Energy Dependent Boundary Conditions and the Few-Body Scattering Problem *

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Abstract

An exactly solvable problem with energy dependent interaction is investigated in the present paper. The selfadjoint model operator describes the scattering problem for three one dimensional particles. It is shown that this problem is equivalent to the diffraction problem in the sector with energy dependent boundary conditions. The problem is solved with the help of the Sommerfeld-Maluzhinetz representation, which transforms the partial differential equation for the eigenfunctions to a functional equation on the integral densities. The solution of the functional equation can be constructed explicitly in the case of identical particles. The three body scattering matrix describing rearrangement and excitation processes is represented in terms of analytic functions.

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1 Introduction

Energy dependent interactions play an important role in the modern mathematical physics. Such interactions allow to model complicated physical phenomena and solve the problem exactly at the same time. A disadvantage of these problems is that they are usually described by nonselfadjoint operators or even by operator bundles. The corresponding eigenfunctions do not satisfy orthogonality and completeness properties. This produces additional difficulties during the investigation of these problems and limits the number of the phenomena which can be described in habitually terms. We show that some of these problems can be solved by considering operators in certain extended Hilbert spaces. In this approach operator bundles with energy dependent interactions appear as restrictions of selfadjoint operators. Resolvents of the operators with energy dependent interactions can be calculated with the help of M.G.Krein formula [43].

A wide class of such operators is well known under the name of operators with zero-range (or delta functional) potentials. The interaction in such problems is described by boundary conditions on some low dimensional manifolds. The most complete set of these problems has been collected in the monographs by S.Albeverio et al. [5] and Yu.N.Demkov and V.N.Ostrovsky [10]. Similar problems were studied with the help of the method of point interactions with internal structure. This method leads to a new class of exactly solvable Schrödinger operators with a richer spectral structure [40, 41]. Selfadjoint operators describing physical phenomena are defined in the orthogonal sum of standard Hilbert spaces and certain internal spaces, describing the interaction. Applications of the discussed methods to the two body problem were considered by V.M.Adamyan and B.S.Pavlov [1]. A similar scattering problem for three particles in the three dimensional space has been studied by B.S.Pavlov, Yu.A.Kuperin, K.A.Makarov, S.P.Merkuriev and A.K.Motovilov [19, 20, 21, 29, 30, 42]. This investigation was inspired by the papers [47, 36, 37, 49, 3, 4], where the system of three particles in three dimensional space interacted via delta potential has been studied. The present paper is devoted to the three-body problem in one space dimension.

The few-body scattering problems form a wide class of complicated quantum mechanical problems [35, 39, 44]. Some of the difficulties appear already at the level of the three-body operator. Such operators describe the following processes: rearrangement $(12)+3 \to 1+(23)$, breakup $(12)+3 \to 1+2+3$, capture $1+2+3 \to (12)+3$ and excitation $(12)+3 \to (12)^*+3$. The corresponding scattering matrix describes the interaction between several asymptotic channels. As the standard scattering problem leads to complicated calculations, exactly solvable models should play an important role in the investigation of these phenomena.

An application of the discussed method of boundary conditions to the case of few-body problems leads to a wide class of operators which can be studied exactly. Unfortunately the simplest models do not give a possibility to describe complicated phenomena. The complexity of the model problem increases with the number of phenomena which can be described by the model. Several systems of one-dimensional particles with interaction of this type have been analyzed. The investigation of this three-body problem was started from the simplest problems such as the system of identical particles [13, 50], the system of impenetrable particles [14], the system of two particles interacting with a wall [2, 28]. Some of the solutions were expressed in terms of the elementary functions, for example in the case of identical particles. The scattering solution can be constructed with the help of the Bethe Ansatz [12] in this case. More realistic problems describing nonidentical particles lead to complicated equations for the eigenfunctions. These equations can be solved using certain integral transformations. Using the Sommerfeld-Maluzhinetz integral representation [31, 32] one transfers the partial differential equation into a difference equation for some analytic function. The solution of this equation can be constructed with the help of special functions [2, 14, 15, 25, 26, 27, 28], and it can be expressed in terms of elementary functions when the problem has some symmetry properties. Usually this case coincides with the one for which the solution can be presented by the Bethe Ansatz. The standard delta functional interaction defines the unique two-body bound state or resonance. The corresponding model can not be used to describe collisional deexcitation processes. The two-cluster and three-cluster channels are orthogonal in the case of equal particles. Breakup and capture processes are forbidden in this model.

The case of identical particles is studied in the present paper. The model operator is constructed following the general scheme suggested by B.S.Pavlov. This is the first application of this scheme to the soluble problem of three-body scattering in dimension one with nontrivial two-body interaction (see [18] where a nontrivial three-body interaction has been introduced). The model constructed has a more realistic scattering matrix and richer structure of the spectrum than the model in [8]. In particular it permits to describe rearrangement processes. The model operator is a selfadjoint perturbation of the "asymptotic" Hamiltonian, describing the free motion in the system of three particles. The model operator describes a system of three arbitrary particles, but later we confine our consideration to the case of identical particles in order to be able to express the solution of the scattering problem in terms of elementary functions. The solution of the two-body scattering problem can be presented by a combination of plane waves. The problem has an arbitrary number of bound states and resonances. The solution of the three-body scattering problem can not be presented in terms of a Bethe Ansatz and it is calculated using the Sommerfeld-Maluzhinetz transformation. The equation for the eigenfunctions is equivalent to the Helmholtz equation in the sector with energy dependent boundary conditions. The diffraction problem is transformed to a functional equation, which is solved exactly. Analytical properties of the solutions of the functional equation are investigated. Singularities of these solutions are determined by the two-body bound states and resonances. The analytical solution of the functional equation yields the analytical two and three body scattering matrices. Simple formulae for these scattering matrices give us the possibility to investigate the relations between the two- and three-body spectral characteristics. Some of the results presented here were discussed by the author in [22]. Our model can be used in statistical physics calculations in order to investigate the relations between the spectral characteristics of two-body operators and the thermodynamic parameters. Such investigations have been started in [24]. Developed methods can be applied to the study of diffraction problems in the domains with singularities.

The model operator is presented by certain block operators acting in the orthogonal sum of Hilbert spaces. The interaction between the components is determined by boundary conditions, which can be considered as antidiagonal singular operators. Thus the operator constructed is close to the set of matrix selfadjoint operators studied recently by V.M.Adamyan, F.V.Atkinson, H.Langer, R.Mennicken and A.Shkalikov [6].

The paper is organized as follows. The two-body Schrödinger operator is constructed in Section 2. The two-body scattering matrix is calculated. Relations with the standard Schrödinger operator are discussed here following the paper [23]. In Section 3 a generalization of the model to the three-body case is considered. The symmetric three body operator is constructed. A selfadjoint extension of this operator is calculated with the help of the von Neumann theory. The equation for the deficiency elements is transformed into the vector difference equation for an analytic function in Section 3. The symmetries of the corresponding equations are discussed. Investigation of a special invariant basis leads to a system of independent two-dimensional difference equations. It is shown that these equations can be decoupled in the case of indistinguishable particles. The solution of the difference equation is obtained in Section 4 for this system. Properties of the deficiency elements are studied in Section 5. A selfadjoint extension of the symmetric three-body operator is constructed in Section 6. The three-body scattering matrix is calculated, it is expressed in terms of elementary functions. The scattering solution is presented by a combination of plane waves constructed in the form of the Bethe Ansatz plus a certain outgoing wave. The outgoing wave is equal to the limit of the deficiency element calculated earlier, when the spectral parameter λ approaches the real line. Relations between the spectral properties of the two and three-body operators are discussed.

2 The two-body Hamiltonian

This section is devoted to the construction of the model operator describing the two-body problem on the line. We first recall some standard facts concerning the two body Schrödinger operator with the interaction introduced by usual potential. We concentrate our attention to the properties of the corresponding scattering data. The main part of this section is devoted to the construction of the model for a two body operator. The two body quantum mechanical problem contains the following asymptotic channels

two noninteracting particles;

two particles in a bound state.

The model operator is defined as a selfadjoint perturbation of the orthogonal sum of the Hamiltonians describing each asymptotic channel. These operators are the two dimensional Laplace operator and the one dimensional matrix second derivative operator with a diagonal threshold matrix. The entries of the latter matrix coincide with the energies of the two body bound states. The eigenfunctions of the model operator corresponding to the discrete and continuous spectra are calculated explicitly. The scattering matrix is expressed in terms of elementary functions. We discuss how to select model operators with the standard properties of the scattering data.

2.1 Interaction determined by a potential

The Schrödinger operator describing two one dimensional quantum particles with equal masses has the following form

$$\mathcal{A}_V = -\frac{1}{2}\left(\frac{d^2}{dr_1^2} + \frac{d^2}{dr_2^2}\right) + V(\mid r_1 - r_2 \mid), \tag{2.1}$$

where r_1, r_2 denote the coordinates of the particles. Here the interaction is determined by a potential V which depends only on the distance between the particles. The center of mass motion can be separated using Jacobi coordinates

$$x_{12} = r_1 - r_2, \quad y_{12} = \frac{r_1 + r_2}{2},$$

the Schrödinger operator can be decomposed as follows

$$\mathcal{A}_V = -\frac{1}{4}\frac{\partial^2}{\partial y_{12}^2} \times I + I \times A_V,$$

$$A_V = -\frac{d^2}{dx_{12}^2} + V(\mid x_{12} \mid). \tag{2.2}$$

The operator A_V has been studied as a selfadjoint operator in the Hilbert space $L_2(\mathbf{R})$ for potentials with the finite first momentum [11, 33, 9, 34]:

$$\int_{-\infty}^{+\infty} |x\, V(x)| dx < \infty. \tag{2.3}$$

The scattering problem for the two-body Schrödinger operator is formulated with the unperturbed operator equal to the second derivative operator

$$A_0 = -\frac{d^2}{dx_{12}^2},$$

defined on the standard domain $Dom(A_0) = W_2^2(\mathbf{R})$. The unperturbed and perturbed operators have the same branch of absolutely continuous spectrum $[0, \infty)$. The perturbed operator A_V with the interaction V can have some additional negative eigenvalues - two-body bound states. Let $f_-(x,k), f_+(x,k), k \in \mathbf{R}\setminus\{0\}$, be the solutions of the equation $A_V f = k^2 f$ in the generalized sense with the following asymptotics

$$f_-(x,k) \sim e^{ikx}, \quad x \to +\infty,$$

$$f_+(x,k) \sim e^{-ikx}, \quad x \to -\infty. \tag{2.4a}$$

The solutions $f_j(k,x)$ are asymptotic to sums of exponentials as $x \to \mp\infty$

$$f_-(x,k) \sim \frac{1}{T_-(k)} e^{ikx} + \frac{R_-(k)}{T_-(k)} e^{-ikx}, \;\; x \to -\infty,$$

$$f_+(x,k) \sim \frac{1}{T_+(k)} e^{-ikx} + \frac{R_+(k)}{T_+(k)} e^{ikx}, \;\; x \to +\infty. \tag{2.4b}$$

The matrix

$$S(k) = \begin{pmatrix} T_+(k) & R_-(k) \\ R_+(k) & T_-(k) \end{pmatrix} \tag{2.5}$$

is called the scattering matrix. This matrix is unitary

$$\begin{gathered} |T_-|^2 + |R_-|^2 = 1 = |T_+|^2 + |R_+|^2, \\ T_-(k)R_+(-k) + R_-(k)T_+(-k) = 0. \end{gathered}$$

One can prove that the transition coefficients coincide and that the following asymptotics for the coefficients of the scattering matrix are valid [11] when $k \to \infty$

$$\begin{aligned} &T_+(k) = T_-(k) = 1 + O(\tfrac{1}{|k|}); \\ &R_-(k) = O(\tfrac{1}{|k|}); R_+(k) = O(\tfrac{1}{|k|}). \end{aligned} \tag{2.6}$$

The following estimates are valid in the low-energies domain

$$T_\pm(k) = O(k), \;\; R_\pm(k) = -1 + O(k) \;\; k \to 0. \tag{2.7}$$

These asymptotics will be called "standard" ones in what follows. The model operators which will be constructed in the next section define unitary scattering matrices. But the coefficients of these matrices do not necessarily have the standard asymptotics. In order to make the model realistic we confine our consideration to the model operators with the standard asymptotics of the scattering matrix (see Section 2.5).

2.2 The model operator

The model two-body operator is constructed as a perturbation of the operator $\mathcal{A}_0^2$ describing possible asymptotic channels for the two-body problem. The operator $\mathcal{A}_0^2$ is equal to the orthogonal sum of two operators $\mathcal{A}_0^2 = -\frac{1}{2}\left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2}\right) \oplus -\frac{1}{4}\frac{d^2}{dy_{12}^2} + A_{12}$. The first operator in this orthogonal sum acts in the Hilbert space $L_2(\mathbf{R}^2)$ and describes two noninteracting particles. The second operator describes two coupled particles moving together. The energies of the bound states are equal to the eigenvalues of the finite dimensional selfadjoint matrix A_{12} acting in the finite dimensional space H_{12}. We suppose that the eigenvalues of A_{12} are negative. The second operator acts in the Hilbert space $L_2(\mathbf{R}, H_{12})$. The standard separation of the center of mass motion gives the following operator

$$A_0^2 = A_{1,2} \oplus A_{12}, A_{1,2} = -\frac{d^2}{dx_{1,2}^2} \tag{2.8}$$

which acts in the orthogonal sum of the Hilbert spaces $H^2 = L_2(\mathbf{R}) \oplus H_{12}$. The unperturbed operator for the scattering problem can be chosen equal to the A_0^2.

The perturbed operator can be constructed by restricting first the operator A_0^2 to a certain symmetric operator and then extending it to another selfadjoint operator. The interaction between the channels will be introduced using some generalized boundary conditions. The restriction of the operator $A_{1,2} \rightarrow A_{1,20}$ to the domain

$$Dom(A_{1,20}) = \left\{ u \in W_2^2(\mathbf{R}), u(0) = 0, u'(0) = 0 \right\}$$

is a symmetric operator with the deficiency indices $(2,2)$. The adjoint operator is defined by the same differential expression on the domain

$$Dom(A_{1,20}^*) = \left\{ u \in W_2^2(\mathbf{R}\backslash\{0\}) \right\}.$$

The boundary form of the adjoint operator is equal to
$u, v \in Dom(A_{1,20}^*)$

$$< A_{1,20}^* u, v >_{L_2} - < u, A_{1,20}^* v >_{L_2} = \tag{2.9}$$

$$= \left(\left[\frac{du}{dx}\right] < \bar{v} > + < \frac{du}{dx} > [\bar{v}] - < u > \left[\frac{d\bar{v}}{dx}\right] - [u] < \frac{d\bar{v}}{dx} > \right) |_{x=0},$$

where $[*]$ and $< * >$ denote the jump and the mean value of function at the origin

$$(2.10) \qquad \begin{array}{rcl} [f(x)] & \equiv & f(x+0) - f(x-0), \\ < f(x) > & \equiv & \frac{f(x+0)+f(x-0)}{2}. \end{array}$$

We restrict the operator A_{12} to the operator A_{120} defined on the domain $\{u_{12} \in H_{12} : \langle u_{12}, \theta \rangle = 0\}$. The restricted operator is a symmetric but not densely defined operator in H_{12}. Thus one cannot use directly the von Neumann theory to construct the selfadjoint extensions of the operator A_{120}. The restricted total operator A^2_{00} is defined by the orthogonal sum of the symmetric operators $A^2_{00} = A_{1,20} \oplus A_{120}$ on the domain

$$Dom(A^2_{00}) = \{(u_{1,2}, u_{12}) \in H^2 : u_{1,2} \in W^2_2(\mathbf{R}), u_{1,2}(0) = 0, u'_{1,2}(0) = 0; \langle u_{12}, \theta \rangle = 0\}.$$

We define the perturbed operator describing the interacting particles as a certain selfadjoint extension of the operator A^2_{00}.

Theorem 2.1 *Let the real parameters a, b, c, d satisfy the following equality*

$$(2.11) \qquad \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1,$$

$\theta \in H_{12}$. *The operator*

$$(2.12) \quad A^2 U = A^2 \begin{pmatrix} u_{1,2} \\ u_{12} \end{pmatrix} = \begin{pmatrix} A^*_{1,20} u_{1,2} \\ A_{12} u_{12} + \left(a \left[\frac{du_{1,2}}{dx} \right] + b < u_{1,2} > \right) |_{x=0} \, \theta \end{pmatrix},$$

defined on the domain of functions from $Dom(A^*_{1,20}) \oplus H_{12}$ satisfying the boundary conditions

$$\left(c \left[\frac{du_{1,2}}{dx} \right] + d < u_{1,2} > \right) |_{x=0} = < u_{12}, \theta >$$

$$(2.13) \qquad [u_{1,2}] \, |_{x=0} = 0$$

is a selfadjoint extension of the operator A^2_{00}.

Remark. The operator A^2 will be called the perturbed model two-body operator in the sequel. Similar two-body operators has been suggested first by K.Makarov [29, 30].

P r o o f. Consider first any element U from the domain of the operator A^2_{00}. The operators A^2 and A^2_{00} map this element to one and the same element of the Hilbert space $H^2 = L_2(\mathbf{R}) \oplus H_{12}$. It follows that the operator A^2 is an extension of the operator A^2_{00}.

We are going to prove that the operator A^2 is densely defined. Let U be a given element from the Hilbert space $H^2 \ni U = (u_{1,2}, u_{12})$. Consider an arbitrary element $(\tilde{u}_{1,2}, u_{12})$ from the domain of the operator A^2. The difference $(u_{1,2}, u_{12}) - (\tilde{u}_{1,2}, u_{12}) = (u_{1,2} - \tilde{u}_{1,2}, 0)$ belongs to the space $L_2(\mathbf{R}) \subset H^2$. The restricted operator $A_{1,20}$ is densely defined, thus for every given $\epsilon > 0$ there exists function $\hat{u}_{1,2}$ from the domain of the operator $A_{1,20}$ such that $\| u_{1,2} - \tilde{u}_{1,2} - \hat{u}_{1,2} \|_{L_2} < \epsilon$. This implies that $\| (u_{1,2}, u_{12}) - (\tilde{u}_{1,2} + \hat{u}_{1,2}, u_{12}) \| < \epsilon$ and $(\tilde{u}_{1,2} + \hat{u}_{1,2}, u_{12})$ belongs to the domain of the operator A^2. Thus the operator A^2 is densely defined.

