int_0^1 \int_0^\infty g'(s) \left| \widehat{\eta}_x^t(s) \right|^2 ds dx.$$

**Proof.** Differentiating $J(t)$, we obtain

$$\begin{aligned} \frac{dJ(t)}{dt} &= \rho_2 \int_0^1 \Psi_{tt}(\phi_x + \Psi) dx + \rho_2 \int_0^1 \Psi_t (\phi_x + \Psi)_t dx \\ &+ \frac{\rho_1 \widehat{b}}{K} \int_0^1 \Psi_x \phi_{tt} dx + \frac{\rho_1}{K} \int_0^1 \phi_t \int_0^\infty g(s) \widehat{\eta}_{tx}^t(s) ds dx \\ &+ \frac{\rho_1 \widehat{b}}{K} \int_0^1 \Psi_{tx} \phi_t dx + \frac{\rho_1}{K} \int_0^1 \phi_{tt} \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds dx. \end{aligned}$$

By using equations (2.1), we find

$$\begin{aligned} \frac{dJ(t)}{dt} &= \int_0^1 (\phi_x + \Psi) \Big[ \widehat{b} \Psi_{xx} + \int_0^\infty g(s) \widehat{\eta}_{xx}^t(x,s) ds - K(\phi_x + \Psi) - \beta \theta_{tx} \Big] dx \\ &+ \rho_2 \int_0^1 \Psi_t^2 dx + \widehat{b} \int_0^1 \Psi_x (\phi_x + \Psi)_x dx \\ &+ (\frac{\rho_1 \widehat{b}}{K} - \rho_2) \int_0^1 \Psi_{tx} \phi_t dx + \frac{\rho_1}{K} \int_0^1 \phi_t(t) \int_0^\infty g(s) (\Psi_t - \widehat{\eta}_s^t)_x(s) ds dx \\ &+ \int_0^1 (\phi_x + \Psi)_x \int_0^\infty g(s) \widehat{\eta}_x^t(s) ds dx, \end{aligned}$$

and exploiting (2.20), we conclude

$$\begin{aligned} \frac{dJ(t)}{dt} &= -K \int_0^1 (\phi_x + \Psi)^2 dx + \rho_2 \int_0^1 \Psi_t^2 dx \\ &+ \frac{\rho_1}{K} \int_0^1 \phi_t \int_0^\infty g^{'}(s) \widehat{\eta}_x^t(s) ds dx + \left[ \widehat{b} \phi_x \Psi_x \right]_{x=0}^{x=1} \\ &+ \Big[ \phi_x(x,t) \int_0^\infty g(s) \widehat{\eta}_x^t(x,s) \Big]_{x=0}^{x=1} - \beta \int_0^1 (\phi_x + \Psi) \theta_{xt} dx. \end{aligned}$$

Young's inequality gives (2.21).

Next, to handle the boundary terms appearing in (2.21), we make use of the function $q(x) = 2 - 4x$, $x \in [0,1]$. Consequently, we have the following results.

**Lemma 2.8.** *Let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* $(2.1)-(2.4)$. *Then we have, for any* $\lambda_3 > 0$ *and* $\varepsilon_3 > 0$,

$$\begin{aligned}
&\Big[\phi_x(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s,x))\Big]_{x=0}^{x=1} \\
&\leq \frac{-\varepsilon_3}{K}\frac{d}{dt}\int_0^1 \rho_1 q(x)\phi_t\phi_x dx + 3\varepsilon_3\int_0^1 \phi_x^2 dx + \frac{2\rho_1\varepsilon_3}{K}\int_0^1 \phi_t^2 dx \\
&- \frac{1}{4\varepsilon_3}\frac{d}{dt}\int_0^1 \rho_2 q(x)\Psi_t(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds)dx \\
&+ \frac{1}{\varepsilon_3}\Big(\widehat{b}^2 + \frac{\widehat{b}^2}{8\lambda_3} + \frac{\widehat{b}^2\lambda_3}{2} + \varepsilon_3^2\Big)\int_0^1 \Psi_x^2 dx + \frac{\beta^2}{4\varepsilon_3\lambda_3}\int_0^1 \theta_{xt}^2 dx \qquad (2.22)\\
&+ \frac{G_0}{4\varepsilon_3}\Big(4 + \frac{1}{2\lambda_3} + 2\lambda_3\Big)\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}^t_x(s)\right|^2 dsdx \\
&+ \frac{\rho_2}{4\varepsilon_3}(2b+\varepsilon_3)\int_0^1 \Psi_t^2 dx + \frac{\lambda_3 K^2}{\varepsilon_3}\int_0^1(\phi_x+\Psi)^2 dx \\
&- \rho_2 g(0)C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}^t_x(s)\right|^2 dsdx.
\end{aligned}$$

