61. *Energy Decay of Solutions of Dissipative Wave Equations*

By Akitaka MATSUMURA
Department of Applied Mathematics and Physics,
Faculty of Engineering, Kyoto University

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1. Introduction. We shall investigate the energy decay of the solutions to the following Cauchy problem;

$$(1)\qquad \begin{cases} L(u)=u_{tt}-\Delta u+a(x,t)u_t=0, & x\in R^n,\ t\geq 0,\\ u(x,0)=\phi(x)\in C_0^\infty, & u_t(x,0)=\psi(x)\in C_0^\infty, \end{cases}$$

where $a(x,t)\in\mathscr{B}^1$ *), $a(x,t)\geq 0$ and Δ=Laplacian in R^n. Rauch and Taylor [3] showed that, if $a(x,t)\equiv a(x)$ and $a(x)$ has compact support, the energy $E(t)$ defined by

$$E(t)=\int_{R^n}|u_t(t)|^2+|\nabla u(t)|^2\,dx \qquad (\nabla\text{; gradient in } R^n)$$

for the solutions of (1) does not decay as t goes to infinity. More generally, Mochizuki [2] showed that, if $0\leq a(x,t)\leq c(1+|x|)^{-1-\delta}$ for some positive constants c and δ ($n\neq 2$), $E(t)\not\to 0$ as $t\to+\infty$. On the other hand, we have from the usual energy estimates that if $a(x,t)\geq$Const. >0 and $a_t(x,t)\leq 0$, $E(t)$ decays like $0(t^{-1})$. In this paper we give more general conditions which guarantee the decay of $E(t)$ and an application to the nonlinear wave equations. Now, letting m be a positive constant, we list up the assumptions:

(A–1) There exist some positive constants r, K and ε such that

$$\operatorname{supp}\phi(x)\cup\operatorname{supp}\psi(x)\subset\{x\in R^n\,|\,|x|\leq r\},$$

$$\min_{|x|\leq mt+r} a(x,t)\geq (K+\varepsilon t)^{-1} \qquad \text{for all } t\geq 0,$$

$$\max_{|x|\leq mt+r} a_t(x,t)\leq \varepsilon^2(2\gamma^2+6\gamma+3)(2+\gamma)^{-1}(K+\varepsilon t)^{-2} \qquad \text{for all } t\geq 0$$

where $\gamma=(3\varepsilon-2+\sqrt{9\varepsilon^2-4\varepsilon+4})/2$.

(A–2) $a(x,t)$ belongs to $\mathscr{B}^{k+1}$ ($k=1,2,\cdots$) and satisfies

$$\max_{|x|\leq mt+r}\sum_{i=1}^{k}\left|\left(\frac{\partial}{\partial t}\right)^i a(x,t)\right|\leq \text{Const.}(1+t)^{-1} \qquad \text{for all } t\geq 0.$$

(A–3) $a(x,t)\equiv(K+\varepsilon t)^{-1}$ for some positive constants K and ε.

Then we have the following

Theorem 1. *Suppose* (A–1) *with* $m=1$. *Then the energy* $E(t)$ *for the solutions of* (1) *decays like* $0(t^{-2/(2+\gamma)})$. *Furthermore suppose* (A–2) (*resp.* (A–3)) *with* $m=1$. *Then the solutions of* (1) *satisfy*

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*) $\mathscr{B}^k$ is the set of all functions defined on $R^n\times[0,+\infty)$ such that all their partial derivatives of order $\leq k$ exist and are continuous and bounded.

$$\left\|\left(\frac{\partial}{\partial t}\right)^{k+1}u(t)\right\|_0^2+\sum_{i=0}^{k}\left\|\left(\frac{\partial}{\partial t}\right)^{i}\nabla u(t)\right\|_{k-i}^2\leq \text{Const.}(1+t)^{-2/(2+\gamma+\theta)}$$

$$\left(\text{resp.}\ \begin{array}{ll}\leq \text{Const.}(1+t)^{-2/(2\varepsilon+\theta)} & \textit{for } \varepsilon>2^{-1}\\ \leq \text{Const.}(1+t)^{-2/(1+\theta)} & \textit{for } \varepsilon\leq 2^{-1}\end{array}\right)$$

where θ is any fixed positive number and $\|\cdot\|_i$ denotes the usual $H^i(R^n)$ norm.