We calculate now the boundary form of the operator (11) on the functions from $Dom(A^*_{1,20}) \oplus H_{12}$

$$
\begin{aligned}
& < A^2 U, V >_H - < U, A^2 V >_H = \\
& = \left(\left[\frac{du_{1,2}}{dx} \right] \overline{< v_{1,2} >} + < \frac{du_{1,2}}{dx} > \overline{[v_{1,2}]} \right. \\
& \left. - < u_{1,2} > \overline{\left[\frac{dv_{1,2}}{dx} \right]} - [u_{1,2}] \overline{< \frac{dv_{1,2}}{dx} >} \right) |_{x=0} \\
& + \left(a \left[\frac{du_{1,2}}{dx} \right] + b < u_{1,2} > \right) |_{x=0} \overline{< v_{12}, \theta >} \\
& - < u_{12}, \theta > \overline{\left(a \left[\frac{dv_{1,2}}{dx} \right] + b < v_{1,2} > \right)} |_{x=0}
\end{aligned}
\tag{2.14}
$$

This boundary form vanishes on the domain of the operator A^2
(2.15)

$$
\begin{aligned}
< A^2U, V > - < U, A^2V > \; &= \Big(\left[\frac{du_{1,2}}{dx}\right] \overline{< v_{1,2} >} - < u_{1,2} > \overline{\left[\frac{dv_{1,2}}{dx}\right]} + \\
&+ac \left[\frac{du_{1,2}}{dx}\right] \overline{\left[\frac{dv_{1,2}}{dx}\right]} + bc < u_{1,2} > \overline{\left[\frac{dv_{1,2}}{dx}\right]} \\
&+ad \left[\frac{du_{1,2}}{dx}\right] \overline{< v_{1,2} >} + bd < u_{1,2} > \overline{< v_{1,2} >} \\
&-ac \left[\frac{du_{1,2}}{dx}\right] \overline{\left[\frac{dv_{1,2}}{dx}\right]} - ad < u_{1,2} > \overline{\left[\frac{dv_{1,2}}{dx}\right]} \\
&-cb \left[\frac{du_{1,2}}{dx}\right] \overline{< v_{1,2} >} - bd < u_{1,2} > \overline{< v_{1,2} >} \Big) \,|_{x=0} = \\
&= \left(\left[\frac{du_{1,2}}{dx}\right] < \overline{v_{1,2}} > - < u_{1,2} > \overline{\left[\frac{dv_{1,2}}{dx}\right]} \right) |_{x=0} \left(1 + \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right) \\
&= 0
\end{aligned}
$$

Thus the operator A^2 is a symmetric extension of the operator A^2_{00}. The adjoint operator is defined by the same formula (2.12) and its domain is a subset of $Dom(A^*_{1,20}) \oplus H_{12}$. If an element $U = (u_{1,2}, u_{12}) \in Dom(A^*_{1,20}) \oplus H_{12}$ belongs to the domain of the adjoint operator then the boundary form (1.14) should be equal to zero for any $V \in Dom(A^2)$. Consider elements V such that $< v_{1,2} > |_{x=0} = \left[\frac{dv_{1,2}}{dx}\right] |_{x=0} =< v_{12}, \theta >= 0$. The boundary form for such V is equal to $-[u_{1,2}] < \overline{\frac{dv_{1,2}}{dx}} >$ and it follows that every function $u_{1,2}$ must be continuous at the origin $[u_{1,2}] = 0$. Consider now elements V such that $< v_{12}, \theta >= 0$. Similar calculations show that the boundary values of U should satisfy the first condition (2.13). It follows that the adjoint operator A^{2*} has the same domain as the operator A^2 and thus it is selfadjoint. □

The operators A^2 and A^2_0 are in general two different selfadjoint extensions of the symmetric operator A^2_{00}. The set of constructed operators A^2 does not coincide with the set of all selfadjoint extensions of the operator A^2_{00}. The advantage of the method presented is that the operator A^2 is defined explicitly. One can use the fact that the operators A^2_0 and A^2 are two selfadjoint extensions of the same symmetric operator A^2_{00} to calculate the resolvent of A^2.

2.3 The resolvent

The resolvent of the perturbed operator for all $\lambda, \Im\lambda \neq 0$ can be calculated using the modified Kreins formula and the resolvent of the unperturbed operator

$$\mathcal{R}_{A_0^2}(\lambda) = \mathcal{R}_{A_{1,2}}(\lambda) \oplus \mathcal{R}_{A_{12}}(\lambda).$$

The resolvent of the operator $A_{1,2}$ is the integral operator with the kernel $r_{A_{1,2}}(\lambda, x, y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}}$. The branch of the square root is fixed by the condition $\Im\lambda > 0 \Rightarrow \Im\sqrt{\lambda} > 0$. The resolvent of the operator A_{12} coincides with the following matrix $\mathcal{R}_{A_{12}}(\lambda) = (A_{12} - \lambda)^{-1}$.

The following function will play an important role in the sequel:

$$D(\lambda) = \frac{b < R_{A_{12}}(\lambda)\theta, \theta > +d}{a < R_{A_{12}}(\lambda)\theta, \theta > +c}. \tag{2.16}$$

The function $\mathbf{R}(\lambda) =< R_{A_{12}}(\lambda)\theta, \theta >$ is analytic in the upper halfplane $\Im\lambda > 0$ and has positive imaginary part there. The real constants a, b, c, d define a conformal map of the upper halfplane onto itself due to the conditions (2.11). It follows that the function $D(\lambda)$ is analytic in the upper half plane and has positive imaginary part there.

Lemma 2.1 *The resolvent of the perturbed operator* $\mathcal{R}_{A^2}(\lambda) = (A^2 - \lambda)^{-1}$ *is the matrix operator of the form*

$$\mathcal{R}_{A^2}(\lambda) = \mathcal{R}_{A_0^2}(\lambda) + \begin{pmatrix} \Delta R_{(1,2)(1,2)}(\lambda) & \Delta R_{(1,2)(12)}(\lambda) \\ \Delta R_{(12)(1,2)}(\lambda) & \Delta R_{(12)(12)}(\lambda) \end{pmatrix}. \tag{2.17}$$

The operators $\Delta R_{(1,2)(1,2)}(\lambda), \Delta R_{(12)(1,2)}(\lambda)$ *are the integral operators with the following kernels*

$$\Delta r_{(1,2)(1,2)}(\lambda, x, y) = -\frac{D(\lambda)}{D(\lambda) + 2i\sqrt{\lambda}}\, e^{i\sqrt{\lambda}|y|}\, \frac{e^{i\sqrt{\lambda}|x|}}{2i\sqrt{\lambda}}, \tag{2.18}$$

$$\Delta r_{(12)(1,2)}(\lambda, y) = \frac{aD(\lambda) - b}{D(\lambda) + 2i\sqrt{\lambda}}\, e^{i\sqrt{\lambda}|y|}(A_{12} - \lambda)^{-1}\theta. \tag{2.19}$$

The operators $\Delta R_{(1,2)(12)}(\lambda), \Delta R_{(12)(12)}(\lambda)$ *are equal to*

$$\Delta R_{(1,2)(12)}(\lambda) = e^{i\sqrt{\lambda}|x|}\, \frac{< (A_{12} - \lambda)^{-1} *, \theta >}{(2ai\sqrt{\lambda} + b)\mathbf{R}(\lambda) + 2ci\sqrt{\lambda} + d}, \tag{2.20}$$

$$\Delta R_{(12)(12)}(\lambda) = -(2ai\sqrt{\lambda} + b)\frac{< (A_{12} - \lambda)^{-1} *, \theta > (A_{12} - \lambda)^{-1}\theta}{(2ai\sqrt{\lambda} + b)\mathbf{R}(\lambda) + 2ci\sqrt{\lambda} + d}. \tag{2.21}$$

P r o o f . Consider an arbitrary $F \in H^2$. Let $\mathcal{R}_{A^2}(\lambda)F = G$. This implies that $G \in Dom(A^2)$ and $F = (A^2 - \lambda)G$. The last equation can be written for the components as follows

$$(-\frac{d^2}{dx^2} - \lambda)g_{1,2}(x) = f_{1,2}(x);$$

$$A_{12}g_{12} + \left(a\left[\frac{dg_{1,2}}{dx}\right] + b < g_{1,2} >\right)|_{x=0}\theta - \lambda g_{12} = f_{12}. \tag{2.22}$$

We apply the operator $R_{A_{12}}(\lambda)$ to the left and right hand sides of the second equation

(2.23)$g_{12} = R_{A_{12}}(\lambda)f_{12} - \left(a\left[\frac{dg_{1,2}}{dx}\right] + b < g_{1,2} >\right)|_{x=0}R_{A_{12}}(\lambda)\theta.$

The projection on the element θ gives the following relation
(2.24)
$< g_{12}, \theta >=< R_{A_{12}}(\lambda)f_{12}, \theta > - \left(a\left[\frac{dg_{1,2}}{dx}\right] + b < g_{1,2} >\right)|_{x=0} < R_{A_{12}}(\lambda)\theta, \theta > .$

Every solution to (2.22) which is continuous at the origin is given by

$$g_{1,2} = R_{A_{1,2}}(\lambda)f_{1,2} + qe^{i\sqrt{\lambda}|x|},$$

where q is a parameter which will be fixed later. The boundary values of the function $g_{1,2}$ at the origin are equal to

$$\begin{array}{rcl} < g_{1,2} > |_{x=0} & = & \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda}|y|}}{2i\sqrt{\lambda}} f_{1,2}(y)dy + q; \\ \left[\frac{dg_{1,2}}{dx}\right]|_{x=0} & = & 2i\sqrt{\lambda}q. \end{array}$$

The element G belongs to the domain of the operator A^2 and satisfies the boundary conditions (2.13). It follows from (2.24) that the boundary values of $g_{1,2}$ should satisfy the following equation

$$(c+ < R_{A_{12}}(\lambda)\theta, \theta > a)\left[\frac{dg_{1,2}}{dx}\right]|_{x=0} + (d+ < R_{A_{12}}(\lambda)\theta, \theta > b) < g_{1,2} > |_{x=0} =$$

$$=< R_{A_{12}}(\lambda)f_{12}, \theta > .$$

The parameter q can be calculated now

$$
\begin{aligned}
q &= \frac{1}{2i\sqrt{\lambda}+D(\lambda)} \times \\
&\times \left(\frac{1}{c+ < R_{A_{12}}(\lambda)\theta, \theta > a} < R_{A_{12}}(\lambda) f_{12}, \theta > - D(\lambda) \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda}|y|}}{2i\sqrt{\lambda}} f_{1,2}(y) dy \right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
g_{12} &= R_{A_{12}}(\lambda) f_{12} + \frac{2i\sqrt{\lambda}(aD(\lambda)-b)}{2i\sqrt{\lambda}+D(\lambda)} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda}|y|}}{2i\sqrt{\lambda}} f_{1,2}(y) dy R_{A_{12}}(\lambda)\theta \\
&- \frac{a2i\sqrt{\lambda}+b}{2i\sqrt{\lambda}(c+\mathbf{R}(\lambda)a)+b\mathbf{R}(\lambda)+d} < R_{A_{12}}(\lambda) f_{12}, \theta > R_{A_{12}}(\lambda)\theta, \\
g_{1,2} &= R_{A_{1,2}}(\lambda) f_{1,2} + \frac{e^{i\sqrt{\lambda}|x|}}{2i\sqrt{\lambda}+D(\lambda)} \times \\
&\times \left(\frac{1}{c+ < R_{A_{12}}(\lambda)\theta, \theta > a} < R_{A_{12}}(\lambda) f_{12}, \theta > - D(\lambda) \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\lambda}|y|}}{2i\sqrt{\lambda}} f_{1,2}(y) dy \right).
\end{aligned}
$$

Formulas (2.18-2.21) follow from the last two equations. □

2.4 Spectrum, eigenfunctions, scattering matrix

The singularities of the resolvent $\mathcal{R}_{A^2}(\lambda)$ are situated at the points which satisfy the equation $D(\lambda)+2i\sqrt{\lambda}=0$. They correspond to the eigenvalues of the operator A^2. The absolutely continuous spectrum of the operator is determined by the discontinuity of the resolvent on the positive part of the real axis due to the discontinuity of the function $\sqrt{\lambda}$ there.

The discrete spectrum eigenfunctions are solutions of the equation $A^2\Psi_s = \lambda_s\Psi_s$, where $\lambda_s > 0,\ s = 1, 2, ..., N^2_{bs}$, are the negative real solutions of the equation

$$2i\sqrt{\lambda_s} = -D(\lambda_s). \tag{2.25}$$

The eigenfunctions can be explicitly calculated

$$\Psi_s = c_s \begin{pmatrix} \psi^s_{1,2}(x) \\ \psi^s_{12} \end{pmatrix}, \tag{2.26}$$

$$\psi^s_{1,2}(x) = e^{-\chi_s|x|}, \chi_s = -i\sqrt{\lambda_s} > 0,$$
$$\psi^s_{12} = -(-2a\chi_s + b)(A_{12} + \chi_s^2)^{-1}\theta.$$

The constant c_s can be determined from the normalizing condition $\| \Psi_s \|= 1$

$$c_s = \left(\frac{1}{\chi_s} + (-2a\chi_s + b)^2 \parallel (A_{12} + \chi^2)^{-1}\theta \parallel^2 \right)^{-1/2}. \tag{2.27}$$

The continuous spectrum eigenfunctions $\Psi = (\psi_{1,2}, \psi_{12})$ are generalized solutions of the following equation

$$\begin{pmatrix} -\frac{d^2}{dx^2}\psi_{1,2} \\ A_{12}\psi_{12} + \left(a \left[\frac{d\psi_{1,2}}{dx}\right] + b < \psi_{1,2} > \right) |_{x=0}\ \theta \end{pmatrix} = \lambda \begin{pmatrix} \psi_{1,2} \\ \psi_{12} \end{pmatrix}, \tag{2.28}$$

satisfying the boundary conditions (2.13). Equation (2.28) can be reduced to the usual one dimensional Schrödinger equation on the axis with energy dependent boundary conditions at the origin. This reduction is similar to the one carried out in the proof of Lemma 2.2. The second of equations (2.28)

$$(a \left[\frac{d\psi_{1,2}}{dx}\right] + b < \psi_{1,2} >) |_{x=0}\ \theta + A_{12}\psi_{12} = \lambda\psi_{12}$$

can be solved as follows

$$\psi_{12} = -\left(a \left[\frac{d\psi_{1,2}}{dx}\right] + b < \psi_{1,2} > \right) |_{x=0}\ (A_{12} - \lambda)^{-1}\,\theta.$$

Substitution into the boundary conditions (2.13) gives the following energy dependent boundary conditions for the component $\psi_{1,2}$

$$\frac{\left[\frac{d\psi_{1,2}}{dx}\right]}{< \psi_{1,2} >} |_{x=0} = -\frac{b\mathbf{R}(\lambda) + d}{a\mathbf{R}(\lambda) + c} \equiv -D(\lambda),$$

$$[\psi_{1,2}] |_{x=0} = 0. \tag{2.29}$$

The multiplicity of the continuous spectrum is equal to 2. As in (2.4a), (2.4b) the following representations for the eigenfunctions can be used

$$\Psi_\pm(\lambda) = \frac{1}{2\sqrt{\pi k}} \begin{pmatrix} \psi_{\pm 1,2}(x) \\ \psi_{\pm 12} \end{pmatrix}, \quad \lambda = k^2$$

$$\psi_{-1,2}(\lambda,x)=\begin{cases} e^{ikx}+R_-(k)e^{-ikx}; & x<0\\ T_-(k)e^{ikx}; & x>0\end{cases}$$

$$\psi_{+1,2}(\lambda,x)=\begin{cases} T_+(k)e^{-ikx}; & x<0\\ e^{-ikx}+R_+(k)e^{ikx}; & x>0\end{cases} \tag{2.30}$$

The left and right reflection and transition coefficients are identical due to the symmetry of the problem

$$R_-(k)=R_+(k)\equiv R(k),\quad T_-(k)=T_+(k)\equiv T(k).$$

The transition and reflection coefficients are calculated from the energy dependent boundary conditions (2.29)

$$\begin{aligned} T(k) &= \tfrac{2ik}{D(\lambda)+2ik}\\ R(k) &= \tfrac{-D(\lambda)}{D(\lambda)+2ik}. \end{aligned} \tag{2.31}$$

The components $\psi_{\pm 12}(\lambda)$ of the eigenfunctions are identical

$$\psi_{\pm 12}(\lambda)=\psi_{12}(\lambda)=i\frac{\sqrt[4]{\lambda}}{\sqrt{\pi}}\frac{aD(\lambda)-b}{D(\lambda)+2i\sqrt{\lambda}}(A_{12}-\lambda)^{-1}\theta \tag{2.32}$$

The reflection and transition coefficients form the unitary scattering matrix

$$S(k)=\begin{pmatrix} T(k) & R(k)\\ R(k) & T(k)\end{pmatrix}. \tag{2.33}$$

The unitarity of the scattering matrix we calculated can be proven directly using the fact that the function $D(k^2)$ is real for the real values of the parameter k.

The discrete spectrum eigenfunctions Ψ_s and continuous spectrum eigenfunctions $\Psi_\pm(\lambda)$ define the spectral decomposition of the operator A^2 :

Theorem 2.2 *Let $F=(f_{1,2},f_{12}), G=(g_{1,2},g_{12})\in H^2$ have infinitely differentiable outside the origin components $f_{1,2},g_{1,2}$ with compact support, then the following formula is valid*

(2.34)

$$\begin{aligned} <F,G>_{H^2} &= \textstyle\sum_{s=1}^{N_{bs}^2}<F,\Psi_s>_{H^2}<\Psi_s,G>_{H^2}\\ &\quad +\textstyle\sum_{\alpha=\pm}\int_0^\infty d\lambda\left(\int_{-\infty}^{+\infty} f_{1,2}(x)\overline{\psi_{\alpha 1,2}}(x)dx+<f_{12},\psi_{\alpha 12}>\right)\\ &\quad \times\left(\int_{-\infty}^{+\infty}\psi_{\alpha 1,2}(x)\overline{g_{1,2}}(x)dx+<\psi_{\alpha 12},g_{12}>\right). \end{aligned}$$

Moreover if $F \in \mathrm{Dom}\,(A^2)$ *then*

(2.35)

$$\begin{aligned} < A^2F, G >_{H^2} &= \sum_{s=1}^{N_{bs}^2} \lambda_s < F, \Psi_s >_{H^2} < \Psi_s, G >_{H^2} \\ &+ \sum_{\alpha=\pm} \int_0^\infty \lambda d\lambda \left(\int_{-\infty}^{+\infty} f_{1,2}(x) \overline{\psi_{\alpha 1,2}}(x) dx + < f_{12}, \psi_{\alpha 12} > \right) \\ &\times \left(\int_{-\infty}^{+\infty} \psi_{\alpha 1,2}(x) \overline{g_{1,2}}(x) dx + < \psi_{\alpha 12}, g_{12} > \right). \end{aligned}$$

The theorem can be proven integrating the resolvent of the operator A^2 over the contour surrounding the discrete and continuous spectra.

2.5 Restrictions on the model

Only the model operators with the standard behavior of the scattering matrix (2.4) will be considered in what follows. The function $D(\lambda)$ is a rational function. It is analytic in the upper halfplane and has positive imaginary part there. It is real on the real axis. Every such function has the following asymptotics at infinity $D(\lambda) = c_1\lambda + c_0 + O(\frac{1}{\lambda}), c_1, c_0 \in \mathbf{R}, c_1 \geq 0$. The transition coefficient $T(k)$ tends to one at infinity only if the linear term in the asymptotics is absent ($c_1 = 0$). Only the model operators with the perturbation determined by the zero parameter d possess such property. The reflection coefficient tends to zero at infinity in this case and the scattering matrix has the standard behavior at infinity (2.6). The scattering matrix (2.7) has standard zero energy behavior if no zero energy bound state is present

$$D(0) \neq 0. \tag{2.36}$$

In the sequel we are going to consider only the model operators with standard behavior of the scattering matrix.

The singularities of the scattering matrix are situated on the positive part of the imaginary axis on the k - plane and in the lower half-plane ($k = \sqrt{\lambda}$). These singularities correspond to the bound states and resonances respectively. We are going to consider the case $a = d = 0$. [1] The number of the eigenvalues of the perturbed and unperturbed operators coincide in this case.

[1] We shall present formulas generally for nonzero a and d, but the final result will be proven for a and d equal to zero.

Lemma 2.2 *Let $a = d = 0$ and all the eigenvalues of A_{12} be negative. The equation*

$$D(k^2) + 2ik = 0 \tag{2.37}$$

has exactly $N_{12} = dimH_{12}$ solutions in the upper halfplane $\Im k > 0$. All these solutions are situated on the imaginary axis.