**Proof.** By using Young's inequality, we easily see that, for $\varepsilon_3 > 0$,

$$\begin{aligned}
&\Big[\phi_x(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(x,s))\Big]_{x=0}^{x=1} \\
&\leq \varepsilon_3\left[\phi_x^2(1,t) + \phi_x^2(0,t)\right] + \frac{1}{4\varepsilon_3}\Big(\widehat{b}\Psi_x(0,t) + \int_0^\infty g(s)\widehat{\eta}^t_x(0,s)ds\Big)^2 \qquad (2.23)\\
&+ \frac{1}{4\varepsilon_3}\Big(\widehat{b}\Psi_x(1,t) + \int_0^\infty g(s)\widehat{\eta}^t_x(1,s)ds\Big)^2.
\end{aligned}$$

By exploiting

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big) \\
&= \int_0^1 \rho_2 q(x)\Psi_{tt}\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big) \\
&+ \int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_{tx} + \int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)ds\Big)
\end{aligned}$$

and equation $(2.1)_2$ we get

$$\frac{d}{dt}\int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big)$$

$$\begin{aligned} = \quad & \int_0^1 q(x)\Big(\widehat{b}\Psi_{xx} + \int_0^\infty g(s)\widehat{\eta}^t_{xx}(x,s)ds - K(\phi_x+\Psi) - \beta\theta_{tx}\Big) \\ & \times\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big)dx \\ & + \int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_{tx} + \int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)ds\Big). \end{aligned} \tag{2.24}$$

Simple calculation shows that

$$\begin{aligned} & \int_0^1 q(x)\Big(\widehat{b}\Psi_{xx} + \int_0^\infty g(s)\widehat{\eta}^t_{xx}(x,s)ds\Big)\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big)dx \\ & = -\frac{1}{2}\int_0^1 q^{'}(x)\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(x,s)ds\Big)^2 dx \\ & + \Big(\frac{q(x)}{2}\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big)^2\Big)_{x=0}^{x=1}. \end{aligned} \tag{2.25}$$

The last term in (2.24) can be treated as follows:

$$\begin{aligned} & \int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_{tx} + \int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)ds\Big) \qquad (2.26) \\ & = \rho_2\widehat{b}\int_0^1 q(x)\Psi_t\Psi_{tx}dx + \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)dsdx \\ & = -\frac{\rho_2\widehat{b}}{2}\int_0^1 q^{'}(x)\Psi_t^2dx + \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)dsdx \\ & = -\frac{\rho_2\widehat{b}}{2}\int_0^1 q^{'}(x)\Psi_t^2dx + \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g(s)(\Psi_{tx}(t) - \widehat{\eta}^t_{sx}(s))dsdx \\ & = -\frac{\rho_2\widehat{b}}{2}\int_0^1 q^{'}(x)\Psi_t^2dx + g_0\rho_2\int_0^1 q(x)\Psi_t\Psi_{tx} - \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g(s)\widehat{\eta}^t_{sx}(s)dsdx \\ & = -\frac{\rho_2(\widehat{b}+g_0)}{2}\int_0^1 q^{'}(x)\Psi_t^2dx + \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g'(s)\widehat{\eta}^t_x(s)dsdx. \end{aligned}$$

Inserting (2.25) and (2.26) into (2.24), we arrive at

$$\begin{aligned} & \Big(\widehat{b}\Psi_x(0,t) + \int_0^\infty g(s)\widehat{\eta}^t_x(0,s)ds\Big)^2 + \Big(\widehat{b}\Psi_x(1,t) + \int_0^\infty g(s)\widehat{\eta}^t_x(1,s)ds\Big)^2 \\ = \quad & -\frac{d}{dt}\int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}^t_x(s)ds\Big) \end{aligned}$$

$$
\begin{aligned}
&+2\int_0^1 \left(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(x,s)ds\right)^2 dx \\
&-K\int_0^1 q(x)(\phi_x+\Psi)\left(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds\right)dx \\
&-\beta\int_0^1 q(x)\theta_{tx}\left(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds\right)dx \\
&+2\rho_2(\widehat{b}+g_0)\int_0^1 \Psi_t^2 dx + \rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g^{'}(s)\widehat{\eta}_x^t(s)dsdx.
\end{aligned} \tag{2.27}
$$

