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Energy Loss and Radiation of a Gyrating Charged Particle in a Magnetic Field

—— *Non-Ionized Medium* ——

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The Fourier series expansions are used to obtain the expressions for the components of the electromagnetic field at an arbitrary point of observation and for the total energy loss of a gyrating charged particle in a non-ionized medium having a uniform magnetic field. For a non-relativistic particle, it is shown that the total energy loss is split into the collision loss, whose formula is found to be the familiar one for linear motion, and the loss due to cyclotron radiations. The relative magnitude of the latter to the former is less than $(\omega_0/\omega_p)^2$, where ω_0 is the cyclotron frequency and $\omega_p{}^2=4\pi n_e e^2/m_e$ where n_e and m_e are the density and mass of electrons in the medium. In the relativistic case, we get the explicit formula of the polarization loss, depending upon the external magnetic field, and of the losses due to the Čerenkov and synchrotron radiations. The spectral and angular distributions of these two radiations are discussed.

§ 1. Introduction

The theoretical study of the energy loss and radiation of a charged particle passing through a medium, especially a plasma, having a magnetic field is important with respect to some astrophysical problems. As is well known, a plasma in a magnetic field has a tensor dielectric constant and behaves like an optically-active anisotropic medium.[1] Therefore, the treatment of a magnetoplasma is much more complicated than that of a non-ionized medium. Studies on the energy dissipation of a charged particle moving parallel to the magnetic field were made by several authors.[2] But for the motion perpendicular to the magnetic field the previous research has been limited to the radiation loss[3] and the calculation of the collision loss has not yet been made. This can be said for the non-ionized medium as well as for the magnetoactive plasma. Although in this paper we shall confine ourselves to the former case, our present work may be a useful step towards the latter investigation.

A charged particle moving uniformly loses its energy by interacting with the medium through which it passes. This loss can be divided into two parts;[4] namely, one is due to close collisions, which are binary, and the other due to

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distant collisions consists of the polarization loss and the loss due to Čerenkov radiation. For a spiralling particle, however, the cyclotron or synchrotron radiations become an additional cause of the energy dissipations.

The motion of a charged particle in the direction of the magnetic field is not affected by this field, so that we can use the familiar formula of the energy loss for linear motion. Hence it is sufficient to consider only the circular motion perpendicular to the magnetic field. We may assume the distortion of the orbit to be negligible, the velocity of the particle being large.

First, in § 2, the potentials of the electromagnetic field produced by a gyrating charged particle will be derived. The calculation is rather tedious and the previous ones are limited to the value in the wave zone. Here the value at an arbitrary point of observation is exactly evaluated. On account of the circular motion, the quantities concerning the field are resolved into harmonic components by the Fourier series expansion.[5] In § 3 the general formula of the total energy loss, including the radiation loss, per unit time is derived from the work of the reaction force of the medium on the particle.[6] This expression is not of a convenient form and it is impossible to get numerical values unless adequate approximations are used.

In the non-relativistic case, we can use some approximations with high accuracy and it is shown in § 4 that the total energy loss is the sum of the collision loss and the loss due to cyclotron radiations. As is expected from the fact that the Larmor radius is extremely larger than the adiabatic limit of the impact parameter, the value of the collision loss is the same one as for the linear motion.

For the relativistic case, applying the asymptotic formulas of the Bessel functions with large order and large argument, it is found that the polarization loss depends on the magnitude of the external magnetic field and the emission of the Čerenkov radiation may be possible in addition to the synchrotron radiation (§ 5). And the spectral and angular distributions of these two radiations will be discussed. In the Appendix, another treatment of the same problem by the method of the Poynting vector is presented.

§ 2. The electromagnetic field of a gyrating charged particle

If only the radiation emitted by the particle is considered, it is sufficient to know the electromagnetic field in the wave zone and it can be done easily. However, as we shall deal with collisions, we must find the value of the field near the orbit of the particle.

Let us consider a charged particle with mass m_0, charge q and velocity $\boldsymbol{v}_0$ gyrating in a uniform magnetic field H_{ex} which is directed along the z axis. The angular velocity ω_0 (i.e. the Larmor frequency) of the particle is $(qH_{ex}/m_0c)\times\sqrt{1-v_0^2/c^2}$ and the radius of the orbit, $r_0=v_0/\omega_0$. The components of the radius vector $\boldsymbol{r}_0$ and of the velocity $\boldsymbol{v}_0$ of the particle are in the cylindrical coordinate (r, φ, z)

$$\boldsymbol{r}_0=(r_0,\ \varphi_0=\omega_0 t,\ 0),\quad \boldsymbol{v}_0=(0,\ v_0,\ 0).$$

The field quantities, such as the scalar and vector potentials, ϕ and $\boldsymbol{A}$, are analysed into the Fourier series respectively

$$\phi=\sum_{n=-\infty}^{\infty}\phi_n e^{-in\omega_0 t},\quad \boldsymbol{A}=\sum_{n=-\infty}^{\infty}\boldsymbol{A}_n e^{-in\omega_0 t}. \tag{2·1}$$

The charge and current densities of the particle are

$$\left.\begin{aligned}\rho&=q\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\sum_{n=-\infty}^{\infty}\rho_n e^{-in\omega_0 t},\\ \boldsymbol{j}&=\rho\boldsymbol{v}_0=\sum_{n=-\infty}^{\infty}\boldsymbol{j}_n e^{-in\omega_0 t},\end{aligned}\right\} \tag{2·2}$$

where the δ function may be expressed by the cylindrical coordinate as

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\delta(r-r_0)\delta(\varphi-\omega_0 t)\delta(z)/r. \tag{2·3}$$

From the Maxwell equations ϕ_n and A_n obey the next equations:

$$\Delta\phi_n+\frac{n^2\omega_0^2\epsilon_n}{c^2}\phi_n=-\frac{4\pi}{\epsilon_n}\rho_n, \tag{2·4}$$

$$\Delta A_n+\frac{n^2\omega_0^2\epsilon_n}{c^2}A_n=-\frac{4\pi}{c}j_n, \tag{2·5}$$

where ϵ_n is the dielectric constant of the medium and a function of frequency $n\omega_0$. ρ_n and j_n are easily obtained as

$$\rho_n=\frac{q\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0}\delta(\boldsymbol{r}-\boldsymbol{r}_0)e^{in\omega_0 t}dt=\frac{q}{2\pi}\frac{\delta(r-r_0)}{r}\delta(z)e^{in\varphi}, \tag{2·6}$$

$$\boldsymbol{j}_n=\boldsymbol{v}_0\rho_n \tag{2·7}$$

The solutions of (2·4) and (2·5) are expressed as follows,

$$\phi_n(\boldsymbol{r})=\int\frac{\exp[in\omega_0\sqrt{\epsilon_n}d/c]\rho_n(\boldsymbol{r}')}{\epsilon_n d}d\boldsymbol{r}', \tag{2·8}$$

$$\boldsymbol{A}_n(\boldsymbol{r})=\int\frac{\exp[in\omega_0\sqrt{\epsilon_n}d/c]\boldsymbol{j}_n(\boldsymbol{r}')}{cd}d\boldsymbol{r}', \tag{2·9}$$

where $d=|\boldsymbol{r}-\boldsymbol{r}'|$.