P r o o f. The solutions of the equation on the physical sheet are situated on the imaginary axis because the functions $D(\lambda)$ and $2i\sqrt{\lambda}$ have imaginary parts with the same sign on the λ-plane outside the real axis, where these functions are real. The function $D(\lambda) = b^2\mathbf{R}(\lambda)$ considered on the real axis is a continuous increasing function on each interval not containing the singularities which coincide with the eigenvalues of the operator A_{12}. The number of the singularities on the negative halfaxis coincides with N_{12}, since all the eigenvalues of the operator A_{12} are negative. The function $2i\sqrt{\lambda}$ is a negative increasing function. It follows that the equation has exactly $dimH_{12}$ solutions in the upper halfplane $\Im k > 0$ and all these solutions are situated on the imaginary axis. □

The constant b can be considered as a perturbation parameter. The eigenvalues of the operator A^2 tend to the eigenvalues of the operator A_{12} in the limit $b \to 0$.

Equation (2.37) has exactly $2N_{12}+1$ solutions since all these solutions are roots of a polynomial in k of degree $2N_{12}+1$. The number of the solutions on the nonphysical sheet $\Im k < 0$ is equal to $N_{12} + 1$. We suppose that all these solutions are situated on the imaginary axis. This is true if the parameter b is small. In the general situation only $N_{12} - 1$ solutions are situated on the imaginary axis in the lower halfplane and two solutions can have nontrivial real part.

The discrete part of the operator A^2 will be denoted by $A^2_d = A^2\mathbf{P}_d$, where $\mathbf{P}_d$ is the discrete spectrum projection for the operator A^2. The discrete spectrum eigenfunctions $\Psi_s, s = 1, 2, ..., N_{12}$, form an orthogonal basis in the finite dimensional subspace $H^2_d = \mathbf{P}_d H^2$. The operator A^2_d is a diagonal operator in this basis

$$A^2_d \sum g_s \Psi_s = \sum (-\chi^2_s) g_s \Psi_s.$$

We note that the operator A^2_d is unitary equivalent to an operator in H_{12} since the spaces H_{12} and H^2_d have the same dimension.

The constructed model two-body problem is exactly solvable in the following sense: the eigenfunctions and scattering data can be expressed in terms of elementary functions. Different scattering channels are orthogonal as in the usual two-body scattering problem. The channel Hamiltonians which were used in the construction of the problem are not the channel Hamiltonians for the perturbed operator because the two-body discrete spectrum changes during the perturbation. The channel Hamiltonian corresponding to the cluster decomposition (two particles in a bound state) can be defined in the space $L_2(\mathbf{R}, H_{12})$ by the following expression

$$\tilde{\mathcal{A}}_{12} = -\frac{d^2}{dy_{12}^2} + A_d^2.$$

The space $L_2(\mathbf{R}, H_{12})$ can be embedded into the space $\mathcal{H}^2$ with the help of the discrete spectrum eigenfunctions of the operator A^2 as it has been described in [44].

3 Three-body Hamiltonian

This section is devoted to the construction of the three-body symmetric operator with nontrivial two-body interactions. A rich symmetry group of the operator will be described. Selfadjoint extensions of the symmetric operator will be constructed using the von Neumann theory in Section 6. Sommerfeld-Maluzhinetz integral transformation will be used in this section to transform the differential equation on the deficiency elements into the difference functional equation. We restrict our consideration later on to the case of indistinguishable particles. The corresponding simplified functional equation will also be derived.

3.1 Symmetric three-body operator

The model three-body operator will be defined as a perturbation of the orthogonal sum of operators describing possible asymptotic channels for the system of three particles. These asymptotic channels are

three noninteracting particles (operator $\mathcal{A}_{1,2,3}$);

two particles in a bound state and the third particle free (operators $\mathcal{A}_{12,3}, \mathcal{A}_{23,1}, \mathcal{A}_{31,2}$);

three particles in a bound state (operator $\mathcal{A}_{123}$).

We restrict our consideration to the case of particles with equal masses. Let (α, β, γ) be a cyclic permutation of the numbers $(1, 2, 3)$ then the three-body Jacobi coordinates can be written as follows

$$
\begin{aligned}
y_{123} &= \tfrac{1}{3}(r_1 + r_2 + r_3), \\
x_{\alpha\beta} &= (r_\alpha - r_\beta), \\
x_{\alpha\beta,\gamma} &= \sqrt{\tfrac{4}{3}}\left(\tfrac{1}{2}(r_\alpha + r_\beta) - r_\gamma\right).
\end{aligned} \tag{3.1}
$$

Then we define the channel Hamiltonians as follows

$$
\begin{aligned}
\mathcal{A}_{1,2,3} &= -\tfrac{1}{2}\left(\tfrac{\partial^2}{\partial r_1^2} + \tfrac{\partial^2}{\partial r_2^2} + \tfrac{\partial^2}{\partial r_3^2}\right) && \text{in} \quad \mathcal{H}_{1,2,3} = L_2(\mathbf{R}^3), \\
\mathcal{A}_{\alpha\beta,\gamma} &= -\left(\tfrac{1}{6}\tfrac{\partial^2}{\partial y_{123}^2} + \tfrac{\partial^2}{\partial x_{\alpha\beta,\gamma}^2}\right) + A_{\alpha\beta} && \text{in} \quad \mathcal{H}_{\alpha\beta,\gamma} = L_2(\mathbf{R}^2, H_{\alpha\beta}), \\
\mathcal{A}_{123} &= -\tfrac{1}{6}\tfrac{\partial^2}{\partial y_{123}^2} + A_{123} && \text{in} \quad \mathcal{H}_{123} = L_2(\mathbf{R}, H_{123}).
\end{aligned} \tag{3.2}
$$

Here the operators $A_{\alpha\beta}, A_{123}$ are certain selfadjoint matrices in the finite dimensional Hilbert spaces $H_{\alpha\beta}, H_{123}$ respectively. The unperturbed operator $\mathcal{A}_0^3$ is the orthogonal sum of the asymptotic channel Hamiltonians:

$$
\mathcal{A}_0^3 = \mathcal{A}_{1,2,3} \oplus \mathcal{A}_{12,3} \oplus \mathcal{A}_{23,1} \oplus \mathcal{A}_{31,2} \oplus \mathcal{A}_{123} \tag{3.3}
$$

acting in the Hilbert space

$$
\mathcal{H}^3 = \mathcal{H}_{1,2,3} \oplus \mathcal{H}_{12,3} \oplus \mathcal{H}_{23,1} \oplus \mathcal{H}_{31,2} \oplus \mathcal{H}_{123}. \tag{3.4}
$$

The operator $\mathcal{A}_0^3$ can be decomposed into the tensor sum

$$
\mathcal{A}_0^3 = -\frac{1}{6}\frac{\partial^2}{\partial y_{123}^2} \times I + I \times A_0^3,
$$

which corresponds to the separation of the center of mass motion. The three body operator with the separated center of mass motion will be investigated. The corresponding unperturbed operator is equal to the orthogonal sum of the channel operators

$$
A_0^3 = A_{1,2,3} \oplus A_{12,3} \oplus A_{23,1} \oplus A_{31,2} \oplus A_{123}. \tag{3.5}
$$

It acts in the orthogonal sum of the Hilbert spaces

$$
H^3 = H_{1,2,3} \oplus H_{12,3} \oplus H_{23,1} \oplus H_{31,2} \oplus H_{123}. \tag{3.6}
$$

The operators used in the decomposition of the operator A_0^3 are equal to

$$(3.7)\quad \begin{array}{rcl} A_{1,2,3} & = & -\left(\frac{\partial^2}{\partial x_{12}^2}+\frac{\partial^2}{\partial x_{12,3}^2}\right) = -\left(\frac{\partial^2}{\partial x_{23}^2}+\frac{\partial^2}{\partial x_{23,1}^2}\right) \\ & = & -\left(\frac{\partial^2}{\partial x_{31}^2}+\frac{\partial^2}{\partial x_{31,2}^2}\right) \text{ in } H_{1,2,3} = L_2(\mathbf{R}^2); \\ A_{\alpha\beta,\gamma} & = & -\frac{\partial^2}{\partial x_{\alpha\beta,\gamma}^2} + A_{\alpha\beta} \text{ in } H_{\alpha\beta,\gamma} = L_2(\mathbf{R}, H_{\alpha\beta}). \end{array}$$

The scalar product in the Hilbert space H^2 will be denoted by $\ll \cdot,\cdot \gg$. We are going to restrict our consideration to the case of the trivial space H_{123} for simplicity.

The interaction between the channels will be introduced by restricting first the operator A_0^3 to a certain symmetric operator A_{00}^3 and constructing its different selfadjoint extension. The operator $A_{1,2,3}$ describes the free motion of particles on the plane $\Lambda = \{r \in \mathbf{R}^3 \mid r_1+r_2+r_3 = 0\}$. In analogy to Section 2 the interaction with the cluster operator $A_{\alpha\beta,\gamma}$ should be introduced on the line ℓ_γ, where the coordinates of the particles α and β coincide $r_\alpha = r_\beta$. These lines ℓ_1, ℓ_2, ℓ_3 divide the plane Λ onto six equal sectors. The point of the intersection of these lines needs very careful consideration. Thus on the first step only the functions with the support separated from the origin will be considered. The operator defined on such functions will be symmetric only but not selfadjoint. In order to introduce the two-body interaction we restrict the operator $A_{1,2,3} \to A_{1,2,30}$ to the set of the smooth functions, vanishing in a neighborhood of the lines $\ell_\gamma, \gamma = 1,2,3$. The adjoint operator is defined on the domain $W_2^2(\Lambda \setminus \{\ell_\gamma\})$. Functions from this domain can have singularities on the lines ℓ_γ. Let us denote by $\Lambda_1, \Lambda_2, ..., \Lambda_6$ the six sectors on the Λ-plane. Thus we introduce the following subspace of bounded functions

Definition 3.1 *The subspace $C_0^\infty \subset W_2^2(\Lambda \setminus \{\ell_\gamma\})$ consists of all infinitely differentiable outside the lines ℓ_γ bounded functions with compact support separated from the origin.*

The support of a function from the defined subspace is not necessarily separated from the screens ℓ_γ. The functions from C_0^∞ can be discontinuous on the lines ℓ_γ but the boundary values of the functions and their normal derivatives from the both sides of the lines exist and are absolutely continuous functions with compact support.

Lemma 3.1 *Let $u_{1,2,3}, v_{1,2,3} \in C_0^\infty$. Then the boundary form of the operator A_{00}^{3*} is equal to*

$$\ll A^*_{1,2,30} u_{1,2,3}, v_{1,2,3} \gg - \ll u_{1,2,3}, A^*_{1,2,30} v_{1,2,3} \gg = \tag{3.8}$$

$$= \sum_{\gamma=1}^{3} \int dx_{\alpha\beta,\gamma} \left\{ \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] \overline{< v_{1,2,3} >} + < \frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} > \overline{[v_{1,2,3}]} - \right.$$

$$\left. - < u_{1,2,3} > \overline{\left[\frac{\partial v_{1,2,3}}{\partial x_{\alpha\beta}} \right]} - [u_{1,2,3}] \overline{< \frac{\partial v_{1,2,3}}{\partial n_\gamma} >} \right\} |_{x_{\alpha\beta}=0} \; .$$

where the sum is taken over all cyclic permutations (α, β, γ) of the numbers $(1, 2, 3)$ parameterized by the number γ.

P r o o f. The lemma can be proven by integrating by parts in the domain $\Lambda \setminus \{\ell_\gamma\}$ which is possible because the functions $u_{1,2,3}, v_{1,2,3}$ are twice continuously differentiable outside the lines ℓ_γ. The boundary values $\frac{\partial u_{1,2,3}}{\partial x_{\alpha,\beta}}, u_{1,2,3}, \frac{\partial v_{1,2,3}}{\partial x_{\alpha,\beta}}, v_{1,2,3}$ exist and they are continuous functions with compact support on every line ℓ_γ. Thus all integrals in formula (3.8) converge. $\square$

Theorem 3.1 *The operator*

$$\begin{aligned} A^3 U^3 &= A^3 \begin{pmatrix} u_{1,2,3} \\ u_{\alpha\beta,\gamma} \end{pmatrix} \\ &= \begin{pmatrix} A^*_{1,2,30} u_{1,2,3} \\ A_{\alpha\beta,\gamma} u_{\alpha\beta,\gamma} + \left(a_{\alpha\beta,\gamma} \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] + b_{\alpha\beta,\gamma} < u_{1,2,3} > \right) |_{\ell_\gamma} \; \theta_{\alpha\beta,\gamma} \end{pmatrix} \end{aligned} \tag{3.9}$$

defined on the domain of functions from $\mathbf{C}_0^\infty = C_0^\infty \oplus \sum_{\gamma=1}^{3} C_0^\infty(\mathbf{R} \setminus \{0\}, H_{\alpha,\beta})$ satisfying the boundary conditions

$$\begin{aligned} \left(c_{\alpha\beta,\gamma} \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] + d_{\alpha\beta,\gamma} < u_{1,2,3} > \right) |_{\ell_\gamma} &= < u_{\alpha\beta,\gamma}, \theta_{\alpha\beta,\gamma} >, \\ [u_{1,2,3}]|_{\ell_\gamma} &= 0 \end{aligned} \tag{3.10}$$

is symmetric if the real parameters a, b, c, d satisfy the following condition

$$det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1$$

and $\theta_{\alpha\beta,\gamma} \in H_{\alpha\beta}$.

P r o o f . Consider two arbitrary elements $U, V \in \mathbf{C}_0^\infty$. The boundary form of the operator A^3 is equal to

$$
\begin{array}{c}
< A^3 U, V >_{H^3} - < U, A^3 V >_{H^3} = \\
= \ll A^*_{1,2,30} u_{1,2,3}, v_{1,2,3} \gg - \ll u_{1,2,3}, A^*_{1,2,30} v_{1,2,3} \gg \\
+ \sum_{\gamma=1}^3 \int dx_{\alpha\beta,\gamma} \left\{ \left(a_{\alpha\beta,\gamma} \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] + b_{\alpha\beta,\gamma} < u_{1,2,3} > \right) |_{x_{\alpha\beta}=0} < \theta_{\alpha\beta,\gamma}, v_{\alpha\beta,\gamma} > \right. \\
\left. - < u_{\alpha\beta,\gamma}, \theta_{\alpha\beta,\gamma} > \overline{\left(a_{\alpha\beta,\gamma} \left[\frac{\partial v_{1,2,3}}{\partial x_{\alpha\beta}} \right] + b_{\alpha\beta,\gamma} < v_{1,2,3} > \right) |_{x_{\alpha\beta}=0}} \right\} = \\
= \sum_{\gamma=1}^3 \int dx_{\alpha\beta,\gamma} \left\{ \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] \overline{< v_{1,2,3} >} |_{x_{\alpha\beta}=0} + < \frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} > \overline{[v_{1,2,3}]} |_{x_{\alpha\beta}=0} \right. \\
- < u_{1,2,3} > \overline{\left[\frac{\partial v_{1,2,3}}{\partial x_{\alpha\beta}} \right]} |_{x_{\alpha\beta}=0} - [u_{1,2,3}] \overline{< \frac{\partial v_{1,2,3}}{\partial n_\gamma} >} |_{x_{\alpha\beta}=0} \\
+ \left(a_{\alpha\beta,\gamma} \left[\frac{\partial u_{1,2,3}}{\partial x_{\alpha\beta}} \right] + b_{\alpha\beta,\gamma} < u_{1,2,3} > \right) |_{x_{\alpha\beta}=0} < \theta_{\alpha\beta,\gamma}, v_{\alpha\beta,\gamma} > \\
\left. - < u_{\alpha\beta,\gamma}, \theta_{\alpha\beta,\gamma} > \overline{\left(a_{\alpha\beta,\gamma} \left[\frac{\partial v_{1,2,3}}{\partial x_{\alpha\beta}} \right] + b_{\alpha\beta,\gamma} < v_{1,2,3} > \right) |_{x_{\alpha\beta}=0}} \right\}.
\end{array}
$$

The integrated expression vanishes at every point on the lines ℓ_γ due to the conditions (3.10). It follows that the operator A^3 on $\mathbf{C}_0^\infty$ is symmetric. □

The operator A^3 with the domain $\mathbf{C}_0^\infty$ is not selfadjoint and its selfadjoint extensions can be described in terms of the deficiency elements. The deficiency elements will be calculated in the following sections. We discuss first the symmetries of the constructed operator.

3.2 The symmetry group

We consider in the sequel the system of identical particles. In terms of the constructed model it means that the operators $A_{\alpha\beta,\gamma}$, the constants $a_{\alpha\beta,\gamma}, b_{\alpha\beta,\gamma}, c_{\alpha\beta,\gamma}, d_{\alpha\beta,\gamma}$ and the vectors $\theta_{\alpha\beta,\gamma}$ are equal. The function $u_{1,2,3}$ can be considered in three different coordinate systems related to the three cluster decompositions of the three particles. We denote the corresponding functions by the index $1, 2$ or 3 in such a way that

$$u^3_{1,2,3}(x_{12}, x_{12,3}) = u^1_{1,2,3}(x_{23}, x_{23,1}) = u^2_{1,2,3}(x_{31}, x_{31,2}).$$

The symmetries of the system of three identical particles interacting via even potential are described by the dihedral group D_{12} [16] generated by two elements s and t such that

$$s^6 = 1, \;\; t^2 = 1, \;\; tst = s^{-1}.$$

The constructed model symmetric operator has the same symmetry group. The element s of order 6 corresponds to the rotation of the plane Λ on the angle $\pi/3$

$$(3.11)\qquad sU = V \Rightarrow \begin{pmatrix} v^{\gamma}_{1,2,3}(x_{\alpha\beta}, x_{\alpha\beta,\gamma}) \\ v_{\alpha\beta,\gamma}(x_{\alpha\beta,\gamma}) \\ v_{\beta\gamma,\alpha}(x_{\beta\gamma,\alpha}) \\ v_{\gamma\alpha,\beta}(x_{\gamma\alpha,\beta}) \end{pmatrix} = \begin{pmatrix} u^{\alpha}_{1,2,3}(-x_{\alpha\beta}, -x_{\alpha\beta,\gamma}) \\ u_{\beta\gamma,\alpha}(-x_{\alpha\beta,\gamma}) \\ u_{\gamma\alpha,\beta}(-x_{\beta\gamma,\alpha}) \\ u_{\alpha\beta,\gamma}(-x_{\gamma\alpha,\beta}) \end{pmatrix}.$$

The element t can be chosen equal to the operator Z_γ of the transposition of the particles α and β

$$(3.12)\qquad tU = Z_\gamma U = V \Rightarrow \begin{pmatrix} v^{\gamma}_{1,2,3}(x_{\alpha\beta}, x_{\alpha\beta,\gamma}) \\ v_{\alpha\beta,\gamma}(x_{\alpha\beta,\gamma}) \\ v_{\beta\gamma,\alpha}(x_{\beta\gamma,\alpha}) \\ v_{\gamma\alpha,\beta}(x_{\gamma\alpha,\beta}) \end{pmatrix} = \begin{pmatrix} u^{\gamma}_{1,2,3}(-x_{\alpha\beta}, x_{\alpha\beta,\gamma}) \\ u_{\alpha\beta,\gamma}(x_{\alpha\beta,\gamma}) \\ u_{\gamma\alpha,\beta}(x_{\beta\gamma,\alpha}) \\ u_{\beta\gamma,\alpha}(x_{\gamma,\alpha,\beta}) \end{pmatrix}.$$

The transpositions Z_γ generate the subgroup of permutations $\mathcal{P}_3$, which consists of 6 elements [16].