Now, we estimate the terms in the right-hand side of (2.27), Hölder's and Young's inequalities, as follows:
The second term

$$
\begin{aligned}
&2\int_0^1 \left(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(x,s)ds\right)^2 dx \\
\leq\ &4\widehat{b}^2\int_0^1 \Psi_x^2 dx + 4G_p\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.
\end{aligned} \tag{2.28}
$$

The third term

$$
\begin{aligned}
&\Big|K\int_0^1 q(x)(\phi_x+\Psi)(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds)dx\Big| \\
&\leq 2K\Big|\int_0^1 (\phi_x+\Psi)(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds)dx\Big| \\
&\leq 4K^2\lambda_3\int_0^1(\phi_x+\Psi)^2dx + \frac{1}{4\lambda_3}\int_0^1\left(\widehat{b}\Psi_x+\int_0^\infty g(s)\widehat{\eta}_x^t(s)ds\right)^2dx, \\
&\leq 4K^2\lambda_3\int_0^1(\phi_x+\Psi)^2dx + \frac{\widehat{b}^2}{2\lambda_3}\int_0^1\Psi_x^2dx + \frac{G_p}{2\lambda_3}\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx.
\end{aligned} \tag{2.29}
$$

The fourth term

$$
\begin{aligned}
&\Big|\beta\int_0^1 q(x)\theta_{tx}(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds)dx\Big| \\
\leq\ &\frac{\beta^2}{\lambda_3}\int_0^1\theta_{tx}^2dx + 2\widehat{b}^2\lambda_3\int_0^1\Psi_x^2dx + 2G_p\lambda_3\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx.
\end{aligned} \tag{2.30}
$$

The last term

$$
\Big|\rho_2\int_0^1 q(x)\Psi_t\int_0^\infty g^{'}(s)\widehat{\eta}_x^t(s)dsdx\Big| \tag{2.31}
$$

$$\leq \quad \rho_2\varepsilon_3 \int_0^1 \Psi_t^2 dx - \rho_2 g(0)C(\varepsilon_3) \int_0^1 \int_0^\infty g^{'}(s) \left|\widehat{\eta}_x^t(s)\right|^2 dsdx.$$

Inserting (2.28) – (2.31) into (2.27), we obtain

$$\begin{aligned}
&\Big(\widehat{b}\Psi_x(0,t) + \int_0^\infty g(s)\widehat{\eta}_x^t(0,s)ds\Big)^2 + \Big(\widehat{b}\Psi_x(1,t) + \int_0^\infty g(s)\widehat{\eta}_x^t(1,s)ds\Big)^2 \\
&\leq -\frac{d}{dt}\int_0^1 \rho_2 q(x)\Psi_t\Big(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds\Big) \\
&+ \Big(4\widehat{b}^2 + \frac{\widehat{b}^2}{2\lambda_3} + 2\widehat{b}^2\lambda_3\Big)\int_0^1 \Psi_x^2 dx \ + \frac{\beta^2}{\lambda_3}\int_0^1 \theta_{xt}^2 dx \qquad (2.32) \\
&+ G_p\Big(4 + \frac{1}{2\lambda_3} + 2\lambda_3\Big)\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx \\
&+ 2\rho_2\Big(\widehat{b} + g_0 + \frac{\varepsilon_3}{2}\Big)\int_0^1 \Psi_t^2 dx + 4K^2\lambda_3\int_0^1(\phi_x+\Psi)^2 dx \\
&- \rho_2 g(0)C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.
\end{aligned}$$

Similarly, by using equation $(2.1)_1$, we arrive at

$$\begin{aligned}
\frac{d}{dt}\int_0^1 \rho_1 q\phi_t\phi_x dx \quad &\leq \quad -K\left[\phi_x^2(1) + \phi_x^2(0)\right] \qquad (2.33) \\
&+3K\int_0^1 \phi_x^2 dx + K\int_0^1 \Psi_x^2 dx + 2\rho_1\int_0^1 \phi_t^2 dx.
\end{aligned}$$

Hence the assertion of the lemma follows from (2.23), (2.32) and (2.33).

