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ENERGY-MOMENTUM CONSERVATION IN GRAVITY THEORIES*

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ABSTRACT

We discuss general properties of the conservation law associated with a local symmetry. Using Noether's theorem and a generalized Belinfante symmetrization procedure in 3+1 dimensions, a symmetric energy-momentum (pseudo) tensor for the gravitational Einstein-Hilbert action is derived and discussed in detail. In 2+1 dimensions, expressions are obtained for energy and angular momentum arising in the $ISO(2,1)$ gauge theoretical formulation of Einstein gravity. In addition, an expression for energy in a gauge theoretical formulation of the string-inspired 1+1 dimensional gravity is derived and compared with the ADM definition of energy.

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I. INTRODUCTION

The definition of energy and momentum in general relativity has been under investigation for a long time. The problem is to find an expression that is physically meaningful and related to some form of continuity equation,

$$\partial_\mu j^\mu = 0 \tag{1}$$

which leads to a conserved quantity,

$$Q = \int_V dV\, j^0 \tag{2}$$

provided $\int_{\partial V} dS^i j^i$ vanishes at infinity. Therefore to insure conservation of Q, j^i has to satisfy suitable boundary conditions. In other words, to get a conserved quantity from a continuity equation we always need to specify asymptotic behavior.

In field theory, conservation equations are usually related to invariance properties of the action, in which case the conserved current is called a *Noether current*. The Einstein-Hilbert action is invariant under diffeomorphisms, which are local transformations; more specifically, it is invariant under Poincaré transformations, which comprise special diffeomorphisms and can be viewed as "global" transformations.

In 3+1 dimensions, asymptotically Minkowski boundary conditions can be posed, so that we can associate energy, momentum and angular momentum with the Noether charges of global Poincaré transformations. To express the angular momentum solely in terms of an energy-momentum (pseudo) tensor, the energy-momentum (pseudo) tensor needs to be symmetric under interchange of two spacetime indices. Our goal is thus to find an expression for the symmetric energy-momentum (pseudo) tensor, which is conserved as in (1), which is given by the Noether procedure rather than by manipulation of the field equations of motion, and which is derived without any statement about "background" or "asymptotic" metric tensors.

In the 2+1 dimensional Einstein gravity, asymptotically Minkowski boundary conditions are not valid.[1] On the other hand, there is a gauge theoretical formulation of the theory,[2] based on the Poincaré group [$ISO(2,1)$]. The Noether charges associated with the Poincaré group gauge transformations are identified as energy and angular momentum.

In 1+1 dimensions, we consider a gauge theoretical formulation[3] of the string-inspired gravity model[4] and obtain an expression for energy arising from the gauge transformations. Another way of finding an expression for energy in 1+1 dimensions is to use the ADM definition;[5] we compare these two approaches.

In Section II, we analyze in a systematic way general properties of the Noether charge associated with a local symmetry and also symmetrization of the energy-momentum tensor ("improvement").

In Section III, the 3+1 dimensional Einstein-Hilbert action is investigated and a symmetric energy-momentum (pseudo) tensor, as an improved Noether current, is derived and compared with other definitions that have appeared in the literature. Also we remark on the conserved Noether current associated with diffeomorphism invariance.

Since asymptotically Minkowski boundary conditions can not be imposed in 2+1 dimensional Einstein gravity, we obtain in Section IV expressions for energy and angular momentum in the context of the gauge theoretical formulation for the theory.

In Section V, we consider a gauge theoretical formulation of 1+1 dimensional gravity. After getting an expression for energy, we show that it agrees with the ADM energy.

Concluding remarks comprise the final Section VI.

II. CONSERVATION LAWS

The Noether current associated with a local symmetry can always be brought to a form that is identically conserved. This was shown by E. Noether,[6] but unlike the construction of conserved currents associated with a global symmetry, her argument has not found its way into field theory textbooks — so we give a general proof in the Appendix.