The element s generates important cyclic subgroup, namely the group of the central rotations on the plane Λ by the angles $n\pi/3$. As if the operator A^3 commutes with the rotations s^n: $A^3 s^n = s^n A^3$ the Hilbert space is decomposable into the orthogonal sum of Hilbert spaces of functions, which are quasi invariant with respect to the rotations s:

$$(3.13)\qquad sU = e^{-im\pi/3}U, \;\; m = 0, 1, 2, 3, 4, 5.$$

Let us denote by P_m the projector on these quasi invariant elements. Every such element is defined by its values in one of the sectors $\Lambda' = \{x_{12} < 0, x_{31} < 0\}$ on the plane Λ and values of the functions $u_{12,3}$ on the positive halfaxis. The transformation

$$T_m : U \to (u^m_o, u^m_1) \in L_2(\Lambda') \oplus L_2(\mathbf{R}_+, H_{12})$$

$$T_m U = ((P_m u_{1,2,3})|_{\Lambda'}, (P_m u_{12,3})|_{\mathbf{R}_+})$$

is invertible on such functions. The operator $6T_m$ is norm preserving. The Hilbert space H^3 and the operator A^3 can be decomposed as follows

$$H^3 = \oplus \sum_{m=0}^{5} T_m^{-1} H_m, \;\; H_m = L_2(\Lambda') \oplus L_2(\mathbf{R}_+, H_{12});$$

$$A^3 = \oplus \sum_{m=0}^{5} T_m^{-1} A_m T_m.$$

Let us introduce the polar coordinates in such a way that $\Lambda' = \{(r,\varphi)|0 \le r \le \infty, 0 \le \varphi \le \pi/3\}$.

Lemma 3.2 *The operator A_m is defined by the following formula*

$$A_m \begin{pmatrix} u_0^m(r,\varphi) \\ u_1^m(r) \end{pmatrix} = \begin{pmatrix} -\Delta_{r,\varphi} u_0^m \\ \left(-\frac{\partial^2}{\partial r^2} + A_{12}\right) u_1^m + \ell(u_0^m)\theta \end{pmatrix},$$

(3.14)

$$\ell(u_0^m) = \frac{a}{r}\left(\frac{\partial u_0^m}{\partial \varphi}\, |_{\varphi=0} - e^{-im\pi/3} \frac{\partial u_0^m}{\partial \varphi}\, |_{\varphi=\pi/3}\right) + \frac{b}{2}\left(u_0^m\, |_{\varphi=0} + e^{-im\pi/3} u_0^m\, |_{\varphi=\pi/3}\right)$$

on the domain of functions from $T_m \mathbf{C}_0^\infty$ satisfying the boundary conditions

(3.15)

$$< u_1^m, \theta >= \frac{c}{r}\left(\frac{\partial u_0^m}{\partial \varphi}\, |_{\varphi=0} - e^{-im\pi/3} \frac{\partial u_0^m}{\partial \varphi}\, |_{\varphi=\pi/3}\right) + \frac{d}{2}\left(u_0^m\, |_{\varphi=0} + e^{-im\pi/3} u_0^m\, |_{\varphi=\pi/3}\right),$$

$$u_0^m\, |_{\varphi=0} = e^{-im\pi/3} u_0^m\, |_{\varphi=\pi/3}\ . \tag{3.16}$$

P r o o f . Consider any $U \in T_m \mathbf{C}_0^\infty$. Then $A_m = T_m A^3 T_m^{-1}$. The domain of the operator coincides with the set $T_m Dom(A^3)$. The boundary conditions (3.15),(3.16) follow from the boundary conditions (3.10) and the fact that every element $T_m^{-1} U$ satisfies (3.13). Similarly formula (3.14) for the operator A_m follows from (3.9) and (3.13). □

3.3 Deficiency elements and Sommerfeld-Maluzhinetz transformation

The deficiency elements for the operator A_m are solutions of the equation

$$A_m^* G^m = \lambda G^m, \ \Im\lambda > 0, \tag{3.17}$$

$$k = \sqrt{\lambda}, \ \Im k \ge 0,$$

$$G^m = (g_0^m, g_1^m).$$

The operator A_m^* is defined by the same differential expression (3.14) on the set of functions from $W_2^2(\Lambda') \oplus W_2^2(\mathbf{R}_+, H_{12})$ satisfying the boundary conditions (3.15),(3.16). We consider first only bounded functions continuously differentiable outside the lines ℓ_γ which are from the domain of A_m^*. These functions are not necessarily equal to zero at the origin. The set of deficiency elements from this class is not trivial. The Sommerfeld-Maluzhinetz integral representation [38, 31, 32] will be used to solve the system of equations (3.17). We consider here the limit case where k is real and positive. This limit of the deficiency element will be used to calculate the eigenfunctions of the extended three-body operator.

We suppose that the components of the function G^m can be presented by the following integrals over the plane waves

$$g_0^m(k) = \frac{1}{2\pi i}\int_\Gamma e^{ikr\cos\alpha}\left\{\tilde{g}_+^m(\alpha+\varphi) + \tilde{g}_-^m(\alpha+\pi/3-\varphi)\right\}d\alpha, \tag{3.18}$$

$$g_1^m(k) = \frac{1}{2\pi i}\int_\Gamma e^{ikr\cos\alpha}\tilde{g}_1^m(\alpha)d\alpha,$$

where Γ is a contour in the complex plane α. The contour goes to infinity for real positive k in the upper half-plane in the strips

$$(2m-1)\pi < \Re\alpha < 2\pi m, \quad m = 0, 1.$$

The contour Γ is not closed and has two infinite branches. We chose first the contour Γ in such a way, that no singularity of the density function is situated inside the contour. It is possible because the singularities are situated at a finite distance from the real axis. This assumption will be justified later when the integral density will be calculated. The integral densities are supposed to be analytic functions in α in the region of deformation of the contour Γ. Moreover we are going to carry out integration by parts during these calculations. We assume that the contribution of the boundary terms is equal to zero. These calculations will be justified later for the calculated solution only. The contour Γ will be chosen in a special way later, but at this time we fix some contour Γ_1 , which is situated in the upper half-plane $\Im\alpha > Q, Q > 0$. The positive real number Q will be determined after the calculation of the solution.

The component g_0^m satisfies the first equation (3.17) due to the special dependence of the integral density on the angle φ. The proof of this fact can be carried out by integration by parts which is possible due to our assumption. Integration by parts in the second equation (3.17) for the component g_1^m gives the following equations for the integral densities $\tilde{g}_1^m$

$$(3.19) \qquad (A_{12} - \lambda \sin^2 \alpha)\tilde{g}_1^m =$$

$$= -\Big\{aik \sin\alpha \left[\tilde{g}_+^m(\alpha) - \tilde{g}_-^m(\alpha + \pi/3) - e^{-im\pi/3}(\tilde{g}_+^m(\alpha+\pi/3) - \tilde{g}_-^m(\alpha))\right]$$
$$+ b/2 \left[\tilde{g}_+^m(\alpha) + \tilde{g}_-^m(\alpha + \pi/3) + e^{-im\pi/3}(\tilde{g}_+^m(\alpha+\pi/3) + \tilde{g}_-^m(\alpha))\right]\Big\}\,\theta.$$

Solving the matrix equation (3.19) we get one of the solutions of the equation. But the general solution contains an additional decreasing exponent which belongs to the kernel of the Sommerfeld-Maluzhinetz transformation

$$(3.20) \qquad \exp\left(i\sqrt{\lambda - A_{12}}r\right) h.$$

Here h is a vector from H_{12}. It parametrizes the solution. The matrix exponential function can be presented by the Sommerfeld integral with the following integral density (see [31, 32])

$$(3.21) \qquad -\frac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}} h.$$

The general solution of the second equation (3.17) in the Sommerfeld representation is equal to

$$(3.22) \qquad \begin{aligned} \tilde{g}_1^m(\alpha) \;=\; & -\tfrac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}} h \\ & -\Big\{aik \sin\alpha \left[\tilde{g}_+^m(\alpha) - \tilde{g}_-^m(\alpha + \pi/3) - e^{-im\pi/3}(\tilde{g}_+^m(\alpha+\pi/3) - \tilde{g}_-^m(\alpha))\right] \\ & + b/2 \left[\tilde{g}_+^m(\alpha) + \tilde{g}_-^m(\alpha + \pi/3) + e^{-im\pi/3}(\tilde{g}_+^m(\alpha+\pi/3) + \tilde{g}_-^m(\alpha))\right]\Big\} \\ & \times (A_{12} - \lambda\sin^2\alpha)^{-1}\theta. \end{aligned}$$

We exclude the component g_1 by projecting the solution onto the element θ. The following difference equation on the integral densities $\tilde{g}_\pm^m(\alpha)$ is obtained

$$(3.23) \qquad e^{-im\pi/3}\tilde{g}_+^m(\alpha+\pi/3) + \tilde{g}_-^m(\alpha+\pi/3) = \Pi(\alpha)\left(\tilde{g}_+^m(\alpha) + e^{-im\pi/3}\tilde{g}_-^m(\alpha)\right) + 2f(\alpha),$$

where the following notations are used

$$(3.24) \qquad \Pi(\alpha) = \frac{2ik\sin\alpha + \mathbf{D}(\lambda\sin^2\alpha)}{2ik\sin\alpha - \mathbf{D}(\lambda\sin^2\alpha)} = \left(T(k\sin\alpha) + R(k\sin\alpha)\right)^{-1},$$

$$f(h,\alpha) = \frac{< \frac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}} h, \theta >}{(2aik\sin\alpha - b)\mathbf{R}(\lambda\sin^2\alpha) + 2cik\sin\alpha - d}.$$

The two-body transition and reflection coefficients appear in the latter formula. [2] We get the second difference equation from the second boundary condition

$$e^{-im\pi/3}\tilde{g}^m_+(\alpha+\pi/3) - \tilde{g}^m_-(\alpha+\pi/3) = \tilde{g}^m_+(\alpha) - e^{-im\pi/3}\tilde{g}^m_-(\alpha). \tag{3.25}$$

These equations can be written in the vector form for the two component functions $\tilde{g}^m(\alpha) = (\tilde{g}^m_+(\alpha), \tilde{g}^m_-(\alpha))$

(3.26)

$$\begin{aligned}\tilde{g}^m(\alpha+\pi/3) &= \tfrac{1}{2}\begin{pmatrix} e^{im\pi/3}(\Pi(\alpha)+1) & \Pi(\alpha)-1 \\ \Pi(\alpha)-1 & e^{-im\pi/3}(\Pi(\alpha)+1) \end{pmatrix}\tilde{g}^m(\alpha) \\ &\quad + f(\alpha)\begin{pmatrix} e^{im\pi/3} \\ 1 \end{pmatrix}.\end{aligned}$$

We are able to obtain the solution for these difference equations in terms of elementary functions only for $m = 0, 3$. The solution to the difference equation for general m will be presented in one of the future publications. The cases $m = 0, 3$ correspond to the system of particles with the wave function symmetric or antisymmetric with respect to the transpositions of the particles (boson or fermion systems correspondingly). The eigenbasis of the matrix

$$\frac{1}{2}\begin{pmatrix} e^{im\pi/3}(\Pi(\alpha)+1) & \Pi(\alpha)-1 \\ \Pi(\alpha)-1 & e^{-im\pi/3}(\Pi(\alpha)+1) \end{pmatrix}$$

is independent of α in this case and the matrix system of the difference equations can be reduced to two independent ordinary difference equations.

3.4 Functional equation for identical particles

We consider in the sequel the case where the wave function of three particles has boson symmetry. The case of fermions leads to the trivial interaction between the particles because every continuous antisymmetric function is equal to zero at the origin. As a result of this the three-body and two-body channels for the fermion problem would be separated in our model.

[2] The vector h can be considered as a parameter in the latter formula. We are going to use the simplified notation $f(\alpha)$ in the following sections except Section 7.

Thus we restrict our consideration to the system of bosons. To obtain the three-body boson Hamiltonian we restrict the operator A^3 to the set H_b^3 of functions invariant with respect to the subgroup of transpositions $\mathcal{P}_3$. The corresponding subset of $\mathbf{C}_0^\infty$ will be denoted by $\mathbf{C}_{0b}^\infty$. Every element U symmetric with respect to the permutation of the particles is determined by its values in the sector Λ' and values of the function $u_{12,3}$ and $u_{23,1}$ on the positive and negative halfaxes respectively. Thus the transformation

$$P : H_b^3 \to L_2(\Lambda') \oplus (\mathbf{R}_+, H_{12}) \oplus (\mathbf{R}_+, H_{12})$$

$$PU = \begin{pmatrix} u_{1,2,3}(x_{12}, x_{12,3})|_{\Lambda'} \\ \frac{1}{2}u_{12,3}(x_{12,3})|_{\mathbf{R}_+} \\ \frac{1}{2}u_{23,1}(-x_{23,1})|_{\mathbf{R}_+} \end{pmatrix}$$

is invertible on bosonic elements. The operator $6P$ preserves the norm of the element.

Lemma 3.3 *The operator $A_b = PA^3P^{-1}$ is defined by the following differential expression*

(3.27)

$$A_b \begin{pmatrix} u_0(r,\varphi) \\ u_1(r) \\ u_2(r) \end{pmatrix} = \begin{pmatrix} -\Delta_{r,\varphi} u_0 \\ \left(-\frac{\partial^2}{\partial r^2} + A_{12}\right) u_1 + \left(\frac{2a}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=0} + b u_0 \mid_{\varphi=0}\right)\theta \\ \left(-\frac{\partial^2}{\partial r^2} + A_{12}\right) u_2 + \left(-\frac{2a}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=\pi/3} + b u_0 \mid_{\varphi=\pi/3}\right)\theta \end{pmatrix}$$

on the domain of functions from $P\mathbf{C}_{0b}^\infty$ satisfying the boundary conditions

$$< u_1, \theta > = \frac{c}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=0} + \frac{d}{2} u_0 \mid_{\varphi=0},$$

(3.28)
$$< u_2, \theta > = -\frac{c}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=\pi/3} + \frac{d}{2} u_0 \mid_{\varphi=\pi/3} .$$

P r o o f . The proof of the lemma is quite similar to the proof of Lemma 3.4. The domain of A_b consists of all U such that $P^{-1}U \in Dom(A^3)$. The boundary conditions (3.28) follow from this inclusion. □

The operator A_b is symmetric and it commutes with the reflection operator with respect to the bisector:

$$Y \begin{pmatrix} u_0(r,\varphi) \\ u_1(r) \\ u_2(r) \end{pmatrix} = \begin{pmatrix} u_0(r, \pi/3 - \varphi) \\ u_2(r) \\ u_1(r) \end{pmatrix}.$$

Consider the transformations $P_s(P_a)$ of all symmetric (antisymmetric) elements from $P\mathbf{C}_{0b}^{\infty}$ to $L_2(\Lambda'') \oplus L_2(\mathbf{R}_+, H_{12})$ (Λ'' denotes the sector on the plane Λ with the angle $\pi/6$: $\Lambda'' = \{(r,\varphi) \mid 0 \le r < \infty, 0 \le \varphi \le \pi/6\}$). These transformations are invertible and the operator A_b can be decomposed as follows

$$A_b = P_s^{-1} A_s P_s \oplus P_a^{-1} A_a P_a.$$

Lemma 3.4 *The operators A_s and A_a are defined by the following formula*

$$A_{s,a} \begin{pmatrix} u_0(r,\varphi) \\ u_1(r) \end{pmatrix} = \begin{pmatrix} -\Delta_{r,\varphi} u_0 \\ \left(-\frac{\partial^2}{\partial r^2} + A_{12}\right) u_1 + \left(\frac{2a}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=0} + b u_0 \mid_{\varphi=0}\right)\theta \end{pmatrix}. \tag{3.29}$$

on the domain of functions from $P_s\mathbf{C}_{0b}^{\infty}$ and $P_a\mathbf{C}_{0b}^{\infty}$ satisfying the following boundary conditions

$$\begin{array}{ll} A_s : & < u_1, \theta >= \frac{c}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=0} + \frac{d}{2} u_0 \mid_{\varphi=0} \\ & \frac{\partial u_0}{\partial \varphi} \mid_{\varphi=\pi/6} = 0; \\ & \\ A_a & < u_1, \theta >= \frac{c}{r}\frac{\partial u_0}{\partial \varphi} \mid_{\varphi=0} + \frac{d}{2} u_0 \mid_{\varphi=0} \\ & u_0 \mid_{\varphi=\pi/6} = 0. \end{array} \tag{3.30}$$

The proof is quite similar to the proof of Lemmas 3.4 and 3.5.

The boundary conditions on the line $\varphi = 0$ for the functions from the domain of the operators A_s and A_a coincide.

The operators A_a and A_s are symmetric but not selfadjoint. The bounded deficiency elements for the operators A_s and A_a can be presented using the Sommerfeld-Maluzhinetz transformation by the following integrals

$$\begin{array}{rcl} g_0^s & = & \frac{1}{2\pi i}\int_\Gamma e^{ikr\cos\alpha}\left\{\tilde{g}_0^s(\alpha+\varphi) + \tilde{g}_0^s(\alpha+\pi/3-\varphi)\right\} d\alpha \\ g_0^a & = & \frac{1}{2\pi i}\int_\Gamma e^{ikr\cos\alpha}\left\{\tilde{g}_0^a(\alpha+\varphi) - \tilde{g}_0^a(\alpha+\pi/3-\varphi)\right\} d\alpha \\ g_1^{s,a} & = & \frac{1}{2\pi i}\int_\Gamma e^{ikr\cos\alpha}\tilde{g}_1^{s,a} d\alpha \end{array} \tag{3.31}$$

The functions g_0^s (g_0^a) satisfy the Helmholtz equation and Neumann (Dirichlet) boundary condition on the line $\varphi = \pi/6$ respectively. Functional equations for the meromorphic functions $\tilde{g}_0^s, \tilde{g}_0^a$ can be derived from the difference equations (3.23) by the following substitution

$$\begin{array}{ll} A_s : & m = 6, \\ & \tilde{g}_+^6 = \tilde{g}_-^6 = \tilde{g}_0^s; \\ A_a : & m = 3, \\ & \tilde{g}_+^3 = -\tilde{g}_-^3 = \tilde{g}_0^a. \end{array}$$

We get the following functional difference equations

$$\tilde{g}_0^s(\alpha + \pi/3) = \Pi(\alpha)\tilde{g}_0^s(\alpha) + f(\alpha)$$

$$\tilde{g}_0^a(\alpha + \pi/3) = -\Pi(\alpha)\tilde{g}_0^a(\alpha) - f(\alpha) \tag{3.32}$$

Here the function $\Pi(\alpha)$ is defined in (3.24). The solutions of these equations will be derived in the next section.

4 Solution of the Functional Equations

The difference equations (3.32) are investigated in the present section. These two equations have similar structure and can be solved using the same method. The solution of the first equation will be discussed in detail. The final formula for the solution of the second equation will be presented.