Let us introduce the functional

$$\mathcal{K}(t) := -\rho_1\int_0^1 \phi_t\phi dx - \rho_2\int_0^1 \Psi_t\Psi dx.$$

It easily follows, by using $\int_0^1 \Psi^2 dx \leq \int_0^1 \Psi_x^2 dx$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{K}(t) \quad &\leq \quad -\rho_1\int_0^1 \phi_t^2 dx - \rho_2\int_0^1 \Psi_t^2 dx + (\widehat{b} + \tfrac{3}{2})\int_0^1 \Psi_x^2 dx \qquad (2.34) \\
&+K\int_0^1 \phi_x^2 dx + \frac{\beta^2}{2}\int_0^1 \theta_{tx}^2 dx - \int_0^1 \Psi_x\int_0^\infty g(s)\widehat{\eta}_x^t(s)dsdx.
\end{aligned}$$

By using (2.11), we obtain, for any $\varepsilon_3 > 0$,

$$\frac{d}{dt}\mathcal{K}(t) \leq -\rho_1\int_0^1 \phi_t^2 dx - \rho_2\int_0^1 \Psi_t^2 dx + (\widehat{b} + \tfrac{3}{2} + \varepsilon_3)\int_0^1 \Psi_x^2 dx \qquad (2.35)$$

$$+ K\int_0^1 \phi_x^2 dx + \frac{\beta^2}{2}\int_0^1 \theta_{tx}^2 dx + C(\varepsilon_3)G_p\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.$$

Let us set

$$\Theta(t) := \int_0^1 (\rho_3\theta_t\theta + \frac{k}{2}\theta_x^2 + \gamma\Psi_x\theta)dx.$$

**Lemma 2.9.** *Let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* $(2.1) - (2.4)$. *Then we have, for any* $\varepsilon_2 > 0$,

$$\frac{d}{dt}\Theta(t) \le -\delta\int_0^1 \theta_x^2 dx + (\rho_3 + \frac{\gamma^2}{4\varepsilon_2})\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \Psi_x^2 dx. \tag{2.36}$$

**Proof.** We differentiate $\Theta(t)$ and use $(2.1)_3$ to obtain

$$\frac{d}{dt}\Theta(t) = \rho_3\int_0^1 \theta_t^2 dx - \delta\int_0^1 \theta_x^2 dx + \gamma\int_0^1 \Psi_x\theta_t dx.$$

Young's inequality then yields (2.36).

To finalize the proof of Theorem 2.1, we define the Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) := NE(t) + N_1I_1 + N_2I_2 + J(t) + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x dx \tag{2.37}$$
$$+ \frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\Psi_t(\widehat{b}\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(s)ds)dx + \mu\mathcal{K}(t) + \Theta(t).$$

Consequently, by using $(2.9), (2.13), (2.17), (2.21), (2.22), (2.34), (2.36)$,

$$\int_0^1 \theta_t^2 dx \le \int_0^1 \theta_{tx}^2 dx,$$

and

$$\int_0^1 \phi_x^2 dx \le 2\int_0^1 (\phi_x + \Psi)^2 dx + 2\int_0^1 \Psi_x^2 dx$$

we get

$$\frac{d}{dt}\mathcal{L}(t) \le -\left[N\beta k - C_1\right]\int_0^1 \theta_{tx}^2 dx + \Lambda_1\int_0^1 \Psi_x^2 dx + \Lambda_2\int_0^1 \phi_t^2 dx \tag{2.38}$$
$$+ \Lambda_3\int_0^1 \Psi_t^2 dx + \Lambda_4\int_0^1 (\phi_x + \Psi)^2 dx$$
$$+ \left[N\frac{\gamma}{2} - C_2\right]\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx - \delta\int_0^1 \theta_x^2 dx$$

$$+ (\rho_3 + \frac{\gamma^2}{4\varepsilon_2}) \int_0^1 \theta_t^2 dx + C_3 \int_0^1 \int_0^\infty g^p(s) \left|\widehat{\eta}_x^t(s)\right|^2 dsdx,$$

where $C_1, C_2$ and $C_3$ are positive constants independent of $N$ and

$$\begin{aligned}
\Lambda_1 &= N_1(-\frac{\widehat{b}}{2} + \lambda_1) + N_2\varepsilon_2\widehat{b}^2 + \varepsilon_2 + 2(3\varepsilon_3 + \mu K) \\
&\quad + \frac{1}{\varepsilon_3}(\widehat{b}^2 + \frac{\widehat{b}^2}{8\lambda_3} + \frac{\widehat{b}^2\lambda_3}{2} + \frac{\varepsilon_3^2}{4}) + \mu(\widehat{b} + \frac{3}{2} + \varepsilon_3) \\
\Lambda_2 &= N_1\varepsilon_1\rho_1 + \varepsilon_3 + \frac{2\rho_1\varepsilon_3}{K} - \mu\rho_1 \\
\Lambda_3 &= N_1(\rho_2 + \frac{\rho_1}{4\varepsilon_1}) - N_2\frac{g_0\rho_2}{2} + \rho_2 + \frac{1}{4\varepsilon_3}(2\rho_2 b + \rho_2\varepsilon_3) - \mu\rho_2 \\
\Lambda_4 &= \frac{K^2\lambda_3}{\varepsilon_3} - \frac{K}{2} + N_2\varepsilon_2 K^2 + 2(3\varepsilon_3 + \mu K).
\end{aligned}$$