Integrations of (2·8) and (2·9) with respect to r' and z' give us

$$\phi_n=\frac{qe^{in\varphi}}{2\pi\epsilon_n}\int_{-\pi}^{\pi}\frac{\exp i[k_n\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}+n\chi]}{\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}}d\chi, \tag{2·10}$$

$$A_{nr}=-\frac{qv_0e^{in\varphi}}{2\pi c}\int_{-\pi}^{\pi}\frac{\exp i[k_n\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}+n\chi]}{\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}}\sin\chi\, d\chi, \tag{2·11}$$

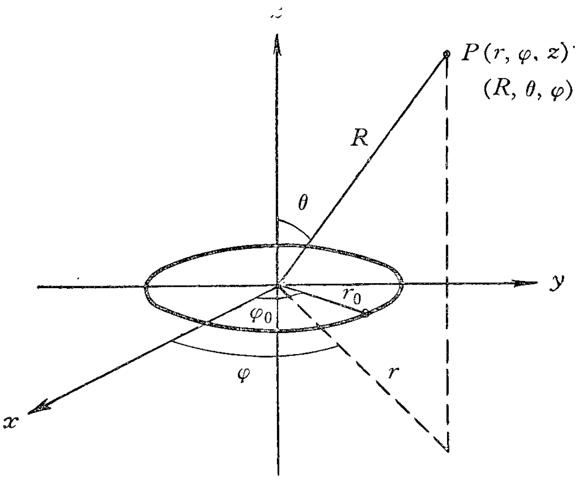

Fig. 1. Geometry of the source and field points

$$A_{n\varphi}=\frac{qv_0e^{in\varphi}}{2\pi c}\int_{-\pi}^{\pi}\frac{\exp i[k_n\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}+n\chi]}{\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}}\cos\chi\,d\chi \qquad (2\cdot12)$$

$$A_{nz}=0, \qquad (2\cdot13)$$

where $\chi=\varphi-\varphi'$, $R^2=r^2+z^2$, $r=R\sin\theta$ and $k_n=n\omega_0\sqrt{\epsilon_n}/c$ is the wave number.

If we consider only the wave zone, namely $R\gg r_0$, the above integrands are easily approximated (see the Appendix).[5),7)] But as we need to consider the case of $R\sim r_0$, it is necessary to use the expansion:

$$\frac{\exp[ik_n\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}]}{\sqrt{R^2+r_0^2-2r_0R\sin\theta\cos\chi}}$$
$$=ik_n\sum_{m=0}^{\infty}(2m+1)P_m(\sin\theta\cos\chi)\begin{cases}j_m(k_nr_0)h_m^{(1)}(k_nR) & R>r_0\\ j_m(k_nR)h_m^{(1)}(k_nr_0) & R<r_0,\end{cases} \qquad (2\cdot14)$$

where j_m and $h_m^{(1)}$ are respectively the spherical Bessel function and the spherical Hankel function of the first kind and of order m.

Substituting (2·14) in (2·10), (2·11) and (2·12), and integrating over the angle χ, we obtain

$$\phi_n=\frac{4\pi ik_nqe^{in\varphi}}{\epsilon_n}\sum_{m=|n|}^{\infty}Y_{mn}(\theta,0)Y_{mn}\left(\frac{\pi}{2},0\right)\begin{cases}j_m(k_nr_0)h_m^{(1)}(k_nR) & R>r_0\\ j_m(k_nR)h_m^{(1)}(k_nr_0) & R<r_0,\end{cases} \qquad (2\cdot15)$$

$$A_{nr}=\pm\frac{2\pi qv_0k_ne^{in\varphi}}{c}$$
$$\times\begin{cases}0, & \text{for } n=0\\ \left[\sum_{m=|n|-1}^{\infty}Y_{m,|n|-1}(\theta,0)Y_{m,|n|-1}\left(\frac{\pi}{2},0\right)\begin{cases}j_m(k_nr_0)h_m^{(1)}(k_nR) & R>r_0\\ j_m(k_nR)h_m^{(1)}(k_nr_0) & R<r_0\end{cases}\right. & \\ \left.-\sum_{m=|n|+1}^{\infty}Y_{m,|n|+1}(\theta,0)Y_{m,|n|+1}\left(\frac{\pi}{2},0\right)\begin{cases}j_m(k_nr_0)h_m^{(1)}(k_nR) & R>r_0\\ j_m(k_nR)h_m^{(1)}(k_nr_0) & R<r_0\end{cases}\right], & \\ & \text{for } |n|\geq1\end{cases} \qquad (2\cdot16)$$

$$A_{n\varphi}=\frac{2\pi i q v_0 k_n e^{in\varphi}}{c}$$

$$\times\begin{cases} 2\sum_{m=1}^{\infty} Y_{m1}(\theta,0)\,Y_{m1}\left(\frac{\pi}{2},0\right) j_m(0)\,h_m^{(1)}(0), & \text{for } n=0 \\ \left[\sum_{m=|n|-1}^{\infty} Y_{m,|n|-1}(\theta,0)\,Y_{m,|n|-1}\left(\frac{\pi}{2},0\right)\begin{cases} j_m(k_n r_0)\,h_m^{(1)}(k_n R) & R>r_0 \\ j_m(k_n R)\,h_m^{(1)}(k_n r_0) & R<r_0 \end{cases}\right. \\ \left. +\sum_{m=|n|+1}^{\infty} Y_{m,|n|+1}(\theta,0)\,Y_{m,|n|+1}\left(\frac{\pi}{2},0\right)\begin{cases} j_m(k_n r_0)\,h_m^{(1)}(k_n R) & R>r_0 \\ j_m(k_n R)\,h_m^{(1)}(k_n r_0) & R<r_0 \end{cases}\right], \\ \qquad\qquad\qquad\qquad \text{for } |n|\geq 1. \end{cases} \tag{2·17}$$

From $\phi_n(\boldsymbol{r})$ and $\boldsymbol{A}_n(\boldsymbol{r})$ just obtained above, the electromagnetic field $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$, at an arbitrary point of observation $\boldsymbol{r}$, may be easily calculated.