As one of the applications to the quasilinear strictly hyperbolic equations, we consider the following Cauchy problem;

$$(2)\quad \begin{cases} u_{tt}-\sum_{i=1}^{n}(1+\sigma_i(u_{x_i}))u_{x_ix_i}+a(x,t)u_t=0, & x\in R^n,\ t\geq 0,\\ u(x,0)=\phi(x)\in C_0^\infty,\ u_t(x,0)=\psi(x)\in C_0^\infty, \end{cases}$$

where $\sigma_i(\tau)$ belongs to $C^\infty(R^1)$ and satisfies that for $k\geq 0$ and $\tau\in R^1$

$$\left|\left(\frac{d}{d\tau}\right)^k\sigma_i(\tau)\right|\leq \text{Const.}\ |\tau|^{\max(q_i-k,0)}\qquad (q_i>0).$$

For the strict hyperbolicity of (2), see (8) and (9) below.

If $a(x,t)\equiv a(x)\geq\text{Const.}>0$, our arguments in [1] with a slight modification are applicable to (2). Now putting $s=[(n/2)]+2$ and $\nu=\|\phi\|_{s+1}+\|\psi\|_s$, we have the following

Theorem 2. *Suppose* (A–1) *and* (A–2) (*resp.* (A–3)) *with $m=2$ and $k=s$. Moreover suppose $q_i\geq 2+\gamma+\theta$ (resp. $q_i\geq 2\varepsilon+\theta$ if $\varepsilon>2^{-1}$, $q_i\geq 1+\theta$ if $\varepsilon\leq 2^{-1}$) ($1\leq i\leq n$) for some positive constant θ. Then there exists a positive constant ν_0 such that* (2) *has a unique C^2-global solution for $0<{}^\forall\nu\leq\nu_0$ and $E(t)$ decays like $0(t^{-2/(2+\gamma+\theta)})$ (resp. $0(t^{-2/(2\varepsilon+\theta)})$ for $\varepsilon>2^{-1}$, $0(t^{-2/(1+\theta)})$ for $\varepsilon\leq 2^{-1}$).*

2. Proof of Theorem 1. Putting $v=(1+\delta t)^p u$ ($\delta>0$, $p>0$), we have

$$\tilde{L}(v)=(1+\delta t)^p L((1+\delta t)^{-p}v)$$
$$=v_{tt}-\Delta v+A(t)v=0$$

where

$$A(t)v=(a-2\delta p(1+\delta t)^{-1})v_t+\delta p(1+\delta t)^{-1}(\delta(p+1)(1+\delta t)^{-1}-a)v.$$

Calculating

$$\int\tilde{L}(v)(v_t+\lambda(1+\delta t)^{-1}v)dx=\frac{d}{dt}\int\frac{1}{2}B(v)dx+\int C(v)dx\qquad(\lambda>0),$$

we have

$$B(v)=v_t^2+|\nabla v|^2+2\lambda(1+\delta t)^{-1}vv_t$$
$$+(1+\delta t)^{-1}\{(\lambda-\delta p)a+\delta(1+\delta t)^{-1}(\delta p(p+1)+\lambda(1-2p))\}v^2,$$
$$C(v)=(a-(2\delta p+\lambda)(1+\delta t)^{-1}v_t^2+\lambda(1+\delta t)^{-1}|\nabla v|^2$$
$$+\delta(1+\delta t)^{-2}\{2^{-1}(\lambda-2\lambda p-p\delta)a+\delta(1+\delta t)^{-1}(\lambda(p^2-p+1)$$
$$+\delta p(p+1))\}v^2+2^{-1}(1+\delta t)^{-1}(\delta p-\lambda)a_tv^2.$$

In the above equalities, we choose δ, λ and p as

$$p=\lambda(2\lambda+\delta)^{-1},\quad \delta=\varepsilon K^{-1},\quad K^{-1}=\lambda(2\lambda+3\delta)(2\lambda+\delta)^{-1}+\lambda\alpha$$

where α is a fixed nonnegative number. Then we note $p^{-1}=2+\gamma+0(\sqrt{\alpha})$ where γ is as in (A–1). Now, noting that $v(x,t)$ is supported in $|x|\leq r+t$, we have from (A–2) that for $|x|\leq r+t$

$$B(v)\geq\delta(2\lambda+3\delta)^{-1}v_t^2+|\nabla v|^2+2\lambda\delta^3(2\lambda+\delta)^{-2}(1+\delta t)^{-2}v^2, \tag{3}$$

$$\begin{aligned}C(v)\geq{}&\alpha\lambda(1+\delta t)^{-1}v_t^2+\lambda(1+\delta t)^{-1}|\nabla v|^2\\&+\frac{9}{2}\alpha\varepsilon\lambda^3\delta^2(2+\gamma)^{-1}(2\lambda+\delta)^{-1}(1+\delta t)^{-3}v^2.\end{aligned} \tag{4}$$