To illustrate the result in a special example, let us consider the Maxwell-scalar system, with a Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^{\mu}\phi)^{*}D_{\mu}\phi \tag{3}$$

where $D_{\mu}\phi \equiv (\partial_{\mu} + ieA_{\mu})\phi$ and $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The Lagrangian $\mathcal{L}$ is invariant under a local $U(1)$ gauge symmetry.

$$\begin{aligned} \phi'(x) &= e^{-ie\theta(x)}\phi(x), \ \ \delta\phi = -ie\theta\phi \\ A'_{\mu}(x) &= A_{\mu}(x) + \partial_{\mu}\theta(x), \ \ \ \delta A_{\mu} = \partial_{\mu}\theta \end{aligned} \tag{4}$$

The associated Noether current

$$\begin{aligned} j^{\mu} &= \frac{\partial\mathcal{L}}{\partial\partial_{\mu}A_{\nu}}\delta A_{\nu} + \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_{\mu}\phi^{*}}\delta\phi^{*} \\ &= -F^{\mu\nu}\partial_{\nu}\theta - (D^{\mu}\phi)^{*}ie\theta\phi + D^{\mu}\phi ie\theta\phi^{*} \end{aligned} \tag{5}$$

can be written with use of the equation of motion

$$\partial_{\nu}F^{\nu\mu} = 2e\,\mathrm{Im}\left[(D^{\mu}\phi)^{*}\phi\right] \tag{6}$$

as

$$j^{\mu} = \partial_{\nu}\left(F^{\nu\mu}\theta\right) \tag{7}$$

which is certainly identically conserved, regardless whether $F^{\mu\nu}$ satisfies the field equations, since the quantity in the parenthesis of (7) is antisymmetric under the interchange of the indices μ and ν.

The Noether charge is constructed as a volume integral of the time component j^{0}.

$$Q = \int_{V} dV \partial_{i}\left[F^{i0}(x)\theta(x)\right] = \int_{\partial V} dS^{i}F^{i0}(x)\theta(x) \tag{8}$$

Without suitable boundary conditions, this charge either diverges or vanishes, and in general does not lead to a conserved quantity. Moreover, even if we get a finite value for Q with some $\theta(x)$, the time dependence of Q is completely determined by the specified boundary condition.

An example of boundary conditions for (7) is

$$\begin{aligned} F^{0i} &\sim o(\frac{1}{r^2}) \\ \partial_0 F^{0i} &\sim o(\frac{1}{r^3}) \end{aligned} \qquad \text{as } r \rightarrow \infty \tag{9}$$

The first condition gives finite Q when θ is constant at infinity, and the second condition ensures that Q is time independent. The asymptotic condition that θ be constant can be extended through all space, thereby arriving at a Noether formula for the total charge arising from a global transformation.

Next, let us review the symmetrization procedure of the energy-momentum tensor which was originally presented by Belinfante,[7] and which is always available in a Poincaré invariant theory. Here, we generalize his method to the case that the Lagrangian contains second derivatives, as is true of the Einstein-Hilbert action.

Thus, consider

$$I = \int_\Omega dx\, \mathcal{L}\,(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) \tag{10}$$

where ϕ is a multiplet of fields, and suppose I is invariant under Poincaré transformations. Under the infinitesimal action of these transformations, coordinates and fields transform respectively by

$$\begin{aligned} x^\mu \rightarrow x'^\mu &= x^\mu - \epsilon^\mu{}_\nu x^\nu - a^\mu \\ \phi(x) \rightarrow \phi'(x') &= L\,\phi(x) \end{aligned} \tag{11}$$

where L, containing the spin matrix S ($L = 1 + \frac{1}{2}\epsilon^\mu{}_\nu S^\nu{}_\mu$), is a representation of the Lorentz group and the constants $\epsilon^\mu{}_\nu$, $S^\nu{}_\mu$ satisfy the following relations.

$$\begin{aligned} &\epsilon^0{}_0 = S^0{}_0 = 0 \\ &\epsilon^0{}_i = \epsilon^i{}_0,\ S^0{}_i = S^i{}_0 \\ &\epsilon^i{}_j = -\epsilon^j{}_i,\ S^i{}_j = -S^j{}_i \end{aligned} \tag{12}$$

To derive the Noether current, let us consider the variation of the Lagrange density under the transformations (11),

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\,\partial_\mu \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi}\,\partial_\mu \partial_\nu \delta\phi \tag{13}$$

where $\delta\phi$ denotes $\phi'(x) - \phi(x)$. Since the action is Poincaré invariant by hypothesis, $\delta\mathcal{L}$ can be written as a total derivative without using the equations of motion.

$$\begin{aligned} &\delta\mathcal{L} = \partial_\mu\,(f^\mu \mathcal{L}) \\ &f^\mu \equiv \epsilon^\mu{}_\nu\, x^\nu + a^\mu \end{aligned} \tag{14}$$