At this point, we have to choose our constants very carefully. First, we take $\lambda_3 = \varepsilon_3^2$, $\mu = 1/16$, and $\lambda_1 < \widehat{b}/4$, then we pick

$$\varepsilon_3 \leq \min(\frac{K}{4(K^2+6)}, \frac{K\rho_1}{32(K+2\rho_1)}).$$

Once $\varepsilon_3$ and $\mu$ (hence $\lambda_3$) are fixed, we then choose $N_1$ so large that

$$N_1\frac{\widehat{b}}{8} > \frac{1}{\varepsilon_3}(\widehat{b} + \frac{\widehat{b}^2}{8\lambda_3} + \frac{\widehat{b}^2\lambda_3}{2} + \frac{\varepsilon_3^2}{4}) + 2(3\varepsilon_3 + \mu K) + \mu(\widehat{b} + \frac{3}{2} + \varepsilon_3).$$

After that, we pick $\varepsilon_1$ small enough so that $\varepsilon_1 \leq 1/64N_1$ and $N_2$ large enough so that

$$N_2\frac{g_0\rho_2}{2} > N_1(\rho_2 + \frac{\rho_1}{4\varepsilon_1}) + \rho_2 + \frac{1}{4\varepsilon_3}(2\rho_2 b + \rho_2\varepsilon_3) - \mu\rho_2.$$

Now, we pick $\varepsilon_2$ so small that

$$\varepsilon_2 < \min\left\{\frac{16}{N_2K}, \frac{N_1\widehat{b}}{16(N_2\widehat{b}^2+1)}\right\}.$$

Finally, after fixing all constants, we choose $N$ large enough so that

$$N > \max\left\{\frac{C_1}{\beta k}, \frac{2(C_2 + C_3/k_0)}{\gamma}\right\}.$$

Consequently, there exists $\sigma_1 > 0$ such that (2.38) takes the form

$$\frac{d}{dt}\mathcal{L}(t) \leq -\sigma_1\Big[\int_0^1 (\theta_t^2 + \theta_x^2 + \Psi_x^2 + \Psi_t^2 + \phi_t^2 + (\phi_x + \Psi)^2)dx$$

$$+\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big]. \tag{2.39}$$

On the other hand, we can choose $N$ even larger if needed so that

$$\beta_1 \le \mathcal{L}(t) \le \beta_2 E(t) \tag{2.40}$$

for two positive constants $\beta_1, \beta_2$. We distinguish two cases.

**Case 1**: $p = 1$. A combination of (2.39) and (2.40) leads to

$$\frac{d}{dt}\mathcal{L}(t) \le -\xi\mathcal{L}(t). \tag{2.41}$$

A simple integration of (2.41) over $(0, t)$ and use of (2.39) lead to (2.7).

**Case 2:** $p > 1$. We use (2.5) and (2.9) to get

$$\begin{aligned}
E^{2p-1}(t) &\le C(E(0))^{2p-2}\int_0^1(\phi_t^2+\Psi_t^2+|\phi_x+\Psi|^2+\Psi_x^2+\theta_t^2+\theta_x^2)dx \\
&\quad +C\Big[\int_0^1\int_0^\infty g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big]^{2p-1} \qquad (2.42)\\
&\le C(E(0))^{2p-2}\int_0^1(\phi_t^2+\Psi_t^2+|\phi_x+\Psi|^2+\Psi_x^2+\theta_t^2+\theta_x^2)dx \\
&\quad +CC_0\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.
\end{aligned}$$

A combination of (2.39), (2.40) and (2.42) gives

$$\mathcal{L}'(t) \le -cE^{2p-1}(t) \le -c\mathcal{L}^{2p-1}(t). \tag{2.43}$$

A simple integration of (2.43) leads to

$$\mathcal{L}(t) \le \frac{C}{(t+1)^{1/(2p-2)}}. \tag{2.44}$$

To obtain (2.8), we observe that

$$\begin{aligned}
&\int_0^1\int_0^t g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx = \int_0^1\int_0^t g(s)\left|\widehat{\eta}_x^t(s)\right|^{\frac{2}{p}}\left|\widehat{\eta}_x^t(s)\right|^{\frac{2(p-1)}{p}} dsdx \\
&\le \Big(\int_0^1\int_0^t \left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big)^{(p-1)/p}\Big(\int_0^1\int_0^t g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big)^{1/p} \qquad (2.45)
\end{aligned}$$