So we got the first part of Theorem 1 easily from (3) and (4) with $\alpha=0$. For the proof of the second part, let α be any fixed positive number. Putting $(\partial/\partial t)^i v=v^i$ and $(\partial/\partial t)^i A(t)=A^i(t)$ $(i\geq 0)$, we have

$$\left(\frac{\partial}{\partial t}\right)^i\tilde{L}(v)=\tilde{L}(v^i)+\sum_{j=1}^{i}\binom{i}{j}A^j(t)v^{i-j}\qquad(i\geq 1).$$

Now it follows from (A–2) that for $\forall\theta>0$ and $\exists C_i(\theta)$ (constants)

$$\begin{aligned}&\left|\left(\sum_{j=1}^{i}\binom{i}{j}A^j(t)v^{i-j}\right)(v^{i+1}+\lambda(1+\delta t)^{-1}v^i)\right|\\&\quad\leq\theta(1+\delta t)^{-1}|v^{i+1}|^2+C_i(\theta)(1+\delta t)^{-1}\left(\sum_{j=1}^{i}|v^j|^2+(1+\delta t)^{-2}v^2\right)\\&\hspace{22em}(1\leq i\leq k).\end{aligned}$$

Let β_i $(0\leq i\leq k)$ be a positive constant. Then, from (4) and (5), there exists some positive constant c such that

$$\begin{aligned}0&=\sum_{i=0}^{k}\beta_i\int\left(\left(\frac{\partial}{\partial t}\right)^i\tilde{L}(v)\right)(v^{i+1}+\lambda(1+\delta t)^{-1}v^i)dx\\
&\geq\frac{d}{dt}\left(\sum_{i=0}^{k}\beta_i\int\frac{1}{2}B(v^i)dx\right)+\sum_{i=0}^{k}c\beta_i(1+\delta t)^{-1}|v^{i+1}|^2\,dx\\
&\quad+\int c\beta_0(1+\delta t)^{-3}v^2dx-\int\sum_{i=0}^{k}\theta\beta_i(1+\delta t)^{-1}|v^{i+1}|^2\,dx\\
&\quad-\int\sum_{i=1}^{k}\beta_iC_i(\theta)(1+\delta t)^{-1}\left(\sum_{j=1}^{i}|v^j|^2+(1+\delta t)^{-2}v^2\right)dx\\
&\geq\frac{d}{dt}\left(\sum_{i=0}^{k}\beta_i\int\frac{1}{2}B(v^i)dx\right)+\int\beta_k(c-\theta)(1+\delta t)^{-1}|v^{k+1}|^2\,dx\\
&\quad+\int\sum_{i=0}^{k-1}(1+\delta t)^{-1}\left((c-\theta)\beta_i-\sum_{j=i+1}^{k}\beta_jC_j(\theta)\right)|v^{i+1}|^2\,dx\\
&\quad+\int\left(c\beta_0-\sum_{j=1}^{k}\beta_jC_j(\theta)\right)(1+\delta t)^{-3}v^2dx.\end{aligned}$$

Now we choose θ and β_i as

$$c-\theta>0,\ (c-\theta)\beta_i-\sum_{j=i+1}^{k}\beta_jC_j(\theta)>0\qquad\text{for } 0\leq i\leq k-1.$$

Thus we have

$$\frac{d}{dt}\left(\sum_{i=0}^{k}\beta_i\int\frac{1}{2}B(v^i)dx\right)\leq 0. \tag{6}$$

Hence the second part of Theorem 1 follows from (3), (6) and the estimates for

$$\|\Delta v^m\|_j=\|v^{m+2}+\sum_{i=0}^{m}\binom{m}{i}A^i(t)v^{m-i}\|_j\qquad\text{for } 0\leq m+j\leq k-1.$$

Finally, for (A–3), we can give a proof in the same way as above by choosing $\delta=\varepsilon K^{-1}$, $\lambda=\alpha\delta$ and $p=(2\varepsilon+\theta)^{-1}$ for $\varepsilon>2^{-1}$, $p=(1+\theta)^{-1}$ for $\varepsilon\leq 2^{-1}$.