On the other hand, using the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} = 0 \tag{15}$$

we can rewrite (13) as a total derivative.

$$\delta \mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \partial_\nu \delta\phi - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \delta\phi \right] \tag{16}$$

Equating the above two expressions for $\delta\mathcal{L}$, (14) and (16), we arrive at a conservation equation.

$$\partial_\mu \left[-f^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \partial_\nu \delta\phi - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \delta\phi \right] = 0 \tag{17}$$

Inserting now the variation, see (11),

$$\delta\phi = f^\mu \partial_\mu \phi + \frac{1}{2} \epsilon^\mu{}_\nu S^\nu{}_\mu \, \phi \tag{18}$$

into (17), we get

$$\partial_\mu \left[f^\alpha \Theta_C{}^\mu{}_\alpha + \frac{1}{2} \epsilon^\alpha{}_\beta \, L^{\mu\beta}{}_\alpha \right] = 0 \tag{19}$$

where $\Theta_C{}^\mu{}_\alpha$ is the unsymmetric, canonical energy-momentum tensor,

$$\Theta_C{}^\mu{}_\alpha = -\delta^\mu_\alpha \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\alpha \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \partial_\nu \partial_\alpha \phi - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \partial_\alpha \phi \tag{20}$$

and

$$\begin{aligned}
L^{\mu 0}{}_0 &= 0 \\
L^{\mu i}{}_0 &= L^{\mu 0}{}_i \\
&= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \, S^i{}_0 \, \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \, S^i{}_0 \, \partial_\nu \phi - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \, S^i{}_0 \phi + \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_0 \phi} \, \partial_i \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_i \phi} \, \partial_0 \phi \right) \\
L^{\mu i}{}_j &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \, S^i{}_j \phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \, S^i{}_j \partial_\nu \phi - \partial_\nu \, \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} \, S^i{}_j \phi - \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_j \phi} \, \partial_i \phi - \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_i \phi} \, \partial_j \phi \right)
\end{aligned} \tag{21}$$

As it is seen, Lorentz invariance of the action needs no reference to a background Minkowski metric. But formulas (12) and (21) can be presented compactly by moving indices with the help of the flat metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$. Thus with the definitions $\epsilon_{\alpha\beta} = \eta_{\alpha\mu} \epsilon^\mu{}_\beta$, $S^{\alpha\beta} = S^\alpha{}_\mu \eta^{\mu\beta}$ and $L^{\mu\alpha\beta} = L^{\mu\alpha}{}_\nu \eta^{\nu\beta}$, we see that the newly defined quantities are antisymmetric in α and β.

Next we define $h^{\mu\beta\alpha}$ as

$$h^{\mu\beta\alpha} = \frac{1}{2} \left[L^{\mu\beta\alpha} - L^{\alpha\beta\mu} - L^{\beta\alpha\mu} \right] \tag{22}$$

so that it is antisymmetric in μ and α and $\frac{1}{2}\epsilon_{\alpha\beta}L^{\mu\beta\alpha}$ is identical to $\epsilon_{\alpha\beta}h^{\mu\beta\alpha}$. Using these properties of $h^{\mu\beta\alpha}$ and $\epsilon_{\alpha\beta} = -\partial_\alpha f_\beta$ [β is lowered with $\eta_{\alpha\beta}$], we finally get

$$\partial_\mu \left[f_\alpha \left(\Theta_C{}^{\mu\alpha} + \partial_\nu h^{\mu\alpha\nu} \right) \right] = 0 \tag{23}$$

[α is raised with $\eta^{\alpha\beta}$.] Upon taking $f_\alpha = a_\alpha$, we arrive at the conserved energy-momentum tensor.

$$\Theta^{\mu\nu} = \Theta_C{}^{\mu\nu} + \partial_\alpha h^{\mu\nu\alpha} \tag{24}$$

To prove that $\Theta^{\mu\nu}$ is symmetric, take f_α to be $\epsilon_{\alpha\beta}x^\beta$. Since (23) holds for arbitrary antisymmetric $\epsilon_{\alpha\beta}$, it follows that

$$\partial_\mu \left[x^\alpha \Theta^{\mu\nu} - x^\nu \Theta^{\mu\alpha} \right] = 0 \tag{25}$$

The conservation law $\partial_\mu \Theta^{\mu\nu} = 0$ and (25) imply that $\Theta^{\mu\nu}$ is symmetric.