§ 3. General formula of energy loss

There are some ways to estimate the energy loss arising from distant collisions. Among these methods, the use of the Poynting vector is not convenient but intricate especially in our case, because we must evaluate the energy flux out of the toroidal surface having the circular orbit of the particle as axis. The simplest way to calculate the energy loss is, as usual, to get the work done by the reaction force of the medium on the particle.[6] This method gives us the energy loss per unit time as follows,

$$-\frac{dW}{dt}=-q(\boldsymbol{v}_0\boldsymbol{E})_{\boldsymbol{r}=\boldsymbol{v}_0 t}=-qv_0E_\varphi(r_0,\,\omega_0 t,\,0). \tag{3·1}$$

Really this formula represents the total energy loss per unit time, including the contribution of close collisions. However, it must be noticed that whether Eq. (3·1) includes the close collision loss or not depends on the choice of the upper limit of order m, which corresponds to the minimum impact parameter (see the next section).

It is obvious from (3·1) that we have only to know E_φ whose harmonic component is given by

$$E_{n\varphi}=-\frac{1}{R\sin\theta}\frac{\partial\phi_n}{\partial\varphi}+\frac{ik_n}{\sqrt{\epsilon_n}}A_{n\varphi}. \tag{3·2}$$

Substituting (2·15) and (2·17) into (3·2), we have

$$E_{n\varphi}(R,\theta,\varphi)=\frac{4\pi n k_n q e^{in\varphi}}{\epsilon_n R\sin\theta}\sum_{m=|n|}^{\infty} Y_{mn}(\theta,0)\,Y_{mn}\left(\frac{\pi}{2},0\right)\begin{cases} j_m(k_n r_0)\,h_m^{(1)}(k_n R) & R>r_0 \\ j_m(k_n R)\,h_m^{(1)}(k_n r_0) & R<r_0 \end{cases}$$

$$-\frac{2\pi q v_0 k_n^2 e^{in\varphi}}{c\sqrt{\epsilon_n}}\left[\sum_{m=|n|-1}^{\infty} Y_{m,|n|-1}(\theta,0)\,Y_{m,|n|-1}\left(\frac{\pi}{2},0\right)\begin{cases} j_m(k_n r_0)\,h_m^{(1)}(k_n R) & R>r_0 \\ j_m(k_n R)\,h_m^{(1)}(k_n r_0) & R<r_0 \end{cases}\right.$$

$$+\sum_{m=|n|+1}^{\infty} Y_{m,|n|+1}(\theta, 0)\, Y_{m,|n|+1}\left(\frac{\pi}{2}, 0\right) \begin{cases} j_m(k_n r_0) h_m^{(1)}(k_n R) & R > r_0 \\ j_m(k_n R) h_m^{(1)}(k_n r_0) & R < r_0 \end{cases}. \tag{3·3}$$

Evidently E_φ is written as

$$E_\varphi = \sum_{n=1}^{\infty} (E_{n\varphi} e^{-in\omega_0 t} + E_{-n\varphi} e^{in\omega_0 t}), \tag{3·4}$$

where $E_{-n\varphi}$ is the complex conjugate of $E_{n\varphi}$ and easily obtained from (3·3).

Using the above relations, finally we get the energy loss of the particle moving in a circle as follows,

$$-\frac{dW}{dt} = -\frac{8\pi q^2 c \beta_0^2}{r_0^2} \sum_{n=1}^{\infty} n^2 \Bigg[\mathrm{Re}\left\{ \frac{1}{\sqrt{\epsilon_n}} \sum_{m=n}^{\infty} Y_{mn}^2\left(\frac{\pi}{2}, 0\right) j_m(n\beta_0\sqrt{\epsilon_n})\, h_m^{(1)}(n\beta_0\sqrt{\epsilon_n}) \right\}$$

$$-\frac{1}{2}\beta_0^2\, \mathrm{Re}\left\{ \sqrt{\epsilon_n} \sum_{m=n-1}^{\infty} Y_{m,n-1}^2\left(\frac{\pi}{2}, 0\right) j_m(n\beta_0\sqrt{\epsilon_n})\, h_m^{(1)}(n\beta_0\sqrt{\epsilon_n}) \right.$$

$$\left. +\sqrt{\epsilon_n} \sum_{m=n+1}^{\infty} Y_{m,n+1}^2\left(\frac{\pi}{2}, 0\right) j_m(n\beta_0\sqrt{\epsilon_n})\, h_m^{(1)}(n\beta_0\sqrt{\epsilon_n}) \right\} \Bigg], \tag{3·5}$$

where $\beta_0 = v_0/c$ and a relation $k_n r_0 = n\beta_0\sqrt{\epsilon_n}$ is used.

The spherical harmonics $Y_{mn}(\pi/2, 0)$ is zero at $m-n=$odd, and when $m-n=2l$ is even its square is

$$Y_{mn}^2\left(\frac{\pi}{2}, 0\right) = \frac{2n+4l+1}{4\pi} \frac{(2l)!\,(2n+2l)!}{2^{2n+4l}\, l!^2 (n+l)!^2}, \quad (l\text{: integer}).$$

Further, there is the following relation,

$$h_m^{(1)}(z) = j_m(z) + i n_m(z),$$

in which $n_m(z)$ is the spherical Neumann function of order m.

In Eq. (3·5), the contributions from the terms with j_m^2 and $n_m j_m$ correspond to the radiation and collision losses respectively, where the collision loss means the polarization loss plus the binary collision loss. Thus we shall divide the total loss as follows,

$$-\frac{dW}{dt} = \left(-\frac{dW}{dt}\right)_{coll} + \left(-\frac{dW}{dt}\right)_{rad}. \tag{3·6}$$

From Eq. (3·5) one cannot say anything about further details about the energy loss Thus, in the following sections we shall derive the explicit formulas, applying the asymptotic formulas for the Bessel functions.

§ 4. Non-relativistic case

For the non-relativistic velocity of the particle ($\beta_0 \ll 1$), the next asymptotic formula for the Bessel functions may be used.[8),9)] That is, for $n\beta_0|\sqrt{\epsilon_n}| \ll 2m-1$ and $\beta_0|\sqrt{\epsilon_n}| \ll 2e^{-1}$, we have

$$j_m(n\beta_0\sqrt{\epsilon_n})\cong\frac{(n\beta_0\sqrt{\epsilon_n})^m}{(2m+1)!!},\quad n_m(n\beta_0\sqrt{\epsilon_n})\cong-\frac{(2m-1)!!}{(n\beta_0\sqrt{\epsilon_n})^{m+1}}. \tag{4·1}$$