4.1 Method of iterations for the functional equation

The coefficients of the functional equation

$$\tilde{g}_0^s(\alpha + \pi/3) = \Pi(\alpha)\tilde{g}_0^s(\alpha) + f(\alpha) \tag{4.1}$$

possess the following properties

$$\Pi(\alpha + \pi) = \Pi(-\alpha) = \Pi^{-1}(\alpha), \tag{4.2}$$

$$\Pi(\alpha + 2\pi) = \Pi(\alpha), \tag{4.3}$$

$$f(\alpha + 2\pi) = f(\alpha). \tag{4.4}$$

The singularities and zeroes of the equation coefficients are situated at finite distance from the real axis. The function $\Pi(\alpha) = (T(k\sin\alpha) + R(k\sin\alpha))^{-1}$ is an unimodular rational function of $k\sin\alpha$. The singularities of the function $T(k)+R(k) = \frac{ik-D(k^2)}{ik+D(k^2)}$ coincide with the two-body bound states and the resonances. We made assumption in section 2.5 that all the two-body resonances are pure imaginary. The zeroes and singularities of $\Pi(\alpha)$ are symmetric with respect to each other in accordance with the property (4.2). Then the singularities and zeroes of $\Pi(\alpha)$ are situated on the lines $\Re\alpha = \pi s, s \in \mathbf{Z}$ in the complex plane α. The zeroes of the function $\Pi(\alpha)$ on the line $\Re\alpha = 0$ will be denoted by $i\gamma_m, m = -N_{12} - 1, ..., -2, -1, 0, 1, 2, ..., N_{12}$ in such a way that positive m correspond to the two-body bound states. Then the zeroes of the function $\Pi(\alpha)$ are situated at the points

$$(-1)^{s+1} i\gamma_m + s\pi, s \in \mathbf{Z},\ m = -N_{12} - 1, ..., -2, -1, 0, 1, ..., N_{12}.$$

The function $\Pi(\alpha)$ possesses a remarkable Blaschke representation. Let us denote by $i\chi_n$ the two-body resonances and bound states on the k-plane in such a way, that positive m correspond to the bound states. Then the following representation holds:

$$\Pi(\alpha) = \Pi_n \frac{ik \sin\alpha + \chi_n}{ik \sin\alpha - \chi_n}.$$

We are going to use this representation in the next section during the calculation of the residues of the functions $\Pi(\alpha)$ at the singular points.

We are going to look for the solutions of the equation which are analytic functions in a neighborhood of infinity, i.e. in the region $\Im\alpha \geq \max|\gamma_m|$. The parameter Q introduced in the previous section can now be chosen equal to $Q = max|\gamma_m|$. The solution of the functional equation can be expressed in terms of the elementary functions using the properties of the coefficients outlined above. The solution has to be a meromorphic function on the plane α.

The general solution of the functional equation is formed as a sum of particular solution of the inhomogeneous equation and general solution of the homogeneous equation. The general solution of the homogeneous equation

$$y(\alpha + \pi/3) = \Pi(\alpha) y(\alpha) \tag{4.5}$$

is represented by the product of one particular solution and arbitrary $\pi/3$ periodic function.

To derive the particular solution of the inhomogeneous equation we iterate this equation 5 times. All solutions of the inhomogeneous equation satisfy this new equation

$$y(\alpha + 2\pi) = y(\alpha) + \sigma(\alpha), \tag{4.6}$$

where

$$\begin{aligned}\sigma(\alpha) &= f(\alpha+5\pi/3)\\ &+\Pi(\alpha+5\pi/3)f(\alpha+4\pi/3)\\ &+\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+\pi)\\ &+\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+2\pi/3)\\ &+\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+\pi/3)\\ &+\Pi(\alpha+\pi)f(\alpha).\end{aligned}$$

The function $\sigma(\alpha)$ is a 2π periodic function. Consequently one of the solutions $y^*(\alpha)$ of the equation (4.6) is equal to

$$y^*(\alpha) = \frac{\alpha}{2\pi}\sigma(\alpha). \tag{4.7}$$

The general solution of the homogeneous equation (4.6) is a 2π periodic function. Hence we arrive to the following Ansatz for the solution of the functional equation (4.1)

$$\tilde{g}_0^s(\alpha) = y^*(\alpha) + y_0(\alpha), \tag{4.8}$$

where $y_0(\alpha)$ is a 2π periodic function. Substitution of this Ansatz (4.8) into the equation (4.1) gives the following equation for the periodic function $y_0(\alpha)$

$$y_0(\alpha+\pi/3) = \Pi(\alpha)y_0(\alpha) + f(\alpha) - \frac{1}{6}\sigma(\alpha+\pi/3). \tag{4.9}$$

Here we have used the fact that the function $\sigma(\alpha)$ satisfies the homogeneous equation

$$\sigma(\alpha+\pi/3) = \Pi(\alpha)\sigma(\alpha). \tag{4.10}$$

The solution of (4.9) can be calculated using the following Ansatz

$$\begin{aligned}y_0(\alpha) &= a_1 f(\alpha+5\pi/3)\\ &+a_2\Pi(\alpha+5\pi/3)f(\alpha+4\pi/3)\\ &+a_3\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+\pi)\\ &+a_4\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+2\pi/3)\\ &+a_5\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+\pi/3)\\ &+a_6\Pi(\alpha+\pi)f(\alpha).\end{aligned}$$

Substitution of this representation into (4.9) gives the following relations for the constants $\{a_j\}_{j=1}^{6}$

$$a_1 = a_6 + 5/6,$$

$$a_j = a_{j+1} + 1/6.$$

So we have got a one parameter set of solutions of the functional equation. Thus the following Theorem has been proven.

Theorem 4.1 *The function*

(4.11)

$$\begin{aligned}
\tilde{g}_0^s(\alpha) \; = \; & (\tfrac{\alpha}{2\pi} + t) f(\alpha + 5\pi/3) \\
& + (\tfrac{\alpha}{2\pi} + t - 1/6)\Pi(\alpha + 5\pi/3) f(\alpha + 4\pi/3) \\
& + (\tfrac{\alpha}{2\pi} + t - 2/6)\Pi(\alpha + 5\pi/3)\Pi(\alpha + 4\pi/3) f(\alpha + \pi) \\
& + (\tfrac{\alpha}{2\pi} + t - 3/6)\Pi(\alpha + 5\pi/3)\Pi(\alpha + 4\pi/3)\Pi(\alpha + \pi) f(\alpha + 2\pi/3) \\
& + (\tfrac{\alpha}{2\pi} + t - 4/6)\Pi(\alpha + 4\pi/3)\Pi(\alpha + \pi) f(\alpha + \pi/3) \\
& + (\tfrac{\alpha}{2\pi} + t - 5/6)\Pi(\alpha + \pi) f(\alpha).
\end{aligned}$$

for every value of the parameter t is a solution of the difference functional equation (4.1), in the region $\Im\alpha > \max|\gamma_m|$.

The set of functions $\tilde{g}_0^s(\alpha)$ does not coincide with the set of all solutions of the functional equation. One can easily write down the complete set of meromorphic solutions of the equation but we are not going to do that here. The one parameter family derived contains the solution we are searching for.

It is necessary to calculate the density $\tilde{g}_1^s(\alpha)$ in order to reconstruct all components of the deficiency element.

Lemma 4.1 *Let $G^s = (g_0^s, g_1^s)$ be a solution of the differential equation for the deficiency elements satisfying the boundary conditions (3.30). Let G^s be presented by the Sommerfeld integrals (3.31) with the density $\tilde{g}_0^s$ given by (4.11). Then the integral density $\tilde{g}_1^s$ is equal to*

$$\begin{aligned}
\tilde{g}_1^s(\alpha) \; = \; & -\frac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}}\frac{h}{2} \\
& + \frac{1}{2}\left\{\tilde{g}_0^s(\alpha)\frac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha) - b)}{2ik\sin\alpha - \mathbf{D}(\lambda\sin^2\alpha)} + f(\alpha)(2aik\sin\alpha - b)\right\} \\
& \times (A_{12} - \lambda\sin^2\alpha)^{-1}\theta.
\end{aligned} \quad (4.12)$$

P r o o f . The integral density can be reconstructed with the help of the equation (3.22) if we put $m = 0$ and $\tilde{g}^s_+(\alpha) = \tilde{g}^s_-(\alpha) = \tilde{g}^s_0(\alpha)$ and $\tilde{g}^s_1(\alpha) = \frac{1}{2}\tilde{g}^m_1(\alpha)$

$$\begin{aligned}\tilde{g}^s_1(\alpha) \;\; = \;\; & -\tfrac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}}\tfrac{h}{2} \\ & -\tfrac{1}{2}\left\{2aik\sin\alpha\left(\tilde{g}^s_0(\alpha) - \tilde{g}^s_0(\alpha+\pi/3)\right) + b\left(\tilde{g}^s_0(\alpha) + \tilde{g}^s_0(\alpha+\pi/3)\right)\right\} \\ & \times(A_{12} - \lambda\sin^2\alpha)^{-1}\theta.\end{aligned}$$

The latter formula can be modified using the fact that $\tilde{g}^s_0(\alpha)$ is a solution of the equation (4.1)

$$\begin{aligned}\tilde{g}^s_1(\alpha) \;\; = \;\; & -\tfrac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}}h/2 \\ & -\tfrac{1}{2}\left\{2aik\sin\alpha\left((1-\Pi(\alpha))\tilde{g}^s_0(\alpha) - f(\alpha)\right) + b\left((1+\Pi(\alpha))\tilde{g}^s_0(\alpha) + f(\alpha)\right)\right\} \\ & \times(A_{12} - \lambda\sin^2\alpha)^{-1}\theta.\end{aligned}$$

Formula (4.12) follows now from (3.14). □

4.2 Analytical properties of the solution

The analytical properties of the solution derived are described by the following:

Theorem 4.2 *The integral densities $\tilde{g}^s_0(\alpha+\varphi) + \tilde{g}^s_0(\alpha+\pi/3-\varphi), \tilde{g}^s_1(\alpha)$ are meromorphic on the whole complex α-plane. The singularities of the function $\tilde{g}^s_0(\alpha+\varphi) + \tilde{g}^s_0(\alpha+\pi/3-\varphi)$ are poles of finite multiplicity at the lattice of points*

$$-\varphi + (-1)^{s+1}i\gamma_m - n\pi/3 + s\pi,\; s = 0, \pm1, \pm2, ...; n = 0, 1, 2,$$

$$-\pi/3 + \varphi + (-1)^{s+1}i\gamma_m - n\pi/3 + s\pi,\; s = 0, \pm1, \pm2, ...; n = 0, 1, 2.$$

The same is true for the density $\tilde{g}^s_1$ with the lattice of points

$$(-1)^{s+1}i\gamma_m - n\pi/3 + s\pi,\; s = 0, \pm1, \pm2, ...; n = 0, 1, 2,$$

$$\pi/3 + (-1)^{s+1}i\gamma_m - n\pi/3 + s\pi,\; s = 0, \pm1, \pm2, ...; n = 0, 1, 2.$$

P r o o f. The singularities of the function $\tilde{g}_0^s(\alpha)$ are caused by the singularities of the functions $\Pi(\alpha)$ and $f(\alpha)$. The singularities of the function $f(\alpha)$ are situated at the points, where the denominator is equal to zero

$$(2aik\sin\alpha - b)\mathbf{R}(\lambda\sin^2\alpha) + 2cik\sin\alpha - d = 0 \Rightarrow 2ik\sin\alpha = \mathbf{D}(\lambda\sin^2\alpha).$$

These singularities coincide with the singularities of the function $\Pi(\alpha)$. Some additional singularities can be caused by the singularities of the numerator $< \frac{k\sin\alpha}{k\cos\alpha-\sqrt{\lambda-A_{12}}}h, \theta >$ but these singularities cancel with the singularities of the denominator. As the result the function $f(\alpha)$ is analytic in a neighborhood of these points. Hence the function $\tilde{g}_0^s(\alpha)$ has singularities at the points

$$..., i\gamma_m - 2\pi/3, i\gamma_m - \pi/3, i\gamma_m, -i\gamma_m + \pi/3, -i\gamma_m + 2\pi/3...$$

The integral density of the Sommerfeld integral for the component $g_0^s(r,\varphi)$ has singularities at the points

$$i\gamma_m - 2\pi/3 - \varphi, i\gamma_m - \pi/3 - \varphi, i\gamma_m - \varphi, -i\gamma_m + \pi/3 - \varphi, -i\gamma_m + 2\pi/3 - \varphi, ...$$

$$i\gamma_m - \pi + \varphi, i\gamma_m - 2\pi/3 + \varphi, i\gamma_m - \pi/3 + \varphi, -i\gamma_m + \varphi, -i\gamma_m + \pi/3 + \varphi, ...$$

The singularities of the function $\tilde{g}_1^s(\alpha)$ are situated at the same points as the singularities of the functions $\tilde{g}_0^s(\alpha), \tilde{g}_0^s(\alpha+\pi/3)$. Additional singularities can appear at the points corresponding to the eigenvalues λ_j of the operator $A_{12} : \lambda\sin^2\alpha_j = \lambda_j$. The function $\mathbf{D}(\lambda\sin^2\alpha)$ is equal to b/a at these points

$$\mathbf{D}(\lambda\sin^2\alpha_j) = \frac{b}{a},$$

if $a \neq 0$ and it has pole of the second order there if $a = d = 0$. The first term in (4.12) has the following singularity near the point α_j

$$-\frac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - A_{12}}}\frac{h}{2} \sim_{\alpha\to\alpha_j} -\frac{k\sin\alpha}{k\cos\alpha - \sqrt{\lambda - \lambda_j}}\frac{h_j}{2}e_j + O(1),$$

where e_j is the eigenvector of A_{12} corresponding to the eigenvalue λ_j. The first term in the square brackets in (4.12) has second order zero at the points $\alpha = \alpha_j$. Thus the second term of (4.12) possesses the following representation

$$\frac{1}{2}\left\{\tilde{g}_0^s(\alpha)\frac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha) - b)}{2ik\sin\alpha - \mathbf{D}(\lambda\sin^2\alpha)} + f(\alpha)(2aik\sin\alpha - b)\right\}(A_{12} - \lambda\sin^2\alpha)^{-1}\theta \sim_{\alpha\to\alpha_j}$$

$$\sim_{\alpha\to\alpha_j} \frac{1}{2}\frac{k\sin\alpha}{k\cos\alpha-\sqrt{\lambda-\lambda_j}}h_j\bar{\theta}_j\frac{\lambda_j-\lambda\sin^2\alpha}{|\theta_j|^2}\frac{1}{\lambda_j-\lambda\sin^2\alpha}\theta_j e_j+O(1).$$

Thus the function $\tilde{g}_1^s(\alpha)$ is bounded in the neighborhood of the point α_j. □

Thus the singularities of all integral densities are situated on finite distance from the real axis. The assumption formulated in Section 3.3 holds for the densities we calculated. The integral densities are analytic everywhere in the region $\Im\alpha > \max|\gamma_m|$, thus the corresponding integrals are solutions of the system of the differential equations for the deficiency elements. However the functions are exponentially increasing ones for large r and consequently do not belong to the Hilbert space even for $\Im\lambda > 0$. The asymptotic behavior of the integrals for $r \to \infty$ will be discussed in the next section. The integration contour should be changed in order to make the functions square integrable.

The solution of the second equation (64) can be derived in the same way (4.13)

$$\begin{aligned}\tilde{g}_0^a(\alpha) \;=\; & -(\tfrac{\alpha}{2\pi}+t)f(\alpha+5\pi/3)\\ & +(\tfrac{\alpha}{2\pi}+t-1/6)\Pi(\alpha+5\pi/3)f(\alpha+4\pi/3)\\ & -(\tfrac{\alpha}{2\pi}+t-2/6)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+\pi)\\ & +(\tfrac{\alpha}{2\pi}+t-3/6)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+2\pi/3)\\ & -(\tfrac{\alpha}{2\pi}+t-4/6)\Pi(\alpha+4\pi/3)\Pi(\alpha+\pi)f(\alpha+\pi/3)\\ & +(\tfrac{\alpha}{2\pi}+t-5/6)\Pi(\alpha+\pi)f(\alpha).\end{aligned}$$

It contains also arbitrary parameter t. The zeroes and the singularities of the function $\tilde{g}_0^a$ are situated at the same points as those of $\tilde{g}_0^s$.

5 Properties of the Deficiency Elements

We discuss here the properties of the solutions of the difference equation derived in the previous section. It will be shown that the Sommerfeld integrals with such densities over the contour Γ_1 are not square integrable functions. Another contour of integration will be chosen. The corresponding integrals will be elements of the Hilbert space for $\lambda : \Im\lambda > 0$. The asymptotic behavior for large $r \to \infty$ and for small $r \to 0$ will be investigated.

5.1 Asymptotics for large r

The asymptotic behavior at infinity will be studied with the help of the steepest descent method. The saddle points for the Sommerfeld integral are $\alpha = 0$ and $\alpha = \pi$. These two critical points define outgoing and incoming spherical waves in the asymptotics of the component g_0^s

$$
\begin{array}{ll}
\alpha = 0: & \frac{1}{2\pi i}\sqrt{\frac{2\pi}{kr}}e^{-i\pi/4}e^{ikr}\left(\tilde{g}_0^s(\varphi) + \tilde{g}_0^s(\pi/3 - \varphi)\right); \\
\alpha = \pi: & \frac{1}{2\pi i}\sqrt{\frac{2\pi}{kr}}e^{i\pi/4}e^{-ikr}\left(\tilde{g}_0^s(\pi + \varphi) + \tilde{g}_0^s(4\pi/3 - \varphi)\right).
\end{array}
\tag{5.1}
$$

The second point defines an exponentially increasing function for k with positive imaginary part. The solution of the difference equation contains free parameter t. This parameter can be chosen in a special way to make the amplitude of the incoming spherical wave equal to zero. Let the parameter t be equal to $-1/6$. Then the amplitude of the incoming spherical wave vanishes.

Lemma 5.1 *If $t = -1/6$ then the solution $\tilde{g}_0^s$ of the difference equation (4.1) satisfies the equation*

$$\tilde{g}_0^s(\pi + \varphi) + \tilde{g}_0^s(4\pi/3 - \varphi) = 0$$

for every φ.

P r o o f. The coefficients of the difference equation possess the following properties

$$
\begin{array}{rcl}
f(-\alpha) & = & \Pi(\alpha + \pi) f(\alpha), \\
\Pi(-\alpha) & = & \Pi(\alpha + \pi).
\end{array}
$$

Then the following calculations can be performed

$$
\begin{array}{rcl}
\tilde{g}_0^s(\pi + \varphi) & = & (\frac{\varphi}{2\pi} + 2/6) f(\varphi + 2\pi/3) \\
& & +(\frac{\varphi}{2\pi} + 1/6)\Pi(\varphi + 2\pi/3) f(\varphi + \pi/3) \\
& & +(\frac{\varphi}{2\pi})\Pi(\varphi + 2\pi/3)\Pi(\varphi + \pi/3) f(\varphi) \\
& & +(\frac{\varphi}{2\pi} - 1/6)\Pi(\varphi + 2\pi/3)\Pi(\varphi + \pi/3)\Pi(\varphi) f(\varphi - \pi/3) \\
& & +(\frac{\varphi}{2\pi} - 2/6)\Pi(\alpha + \pi/3)\Pi(\varphi) f(\varphi - 2\pi/3) \\
& & +(\frac{\varphi}{2\pi} - 3/6)\Pi(\varphi) f(\varphi - \pi) =
\end{array}
$$

$$
(5.2)\qquad \begin{aligned}
&= (\tfrac{\varphi}{2\pi}+2/6)\Pi(-\varphi+\pi/3)f(-\varphi-2\pi/3)\\
&\quad +(\tfrac{\varphi}{2\pi}+1/6)\Pi(-\varphi+2\pi/3)\Pi(-\varphi+\pi/3)f(-\varphi-\pi/3)\\
&\quad +(\tfrac{\varphi}{2\pi})\Pi(-\varphi+3\pi/3)\Pi(-\varphi+2\pi/3)\Pi(-\varphi+\pi/3)f(-\varphi)\\
&\quad +(\tfrac{\varphi}{2\pi}-1/6)\Pi(-\varphi+\pi/3)\Pi(-\varphi+2\pi/3)f(-\varphi+\pi/3)\\
&\quad +(\tfrac{\varphi}{2\pi}-2/6)\Pi(-\alpha+\pi)f(-\varphi+2\pi/3)\\
&\quad +(\tfrac{\varphi}{2\pi}-3/6)f(-\varphi+\pi)=\\
&= -\tilde{g}_0^s(-\varphi+4\pi/3).
\end{aligned}
$$

The proof of the lemma is accomplished. □

The amplitude of the incoming spherical wave for $t=-1/6$ is equal to zero. However the contour Γ_1 can not be deformed to the steepest descent one in the region of the analyticity of the solution $\tilde{g}_0^s(\alpha+\varphi)+\tilde{g}_0^s(\alpha+\pi/3-\varphi)$. The residues at the poles of the integral density would add exponentially increasing terms into the asymptotics. It means that the Sommerfeld integral over the contour Γ_1 is not an element of the Hilbert space. Hence a new contour of the integration must be chosen.