In conclusion, we have derived an expression for a conserved and symmetric energy-momentum tensor for a Poincaré invariant theory whose action may contain second derivatives.

III. GRAVITATIONAL ENERGY-MOMENTUM (PSEUDO) TENSOR IN 3+1 DIMENSIONS

The Einstein-Hilbert action**

$$I = -\frac{1}{16\pi k} \int d^4x \sqrt{-g} R + I_M \tag{26}$$

where k is the gravitational coupling, R the scalar curvature, and I_M denotes a matter action, can be put into a form involving only first derivatives through an integration-by-part of terms involving the second derivatives. The explicit form of the first-derivative action is

$$\bar{I} = -\frac{1}{16\pi k} \int d^4x \sqrt{-g} G + I_M \tag{27}$$

where G is given in terms of the Christoffel connections Γ.

$$G = g^{\mu\nu} \left(\Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} \right) \tag{28}$$

To be specific, let us take the matter action for a massless scalar.

$$I_M = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{29}$$

The same equation of motion follows from I and $\bar{I}$,

$$8\pi k \, T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{30}$$

** Hereafter we use conventions of Landau and Lifschitz, see Ref. [8].

where $T_{\mu\nu}$ is the matter energy-momentum tensor $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I_M}{\delta g^{\mu\nu}}$.

Although the action $\bar{I}$ is conventional in that only first derivatives occur, its integrand G is no longer a scalar. However, both I and $\bar{I}$ are invariant under Poincaré transformations. Therefore we can use the generalized Belinfante method to find an expression for the symmetric energy-momentum (pseudo) tensor arising both from I and $\bar{I}$.

Note that the spin matrix for the metric field $g_{\mu\nu}$ is given by

$$\left(S^{\alpha\beta} g\right)_{\mu\nu} = \left(\eta^{\beta\kappa}\delta^{\alpha}_{\mu} - \eta^{\alpha\kappa}\delta^{\beta}_{\mu}\right) g_{\kappa\nu} + \left(\eta^{\beta\kappa}\delta^{\alpha}_{\nu} - \eta^{\alpha\kappa}\delta^{\beta}_{\nu}\right) g_{\mu\kappa} \tag{31}$$

After some straightforward calculations following the generalized Belinfante method and with a bit more algebra using the equations of motion, we are led to the following symmetric energy-momentum (pseudo) tensor from both I and $\bar{I}$.

$$\Theta^{\mu\nu} = \frac{1}{16\pi k}\partial_\alpha\partial_\beta \left[\sqrt{-g}\left(\eta^{\mu\nu}g^{\alpha\beta} - \eta^{\alpha\nu}g^{\mu\beta} + \eta^{\alpha\beta}g^{\mu\nu} - \eta^{\mu\beta}g^{\alpha\nu}\right)\right] \tag{32}$$

Let us compare the above result to other formulas for the symmetric energy-momentum (pseudo) tensor found in the literature. Although there are many expressions for the gravitational energy and momentum,[8,9,10] there seem to be only two for a *symmetric* energy-momentum (pseudo) tensor. These are obtained by manipulating the Einstein field equation. The first one is discussed by Landau and Lifshitz[8]

$$\Theta'^{\mu\nu} = \frac{1}{16\pi k}\,\partial_\alpha\partial_\beta \left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\alpha\nu}g^{\mu\beta})\right] \tag{33}$$

and the other by Weinberg[10]

$$\Theta''^{\mu\nu} = \frac{1}{16\pi k}\partial_\alpha\partial_\beta \left[\eta^{\mu\nu}A^{\alpha\beta} - \eta^{\alpha\nu}A^{\mu\beta} + \eta^{\alpha\beta}A^{\mu\nu} - \eta^{\mu\beta}A^{\alpha\nu}\right] \tag{34}$$

where $A^{\alpha\beta} = -h^{\alpha\beta} + \frac{1}{2}\eta^{\alpha\beta}\, h^{\gamma}{}_{\gamma}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and indices are raised and lowered with the flat metric.