I) *Collision loss*

Using Eq. (4·1), the series of terms containing $j_m n_m$ in Eq. (3·5) are reduced to

$$\left.\begin{aligned}
&\sum_{m=n}^{\infty}Y_{mn}^2\left(\frac{\pi}{2},\,0\right)j_m(n\beta_0\sqrt{\epsilon_n})n_m(n\beta_0\sqrt{\epsilon_n})=-\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}}\sum_{l=0}^{\infty}\frac{(2l)!\,(2n+2l)!}{2^{2n+4l}\,l!^2(n+l)!^2},\\
&\sum_{m=n-1}^{\infty}Y_{m,n-1}^2\left(\frac{\pi}{2},\,0\right)j_m(n\beta_0\sqrt{\epsilon_n})n_m(n\beta_0\sqrt{\epsilon_n})=-\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}}\sum_{l=0}^{\infty}\frac{(2l)!\,(2n+2l-2)!}{2^{2n+4l-2}l!^2(n+l-1)!^2},\\
&\sum_{m=n+1}^{\infty}Y_{m,n+1}^2\left(\frac{\pi}{2},\,0\right)j_m(n\beta_0\sqrt{\epsilon_n})n_m(n\beta_0\sqrt{\epsilon_n})=-\frac{1}{4\pi n\beta_0\sqrt{\epsilon_n}}\sum_{l=0}^{\infty}\frac{(2l)!\,(2n+2l+2)!}{2^{2n+4l+2}l!^2(n+l+1)!^2}.
\end{aligned}\right\} \tag{4·2}$$

The Stirling formula may be applied to the right-hand side of (4·2) and then we have, for example,

$$\sum_{l=0}^{\infty}\frac{(2l)!\,(2n+2l)!}{2^{2n+4l}\,l!^2(n+l)!^2}\simeq\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{1}{\sqrt{l(n+l)}}. \tag{4·3}$$

The summation may be further replaced by an integral. Although these infinite series or integrals diverge, actually there exists an upper limit of order m or of the integral variable. This fact is based on the following reason.

$Y_{mn}(\theta,\varphi)$ is the eigenfunction of the angular momentum operator, and m and n are the azimuthal and magnetic quantum numbers, respectively, of a spherical wave. The virtual photon accompanying by the particle whose angular dependence will be given by $Y_{mn}(\theta,\varphi)$ has an angular momentum $n\hbar$ along the z axis,[10] and has a frequency $n\omega_0$ as a result of the circular motion of the particle with an angular velocity ω_0. On the other hand, a maximum frequency of the virtual photon exists and is given by v_0/p_{min}, where p_{min} is the lower limit of the impact parameter.* Accordingly, the maxium value of m, which is designated by M, is given by

$$M\omega_0=v_0/p_{min} \tag{4·4}$$

with

$$p_{min}=\begin{cases}qe/m_e v_0^2 & \text{if } qe/\hbar v_0>1\\ \hbar\sqrt{1-\beta_0^2}/m_e v_0 & \text{if } qe/\hbar v_0<1,\end{cases}$$

where $-e$ and m_e are the charge and mass of the electron.

Finally, we obtain

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* If we take as $p_{min}\approx$atomic radius in (4·4), (3·5) represents only the loss due to distant collisions.

$$\text{each expression of } (4\cdot 2) = -\frac{1}{4\pi^2}\frac{1}{n\beta_0\sqrt{\epsilon_n}}\ln\left(\frac{M}{n}\right).$$

Then the formula of the collision loss may be shown to be reduced to

$$-\frac{dW}{dt}\bigg|_{coll} = -\frac{2q^2 v_0}{\pi r_0^2}\sum_{n=1}^{\infty} n\ln\left(\frac{M}{n}\right)\mathrm{Im}\left(\frac{1}{\epsilon_n}-\beta_0^2\right) = \frac{2q^2 v_0}{\pi r_0^2}\sum_{n=1}^{\infty}\frac{\epsilon_{2n}}{|\epsilon_n|^2}n\ln\left(\frac{M}{n}\right), \tag{4·5}$$

where ϵ_{2n} is the imaginary part of ϵ_n and $\epsilon_n = \epsilon_{1n} + i\epsilon_{2n}$. The summation over n in Eq. (4·5) has poles at the resonant frequencies ω_j of $n\epsilon_{2n}/|\epsilon_n|^2$ and the contributions of terms with small n's are completely negligible. In order to compare with the case of the linear motion, we shall replace the sum by the integral over frequency ω as follows,

$$-\frac{dW}{dt}\bigg|_{coll} = \frac{2q^2}{\pi v_0}\int_0^{\infty}\frac{\epsilon_{2\omega}}{|\epsilon_\omega|^2}\ln\left(\frac{M\omega_0}{\omega}\right)\omega\, d\omega = \frac{4\pi n_e q^2 e^2}{m_e v_0}\sum_j f_j \ln\left(\frac{M\omega_0}{\omega_j}\right), \tag{4·6}$$

where ω_j and f_j are respectively the frequency and the strength of the j-th oscillator of the atomic electrons, and where n_e is the electron density of the medium.

When we substitute the value of M defined by (4·4) in the energy loss formula (4·6), it agrees with the well-known one for the linear motion. After all, in the non-relativistic case it can be said that the collision loss for the circular motion may be approximated by the one for the linear motion with sufficient accuracy This should be expected from the fact that the Larmor radius r_0 is extremely larger than the adiabatic limit of the impact parameter $p_{max} = v_0/\omega_j$, because $\omega_0 \ll \omega_j$, however strong the magnetic field may be.

II) Radiation loss

Using the asymptotic form (4·1), the summation over m of the terms containing j_m^2 is reduced, for instance, to

$$\sum_{m=n}^{\infty} Y_{mn}^2\left(\frac{\pi}{2},0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(2n+4l+1)(2l)!(2n+2l)!}{2^{2n+4l}l!^2(n+l)!^2}\frac{(n\beta_0\sqrt{\epsilon_n})^{2n+4l}}{(2n+4l+1)!!^2},$$

where the right series decreases very rapidly with l, and by retaining only the first term we have

$$\sum_{m=n}^{\infty} Y_{mn}^2\left(\frac{\pi}{2},0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi}\frac{(n\beta_0\sqrt{\epsilon_n})^{2n}}{(2n+1)!}.$$

Similarly, we get

$$\sum_{m=n-1}^{\infty} Y_{m,n-1}^2\left(\frac{\pi}{2},0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi}\frac{(n\beta_0\sqrt{\epsilon_n})^{2n-2}}{(2n-1)!}$$

and

$$\sum_{m=n+1}^{\infty} Y_{m,n+1}^2\left(\frac{\pi}{2},0\right) j_m^2(n\beta_0\sqrt{\epsilon_n}) = \frac{1}{4\pi}\frac{(n\beta_0\sqrt{\epsilon_n})^{2n+2}}{(2n+3)!},$$

in which the last one will be omitted, because it is negligibly small compared with the other two.

Thus the contribution from the terms of j_m^2 in (3·5) give us

$$-\frac{dW}{dt}\bigg|_{rad}=\frac{2q^2c\beta_0^2}{r_0^2}\sum_{n=1}^{n}\frac{n(n+1)}{(2n+1)!}\mathrm{Re}\left\{\frac{(n\beta_0\sqrt{\epsilon_n})^{2n}}{\sqrt{\epsilon_n}}\right\}$$

$$=\frac{2q^2v_0}{r_0^2}\sum_{n=1}^{\infty}\frac{n+1}{(2n+1)!}(n\beta_0)^{2n+1}\epsilon_{1n}^{n-1/2}, \qquad (4\cdot7)$$

where the effect of ϵ_{2n} has been neglected. If we put $\epsilon_{1n}=1$ in (4·7), we obtain the well-known formula for the cyclotron radiation in vacuum.