The new contour Γ_2 goes to infinity in the same strips as contour Γ_1 and passes the saddle points $\alpha=0$ and $\alpha=\pi$. It surrounds all corresponding to the resonances singularities in the region $\Im\alpha>0, \pi\geq\Re\alpha\geq 0$ and all corresponding to the bound states singularities in the region $\Im\alpha>0, 2\pi\geq \Re\alpha\geq\pi$. No other singularities are situated inside the contour.

Lemma 5.2 *The asymptotics of the integral*

$$
g_0(r,\varphi)=\frac{1}{2\pi i}\int_{\Gamma_2} e^{ikr\cos\alpha}(\tilde{g}_0^s(\alpha+\varphi)+\tilde{g}_0^s(\alpha+\pi/3-\varphi))d\alpha
$$

is given by

$$
(5.3)\qquad \begin{aligned}
g_0(r,\varphi)\ \sim_{r\to\infty}\ & \tfrac{1}{2\pi i}\sqrt{\tfrac{2\pi}{kr}}e^{-i\pi/4}e^{ikr}\left(\tilde{g}_0^s(\alpha+\varphi)+\tilde{g}_0^s(\alpha+\pi/3-\varphi)\right)\\
&+\textstyle\sum_{m>0}\left\{e^{ikr\cos(i\gamma_m-\varphi)}+e^{ikr\cos(i\gamma_m+\varphi-\pi/3)}\right\}\\
&\times\left(f(\alpha)+\Pi(\alpha+4\pi/3)f(\alpha+\pi/3)\right.\\
&+\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+2\pi/3)\\
&\left.+\Pi(\alpha)\Pi(\alpha+4\pi/3)\Pi(\alpha+5\pi/3)f(\alpha+\pi)\right)|_{\alpha=i\gamma_m}\ 2\tan i\gamma_m\Pi^m,
\end{aligned}
$$

where $\Pi^m=\Pi_{n\neq m}\frac{\chi_m+\chi_n}{\chi_m-\chi_n}$.

P r o o f . If $\varphi\neq 0,\pi/3$ then the asymptotics of the integral is given by the steepest descent method in accordance with Lemma 5.1 and formulas

(5.1). If $\varphi = 0$ or $\varphi = \pi/3$ then the contour Γ_2 cannot be transformed to the steepest descent one without passing through the singularities of the integral density. The asymptotics of the integral for $\varphi = 0$ contains in addition to the spherical outgoing wave (5.1) the outgoing surface waves which are determined by the residues at the points $i\gamma_m, i\gamma_m + 2\pi, m > 0$ for $\varphi = 0$,

$$
\begin{array}{l}
(\sum_{m>0} Res\,(\tilde g_0^s(\alpha+\varphi) + \tilde g_0^s(\alpha+\pi/3-\varphi))\mid_{\alpha=i\gamma_m+2\pi-\varphi} \\
-\sum_{m>0} Res\,(\tilde g_0^s(\alpha+\varphi) + \tilde g_0^s(\alpha+\pi/3-\varphi))\mid_{\alpha=i\gamma_m-\varphi})\, e^{ikr\cos(i\gamma_m-\varphi)} \\
= (\sum_{m>0} Res\,(\tilde g_0^s(\alpha))\mid_{\alpha=i\gamma_m+2\pi} - \sum_{m>0} Res\,(\tilde g_0^s(\alpha))\mid_{\alpha=i\gamma_m})\, e^{ikr\cos(i\gamma_m-\varphi)} \\
= \sum_{m>0} (f(\alpha) + \Pi(\alpha+4\pi/3)f(\alpha+\pi/3) + \Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+2\pi/3) \\
+ \Pi(\alpha)\Pi(\alpha+4\pi/3)\Pi(\alpha+5\pi/3)f(\alpha+\pi))\mid_{\alpha=i\gamma_m} \\
Res\,(\Pi(\alpha+\pi))\mid_{\alpha=i\gamma_m} e^{ikr\cos(i\gamma_m-\varphi)}.
\end{array} \tag{5.4}
$$

The residue of the function $\Pi(\alpha)$ can be calculated using the Blaschke representation

$$
\begin{array}{rcl}
Res\;\Pi(\alpha+\pi)\mid_{\alpha=i\gamma_m} & = & \prod_{n\neq m} \frac{ik\sin\alpha-\chi_n}{ik\sin\alpha+\chi_n}\mid_{\alpha=i\gamma_m} Res\left(\frac{ik\sin\alpha-\chi_m}{ik\sin\alpha+\chi_m}\right)\mid_{\alpha=i\gamma_m} \\
& = & 2\tan i\gamma_m \prod_{n\neq m} \frac{\chi_m+\chi_n}{\chi_m-\chi_n} = 2\tan i\gamma_m \Pi^m.
\end{array}
$$

The calculated residues determine the surface waves in the asymptotics. These functions decrease exponentially inside the sector. Really

$$
\mid e^{ikr\cos(i\gamma_m-\varphi)} \mid\sim e^{-kr(\exp(-\gamma_m)-\exp(\gamma_m))\sin(\varphi)/2}
$$

is an exponentially decreasing function for $\gamma_m > 0, \pi/3 > \varphi > 0$. But this function does not decrease exponentially for $\varphi = 0$ and real λ. The residues at the points $i\gamma_m - \pi/3, i\gamma_m + 5\pi/3$ can be analyzed in the same way. Thus the asymptotics of the integral is given by (5.3). □

A similar method can be applied to investigate the properties of the element $g_1^s(r)$. The difference is that the saddle points do not give the main contribution to the asymptotics in this case.

Lemma 5.3 *The asymptotics of the integral*

$$
g_1^s(r) = \frac{1}{2\pi i}\int_{\Gamma_2} \tilde g_1^s(\alpha) e^{ikr\cos\alpha} d\alpha
$$

is given by the formula

(5.5)

$$
\begin{array}{rcl}
g_1^s(r) & \sim_{r\to\infty} & \sum_{m>0} e^{ikr\cos(i\gamma_m)} \psi_{12}^s \\
& & \times [f(\alpha) + \Pi(\alpha+4\pi/3)f(\alpha+\pi/3) + \Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(\alpha+2\pi/3) + \\
& & +\Pi(\alpha)\Pi(\alpha+4\pi/3)\Pi(\alpha+5\pi/3)f(\alpha+\pi)]\mid_{\alpha=i\gamma_m} \tan i\gamma_m \Pi^m,
\end{array}
$$

where the element ψ^s_{12} is defined by (2.26).

P r o o f . The proof of this Lemma is quite similar to the proof of Lemma 5.2.

Theorem 5.1 *The Sommerfeld integrals $g^s_0(r,\varphi), g^s_1(r)$ of the densities $\tilde{g}^s_0(\alpha+\varphi)+\tilde{g}^s_0(\alpha+\pi/3-\varphi)$ and $\tilde{g}^s_1(\alpha)$ over the contour Γ_2 form the deficiency elements for the operator A_s corresponding for $\lambda = k^2, \Im\lambda > 0$.*

P r o o f. The Sommerfeld integrals over the contour Γ_2 are bounded functions on every compact subset of Λ'' and $\mathbf{R}_+$ respectively. These functions decrease exponentially at infinity for λ with positive imaginary part. It follows that the corresponding functions are elements from the Hilbert space $L_2(\Lambda'') \oplus L_2(\mathbf{R}_+, H_{12})$.

The integrals over the contour Γ_1 are solutions of the differential equations and satisfy the boundary conditions (3.30). This is true because the integration by parts gives no boundary terms, since the function $e^{ik\cos\alpha}$ decreases exponentially in the strips where the contour Γ_1 tends to ∞. The integral over the contour Γ_2 differs from the integral over the original contour Γ_1 by the residues at the points

$$
\begin{array}{lr}
i\gamma_m + 2\pi - \varphi, i\gamma_m + 5\pi/3 - \varphi, i\gamma_m + 4\pi/3 - \varphi, & m > 0; \\
i\gamma_m + 5\pi/3 + \varphi, i\gamma_m + 4\pi/3 + \varphi, i\gamma_m + \pi + \varphi, & \\
 & \\
-i\gamma_m + \pi - \varphi, -i\gamma_m + 2\pi/3 - \varphi, -i\gamma_m + \pi/3 - \varphi, & m \le 0. \\
-i\gamma_m + 2\pi/3 + \varphi, -i\gamma_m + \pi/3 + \varphi, -i\gamma_m + \varphi, &
\end{array}
\tag{5.6}
$$

The residues at the points corresponding to $m > 0$ give the set of the surface waves coming along the boundary of the sector from infinity and going away after two reflections. This set of functions is similar to the set of surface waves which will be obtained in Section 7. The residues for $m \le 0$ correspond to the analogous set of the resonance functions. Both sets of functions satisfy the differential equations and the boundary conditions. The corresponding residues for the component g^s_1 must be taken into account also. It follows that the integrals over the contour Γ_2 form the deficiency elements for the operator A_s. □

The calculated deficiency elements depend on the parameter $h \in H_{12}$. It will be shown in the next section that there exist N_{12} linearly independent deficiency elements. Thus we are going to use the following notation for the deficiency element we constructed $G^s_\lambda(h) \in L_2(\Lambda'') \oplus (\mathbf{R}_+, H_{12})$.

5.2 Boundary values of the deficiency elements

We are going to study the behavior of the calculated functions in a neighborhood of the point zero. The zero and the first components are bounded continuous function there. The boundary values of the first component at the origin will be calculated.

Lemma 5.4 *The boundary values of the component g_1^s at the origin are given by the following formulas*

$$
\begin{aligned}
g_1^s(0) \;=\; & \tfrac{h}{2} \\
& +\tfrac{1}{2}\sum_{m>0}\sum_{n=0}^{3}\Big(A_{12}-\lambda\sin^2(i\gamma_m+\pi+n\pi/3)\Big)^{-1}\theta \\
& \times Res\left\{\tilde g_0^s(\alpha)\tfrac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha)-b)}{2ik\sin\alpha-\mathbf{D}(\lambda\sin^2\alpha)}+f(\alpha)(2aik\sin\alpha-b)\right\}\Big|_{\alpha=i\gamma_m+\pi+n\pi/3} \\
& +\tfrac{1}{2}\sum_{m\le 0}\sum_{n=0}^{3}\Big(A_{12}-\lambda\sin^2(-i\gamma_m+n\pi/3)\Big)^{-1}\theta \\
& \times Res\left\{\tilde g_0^s(\alpha)\tfrac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha)-b)}{2ik\sin\alpha-\mathbf{D}(\lambda\sin^2\alpha)}+f(\alpha)(2aik\sin\alpha-b)\right\}\Big|_{\alpha=-i\gamma_m+n\pi/3}
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
\frac{\partial g_1^s}{\partial r}(0) \;=\; & i\sqrt{\lambda-A_{12}}\tfrac{h}{2} \\
& +\tfrac{1}{2}\sum_{m>0}\sum_{n=0}^{3} ik\cos(i\gamma_m+\pi+n\pi/3)\Big(A_{12}-\lambda\sin^2(i\gamma_m+\pi+n\pi/3)\Big)^{-1}\theta \\
& \times Res\left\{\tilde g_0^s(\alpha)\tfrac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha)-b)}{2ik\sin\alpha-\mathbf{D}(\lambda\sin^2\alpha)}+f(\alpha)(2aik\sin\alpha-b)\right\}\Big|_{\alpha=i\gamma_m2+\pi+n\pi/3} \\
& +\tfrac{1}{2}\sum_{m\le 0}\sum_{n=0}^{3} ik\cos(-i\gamma_m+n\pi/3)\Big(A_{12}-\lambda\sin^2(-i\gamma_m+n\pi/3)\Big)^{-1}\theta \\
& \times Res\left\{\tilde g_0^s(\alpha)\tfrac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha)-b)}{2ik\sin\alpha-\mathbf{D}(\lambda\sin^2\alpha)}+f(\alpha)(2aik\sin\alpha-b)\right\}\Big|_{\alpha=-i\gamma_m+n\pi/3}\,.
\end{aligned}
\tag{5.8}
$$

P r o o f . The value $u(0)$ of a Sommerfeld integral at the point zero is determined by behavior of the integral density $\tilde u(\alpha)$ at infinity. Particularly, the following equation is valid for every even function analytic inside the contour of the integration [31, 32]

$$
u(0)=\frac{1}{i}\lim_{\alpha\to i\infty}\tilde u(\alpha)
\tag{5.9}
$$

Thus the boundary values of the integral over the initial contour Γ_1 with the density

$$
\begin{aligned}
\tilde g_1^s(\alpha) \;=\; & -\tfrac{k\sin\alpha}{k\cos\alpha-\sqrt{\lambda-A_{12}}}\tfrac{h}{2} \\
& +\tfrac{1}{2}\left\{\tilde g_0^s(\alpha)\tfrac{4ik\sin\alpha(a\mathbf{D}(\lambda\sin^2\alpha)-b)}{2ik\sin\alpha-\mathbf{D}(\lambda\sin^2\alpha)}+f(\alpha)(2aik\sin\alpha-b)\right\}(A_{12}-\lambda\sin^2\alpha)^{-1}\theta
\end{aligned}
$$

are equal to $\frac{h}{2}$ and $i\sqrt{\lambda - A_{12}}\frac{h}{2}$ correspondingly. To calculate the boundary values of the first component of the deficiency element g_1^s it is necessary to add the residues at the singular points situated between the contour Γ_1 and Γ_2. We get formulas (5.7) and (5.8). □

All the residues which appeared in (5.7) and (5.8) can be calculated explicitly. The corresponding calculations are presented in the Appendix A.

The boundary values of the first component are linear functions of the vector h. Therefore the following matrices B_s and $\tilde{B}_s$ can be introduced

$$\left\{ \begin{array}{rcl} g_1^s(0) & = & (1 + B_s(k))\frac{h}{2} \\ \frac{\partial g_1^s}{\partial r}(0) & = & (i\sqrt{\lambda - A_{12}} + \tilde{B}_s(k))\frac{h}{2}. \end{array} \right. \tag{5.10}$$

Lemma 5.5 *The matrix $1 + B_s(k)$ is invertible.*

P r o o f . The deficiency elements $G_\lambda^s(h)$ corresponding to different $h \in H_{12}$ have different asymptotics at infinity. Suppose that there exists $h \in H_{12}, h \neq 0$ such that

$$g_1^s(0) = 0. \tag{5.11}$$

Consider the following expression

$$< A_s^* G^s(h), G^s(h) >= k^2 < G^s(h), G^s(h) > .$$

Integrating by parts two times both components we get the following formula

$$< A_s^* G^s(h), G^s(h) >=< G^s(h), A_s^* G^s(h) >= \bar{k}^2 < G^s(h), G^s(h) > . \tag{5.12}$$

The boundary terms from the second component vanish due to the condition (5.11). Similar terms produced by the boundary of Λ'' vanish due to the conditions (3.10). The proof is similar to the proof of the symmetry of the operator A^3 (Theorem 3.3). It follows from (5.12) that $G_\lambda^s(h) = 0$ and therefore $h = 0$. We got the contradiction which proves the Lemma. □

Corollary 5.1 *The boundary values $g_1^s(0)$ span the finite dimensional space H_{12}.*

The matrix $C_s(\lambda)$ connecting the boundary values of the first component can be introduced

$$\frac{\partial g_1}{\partial r}(0) = C_s(\lambda) g_1(0) \Rightarrow \tag{5.13}$$

$$C_s(\lambda) = \left(i\sqrt{\lambda - A} + \tilde{B}_s \right) (1 + B_s)^{-1} .$$

A similar procedure can be also carried out for the antisymmetric deficiency element. The corresponding matrices will be denoted by $B_a, \tilde{B}_a$ and $C_a(\lambda)$.

6 Selfadjoint Three-body Operator

Two N_{12}-dimensional families of deficiency elements for the operators A_s and A_a define $2N_{12} \times 2N_{12}$ family of symmetric extensions of the operator $A_s \oplus A_a$. We describe this family by the boundary conditions at the origin. We prove that the extended symmetric operator is bounded from below. The selfadjoint operator will be defined by the Friedrichs procedure.

Consider the operator $A_b^* = A_s^* \oplus A_a^*$ restricted to the domain

$$D = \mathbf{C}_{0b}^{\infty} + \mathcal{L}\{G_\lambda^s(h), G_\lambda^a(h), G_{\bar\lambda}^s(h), G_{\bar\lambda}^a(h)\}_{h \in H_{12}}.$$

Every symmetric extension of the operator A_b to a subset of D coincides with the restriction of the operator A_b^* to this subset.