and use (2.44) and $\widehat{\eta}_x^t(x,s) = \Psi_x(x,t) - \Psi_x(x,t-s)$ to get

$$\int_0^t\int_0^1 \left|\widehat{\eta}_x^t(s)\right|^2 dxds$$

$$\begin{aligned}
&\leq 2\int_0^t\int_0^1 \left|\Psi_x(x,t)\right|^2 dxds + 2\int_0^t\int_0^1 \left|\Psi_x(x,t-s)\right|^2 dxds \\
&\leq \frac{4}{\gamma\widehat{b}} tE(t) + \frac{4}{\gamma\widehat{b}}\int_0^t E(t-s)ds \\
&\leq \frac{Ct}{(t+1)^{1/(2p-2)}} + C\int_0^t \frac{ds}{(t-s+1)^{1/(2p-2)}} \\
&\leq \frac{C}{(t+1)^{(3-2p)/(2p-2)}} + \frac{2p-2}{3-2p}C\Big[1 - \frac{1}{(t+1)^{(3-2p)/(2p-2)}}\Big] \\
&\leq \Pi, \qquad p < 3/2,
\end{aligned}$$

where $\Pi$ is a constant independent of $t$. Hence, we get

$$\int_0^\infty\int_0^1 \left|\widehat{\eta}_x^t(s)\right|^2 dxds \leq \Pi.$$

Consequently, we have from (2.45)

$$\int_0^1\int_0^\infty g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx \leq \Pi^{(p-1)/p}\Big(\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big)^{1/p}$$

or

$$\Big(\int_0^\infty\int_0^1 g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dxds\Big)^p \leq C\int_0^\infty\int_0^1 g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dxds. \tag{2.46}$$

Similarly to (2.42), we obtain

$$\begin{aligned}
E^p(t) \quad &\leq \quad C\int_0^1 (\phi_t^2 + \Psi_t^2 + |\phi_x + \Psi|^2 + \Psi_x^2 + \theta_t^2 + \theta_x^2)dx \\
&\qquad +C\Big[\int_0^1\int_0^\infty g(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx\Big]^p \qquad (2.47)\\
&\leq \quad C\int_0^1 (\phi_t^2 + \Psi_t^2 + |\phi_x + \Psi|^2 + \Psi_x^2 + \theta_t^2 + \theta_x^2)dx \\
&\qquad +C\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2 dsdx.
\end{aligned}$$

A combination of (2.39), (2.40), and (2.47) yields

$$\mathcal{L}'(t) \leq -cE^p(t) \leq -c\mathcal{L}^p(t). \tag{2.48}$$

A simple integration of (2.48) gives

$$\mathcal{L}(t) \leq \frac{C}{(t+1)^{1/(p-1)}}. \tag{2.49}$$

Again, use of (2.40) leads to (2.8). This completes the proof of Theorem 2.1.

## 3. POLYNOMIAL DECAY $\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$

In this section, we show that in the case of different wave-speed propagation ($\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$), the solution energy $E(t)$ decays at a polynomial rate even if the relaxation function $g$ decays exponentially provided that the initial data are regular enough. Let's define the second-order energy by

$$E_2(t) = E_1(\phi_t, \Psi_t, \theta_t, \widehat{\eta}_t^t), \tag{3.1}$$

where $E_1$ is given in (2.5).

**Theorem 3.1** *Suppose that*

$$\frac{\rho_1}{K} \neq \frac{\rho_2}{b} \tag{3.2}$$

*and let*

$$\begin{aligned} \phi_0, \Psi_0,\ \theta_0 &\in H^2(0,1) \cap H_0^1(0,1), \quad \widehat{\eta}_0^t \in L_g^2(\mathbb{R}^+, H^2(0,1) \cap H_0^1(0,1)), \\ \phi_1, \Psi_1,\ \theta_1 &\in H_0^1(0,1). \end{aligned} \tag{3.3}$$

*Then there exists a positive constants $C$, such that,* for all $t \geq 0$,

$$E(t) \leq Ct^{-1/(2p-1)}, \qquad p \geq 1. \tag{3.4}$$

To prove this result, we need two lemmas.