III) Partition of energy loss

It is readily shown from (4·6) and (4·7) that the radiation loss is less than a fraction $(\omega_0/\omega_p)^2$ of the collision loss, where $\omega_p^2=4\pi n_e e^2/m_e$. In a dense medium, $(\omega_0/\omega_p)^2$ is ordinarily much less than unity and the contribution from radiations to the stopping power may be negligible. As $(\omega_0/\omega_p)^2$ is proportional to H_{ex}^2/n_e, however, the radiation loss in a rarefied gas having a strong magnetic field is not always negligible compared with the collision loss.

The ratio of the collision loss to the radiation loss, designated by $\mathcal{R}$, is as follows:

$$\mathcal{R}=\frac{6\pi n_e m_0^2c^2}{m_e Z^2 H_{ex}^2}\frac{1-\beta_0^2}{\beta_0^3}\ln\left\{\frac{m_e c^2\beta_0^2}{\hbar\bar{\omega}_j\sqrt{1-\beta_0^2}}\right\}, \qquad (4\cdot8)$$

where $q=Ze$ and $\bar{\omega}_j=\sum f_j\ln\omega_j$. For the radiation loss, there has been used the next familiar value in vacuum,

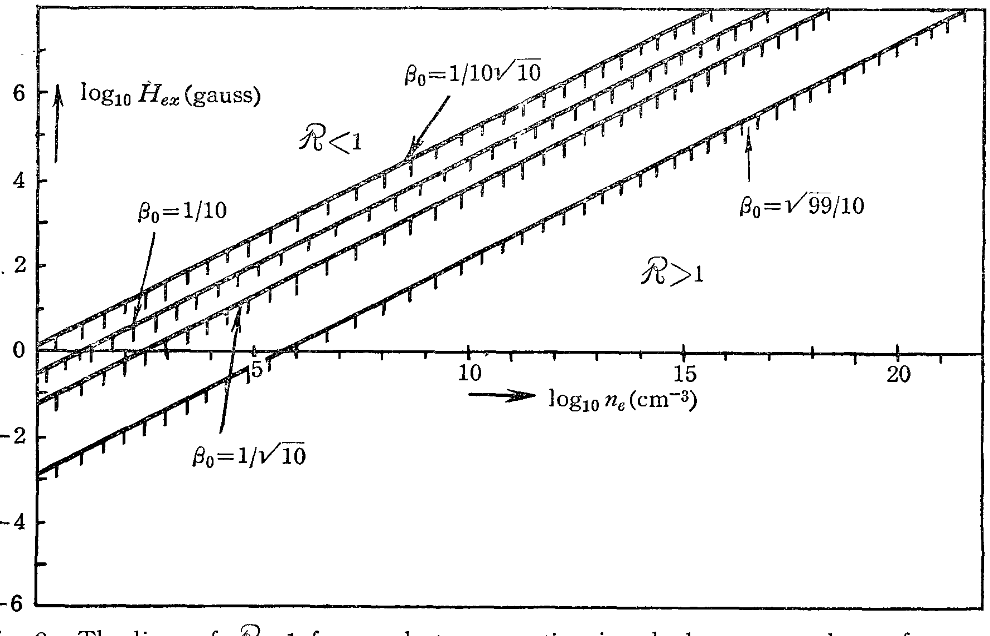

Fig. 2. The lines of $\mathcal{R}=1$ for an electron gyrating in a hydrogen gas, drawn for some values of β_0. The regions under the lines correspond to $\mathcal{R}>1$.

$$-\left.\frac{dW}{dt}\right|_{rad}=\frac{2\omega_0^2 q^2\beta_0^2}{3c(1-\beta_0^2)^2}. \tag{4·9}$$

Fig. 2 shows the region of $\mathcal{R}>1$ in the n_e-H_{ec} plane for an electron gyrating in a hydrogen gas. Eq. (4·8) is not valid for the ultra-relativistic case, because of the H_{ec} dependence of the collision loss (see the next section).

§ 5. Relativistic case

In the relativistic case, we must use the following asymptotic formulas for the spherical Bessel functions with large order and large argument.[8),9)]

Case 1) If $m+\frac{1}{2}>n\beta_0|\sqrt{\epsilon_n}|$, we have

$$\left.\begin{aligned} j_m(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{\exp[(m+\frac{1}{2})(\tanh\alpha_m-\alpha_m)]}{\sqrt{2n\beta_0\sqrt{\epsilon_n}}\sqrt{(2m+1)\tanh\alpha_m}}, \\ n_m(n\beta_0\sqrt{\epsilon_n}) &\simeq -\frac{2\exp[(m+\frac{1}{2})(\alpha_m-\tanh\alpha_m)]}{\sqrt{2n\beta_0\sqrt{\epsilon_n}}\sqrt{(2m+1)\tanh\alpha_m}}, \end{aligned}\right\} \tag{5·1}$$

where

$$\tanh^2\alpha_m=1-\left(\frac{n\beta_0\sqrt{\epsilon_n}}{m+\frac{1}{2}}\right)^2.$$

Case 2) If $m+\frac{1}{2}<n\beta_0|\sqrt{\epsilon_n}|$, we have

$$\left.\begin{aligned} j_m(n\beta_0\sqrt{\epsilon_n}) &\simeq \frac{1}{\sqrt{n\beta_0\sqrt{\epsilon_n}}}\frac{\cos[(m+\frac{1}{2})(\tan\gamma_m-\gamma_m)-\pi/4]}{\sqrt{(m+\frac{1}{2})\tan\gamma_m}}, \\ n_m(n\beta_0\sqrt{\epsilon_n}) &\simeq -\frac{1}{\sqrt{n\beta_0\sqrt{\epsilon_n}}}\frac{\sin[(m+\frac{1}{2})(\tan\gamma_m-\gamma_m)-\pi/4]}{\sqrt{(m+\frac{1}{2})\tan\gamma_m}}, \end{aligned}\right\} \tag{5·2}$$

where

$$\tan^2\gamma_m=\left(\frac{n\beta_0\sqrt{\epsilon_n}}{m+\frac{1}{2}}\right)^2-1.$$

I) Collision loss

Substituting (5·1) in the cross term $j_m n_m$ of (3·5), we obtain the polarization loss, including the close collision loss, in the following way.

$$-\left.\frac{dW}{dt}\right|_{coll}=\frac{2q^2c\beta_0^2}{\pi r_0^2}\sum_{n=1}^{\infty}n\,\mathrm{Re}\frac{i}{\epsilon_n}\sum_{l=0}^{l_{max}}\frac{1}{\sqrt{l(l+n)}\sqrt{1-\left(\frac{n\beta_0\sqrt{\epsilon_n}}{n+2l+\frac{1}{2}}\right)^2}}, \tag{5·3}$$

where we have used (5·1) even for small n because the contributions from the terms with small n's are completely negligible. The summation over l is limited to a certain upper limit l_{max} given by (4·4) and its imaginary part is negligibly small. If $\beta_0|\sqrt{\epsilon_n}|\ll 1$, we get (4·5) from (5·3) again.