Lemma 6.1 *The boundary form of the operator A_b^* for every two elements $U, V \in D$ is equal to*

$$\begin{aligned} < A_b^* U, V > - < U, A_b^* V > \;=\; & < \tfrac{\partial}{\partial r} u_1^s(0), v_1^s(0) > - < u_1^s(0), \tfrac{\partial}{\partial r} v_1^s(0) > \\ & + < \tfrac{\partial}{\partial r} u_1^a(0), v_1^a(0) > - < u_1^a(0), \tfrac{\partial}{\partial r} v_1^a(0) > . \end{aligned} \tag{6.1}$$

P r o o f . The Lemma can be proven by integrations by parts if one takes into account that the functions u_0, u_1 and v_0, v_1 are bounded and twice differentiable. □

Lemma 6.2 *Let $\Im\lambda > 0$. The boundary values $(u_1^s(0), u_1^a(0), \frac{\partial}{\partial r} u_1^s(0), \frac{\partial}{\partial r} u_1^a(0))$ of the deficiency elements $U \in \mathcal{L}\{G_\lambda^s(h), G_\lambda^a(h), G_{\bar\lambda}^s(h), G_{\bar\lambda}^a(h)\}_{h \in H_{12}}$ span the finite dimensional vector space $H_{12} \oplus H_{12} \oplus H_{12} \oplus H_{12}$.*

P r o o f . The subspace $\mathcal{L}\{G^s_\lambda(h), G^a_\lambda(h), G^s_{\bar\lambda}(h), G^a_{\bar\lambda}(h)\}_{h\in H_{12}}$ has dimension $4(dim\ H_{12})$ which coincides with the dimension of the orthogonal sum $H_{12} \oplus H_{12} \oplus H_{12} \oplus H_{12}$. Thus it is sufficient to show that the linear map between the two vector spaces of equal dimension :

$$\eta : \mathcal{L}\{G^s_\lambda(h), G^a_\lambda(h), G^s_{\bar\lambda}(h), G^a_{\bar\lambda}(h)\}_{h\in H_{12}} \to H_{12} \oplus H_{12} \oplus H_{12} \oplus H_{12},$$

$$\eta(U) = (u^s_1(0), u^a_1(0), \frac{\partial}{\partial r}u^s_1(0), \frac{\partial}{\partial r}u^a_1(0))$$

has zero kernel. Let U be an element from the kernel of the map η. Then it has the following representation

$$U = U_\lambda + U_{\bar\lambda},\ U_\lambda \in \mathcal{L}\{G^s_\lambda(h), G^a_\lambda(h)\}_{h\in H_{12}}, U_{\bar\lambda} \in \mathcal{L}\{G^s_{\bar\lambda}(h), G^a_{\bar\lambda}(h)\}_{h\in H_{12}}.$$

Consider the following scalar product

$$< A^*_b(U_\lambda + U_{\bar\lambda}), U_\lambda >= \lambda < U_\lambda, U_\lambda > +\bar\lambda < U_{\bar\lambda}, U_\lambda > .$$

Integrating by parts in the first scalar product we get the following equation

$$< A^*_b(U_\lambda + U_{\bar\lambda}), U_\lambda >=< U_\lambda + U\bar\lambda, A^*_b U_\lambda >= \bar\lambda < U_\lambda, U_\lambda > +\bar\lambda < U_{\bar\lambda}, U_\lambda > .$$

It follows that $< U_\lambda, U_\lambda >= 0$ because λ has nontrivial imaginary part. □

Lemma 6.3 *The restriction A_{bL} of the operator A^*_b to the subspace of functions from D which boundary values span Lagrangian subspace L of $H_{12} \oplus H_{12} \oplus H_{12} \oplus H_{12}$ with respect to the boundary form (6.1) is a symmetric extension of the operator A_b.*

P r o o f. The domain of the operator A_b is in D and their boundary values are trivial. It follows that every restriction of A^*_b to the domain defined by the Lagrangian subspace is an extension of the operator A_b. This restriction defines symmetric operator because the boundary form vanishes on every domain determined by the Lagrangian subspace. □

Lemma 6.4 *The operator A_{bL} is bounded from below .*

P r o o f . The operator A_{bL} is a finite dimensional extension of the operator A_b. Hence it is enough to prove that both operators A_s and A_a are bounded from below. We consider the case where $a = d = 0$ for simplicity. Let $U \in P_s\mathbf{C}^\infty_{0b}$. Then

$$
\begin{array}{l}
< A_s U, U > = \\
\int_{\Lambda''}(-\Delta u_0(r,\varphi))\bar{u}_0(r,\varphi) r dr d\varphi \\
+\int_0^\infty < (-\frac{\partial^2}{\partial r^2} + A_{12})u_1(r), u_1(r) > dr + \int_0^\infty b u_0(r,0) < \theta, u_1(r) > dr \\
= \int_{\Lambda''} |\nabla u_0|^2 r dr d\varphi + \int_0^\infty \frac{1}{r}\frac{\partial}{\partial\varphi} u_0(r,0)\bar{u}(r,0) dr \\
+\int_0^\infty |\frac{\partial}{\partial r} u_1(r)|^2 dr + \int_0^\infty < A_{12}u_1(r), u_1(r) > dr + \int_0^\infty b u_0(r,0) < \theta, u_1(r) > dr \\
= \int_{\Lambda''} |\nabla u_0|^2 r dr d\varphi + \int_0^\infty < A_{12}u_1(r), u_1(r) > dr + \int_0^\infty |\frac{\partial}{\partial r} u_1(r)|^2 dr \\
+b\int_0^\infty 2\Re(< u_1, \theta > \bar{u}_0(r,0)) dr.
\end{array}
$$

The latter integral can be estimated as follows

$$
|2b\int_0^\infty \Re(< u_1(r), \theta > \overline{u_0}(r,0) dr| \leq |b| \left(\| u_1 \|^2 + \frac{1}{\epsilon} \| u_0 \|^2_{L_2} + \epsilon \| \nabla u_0 \|^2_{L_2} \right)
$$

with arbitrary positive ϵ.

The following estimate is valid for the quadratic form of the operator

$$
< A_s U, U > \geq \int_0^\infty < A_{12}u_1(r), u_1(r) > dr - |b| \| u_1 \|^2 - \frac{|b|}{\epsilon} \| u_0 \|^2_{L_2},
$$

provided $\epsilon|b| < 1$. The operator A_s is bounded from below since every term in the latter formula can be estimated by the norm of the element U. □

Now the selfadjoint operator describing the system of three one dimensional particles can be defined as follows. First consider the symmetric extension A_{bL} of the operator A_b to the set of functions from D satisfying the boundary conditions

$$
\left(\begin{array}{c} u'_{s,1}(0) \\ u'_{a,1}(0) \end{array} \right) = \mathbf{Q} \left(\begin{array}{c} u_{s,1}(0) \\ u_{a,1}(0) \end{array} \right), \tag{6.2}
$$

where $\mathbf{Q}$ is a selfadjoint matrix. These boundary conditions determine some Lagrangian subspace in the space of boundary values. Hence the operator L_{bL} is a symmetric and bounded from below operator (Lemmas 6.3 and 6.4). The operator A_{bL} commutes with the symmetry operator with respect to the bisector if the boundary conditions have the following form

$$
\begin{array}{rcl}
u'_{s,1}(0) & = & Q_s u_{s,1}(0), \\
u'_{a,1}(0) & = & Q_a u_{a,1}(0).
\end{array} \tag{6.3}
$$

The boundary conditions (6.3) will be used in the sequel. These boundary conditions describe the symmetric extensions A_{aL} and A_{sL} of the operators A_a and A_s and $A_{bL} = A_{aL} \oplus A_{bL}$. Every symmetric operator which is bounded from below can be extended to a selfadjoint operator using Friedrichs procedure. We are going to keep the same notation A_b for the selfadjoint operator describing the system of three one dimensional particles. The spectral properties and scattering matrix of this operator will be discussed in the next section.

7 Spectrum, Scattering Matrix

The spectral properties of the operators A_s and A_a are quite similar. We are going to study the spectrum and scattering matrix for the operator A_s. The spectrum of the operator A_s consists of the continuous spectrum $[0, \infty)$ corresponding to the processes with three free particles, branches of the continuous spectrum $[-\chi_m^2, \infty), m = 1, 2, ..., N_{12}$, corresponding to the two-body bound states, and probably some eigenvalues. The scattering matrix will be calculated from the asymptotics of the continuous spectrum eigenfunctions of the operator A_s.

7.1 Definition of the scattering matrix

We are going to calculate the continuous spectrum eigenfunctions of the operator A_s. These functions are generalized solutions of the following equation

$$(7.1) \qquad \begin{pmatrix} -\Delta_{r,\varphi} u_0(r,\varphi) \\ -\frac{\partial^2 u_1(r)}{\partial r^2} + A_{12} u_1(r) + b u_0(r,0)\theta \end{pmatrix} = \lambda \begin{pmatrix} u_0(r,\varphi) \\ u_1(r) \end{pmatrix}$$

satisfying the boundary conditions

$$(7.2) \qquad \begin{array}{rcl} < u_1(r), \theta > & = & \frac{c}{r}\frac{\partial u_0(r,0)}{\partial \varphi}; \\ \frac{\partial u_0(r,\pi/6)}{\partial \varphi} & = & 0; \\ \frac{\partial u_1(0)}{\partial r} & = & Q_s u_1(0). \end{array}$$

The function $u_0(r, \varphi)$ is defined in the sector $0 \le r < \infty,\ 0 \le \varphi \le \pi/6$. The function $u_1(r)$ is a vector valued function on $\mathbf{R}_+$ taking values in H_{12}. Both functions $u_0(r, \varphi)$ and $u_1(r)$ are continuous and bounded. The asymptotics for large r of the zero component of every solution of the problem (7.1), (7.2) are equal to the sum of incoming and outgoing plane waves, spherical wave, and surface waves. The continuous spectrum eigenfunctions can be separated onto two classes in accordance to the type of the incoming channel. The eigenfunctions of the first type correspond to the three-body incoming channel. Such eigenfunctions contain only one incoming plane wave in the asymptotics of the zero component. The eigenfunctions of the second type contain only incoming surface wave in the asymptotics of the zero component. In addition to these incoming waves the asymptotics of the zero component contains a set of outgoing waves: plane, spherical and surface ones. This set of eigenfunctions will be called *incoming*. The second complete set of the eigenfunctions, so called *outgoing* set, is determined by the different outgoing waves. The eigenfunctions contain in their asymptotics only one outgoing wave and a set of incoming waves. The scattering matrix can be defined as an operator connecting the spectral representations with respect to these two sets of the eigenfunctions. We are going to define the scattering matrix from the asymptotics of the incoming set of eigenfunctions. The scattering matrix is an integral operator of the form

$$\mathbf{S}(\lambda) = \left\{ \begin{matrix} S_{33} & S_{32} \\ S_{23} & S_{22} \end{matrix} \right\}, \tag{7.3}$$

acting in the space $L_2(0, \pi/6) \oplus K_{12}, \dim K_{12} = N_{12}$. The operator S_{33} is an integral operator with the kernel $s_{33}(\lambda, \varphi, \varphi_0)$. The operator S_{23} is a matrix integral operator with the kernel $s_{23}(\lambda, m, \varphi_0),\ \ m = 1, 2, ..., N_{12}$. The operators S_{32} and S_{22} are operators of multiplication by the matrices $s_{32}(\lambda, \varphi, m), s_{22}(\lambda, m, n)$.

Consider first the incoming eigenfunction determined by the incoming plane wave

$$u_0^{in}(\lambda, \varphi_0, r, \varphi) = \frac{exp(-ikr\cos(\varphi - \varphi_0))}{2\pi\sqrt{2k}}, k^2 = \lambda, 0 < \varphi_0 < \pi/6. \tag{7.4}$$

Then the asymptotics at infinity $r \to \infty$ of the zero component of this function contains the following outgoing waves

$$\begin{gathered} R_{33}(\lambda, \varphi_0)\tfrac{exp(-ikr\cos(\varphi - \varphi_0 - \pi))}{2\pi\sqrt{2k}} + a_{33}(\lambda, \varphi, \varphi_0)\tfrac{1}{2k}\tfrac{e^{-i\pi/4}}{\sqrt{\pi r}}e^{ikr} \\ + \textstyle\sum_{m=1}^{N_{12}} s_{23}(\lambda, m, \varphi_0)\tfrac{c_m}{\sqrt{2\pi}\sqrt[4]{\lambda + \chi_m^2}}e^{ikr\cos(\varphi - i\gamma_m)}. \end{gathered} \tag{7.5}$$

The normalizing constant for the surface wave is determined by the equation (2.27), $-\chi_m^2$ is the energy of the two-body bound state. The real number γ_m is given by the formula $i\gamma_m = \arctan \frac{i\chi_m}{\sqrt{\lambda+\chi_m^2}}$. The scattering amplitude a_{33} and three-body reflection coefficient R_{33} form the kernel s_{33} of the scattering matrix

$$s_{33}(\lambda, \varphi, \varphi_0) = R_{33}(\lambda, \varphi_0)\delta(\varphi - \varphi_0) + a_{33}(\lambda, \varphi_0, \varphi). \tag{7.6}$$

The scattering amplitude s_{23} coincides with the kernel of the operator S_{23}.

Similarly the eigenfunction determined by incoming surface wave

(7.7)
$$u_0^{in}(\lambda, n, r, \varphi) = \frac{c_n}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi_n^2}} e^{-ikr\cos(\varphi - i\gamma_n)},\ n = 1, 2, ..., N_{12}, \lambda > -\chi_n^2$$

has asymptotics at infinity, which contains the following outgoing waves

$$\begin{array}{c} s_{32}(\lambda, \varphi, n)\frac{1}{2k}\frac{e^{-i\pi/4}}{\sqrt{\pi r}}e^{ikr} \\ +\sum_{m=1}^{N_{12}} s_{22}(\lambda, m, n)\frac{c_m}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi_m^2}}e^{ikr\cos(\varphi - i\gamma_m)}. \end{array} \tag{7.8}$$

The scattering amplitudes $s_{32},\ s_{22}$ are equal to the coefficients of the matrix operators $S_{32},\ S_{22}$.

7.2 Calculation of the scattering matrix

The incoming eigenfunctions defined by the three-body plane wave can be presented by the sum of plane waves and the limit of the deficiency element on the real axis

$$u(\lambda, \varphi_0) = u^{plane}(\lambda, \varphi_0) + G_\lambda^s(h(\lambda, \varphi_0)), \tag{7.9}$$

where u^{plane} denotes the set of plane waves and $G_\lambda^s(h(\lambda, \varphi_0))$ is the limit of the deficiency element on the real axis. The set of plane waves is the result of multiple reflections of the incoming plane wave from the boundaries of the sector in accordance to the laws of the geometrical optics. The total number of the reflected waves is equal to 11. The reflection coefficient from the boundary $\varphi = \pi/6$ is equal to 1 for the symmetric functions (to -1 for A_a). The reflection coefficients from the boundary $\varphi = 0$ are determined by the two-body scattering matrix only $P(k_\perp) = T(k_\perp) + R(k_\perp)$. Here $k_\perp$

denotes the perpendicular component of the wave vector. We note that the reflection coefficient $P(k)$ is related to the coefficient $\Pi(\alpha)$ in the difference functional equation as follows $\Pi(\alpha) = P(k\sin\alpha)^{-1}$.

The incoming eigenfunctions determined by the three-body incoming plane wave can be parameterized by the energy $\lambda = k^2, 0 \le \lambda < \infty$ and the angle $\varphi_0, 0 \le \varphi_0 \le \pi/6$ of the incoming wave (7.4). The set of the induced plane waves consists of the twelve waves with the following zero component

(7.10)

$$
\begin{aligned}
u_0^p(\lambda,\varphi_0,r,\varphi) \;=\; & \tfrac{1}{2\pi\sqrt{2k}}\{\exp(-ikr\cos(\varphi-\varphi_0))+\\
&+\Pi^{-1}(\varphi_0)\exp(-ikr\cos(\varphi+\varphi_0))\\
&+\Pi^{-1}(\varphi_0)\exp(-ikr\cos(\varphi-\varphi_0-\pi/3))\\
&+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\exp(-ikr\cos(\varphi+\varphi_0+\pi/3))\\
&+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\exp(-ikr\cos(\varphi-\varphi_0-2\pi/3))\\
&+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\Pi^{-1}(2\pi/3+\varphi_0)\exp(-ikr\cos(\varphi+\varphi_0+2\pi/3))\\
&+\exp(-ikr\cos(\varphi+\varphi_0-\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\exp(-ikr\cos(\varphi-\varphi_0+\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\exp(-ikr\cos(\varphi+\varphi_0-2\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\Pi^{-1}(2\pi/3-\varphi_0)\exp(-ikr\cos(\varphi-\varphi_0+2\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\Pi^{-1}(2\pi/3-\varphi_0)\exp(-ikr\cos(\varphi+\varphi_0+\pi))\\
&+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\Pi^{-1}(2\pi/3+\varphi_0)\exp(-ikr\cos(\varphi-\varphi_0-\pi))\}
\end{aligned}
$$

The first component for the set of plane waves is given by

(7.11)

$$
\begin{aligned}
u_1^p(\lambda,\varphi_0,r) \;=\; & \tfrac{1}{2\pi\sqrt{2k}}\{\exp(-ikr\cos(\varphi_0))\,\psi_{12}(\lambda\sin^2(\varphi_0))\\
&+\Pi^{-1}(\varphi_0)\exp(-ikr\cos(\varphi_0+\pi/3))\,\psi_{12}(\lambda\sin^2(\varphi_0+\pi/3))\\
&+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\exp(-ikr\cos(\varphi_0+2\pi/3))\,\psi_{12}(\lambda\sin^2(\varphi_0+2\pi/3))\\
&+\exp(-ikr\cos(\varphi_0-\pi/3))\,\psi_{12}(\lambda\sin^2(-\varphi_0+\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\exp(-ikr\cos(\varphi_0-2\pi/3))\,\psi_{12}(\lambda\sin^2(-\varphi_0+2\pi/3))\\
&+\Pi^{-1}(\pi/3-\varphi_0)\Pi^{-1}(2\pi/3-\varphi_0)\exp(-ikr\cos(\varphi_0+\pi))\,\psi_{12}(\lambda\sin^2(\varphi_0))\Big\}
\end{aligned},
$$

where the function $\psi_{12}(\lambda)$ is defined in (2.32). Thus the boundary values of the first component at the point zero are equal to

(7.12)

$$\begin{aligned} u_1^p(\lambda,\varphi_0,0) \; &= \; \tfrac{1}{2\pi\sqrt{2k}} \left\{ \, \psi_{12}(\lambda \sin^2(\varphi_0)) \right. \\ &+\Pi^{-1}(\varphi_0)\psi_{12}(\lambda \sin^2(\varphi_0+\pi/3)) \\ &+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\psi_{12}(\lambda \sin^2(\varphi_0+2\pi/3))+ \\ &+\psi_{12}(\lambda \sin^2(-\varphi_0+\pi/3)) \\ &+\Pi^{-1}(\pi/3-\varphi_0)\psi_{12}(\lambda \sin^2(-\varphi_0+2\pi/3)) \\ &\left. +\Pi^{-1}(\pi/3-\varphi_0)\Pi^{-1}(2\pi/3-\varphi_0)\; \psi_{12}(\lambda \sin^2(\varphi_0)) \right\}, \end{aligned}$$

(7.13)

$$\begin{aligned} \tfrac{\partial u_1^p(\lambda,\varphi_0,r)}{\partial r} \, |_{r=0} \; &= \; \tfrac{-ik}{2\pi\sqrt{2k}} \left\{ \cos(\varphi_0)\; \psi_{12}(\lambda \sin^2(\varphi_0)) \right. \\ &+\Pi^{-1}(\varphi_0)\cos(\varphi_0+\pi/3)\psi_{12}(\lambda \sin^2(\varphi_0+\pi/3)) \\ &+\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\cos(\varphi_0+2\pi/3)\psi_{12}(\lambda \sin^2(\varphi_0+2\pi/3))+ \\ &+\cos(\varphi_0-\pi/3)\psi_{12}(\lambda \sin^2(-\varphi_0+\pi/3)) \\ &+\Pi^{-1}(\pi/3-\varphi_0)\cos(\varphi_0-2\pi/3)\psi_{12}(\lambda \sin^2(-\varphi_0+2\pi/3)) \\ &\left. +\Pi^{-1}(\pi/3-\varphi_0)\Pi^{-1}(2\pi/3-\varphi_0)\cos(\varphi_0+\pi)\; \psi_{12}(\lambda \sin^2(\varphi_0)) \right\}. \end{aligned}$$

The boundary values of the first component of the set of plane waves do not satisfy in general the boundary conditions (7.2) at the origin. Hence it is necessary to add some outgoing wave, which can be presented as a limit of the deficiency element on the real axis. Every deficiency element can be parameterized by the vector $h(\lambda,\varphi_0)$. Substitution of the representation (7.9) into the boundary conditions (7.2) gives the following inhomogeneous linear equation on the vector $h(\lambda,\varphi_0)$

$$\frac{\partial u_1^p(\lambda,\varphi_0,r)}{\partial r} \, |_{r=0} + \left(i\sqrt{\lambda - A_{12}} + \tilde{B}_s \right) \frac{h(\lambda,\varphi_0)}{2} =$$

$$= Q_s \left(u_1^p(e,\varphi_0,0) + (1+B_s)\, \frac{h(\lambda,\varphi_0)}{2} \right). \tag{7.14}$$

The vector $h(\lambda,\varphi_0)$ can be calculated as

(7.15)

$$h(\lambda,\varphi_0) = 2\left(i\sqrt{\lambda - A_{12}} + \tilde{B}_s - Q_s\,(1+B_s) \right)^{-1} \left(-\frac{\partial u_1^p}{\partial r} + Q_s u_1^p \right) |_{r=0} \,.$$

The first two components of the scattering matrix can be calculated now from the asymptotics of the solution

(7.16)

$$\begin{array}{rcl} s^s_{33}(\lambda,\varphi,\varphi_0) & = & \delta(\varphi-\varphi_0)\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\Pi^{-1}(2\pi/3+\varphi_0) \\ & & +(-i\sqrt{2k})\left(\tilde g^s_0(h(\lambda,\varphi_0),\varphi)+\tilde g^s_0(h(\lambda,\varphi_0),\pi/3-\varphi)\right); \end{array}$$