Although obtained by totally different methods, Θ' and Θ'' agree with Θ in (32) up to first order in h. The difference between Θ' and Θ is

$$\Theta'^{\mu\nu} - \Theta^{\mu\nu} = \frac{1}{16\pi k}\partial_\alpha\partial_\beta \left[l^{\mu\nu}l^{\alpha\beta} - l^{\mu\beta}l^{\alpha\nu}\right] \sim o(h^2) \tag{35}$$

where $l^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$; while the difference between Θ'' and Θ is

$$\Theta''^{\mu\nu} - \Theta^{\mu\nu} = \frac{1}{16\pi k}\partial_\alpha\partial_\beta \left[\eta^{\mu\nu}B^{\alpha\beta} - \eta^{\alpha\nu}B^{\mu\beta} + \eta^{\alpha\beta}B^{\mu\nu} - \eta^{\mu\beta}B^{\alpha\nu}\right] \sim o(h^2) \tag{36}$$

where $B^{\alpha\beta} = -h^{\alpha\beta} + \frac{1}{2}\eta^{\alpha\beta}\, h^{\gamma}{}_{\gamma} - \sqrt{-g}\, g^{\alpha\beta}$.

The corresponding expression for the energy derived from Θ is at order h

$$E = \int d^3r\ \Theta^{00} = \frac{1}{16\pi k}\int dS^i\left[\partial_i h_{jj} - \partial_j h_{ij}\right] + o(h^2) \tag{37}$$

while the angular momentum reads

$$
\begin{aligned}
J_{ij} &= \int d^3r \, (x^i \Theta^{0j} - x^j \Theta^{0i}) \\
&= \frac{1}{16\pi k} \int dS^k \left[(x^i \partial_0 h_{jk} - x^i \partial_k h_{0j} + \delta_{ki} h_{0j}) - (i \leftrightarrow j) \right] + o(h^2)
\end{aligned}
\tag{38}
$$

We evaluate these expressions on a solution to the Einstein's equation with a rotating point source — the Kerr solution — whose line element has the following large r asymptote.

$$
\begin{aligned}
ds^2 = &\left(1 - \frac{2km}{r} + \mathrm{o}(r^{-2})\right) dt^2 - \left(4kJ\epsilon_{ij3}\frac{x^j}{r^3} + \mathrm{o}(r^{-3})\right) dx^i dt \\
&- \left(1 + \frac{2km}{r} + \mathrm{o}(r^{-2})\right) dx^i dx^i
\end{aligned}
\tag{39}
$$

We find $E = m$ and $J_{ij} = J\,\epsilon_{ij3}$. [$E$, E' and E'' (similarly J_{ij}, J'_{ij} and J''_{ij}) could be different from one another, if the order h terms in (37) and (38) vanish and the terms of order h^2 survive; this of course does *not* happen for the Kerr solution.]

We conclude this Section with comments on several other applications of Noether's method to general relativity. Observe that the action I is diffeomorphism invariant– a symmetry certainly bigger than the (global) Poincaré symmetry. We are naturally led to inquire what is the conserved current associated with this diffeomorphism invariance. From (17), we can read off the expression for the Noether current associated with the diffeomorphism $\delta x^\mu = -f^\mu(x)$, where f^μ is an arbitrary function of x.

$$
j^\mu_f = -f^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi}\partial_\nu \delta\phi - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi}\delta\phi
\tag{40}
$$

Starting from the action I and after some straightforward calculations, we get

$$
\frac{1}{\sqrt{-g}} j^\mu_f = T^\mu{}_\nu f^\nu + \frac{1}{16\pi k}\left[f^\mu R - D_\nu \left(D^\mu f^\nu + D^\nu f^\mu - 2g^{\mu\nu} D_\alpha f^\alpha \right) \right]
\tag{41}
$$

Using the equation of motion (30) and the relation $[D_\nu, D^\mu] f^\nu = R^\mu{}_\nu f^\nu$, we get a remarkably simple expression,

$$
\frac{1}{\sqrt{-g}} j^\mu_f = \frac{1}{16\pi k} D_\nu \left[D^\mu f^\nu - D^\nu f^\mu \right]
\tag{42}
$$

which was first given by Komar[9] *** and is extensively discussed in the literature. In spite of the simple and appealing formula (42) for the current, we encounter the following difficulty in attempting to use it in a definition of energy. For $f^\mu = \delta^\mu_0$, $E_N = \int d^3r \, j^0_f$ gives only half of the expected energy for the Kerr solution (39) [note that E_N is not obtained from a symmetric

*** Komar's formula is actually twice of (42). Presumably, he reached his expression by guesswork, so he did not obtain the factor $\frac{1}{2}$, which comes from the normalization of the action. Later, P. G. Bergmann[11] derived (42) with the correct factor.