3. Proof of Theorem 2. Putting $v=(1+\delta t)^p u$, we may consider the next Cauchy problem;

$$(7)\qquad \begin{cases}\hat{L}(v)\equiv v_{tt}-\sum_{i=1}^{n}(1+\sigma_i((1+\delta t)^{-p}v_{x_i})v_{x_ix_i}+A(t)v=0,\\ v(0)=\phi,\ v_t(0)=\delta p\phi+\psi.\end{cases}$$

First we choose a positive constant μ_1 so that for any $t\geq 0$ and $1\leq i\leq n$

$$(8)\qquad \sup_{x\in R^n}|\sigma_i((1+\delta t)^{-p}w(t))|\leq\frac{1}{2}\qquad \text{if } \|w(t)\|_{[n/2]+1}\leq\mu_1.$$

For the proof it suffices to show the following a-priori estimates: There exist the positive constants μ_0 and $\chi_0(<1)$ such that if $v(x,t)$ satisfies (7) for $0\leq t\leq T$ (any fixed positive number) and

$$(9)\qquad \begin{aligned}&\|v^{s+1}(t)\|_0+\sum_{i=0}^{s}\|\nabla v^i(t)\|_{s-1}\leq\mu\qquad(0\leq\mu\leq\mu_1),\\ &\|v(t)\|_0\leq\mu(1+\delta t),\end{aligned}$$

then $v(x,t)$ satisfies

$$(10)\qquad \begin{aligned}&\|v^{s+1}(t)\|_0+\sum_{i=0}^{s}\|\nabla v^i(t)\|_{s-i}\leq\chi_0\mu,\\ &\|v(t)\|_0\leq\chi_0\mu(1+\delta t)\end{aligned}$$

for $0<\mu\leq\mu_0$ and $0<\nu\leq\nu_0(\mu)$ where $\nu_0(\mu)$ denotes some positive constant depending only on μ and where μ_0 and χ_0 are independent of T. We note that $v(x,t)$ is supported in $|x|\leq r+2t$ from (8) for this case. Then under the assumptions above, choosing $\beta_i(>0)$ similarly as before, there exist the positive constants c_1 and c_2 such that

$$(11)\qquad \begin{aligned}0&=\sum_{i=0}^{s}\beta_i\int\left(\left(\frac{\partial}{\partial t}\right)^i\hat{L}(v)\right)(v^{i+1}+\lambda(1+\delta t)^{-1}v^i)dx\\ &\geq\frac{d}{dt}\left(\sum_{i=0}^{s}\beta_i\int D(v^i)dx\right)\\ &\quad+c_1(1+\delta t)^{-1}(\|v^{s+1}\|_0^2+\sum_{i=0}^{s}\|\nabla v^i\|_0^2+(1+\delta t)^{-2}\|v\|_0^2)\\ &\quad-\mu c_2(1+\delta t)^{-1}\left(\|v^{s+1}\|_0^2+\sum_{i=0}^{s}\|\nabla v^i\|_{s-i}^2+(1+\delta t)^{-2}\|v\|_0^2\right)\end{aligned}$$

where

$$(12)\qquad D(w)=B(w)+\sum_{i=1}^{n}\sigma_i((1+\delta t)^{-p}v_{x_i})|w_{x_i}|^2.$$

On the other hand, estimating

$$\left\|\sum_{i=1}^{n}(1+\sigma_i)v^m_{x_ix_i}\right\|_j=\left\|v^{m+2}-\sum_{i=1}^{n}\sum_{k=1}^{m}\binom{m}{k}v^{m-k}_{x_ix_i}\left(\frac{\partial}{\partial t}\right)^k\sigma_i+\sum_{i=1}^{m}\binom{m}{i}A^i(t)v^{m-i}\right\|_j\qquad\text{for } 0\leq m+j\leq s-1,$$

we have

$$(13)\qquad \sum_{i=0}^{s}\|\nabla v^i\|_{s-i}^2\leq\text{Const.}\left((\|v^{s+1}\|_0^2+\sum_{i=0}^{s}\|\nabla v^i\|_0^2+(1+\delta t)^{-2}\|v\|_0^2\right).$$

So (11) and (13) give

$$(14)\qquad \frac{d}{dt}\left(\sum_{i=0}^{s}\beta_i\int D(v^i)dx\right)\leq 0\qquad\text{for } 0<\mu\leq{}^{\exists}\mu_0.$$

Thus (3), (8), (12), (13) and (14) imply a-priori estimates (10). For more detailed arguments, refer to [1] (Lemma 4 for the estimates of the composite functions and Theorem 2 for the global existence).

References

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