(7.17)

$$\begin{array}{rcl} s^s_{23}(\lambda,m,\varphi_0) & = & \frac{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_m}}{c_m}\left(f(h(\lambda,\varphi_0),\alpha)+\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+\pi/3)\right. \\ & & +\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+2\pi/3) \\ & & \left.+\Pi(\alpha)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+\pi)\right)|_{\alpha=i\gamma_m}\ 2\tan i\gamma_m\Pi^m. \end{array}$$

The eigenfunctions corresponding to the surface waves can be considered in the same way. These eigenfunction can be presented by the following sum

$$u(\lambda,m)=u^{surf}(\lambda,m)+G(\lambda,h(\lambda,m)). \tag{7.18}$$

The eigenfunctions are parameterized by the energy of the incoming surface wave (7.4) $\lambda\in[-\chi^2_m,\infty)$ and the two-body bound state m, which can be represented formally as a plane wave with the complex wave vector

$$u^{in}_0(\lambda,n,r,\varphi)=\frac{c_n}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_n}}e^{-ikr\cos(\varphi-i\gamma_n)}.$$

The set of surface waves can be constructed using the two-body scattering data

(7.19)

$$\begin{array}{rcl} u^{surf}_0(\lambda,m,r,\varphi) & = & \frac{c_n}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_n}}\left\{e^{-ikr\cos(\varphi-i\gamma_m)}+e^{-ikr\cos(\varphi-\pi/3+i\gamma_m)}\right. \\ & & +\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(\varphi+\pi/3-i\gamma_m)} \\ & & +\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(\varphi-2\pi/3+i\gamma_m)} \\ & & +\Pi^{-1}(2\pi/3-i\gamma_m)\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(\varphi+2\pi/3-i\gamma_m)} \\ & & \left.+\Pi^{-1}(2\pi/3-i\gamma_m)\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(\varphi-\pi+i\gamma_m)}\right\}. \end{array}$$

The first component of the set of plane waves can also be calculated

(7.20)

$$\begin{array}{rcl} u^{surf}_1(\lambda,m,r) & = & \frac{c_n}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_n}}\left\{e^{-ikr\cos(i\gamma_m)}\ \frac{\psi^s_{12}}{2}\right. \\ & & +e^{-ikr\cos(-\pi/3+i\gamma_m)}\psi_{12}(\lambda\sin^2(\pi/3-i\gamma_m)) \\ & & +\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(-2\pi/3+i\gamma_m)}\psi_{12}(\lambda\sin^2(2\pi/3-i\gamma_m)) \\ & & \left.+\Pi^{-1}(2\pi/3-i\gamma_m)\Pi^{-1}(\pi/3-i\gamma_m)e^{-ikr\cos(\varphi-\pi+i\gamma_m))}\ \frac{\psi^s_{12}}{2}\right\}, \end{array}$$

where ψ^s_{12} is defined by (2.26). The boundary values of the first component at the origin are given by

$$
\begin{aligned}
u_1^{surf}(\lambda,m,0) \;=\; & \tfrac{c_n}{\sqrt{2\pi}\sqrt[4]{\lambda+\chi_n^2}}\left\{\tfrac{\psi^s_{12}}{2} + \psi_{12}(\lambda\sin^2(\pi/3 - i\gamma_m))\right.\\
& +\Pi^{-1}(\pi/3 - i\gamma_m)\psi_{12}(\lambda\sin^2(2\pi/3 - i\gamma_m))\\
& \left.+\Pi^{-1}(2\pi/3 - i\gamma_m)\Pi^{-1}(\pi/3 - i\gamma_m)\ \tfrac{\psi^s_{12}}{2}\right\},
\end{aligned}
\tag{7.21}
$$

(7.22)

$$
\begin{aligned}
\tfrac{\partial u_1^{surf}(\lambda,m,r)}{\partial r}\,|_{r=0} \;=\; & \tfrac{-ikc_n}{2\sqrt{2\pi}\sqrt[4]{\lambda+\chi_n^2}}\left\{\cos(-i\gamma_m)\ \psi^s_{12}\right.\\
& +\cos(-\pi/3 + i\gamma_m)\psi_{12}(\lambda\sin^2(\pi/3 - i\gamma_m))\\
& +\Pi^{-1}(\pi/3 - i\gamma_m)\cos(-2\pi/3 + i\gamma_m)\psi_{12}(\lambda\sin^2(2\pi/3 - i\gamma_m))\\
& \left.+\Pi^{-1}(2\pi/3 - i\gamma_m)\Pi^{-1}(\pi/3 - i\gamma_m)\cos(-\pi + i\gamma_m)\ \psi^s_{12}\right\}.
\end{aligned}
$$

The vector $h(\lambda,m)$ can be calculated from the boundary conditions
(7.23)

$$
\frac{\partial u_1^{surf}}{\partial r}\,|_{r=0} + \left(i\sqrt{\lambda - A_{12}} + \tilde{B}_s\right)\frac{h(\lambda,m)}{2} = Q_s\left(u_1^{surf}\,|_{r=0} + (1+B_s)\,\frac{h(\lambda,m)}{2}\right) \Rightarrow
$$

(7.24)

$$
h(\lambda,m) = 2\left(i\sqrt{\lambda - A_{12}} + \tilde{B}_s - Q_s\,(1+B_s)\right)^{-1}\left(-\frac{\partial u_1^{surf}}{\partial r} + Q_s u_1^{surf}\right)|_{r=0}\,.
$$

The components S_{22}, S_{32} of the scattering matrix can be calculated from the asymptotics of the constructed eigenfunction
(7.25)

$$
\begin{aligned}
s^s_{22}(\lambda,m,n) \;=\; & \delta_{nm}\Pi^{-1}(2\pi/3 - i\gamma_m)\Pi^{-1}(\pi/3 - i\gamma_m)\\
& +\tfrac{\sqrt{2\pi}\sqrt[4]{\lambda+\chi_m^2}}{c_m}\left(f(h(\lambda,m),\alpha) + \Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+\pi/3)\right.\\
& +\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+2\pi/3)\\
& \left.+\Pi(\alpha)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+\pi)\right)|_{\alpha=i\gamma_n}\\
& \times 2\tan i\gamma_n\Pi^n(-i\gamma_n)\Theta(\lambda+\chi_m^2),
\end{aligned}
$$

$$
s^s_{32}(\lambda,\varphi,m) = -i\sqrt{2k}\left\{\tilde{g}^s_0(h(\lambda,m),\varphi) + \tilde{g}^s_0(h(\lambda,m),\pi/3-\varphi)\right\}\Theta(\lambda).
\tag{7.26}
$$

Here $\Theta(\lambda)$ is the Heaviside function. We succeeded in precise analytical calculation of the whole three-body scattering matrix for the case of identical particles. The following theorem has been proven

Theorem 7.1 *The matrix integral operators $S^s_{33}, S^s_{23}, S^s_{32}, S^s_{22}$ with the kernels*

$$\begin{aligned}
s^s_{33}(\lambda,\varphi,\varphi_0) &= \delta(\varphi-\varphi_0)\Pi^{-1}(\varphi_0)\Pi^{-1}(\pi/3+\varphi_0)\Pi^{-1}(2\pi/3+\varphi_0) \\
&\quad +(-i\sqrt{2k})\left(\tilde g^s_0(h(\lambda,\varphi_0),\varphi)+\tilde g^s_0(h(\lambda,\varphi_0),\pi/3-\varphi)\right); \\
s^s_{23}(\lambda,m,\varphi_0) &= \frac{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_m}}{c_m}\left(f(h(\lambda,\varphi_0),\alpha)+\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+\pi/3)\right. \\
&\quad +\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+2\pi/3) \\
&\quad \left.+\Pi(\alpha)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,\varphi_0),\alpha+\pi)\right)|_{\alpha=i\gamma_m}\,2\tan i\gamma_m\Pi^m; \\
s^s_{32}(\lambda,\varphi,m) &= -i\sqrt{2k}\,\{\tilde g^s_0(h(\lambda,m),\varphi)+\tilde g^s_0(h(\lambda,m),\pi/3-\varphi)\}\,\Theta(\lambda); \\
s^s_{22}(\lambda,m,n) &= \delta_{nm}\Pi^{-1}(2\pi/3-i\gamma_m)\Pi^{-1}(\pi/3-i\gamma_m) \\
&\quad +\frac{\sqrt{2\pi}\sqrt[4]{\lambda+\chi^2_m}}{c_m}\left(f(h(\lambda,m),\alpha)+\Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+\pi/3)\right. \\
&\quad +\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+2\pi/3) \\
&\quad \left.+\Pi(\alpha)\Pi(\alpha+5\pi/3)\Pi(\alpha+4\pi/3)f(h(\lambda,m),\alpha+\pi)\right)|_{\alpha=i\gamma_n} \\
&\quad \times 2\tan i\gamma_n\Pi^n(-i\gamma_n)\Theta(\lambda+\chi^2_m);
\end{aligned}$$

form the scattering matrix

$$S^s=\begin{pmatrix} S^s_{33} & S^s_{32} \\ S^s_{23} & S^s_{22}\end{pmatrix}$$

for the operator A_s.

The energies of the three-body bound states can be calculated as follows. The wave function of the three-body bound state is equal to the limit of some deficiency element on the real axis. Substitution of the boundary values of the deficiency element into the boundary conditions (7.2) gives the dispersion equation for the energy of the three particles bound state

$$\det\left(i\sqrt{\lambda-A_{12}}+\tilde B_s-Q_s(I+B_s)\right)=0. \tag{7.27}$$

The solutions of the dispersion equation are situated on the negative real axis λ. The solutions define the singularities of scattering amplitudes. Equations (7.14) and (7.23) for the vector h cannot be solved at these points.

It should be underlined that additional singularities in the scattering amplitudes are produced by the two-body bound states and resonances. A more rich structure of the three-body bound states can be obtained by adding the three-body space of interaction. The bound state eigenfunctions are orthogonal to the continuous spectrum eigenfunctions.

Thus the investigation of the three-body model scattering problem is accomplished. All eigenfunctions were presented by Sommerfeld integrals, however the scattering matrix was calculated in terms of elementary functions. This was possible due to the simple geometry of the problem. But the scattering matrix for nonidentical particles should contain some special functions. The components s_{23}, s_{32} of the scattering matrix are continuous functions of the angles for all φ. The component s_{33} contains a singularity corresponding to the back scattering. In the case of the two-body zero energy resonance the function $\tilde{g}_0^s(\alpha)$ has a singularity at the origin. As the result, the asymptotics of the Sommerfeld integral can not be calculated by the saddle point method directly for $\varphi = 0$. Hence the scattering amplitude is discontinuous at $\varphi = 0$ in this case. The analytical continuation of the scattering matrix has singularities at the points of discrete spectrum. All these properties of the scattering amplitudes are similar to the properties of amplitudes for the standard three body Hamiltonians with the two-body potentials. Our model can be used for the investigation of the influence of the presence of zero energy eigenstate or resonance on the analytical properties of the corresponding scattering amplitudes [45, 17]. It is important to discuss the propagator estimates for the model constructed [46]. These questions will be studied in one of the future publications.

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Appendix A.

Analytical formulas for the boundary values of the deficiency elements.

We introduce the following linear operators $\rho_m : H_{12} \to \mathbf{C}$

$$m > 0 : \quad \rho_m : h \to Res\tilde{g}_0^s(\alpha) \mid_{\alpha = i\gamma_m + 2\pi} =$$

$$= 2 \tan i\gamma_m \Pi^m \quad \Bigg\{ \left(\frac{i\gamma_m}{2\pi} + \frac{1}{2} \right) \Pi(i\gamma_m + 5\pi/3) \Pi(i\gamma_m + 4\pi/3) \Pi(i\gamma_m) \times \frac{(-1) \langle \frac{k \sin i\gamma_m}{k \cos i\gamma_m + \sqrt{\lambda - A_{12}}} *, \theta \rangle}{(2aik \sin i\gamma_m + b)\mathbf{R}(\lambda \sin^2 i\gamma_m) + 2cik \sin i\gamma_m + d}$$
$$+ \left(\frac{i\gamma_m}{2\pi} + \frac{1}{3} \right) \Pi(i\gamma_m + 5\pi/3) \Pi(i\gamma_m + 4\pi/3) \times \frac{\langle \frac{k \sin(i\gamma_m + 2\pi/3)}{k \cos(i\gamma_m + 2\pi/3) + \sqrt{\lambda - A_{12}}} *, \theta \rangle}{(2aik \sin(i\gamma_m + 2\pi/3) - b)\mathbf{R}(\lambda \sin^2(i\gamma_m + 2\pi/3)) + 2cik \sin(i\gamma_m + 2\pi/3) - d}$$
$$+ \left(\frac{i\gamma_m}{2\pi} + \frac{1}{6} \right) \Pi(i\gamma_m + 4\pi/3) \times \frac{\langle \frac{k \sin(i\gamma_m + \pi/3)}{k \cos(i\gamma_m + \pi/3) + \sqrt{\lambda - A_{12}}} *, \theta \rangle}{(2aik \sin(i\gamma_m + \pi/3) - b)\mathbf{R}(\lambda \sin^2(i\gamma_m + \pi/3)) + 2cik \sin(i\gamma_m + \pi/3) - d}$$
$$+ \left(\frac{i\gamma_m}{2\pi} \right) \frac{\langle \frac{k \sin(i\gamma_m)}{k \cos(i\gamma_m) + \sqrt{\lambda - A_{12}}} *, \theta \rangle}{(2aik \sin(i\gamma_m) - b)\mathbf{R}(\lambda \sin^2(i\gamma_m)) + 2cik \sin(i\gamma_m) - d} \Bigg\},$$

$$m \leq 0 : \quad \rho_m : h \to Res\tilde{g}_0^s(\alpha) \mid_{\alpha = -i\gamma_m + \pi} .$$

Then the boundary values of the deficiency element at the origin are

(A.1)

$$
\begin{aligned}
g_1^s(0) \quad = \quad & \tfrac{h}{2} + \tfrac{1}{2}\textstyle\sum_{m>0} \Big\{ (-2aik\sin(i\gamma_m) + b) \left(A_{12} - \lambda\sin^2(i\gamma_m)\right)^{-1}\theta \\
& +(-2aik\sin(i\gamma_m + 5\pi/3) + b)\Pi(i\gamma_m + 2\pi/3)\left(A_{12} - \lambda\sin^2(i\gamma_m + 5\pi/3)\right)^{-1}\theta \\
& +(-2aik\sin(i\gamma_m + 4\pi/3) + b)\Pi(i\gamma_m + \pi/3) \\
& \times\Pi(i\gamma_m + 2\pi/3)\left(A_{12} - \lambda\sin^2(i\gamma_m + 4\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m + 5\pi/3) + b)\left(A_{12} - \lambda\sin^2(i\gamma_m + 5\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m + 4\pi/3) + b)\Pi(i\gamma_m + 2\pi/3)\left(A_{12} - \lambda\sin^2(i\gamma_m + 4\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m + \pi) + b)\Pi(i\gamma_m + \pi/3) \\
& \times\Pi(i\gamma_m + 2\pi/3)\left(A_{12} - \lambda\sin^2(i\gamma_m + \pi)\right)^{-1}\theta\Big\}\rho_m h \\
& +\tfrac{1}{2}\textstyle\sum_{m\le 0}\Big\{(-2aik\sin(i\gamma_m) + b)\left(A_{12} - \lambda\sin^2(i\gamma_m)\right)^{-1}\theta \\
& +(-2aik\sin(-i\gamma_m + 2\pi/3) + b)\Pi(-i\gamma_m + 5\pi/3)\left(A_{12} - \lambda\sin^2(-i\gamma_m + 2\pi/3)\right)^{-1}\theta \\
& +(-2aik\sin(-i\gamma_m + \pi/3) + b)\Pi(-i\gamma_m + 4\pi/3) \\
& \times\Pi(-i\gamma_m + 5\pi/3)\left(A_{12} - \lambda\sin^2(-i\gamma_m + \pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m + 2\pi/3) + b)\left(A_{12} - \lambda\sin^2(-i\gamma_m + 2\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m + \pi/3) + b)\Pi(-i\gamma_m + 5\pi/3)\left(A_{12} - \lambda\sin^2(-i\gamma_m + \pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m) + b)\Pi(-i\gamma_m + 4\pi/3) \\
& \times\Pi(-i\gamma_m + 5\pi/3)\left(A_{12} - \lambda\sin^2(i\gamma_m)\right)^{-1}\theta\Big\}\rho_m h.
\end{aligned}
$$

$(A.2)$

$$
\begin{aligned}
\frac{\partial g_1^s(r)}{\partial r}\Big|_{r=0} \quad = \quad & i\sqrt{\lambda - A_{12}}\frac{h}{2} \\
& +\tfrac{1}{2}\sum_{m>0}\Big\{ ik\cos(i\gamma_m)(-2aik\sin(i\gamma_m)+b)\left(A_{12}-\lambda\sin^2(i\gamma_m)\right)^{-1}\theta \\
& +(-2aik\sin(i\gamma_m+5\pi/3)+b)\Pi(i\gamma_m+2\pi/3) \\
& \times ik\cos(i\gamma_m+5\pi/3)\left(A_{12}-\lambda\sin^2(i\gamma_m+5\pi/3)\right)^{-1}\theta \\
& +(-2aik\sin(i\gamma_m+4\pi/3)+b)\Pi(i\gamma_m+\pi/3)\Pi(i\gamma_m+2\pi/3) \\
& \times ik\cos(i\gamma_m+4\pi/3)\left(A_{12}-\lambda\sin^2(i\gamma_m+4\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m+5\pi/3)+b) \\
& \times ik\cos(i\gamma_m+5\pi/3)\left(A_{12}-\lambda\sin^2(i\gamma_m+5\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m+4\pi/3)+b)\Pi(i\gamma_m+2\pi/3) \\
& \times ik\cos(i\gamma_m+4\pi/3)\left(A_{12}-\lambda\sin^2(i\gamma_m+4\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(i\gamma_m+\pi)+b)\Pi(i\gamma_m+\pi/3)\Pi(i\gamma_m+2\pi/3) \\
& \times ik\cos(i\gamma_m+\pi)\left(A_{12}-\lambda\sin^2(i\gamma_m+\pi)\right)^{-1}\theta\Big\}\rho_m h \\
& +\tfrac{1}{2}\sum_{m\le 0}\Big\{(-2aik\sin(i\gamma_m)+b)ik\cos(-i\gamma_m+\pi)\left(A_{12}-\lambda\sin^2(i\gamma_m)\right)^{-1}\theta \\
& +(-2aik\sin(-i\gamma_m+2\pi/3)+b)\Pi(-i\gamma_m+5\pi/3) \\
& \times ik\cos(-i\gamma_m+2\pi/3)\left(A_{12}-\lambda\sin^2(-i\gamma_m+2\pi/3)\right)^{-1}\theta \\
& +(-2aik\sin(-i\gamma_m+\pi/3)+b)\Pi(-i\gamma_m+4\pi/3)\Pi(-i\gamma_m+5\pi/3) \\
& \times ik\cos(-i\gamma_m+\pi/3)\left(A_{12}-\lambda\sin^2(-i\gamma_m+\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m+2\pi/3)+b) \\
& \times ik\cos(-i\gamma_m+2\pi/3)\left(A_{12}-\lambda\sin^2(-i\gamma_m+2\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m+\pi/3)+b)\Pi(-i\gamma_m+5\pi/3) \\
& \times ik\cos(-i\gamma_m+\pi/3)\left(A_{12}-\lambda\sin^2(-i\gamma_m+\pi/3)\right)^{-1}\theta \\
& +(2aik\sin(-i\gamma_m)+b)\Pi(-i\gamma_m+4\pi/3)\Pi(-i\gamma_m+5\pi/3) \\
& \times ik\cos(i\gamma_m)\left(A_{12}-\lambda\sin^2(i\gamma_m)\right)^{-1}\theta\Big\}\rho_m h.
\end{aligned}
$$

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