Considering that $\beta_0|\sqrt{\epsilon_n}|$ is nearly equal to unity in the ultra-relativistic case, and replacing the sum by the integral, the sum over l is reduced to

$$\sum_l \cdots \simeq \frac{1}{2}\int^{l_{max}} \frac{n+2l}{l(l+n)}\,dl = \frac{1}{2}\ln\left(\frac{M^2}{4n}-\frac{n}{4}\right) = \frac{1}{2}\ln\left(\frac{m_e^2 v_0^4}{4\hbar^2\omega\omega_0(1-\beta_0^2)}-\frac{\omega}{4\omega_0}\right),$$

where (4·4) has been used. Then the collision loss per unit time in the ultra-relativistic case is found to be

$$-\frac{dW}{dt}\bigg|_{coll} = \frac{4\pi n_e e^2 q^2}{m_e v_0}\sum_j f_j \ln\left(\frac{m_e^2 v_0^4}{4\hbar^2\omega_j\omega_0(1-\beta_0^2)}-\frac{\omega_j}{4\omega_0}\right)^{1/2}. \qquad (5\cdot4)$$

The second term in the logarithmic factor is negligible compared with the first term. Here it should be noted that (5·4) depends on the external magnetic field, in contrast with the case of motion parallel to the magnetic field, and that it is not applicable to the limiting case $H_{ex}\to 0$.

II) Synchrotron radiation

When $m+1/2 > n\beta_0|\sqrt{\epsilon_n}|$, the contributions of the square terms $j_m^2(n\beta_0\sqrt{\epsilon_n})$ in (3·5) give us the energy loss by synchrotron radiations. First, we must apply the recurrence formula and then use the asymptotic one for (3·5).* Here it should be noticed that if $m\gg n$, viz. $\alpha_m\gg 1$, $j_m^2(n\beta_0\sqrt{\epsilon_n})$ is nearly equal to zero; and thus we have only to consider the case of $\alpha_m<1$. Therefore we have

$$\tanh\alpha_m-\alpha_m \simeq -\frac{1}{3}\alpha_m^3 \simeq -\frac{1}{3}\left\{1-\left(\frac{n\beta_0\sqrt{\epsilon_n}}{m+\frac{1}{2}}\right)^2\right\}^{3/2}.$$

Approximately we get the following

$$\frac{1}{2}\beta_0^2\sqrt{\epsilon_n}\left[\sum_{m=n-1} Y_{m,n-1}^2\left(\frac{\pi}{2},0\right)j_m^2(n\beta_0\sqrt{\epsilon_n})+\sum_{m=n+1} Y_{m,n+1}^2\left(\frac{\pi}{2},0\right)j_m^2(n\beta_0\sqrt{\epsilon_n})\right]$$

$$-\frac{1}{\sqrt{\epsilon_n}}\sum_{m=n} Y_{mn}^2\left(\frac{\pi}{2},0\right)j_m^2(n\beta_0\sqrt{\epsilon_n}) \simeq \frac{1}{2}\beta_0\sqrt{\epsilon_n}\,Y_{n-1,n-1}^2\left(\frac{\pi}{2},0\right)j_{n-1}^2(n\beta_0\sqrt{\epsilon_n})$$

$$\simeq \frac{\beta_0 n^{-3/2}}{16\pi\sqrt{\pi}}\,\frac{\exp[-(2n/3)\,(1-\beta_0^2\epsilon_n)^{3/2}]}{\sqrt{1-\beta_0^2\epsilon_n}}, \qquad (5\cdot5)$$

where the remaining terms are found to be almost cancelled, using the recurrence formula and adequate approximations

As we can neglect the imaginary part of ϵ_n in (5·5), we obtain the loss due to the synchrotron radiation as follows,

$$-\frac{dW}{dt}\bigg|_{syn} = \frac{qc\beta_0^3}{2\sqrt{\pi}\,r_0^2}\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{1-\beta_0^2\epsilon_{1n}}}\exp\left[-\frac{2n}{3}(1-\beta_0^2\epsilon_{1n})^{3/2}\right]. \qquad (5\cdot6)$$

For small n, the spectrum represented by (5·6) is not a good approximation. If

* If we directly apply (5·1) for the j_m^2 term in (3·5), we are misled to the wrong result (energy gain) on account of incorrectness of (5·1).

we neglect the dispersion and replace ϵ_{1n} by a certain average ϵ, we have

$$(5\cdot6)=\frac{q^2 c\beta_0^3}{2\sqrt{\pi}\, r_0^2\sqrt{1-\beta_0^2\epsilon}}\sum_{n=1}^{\infty}\sqrt{n}\, e^{-n/n_c},$$

where n_c is defined by

$$\frac{2n_c}{3}(1-\beta_0^2\epsilon)^{3/2}=1.$$

If we put $y=n/n_c$ and approximate the sum by the integral, we obtain

$$-\frac{dW}{dt}=\frac{3\sqrt{3}\, q^2 c\beta_0^3}{4\sqrt{2\pi}\, r_0^2(1-\beta_0^2\epsilon)^{11/4}}\int_0^{\infty}\sqrt{y}\, e^{-y}dy=\frac{3\sqrt{3}\, q^2 c\beta_0^3}{8\sqrt{2}\, r_0^2(1-\beta_0^2\epsilon)^{11/4}}. \quad (5\cdot7)$$

Putting ϵ equal to unity, the above expressions are a little different from the ordinary ones in vacuum given by the method of the Poynting vector (see the Appendix or reference 7)). One of the reasons is due to the poor approximations to the lower harmonics.

III) Čerenkov radiation

When the condition $\beta_0|\sqrt{\epsilon_n}|>1$ is satisfied, it is also possible for a gyrating charged particle to emit the Čerenkov radiation, provided the wavelength is much smaller than the radius of the orbit.