tensor while E in (37) is]. But we can not simply "renormalize" j_f^μ by a factor of two and get universal agreement with previous formulas. This is because if we construct the angular momentum generator from (42) $\int d^3r\, j_f^0$, $f^i = \epsilon^i{}_j x^j$, the expression agrees with that from (38) at order h, and gives the correct answer in the Kerr case. The resolution of this problem is known.[12] One needs to supplement the Einstein-Hilbert action I with a surface term, I_s, which however is not diffeomorphism invariant, but respects some restricted symmetry group, *e.g.* the Poincaré group. One then applies Noether's theorem to $I + I_s$, but of course one is no longer discussing arbitrary diffeomorphisms, but rather the restricted invariances of $I + I_s$. The resulting constants of motion no longer arise from a locally conserved current, since they include a contribution from I_s. With this procedure one can supplement the Komar expression and arrive at the accepted values of energy and angular momentum.

Note that our use of the Belinfante-improved Noether method yields the same symmetric (pseudo) tensor (32), whether or not a surface term is included (*i.e.* whether I or $\bar{I}$ are used).

The action in (26) may alternatively be presented in first-order, Palatini form,

$$\begin{aligned} I =& \frac{1}{64\pi k}\int d^4x \epsilon^{\alpha\beta\gamma\delta}\epsilon_{ABCD} e_\alpha^A e_\beta^B R_{\gamma\delta}^{CD} \\ R_{\gamma\delta}^{CD} \equiv& \partial_\gamma \omega_\delta^{CD} + \omega_\gamma^{CE}\omega_{\delta E}{}^D - (\gamma \leftrightarrow \delta) \end{aligned} \tag{43}$$

where e_μ^A and ω_μ^{AB} are the *Vierbein* and the spin-connection. We assume that the *Vierbein* is invertible, with inverse E_A^μ and $e \equiv \det e_\mu^A$ equals $\sqrt{-g}$. Then, the scalar curvature is $E_A^\mu E_B^\nu R_{\mu\nu}^{AB}$. Since this action is invariant under Poincaré transformations, let us again construct the symmetric energy-momentum (pseudo) tensor. Noting that the spin matrices for e_μ^A and ω_μ^{AB} are given by

$$\begin{aligned} \left(S^{\beta\alpha}e^A\right)_\mu &= \left(\delta_\mu^\beta \eta^{\alpha\nu} - \delta_\mu^\alpha \eta^{\beta\nu}\right) e_\nu^A \\ \left(S^{\beta\alpha}\omega^{AB}\right)_\mu &= \left(\delta_\mu^\beta \eta^{\alpha\nu} - \delta_\mu^\alpha \eta^{\beta\nu}\right) \omega_\nu^{AB} \end{aligned} \tag{44}$$

and going through the generalized Belinfante procedure, one finds that the symmetric tensor vanishes. This is because the added superpotential, needed to symmetrize the nonsymmetric, canonical energy-momentum (pseudo) tensor, exactly cancels it.

IV. ENERGY AND ANGULAR MOMENTUM IN 2+1 DIMENSIONAL GRAVITY

We begin by considering the 2+1 dimensional Einstein-Hilbert action.

$$I = -\frac{1}{16\pi k}\int d^3x \sqrt{g} R \tag{45}$$

Because the Einstein and curvature tensors are equivalent, spacetime is flat outside sources. Therefore, all effects of localized sources are on the global geometry. In the presence of such global effects, spacetime is not asymptotically Minkowski. For example, we can solve the Einstein equation for a rotating point mass (string in 3+1 dimensions). The solution is described by the line element[1]

$$ds^2 = (dt + 4kJd\theta)^2 - dr^2 - (1-4km)^2 r^2 d\theta^2 \tag{46}$$

and there is no coordinate choice in which the asymptote is Minkowski spacetime. The best one can do is to make the line element "locally Minkowski",

$$ds^2 = d\tau^2 - dx'^2 - dy'^2 \tag{47}$$

through the redefinitions

$$\tau = t + 4kJ\theta, \quad x' = r\cos(1-4km)\theta, \quad y' = r\sin(1-4km)\theta \tag{48}$$

However, the angular range of $(1-4km)\theta$ is diminished to $(1-4km)2\pi$, while τ jumps by $8\pi kJ$ whenever the origin is circumnavigated. Such geometry is conical and not globally Minkowski.