**Lemma 3.2.** *Suppose that* (3.3) *holds and let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* $(2.1) - (2.4)$. *Then we have*

$$\frac{dE_2(t)}{dt} = -\beta k \int_0^1 \theta_{ttx}^2 dx + \frac{\gamma}{2} \int_0^1 \int_0^\infty g'(s) \left|\widehat{\eta}_{tx}^t(s)\right|^2 ds dx \leq 0. \tag{3.5}$$

**Proof.** We differentiate equations (2.1) with respect to time and then multiply by $\gamma\phi_{tt}$, $\gamma\Psi_{tt}$, and $\beta\theta_{tt}$ respectively. By integrating over $(0,1)$ and summing up, as in Lemma 2.2, we obtain (3.5).

**Lemma 3.3.** *Suppose that* $(3.2), (3.3)$ *hold and let* $(\phi, \Psi, \theta, \widehat{\eta}^t)$ *be a solution of* $(2.1) - (2.4)$. *Then, for* $\varepsilon_3 > 0$, *we conclude*

$$\begin{aligned} \frac{dJ(t)}{dt} \leq\ & \left[\phi_x (b\Psi_x + \int_0^\infty g(s)\widehat{\eta}_x^t(x,s))\right]_{x=0}^{x=1} - \frac{K}{2}\int_0^1 (\phi_x + \Psi)^2 dx \\ & + \rho_2 \int_0^1 \Psi_t^2 dx + \varepsilon_3 \int_0^1 \phi_t^2 dx + \frac{\beta^2}{2K}\int_0^1 \theta_{tx}^2 dx \\ & - g(0) C(\varepsilon_3) \int_0^1 \int_0^\infty g'(s) \left|\widehat{\eta}_x^t(s)\right|^2 ds dx \end{aligned} \tag{3.6}$$

$$+C(\varepsilon_3)\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}^t_{tx}(s)\right|^2 dsdx.$$

**Proof.** Similarly to the proof of Lemma 2.7, we have

$$\begin{aligned}\frac{dJ(t)}{dt} &= \int_0^1(\phi_x+\Psi)\Big[\widehat{b}\Psi_{xx}+\int_0^\infty g(s)\widehat{\eta}^t_{xx}(x,s)ds-K(\phi_x+\Psi)-\beta\theta_{tx}\Big]dx\\ &+\rho_2\int_0^1\Psi_t^2dx+\widehat{b}\int_0^1\Psi_x(\phi_x+\Psi)_xdx \qquad (3.7)\\ &+(\frac{\rho_1\widehat{b}}{K}-\rho_2)\int_0^1\Psi_{tx}\phi_tdx+\frac{\rho_1}{K}\int_0^1\phi_t(t)\int_0^\infty g(s)(\Psi_t-\widehat{\eta}^t_s)_x(s)dsdx\\ &+\int_0^1(\phi_x+\Psi)_x\int_0^\infty g(s)\widehat{\eta}^t_x(s)dsdx.\end{aligned}$$

Since $\frac{\rho_1}{K}\neq\frac{\rho_2}{b}$ we then have to handle the term $\int_0^1\Psi_{tx}\phi_tdx$. For this, we use the idea of [9] to obtain

$$\begin{aligned}\int_0^1\Psi_{tx}\phi_tdx &= \frac{1}{g_0}\int_0^1\Big(\int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)ds\Big)\phi_tdx \qquad (3.8)\\ &-\frac{1}{g_0}\int_0^1\Big(\int_0^\infty g'(s)\widehat{\eta}^t_x(s)ds\Big)\phi_tdx,\end{aligned}$$

where $g_0=\int_0^\infty g(s)ds$. Therefore, using Young's inequality and (2.11), we estimate the terms of (3.8) as follows:

$$\begin{aligned}&\frac{1}{g_0}(\frac{\rho_1\widehat{b}}{K}-\rho_2)\int_0^1\Big(\int_0^\infty g(s)\widehat{\eta}^t_{tx}(s)ds\Big)\phi_tdx \qquad (3.9)\\ \leq\ &\frac{\varepsilon_3}{2}\int_0^1\phi_t^2dx+C(\varepsilon_3)\int_0^1\int_0^\infty g^p(s)|\widehat{\eta}^t_{tx}(s)|^2dsdx\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{g_0}(\frac{\rho_1\widehat{b}}{K}-\rho_2)\int_0^1\Big(\int_0^\infty g'(s)\widehat{\eta}^t_x(s)ds\Big)\phi_tdx \qquad (3.10)\\ \leq\ &\frac{\varepsilon_3}{2}\int_0^1\phi_t^2dx-g(0)C(\varepsilon_3)\int_0^1\int_0^\infty g'(s)|\widehat{\eta}^t_x(s)|^2dsdx.\end{aligned}$$

By inserting (3.9) and (3.10) into (3.8) and taking into account the estimates of Lemma 2.7, the desired result (3.6) follows.