As it is sufficient to consider only the case of $\gamma_m<1$, we can write as

$$\tan\gamma_m-\gamma_m\simeq\frac{1}{3}\gamma_m^3$$

and thus from $(5\cdot1)$

$$j_m^2(n\beta_0\sqrt{\epsilon_n})\simeq\frac{2}{2m+1}\frac{\cos^2\left\{\frac{2m+1}{6}\gamma_m^3-\frac{\pi}{4}\right\}}{n\beta_0\sqrt{\epsilon_n}\tan\gamma_m}.$$

Therefore we have

$$\sum_{m=n}Y_{mn}^2\left(\frac{\pi}{2},\ 0\right)j_m^2(n\beta_0\sqrt{\epsilon_n})\simeq\frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}}f(n),$$

$$\sum_{m=n-1}Y_{m,n-1}^2\left(\frac{\pi}{2},\ 0\right)j_m^2(n\beta_0\sqrt{\epsilon_n})\simeq\frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}}f(n-1),$$

$$\sum_{m=n+1}Y_{m,n+1}^2\left(\frac{\pi}{2},\ 0\right)j_m^2(n\beta_0\sqrt{\epsilon_n})\simeq\frac{1}{2\pi n\beta_0\sqrt{\epsilon_n}}f(n+1),$$

where the function $f(n)$ is

$$f(n)=\frac{1}{\pi}\sum_l\frac{1}{\sqrt{l(l+n)}}\frac{\cos^2\left(\frac{2n+4l+1}{6}\gamma_{n+2l}^3-\frac{\pi}{4}\right)}{\sqrt{\left(\frac{n\beta_0\sqrt{\epsilon_n}}{n+2l+\frac{1}{2}}\right)^2-1}} \quad (5\cdot8)$$

and

$$f(n) \simeq f(n-1) \simeq f(n+1) \quad \text{(for large } n\text{)},$$

where the upper limit of l is determined by the condition $n\beta_0|\sqrt{\epsilon_n}|>n+2l+1/2$.

Thus the energy loss due to the Čerenkov radiation is expressed as

$$-\frac{dW}{dt}\bigg|_{\check{C}er}=\frac{4q^2v_0}{r_0^2}\sum_{\substack{n\\ \beta_0|\sqrt{\epsilon_n}|>1}} \mathrm{Re}\left\{n\left(\beta_0^2-\frac{1}{\epsilon_n}\right)f(n)\right\}, \tag{5·9}$$

where the sum over n is limited to the case of $\beta_0|\sqrt{\epsilon_n}|>1$ and the imaginary part of $f(n)$ is small enough to be omitted. If we put $\cos^2(\cdots)$ equal to unity in $f(n)$, and considering $\beta_0^2|\sqrt{\epsilon_n}|\gtrsim 1$, $f(n)$ is reduced to

$$f(n)\simeq\frac{1}{\pi}\sum_l\left(1+\frac{l+\frac{1}{2}}{n+l}\right)\frac{1}{\sqrt{2l(n\beta_0\sqrt{\epsilon_n}-n-2l-\frac{1}{2})}}\simeq\frac{1}{4},$$

where the second term in brackets is neglected and the sum is replaced by the integral.

After all, we get the following expression,

$$-\frac{dW}{dt}\bigg|_{\check{C}er}=\frac{q^2v_0}{r_0^2}\sum_{\substack{n\\ \beta_0|\sqrt{\epsilon_n}|>1}} n\left(\beta_0^2-\frac{\epsilon_{1n}}{|\epsilon_n|^2}\right)(1-g(n)), \tag{5·10}$$

where $g(n)$ represents the effect of interference and is given by

$$g(n)=\frac{4}{\pi}\sum_l\frac{\sin^2\left(\frac{2n+4l+1}{6}\gamma^3_{n+2l}-\frac{\pi}{4}\right)}{\sqrt{l(l+n)}\sqrt{\left(\frac{n\beta_0\sqrt{\epsilon_n}}{n+2l+\frac{1}{2}}\right)^2-1}}.$$

Replacing the sum over n by the integral, we have

$$-\frac{dW}{dt}\bigg|_{\check{C}er}=\frac{q^2}{v_0}\int_{\check{C}er.}\left(\beta_0^2-\frac{\epsilon_{1\omega}}{|\epsilon_\omega|^2}\right)(1-g(\omega/\omega_0))\,\omega\, d\omega, \tag{5·11}$$

where $g(\omega/\omega_0)$ is much less than unity for high frequencies. On account of the poor approximation, however, the low frequency part of the spectrum given by (5·10) or (5·11) is not valid. As is mentioned before, the Čerenkov radiation with rather low frequency will be weak or impossible, because of the destructive effect of interference (see the Appendix).

IV) Angular distribution of radiation

As will be shown in the Appendix, most of the synchrotron radiation is concentrated within an angle $\Theta\simeq\pm\sqrt{1-\beta_0^2\epsilon}$, where $\Theta=\pi/2-\theta$. On the other hand, the angular distribution of the Čerenkov radiation is almost like a δ function at $\theta=\cos^{-1}(1/\beta_0\sqrt{\epsilon})$ Fig. 3 and Fig. 4 schematically represent the angular distributions of these two radiations.

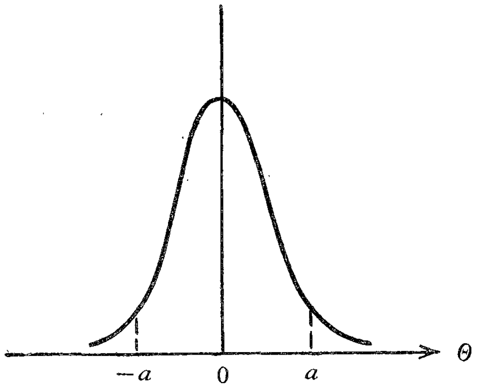

Fig. 3. The angular distribution of the synchrotron radiation. $a \simeq \sqrt{1-\beta_0^2 \epsilon}$.

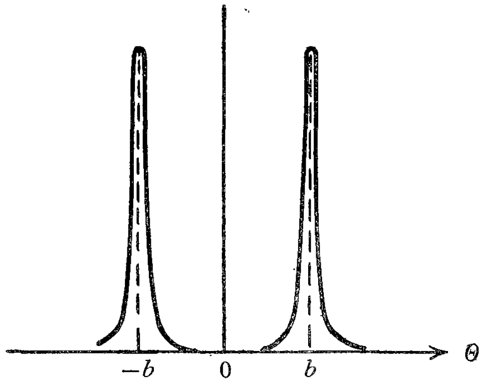

Fig. 4. The angular distribution of the Čerenkov radiation. $b \simeq \cos^{-1}(1/\beta_0 \sqrt{\epsilon})$.

In a dense medium, the two angular distributions may rather be of great difference; in a rarefied gas, however, they may practically be indistinguishable.