For another viewpoint, let us consider this theory as the Poincaré $ISO(2,1)$ gauge theory of gravity.[2] Here, we can exploit the possibility of relating charges associated with gauge transformations to energy and angular momentum.

The commutation relations of the Poincaré $ISO(2,1)$ group are

$$\begin{aligned}&[P_A, P_B] = 0, \quad [J_A, J_B] = \epsilon_{AB}{}^C J_C\\&[J_A, P_B] = \epsilon_{AB}{}^C P_C\end{aligned} \tag{49}$$

where indices are raised or lowered by η_{AB}, and $\epsilon^{012} = 1$. In Poincaré invariant field theories, P_A's are interpreted as translation generators, J_0 is interpreted as angular momentum generator and the two J_i's as boosts.

If we introduce a connection one-form $A = e^A P_A + \omega^A J_A$, where e^A and ω^A are respectively the *Dreibein* and the spin-connection, the curvature two-form is given by

$$\begin{aligned}F &= dA + A^2\\&= (de^A + \epsilon^A{}_{BC}\omega^B e^C)P_A + (d\omega^A + \frac{1}{2}\epsilon^A{}_{BC}\omega^B\omega^C)J_A\end{aligned} \tag{50}$$

The Chern-Simons action for this connection is

$$I = \frac{1}{16\pi k}\int \langle A\, ,\ dA + \frac{2}{3}A^2\rangle \tag{51}$$

with $\langle\ ,\ \rangle$ denoting an invariant bilinear form in the algebra.

$$\langle J_A\ ,\ P_B\rangle = \eta_{AB}, \quad \langle P_A\ ,\ P_B\rangle = \langle J_A\ ,\ J_B\rangle = 0 \tag{52}$$

One verifies that (51) is a first order Palatini action equivalent to (45).

The generator of gauge transformations is also an element of the algebra: $\theta = \alpha^A P_A + \beta^A J_A$ with α^A and β^A being infinitesimal parameters. The variation of A under a gauge transformation is

$$\delta A = d\theta + [A, \theta] \tag{53}$$

Note that the Lagrange density in the action changes under the gauge transformation by a total derivative,

$$\delta \mathcal{L} = \partial_\mu X^\mu \tag{54}$$

where $X^\mu = \frac{1}{16\pi k} \epsilon^{\mu\nu\rho} \langle A_\nu \ , \ \partial_\rho \theta \rangle$. Therefore the Noether current associated with this gauge transformation is

$$j^\mu = \langle \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \ , \ \delta A_\nu \rangle - X^\mu \tag{55}$$

Using the equation of motion ($F = 0$), we get

$$j^\mu = \frac{1}{8\pi k} \epsilon^{\mu\nu\rho} \partial_\nu \langle A_\rho \ , \ \theta \rangle \tag{56}$$

which is an identically conserved current as expected and totally dependent on the choice of gauge function θ.

The solution to $F = 0$, which leads to (46), gives rise to the following *Dreibein* and spin-connection[13]

$$\begin{aligned} e^0 &= dt + \frac{4kJ}{r^2} \mathbf{r} \times d\mathbf{r} \\ \mathbf{e} &= (1 - 4km) d\mathbf{r} + \frac{4km}{r^2} \mathbf{r} \, (\mathbf{r} \cdot d\mathbf{r}) \\ \omega^0 &= \frac{4km}{r^2} \mathbf{r} \times d\mathbf{r} \\ \boldsymbol{\omega} &= 0 \end{aligned} \tag{57}$$

with $x^1 = r\cos\theta$ and $x^2 = r\sin\theta$. We inquire if "charges" coming from (55) and (56) could be identified as energy and angular momentum, with values m and J respectively on the solution (57).

To proceed we must choose a "global" form for θ in (56). A natural choice is to take θ to be 1 along the P^0 direction for defining energy and 1 along the J^0 direction for defining angular momentum. With this one finds

$$E \equiv \frac{1}{8\pi k} \oint dx^i \omega_i^0 = m \, , \quad J \equiv \frac{1}{8\pi k} \oint dx^i e_i^0 = J \tag{58}$$

Another choice for θ could be the following. We recall the relation between a diffeomorphism implemented by a Lie derivative on a gauge potential (connection) and a gauge transformation.[14]

$$\begin{aligned} \delta_f A_\mu = L_f A_\mu &= f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha \\ &= f^\alpha F_{\alpha\mu} + \partial_\mu (f^\alpha A_\alpha) + [A_\mu, f^\alpha A_\alpha] \end{aligned} \tag{59}$$