**Proof of Theorem 3.1.** To finalize the proof of Theorem 3.1, we define the Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) := N\left[E_1(t)+E_2(t)\right]+N_1I_1+N_2I_2+J(t)+\frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x dx$$
$$+\frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\Psi_t(\widehat{b}\Psi_x+\int_0^\infty g(s)\widehat{\eta}_x^t(s)ds)dx+\mu\mathcal{K}(t)+\Theta(t). \quad (3.11)$$

Consequently, by taking the time derivative of $\mathcal{L}(t)$, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\left[N\beta k-C_1\right]\int_0^1 \theta_{tx}^2dx+\Lambda_1\int_0^1\Psi_x^2dx+\overline{\Lambda_2}\int_0^1\phi_t^2dx \\
&+\Lambda_3\int_0^1\Psi_t^2dx+\Lambda_4\int_0^1(\phi_x+\Psi)^2dx \\
&+\left[N\frac{\gamma}{2}-C_2\right]\int_0^1\int_0^\infty g'(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx-\delta\int_0^1\theta_x^2dx \\
&+(\rho_3+\frac{\gamma^2}{4\varepsilon_2})\int_0^1\theta_t^2dx+C_3\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx \\
&+C(\varepsilon_3)\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_{tx}^t(s)\right|^2dsdx \qquad (3.12) \\
&-N\beta k\int_0^1\theta_{ttx}^2dx+N\frac{\gamma}{2}\int_0^1\int_0^\infty g^{'}(s)\left|\widehat{\eta}_{tx}^t(s)\right|^2dsdx,
\end{aligned}$$

where

$$\overline{\Lambda_2}=N_1\varepsilon_1\rho_1+2\varepsilon_3+\frac{2\rho_1\varepsilon_3}{K}-\mu\rho_1.$$

Choosing the constants carefully as in section 2, it is easy to see that, for $\sigma_2>0$, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\sigma_2\Big[\int_0^1(\theta_t^2+\theta_x^2+\Psi_x^2+\Psi_t^2+\phi_t^2+(\phi_x+\Psi)^2)dx \qquad (3.13) \\
&+\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_x^t(s)\right|^2dsdx+\int_0^1\int_0^\infty g^p(s)\left|\widehat{\eta}_{tx}^t(s)\right|^2dsdx\Big].
\end{aligned}$$

Moreover, we can choose $N$ so large that $\mathcal{L}(t)\geq 0$.

We distinguish two cases:

**Case 1.** $p=1$. It is not hard to see that

$$\frac{d}{dt}\mathcal{L}(t)\leq -\alpha E(t). \quad (3.14)$$

Direct integration of (3.14) gives

$$\alpha \int_0^t E(s)ds \leq \mathcal{L}(0) - \mathcal{L}(t) \leq \mathcal{L}(0), \qquad \forall t \geq 0. \tag{3.15}$$

By using (3.11), one can find $\sigma_3 > 0$, such that

$$\mathcal{L}(0) \leq \sigma_3(E_1(0) + E_2(0)), \qquad \forall t \geq 0. \tag{3.16}$$

Hence (3.15) and (3.16) imply

$$\int_0^t E(s)ds \leq C(E_1(0) + E_2(0)), \qquad \forall t \geq 0. \tag{3.17}$$

By noting that

$$\frac{d}{dt}(tE(t)) = E(t) + t\frac{d}{dt}E(t) \leq E(t),$$

a simple integration leads to

$$tE(t) \leq \int_0^t E(s)ds \leq C(E_1(0) + E_2(0)), \qquad \forall t \geq 0.$$

Consequently, we get

$$E(t) \leq \frac{C}{t}(E_1(0) + E_2(0)), \qquad \forall t \geq 0.$$

**Case 2.** $p > 1$. By using (2.43), we obtain

$$\frac{d\mathcal{L}}{dt} \leq -cE^{2p-1}(t)$$

which implies

$$\int_0^t E^{2p-1}(s)ds \leq c(\mathcal{L}(0) - \mathcal{L}(t)) \leq c\mathcal{L}(0), \qquad \forall t \geq 0. \tag{3.18}$$

On the other hand, we have

$$\frac{d}{dt}(tE^{2p-1}(t)) = E^{2p-1}(t) + (2p-1)tE^{2p-2}\frac{d}{dt}E(t) \leq E^{2p-1}(t). \tag{3.19}$$

Similar calculations, using (3.16), (3.18), and (3.19), yield

$$E(t) \leq Ct^{-1/(2p-1)}, \qquad \forall t \geq 0.$$

This completes the proof of Theorem 3.1.

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