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Appendix

The next treatment follows the method of reference 7). The fields of a gyrating charged particle in the wave zone are given from (2·2) as

$$\left.\begin{aligned} -D_\varphi = H_\theta &= \frac{q\beta_0^2}{r_0 R} \sum_{n=-\infty}^{\infty} n\sqrt{\epsilon_n} e^{-in\gamma} J_n'(n\beta_0\sqrt{\epsilon_n}\sin\theta), \\ D_\theta = H_\varphi &= -\frac{iq\beta_0}{r_0 R} \cot\theta \sum_{n=-\infty}^{\infty} n e^{-in\gamma} J_n(n\beta_0\sqrt{\epsilon_n}\sin\theta), \end{aligned}\right\} \quad (\mathrm{A}\cdot 1)$$

where

$$\gamma = \omega_0 t + \frac{\pi}{2} - \varphi - \frac{\omega_0\sqrt{\epsilon_n}}{c} R.$$

If we neglect the absorption, namely ϵ_n is real, we get

$$\left.\begin{aligned} -D_\varphi = H_\theta &= \frac{2q\beta_0^2}{r_0 R} \sum_{n=1}^{\infty} n\sqrt{\epsilon_n} J_n'(n\beta_0\sqrt{\epsilon_n}\sin\theta)\cos n\gamma, \\ D_\theta = H_\varphi &= -\frac{2q\beta_0}{r_0 R} \cot\theta \sum_{n=1}^{\infty} n J_n(n\beta_0\sqrt{\epsilon_n}\sin\theta)\sin n\gamma. \end{aligned}\right\} \quad (\mathrm{A}\cdot 1)'$$

Thus the energy flow per unit time in a solid angle $d\Omega$ is reduced to

$$\left.\begin{aligned} dW &= \sum_{n=1}^{\infty} dW_n \\ dW_n &= \frac{q^2 n^2 \beta_0^2 c}{2\pi r_0^2 \sqrt{\epsilon_n}} [\cot^2\theta J_n^2(n\beta_0\sqrt{\epsilon_n}\sin\theta) + \beta_0^2 \epsilon_n J_n'^2(n\beta_0\sqrt{\epsilon_n}\sin\theta)] d\Omega, \end{aligned}\right\} \quad (\mathrm{A}\cdot 2)$$

where the time average over a period of motion has been taken. Therefore the radiation energy of a frequency $n\omega_0$ per unit time is

$$W_n = \frac{q^2 \beta_0 c n}{r_0^2 \epsilon_n} \left[2\beta_0^2 \epsilon_n J'_{2n}(2n\beta_0\sqrt{\epsilon_n}) + (\beta_0^2 \epsilon_n - 1) \int_0^{2n\beta_0\sqrt{\epsilon_n}} J_{2n}(x)\, dx \right]. \quad (\mathrm{A}\cdot 3)$$

1. Synchrotron radiation ($\beta_0^2 \epsilon_n < 1$)

Neglecting the dispersion, and replacing ϵ_n by a certain average ϵ, the following spectrum is obtained:

$$\left.\begin{aligned} &\text{if } \omega < \omega_c, \\ &\qquad W_n\, dn = \frac{3\sqrt{3}}{2^{4/3}\pi} \frac{q^2 c\beta_0}{r_0^2 \epsilon} \frac{\Gamma(2/3)}{(1-\beta_0^2 \epsilon)^2} y^{1/3} dy, \\ &\text{if } \omega > \omega_c, \\ &\qquad W_n\, dn = \frac{3\sqrt{3}}{4\sqrt{2\pi}} \frac{q^2 c\beta_0}{r_0^2 \epsilon} \frac{1}{(1-\beta_0^2 \epsilon)^2} y^{1/2} e^{-y} dy, \end{aligned}\right\} \quad (\mathrm{A}\cdot 4)$$

where $\omega_c = n_c\omega_0$, $y = n/n_c$ and $n_c = 3/2 \cdot (1-\beta_0^2)^{-3/2}$.

2. Čerenkov radiation ($\beta_0^2 \epsilon_n > 1$)

Using the recurrence formula, and considering the relation

$$\int_0^{\infty} J_\nu(x)\, dx = 1,$$

(A·3) may be rewritten as follows,

$$W_n = \frac{q^2 c\beta_0 n}{r_0^2 \epsilon_n} (\beta_0^2 \epsilon_n - 1) \left[1 - \int_{2n\beta_0\sqrt{\epsilon_n}}^{\infty} J_{2n}(x)\, dx + \frac{2\beta_0\sqrt{\epsilon_n}}{\beta_0^2 \epsilon_n - 1} \{ J_{2n}(2n\beta_0\sqrt{\epsilon_n}) - \beta_0\sqrt{\epsilon_n} J_{2n+1}(2n\beta_0\sqrt{\epsilon_n}) \} \right].$$

For large n, being approximately

$$J_{2n}(2n\beta_0\sqrt{\epsilon_n}) \simeq J_{2n+1}(2n\beta_0\sqrt{\epsilon_n}),$$

we have

$$W_n = \frac{q^2 c\beta_0 n}{r_0^2 \epsilon_n} (\beta_0^2 \epsilon_n - 1) \left[1 - \int_{2n\beta_0\sqrt{\epsilon_n}}^{\infty} J_{2n}(x)\, dx - \beta_0\sqrt{\epsilon_n} J_{2n}(2n\beta_0\sqrt{\epsilon_n}) \right], \quad (\mathrm{A}\cdot 5)$$

where we have used an approximation

$$2(1-\beta_0\sqrt{\epsilon_n})\simeq 1-\beta_0^2\epsilon_n.$$

The second and third terms in brackets represent the destructive effect of interference and may be neglected in the case of $n\gg 1$. Hence, when the wavelength is much smaller than the radius of the orbit, the same spectrum as for the linear motion is obtained.

3. Angular distribution

Neglecting the dispersion, and putting ϵ_n equal to ϵ, the radiation energy emitted in a solid angle $d\Omega$ per unit time is given by

$$dW=\frac{\beta_0^2\epsilon}{4(1-\beta_0^2\epsilon\sin^2\theta)^{7/2}}\left[1+\cos^2\theta-\frac{\beta_0^2\epsilon}{4}(1+3\beta_0^2\epsilon)\sin^4\theta\right]d\Omega. \quad (A\cdot 6)$$

Thus it is necessary that

$$1-\beta_0^2\epsilon\sin^2\theta\geq 0.$$

a) Čerenkov radiation

If we put $\theta=\pi/2+\Theta$, we get the condition

$$\cos\Theta\leq\frac{1}{\beta_0\sqrt{\epsilon}}.$$

The maximum intensity is evidently lying at an angle given by

$$\cos\Theta=\frac{1}{\beta_0\sqrt{\epsilon}}, \quad (A\cdot 7)$$

which is nothing but the Čerenkov relation. As is easily seen from (A·6), the angular distribution is almost like a δ function.

b) Synchrotron radiation

The most part of the radiation is emitted within an angle

$$\Theta\simeq\sqrt{1-\beta_0^2\epsilon}. \quad (A\cdot 8)$$

Thus the angular distributions of these two radiations may be schematically shown by Fig. 3 and Fig. 4.

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