In this theory $F_{\alpha\mu}$ vanishes on shell, and it is natural to identify the gauge transformation generated by $f^\alpha A_\alpha$ with the infinitesimal diffeomorphism f^α. With this choice, we find

$$Q_f = \frac{1}{8\pi k} \oint dx^i \langle A_i \ , \ A_\alpha f^\alpha \rangle \tag{60}$$

which with (57), becomes

$$Q_f = \frac{1}{8\pi k} \oint dx^i \left[\omega_i^0 f^0 + \left(e_i^0 \omega_j^0 + \omega_i^0 e_j^0 \right) f^j \right] \tag{61}$$

For energy we take $f^0 = 1$ and $f^i = 0$, thereby again one finds

$$E = m \tag{62}$$

However for angular momentum, where $f^0 = 0$ and $f^i = \epsilon^{ij} x^j$, one gets $8kmJ$. We do not have an explanation for the dimensionless factor $8km$.

V. 1+1 DIMENSIONAL ENERGY IN GAUGE THEORETICAL FORMULATION

In 1+1 dimensions, the action of string-inspired gravity theory[4] can be written as

$$4\pi G \, I_g' = \int d^2x \sqrt{-g} \, (\eta R - \Lambda) \tag{63}$$

where the "physical" metric is $g_{\mu\nu}/\eta$ while R is the scalar curvature constructed from $g_{\mu\nu}$.

This theory is reformulated as a gauge theory using a centrally extended Poincaré group,[3] whose algebra is

$$[P_a, J] = \epsilon_a{}^b P_b, \quad [P_a, P_b] = \mathcal{B} \epsilon_{ab} I \tag{64}$$

The connection one-form A and the curvature two-form F are explicitly

$$\begin{aligned} A &= e^a P_a + \omega J + \mathcal{B} a I \\ F &= dA + A^2 \\ &= (De)^a \, P_a + d\omega J + \mathcal{B} \left(da + \frac{1}{2} e^a \epsilon_{ab} e^b \right) I \end{aligned} \tag{65}$$

where e^a and ω are the *Zweibein* and the spin-connection respectively. The action

$$4\pi G \, I_g = \int \sum_{A=0}^{3} \eta_A F^A = \int \left[\eta_a \, (De)^a + \eta_2 d\omega + \mathcal{B} \eta_3 \left(da + \frac{1}{2} e^a \epsilon_{ab} e^b \right) \right] \tag{66}$$

is equivalent to I_g' with $\eta_2 = 2\eta$ and is invariant under gauge transformations,

$$\begin{aligned} A &\rightarrow U^{-1} A U + U^{-1} dU \\ H &\rightarrow U^{-1} H U \end{aligned} \tag{67}$$

Here $H = \eta_a P^a - \frac{1}{B}\eta_3 J - \frac{1}{B}\eta_2 I$, and U is the gauge function $e^{\theta^a P_a} e^{\alpha J} e^{\beta I}$ with arbitrary local parameters $\theta^A = (\theta^a, \alpha, \beta)$. Note that the infinitesimal form of a gauge transformation on A is explicitly

$$\begin{aligned} \delta e^a &= -\epsilon^a{}_b \alpha e^b + \epsilon^a{}_b \theta^b \omega + d\theta^a \\ \delta\omega &= d\alpha \\ \delta a &= -\theta^a \epsilon_{ab} e^b + d\beta/\mathcal{B} \end{aligned} \tag{68}$$

The Noether current associated to this gauge transformation is

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial \partial_\mu e^a_\nu}\, \delta e^a_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \omega_\nu}\, \delta\omega_\nu + \frac{\partial \mathcal{L}}{\partial \partial_\mu a_\nu}\, \delta a_\nu \\ &= \frac{\partial \mathcal{L}}{\partial \partial_\mu A^A_\nu}\, \delta A^A_\nu \end{aligned} \tag{69}$$

Using the equations of motion and (68), we get

$$4\pi G\, j^\mu = \epsilon^{\mu\nu} \partial_\nu \left(\eta_a \theta^a + \eta_2 \alpha + \eta_3 \beta\right) = \epsilon^{\mu\nu} \partial_\nu \left(\eta_A \theta^A\right) \tag{70}$$

As anticipated, the current j^μ is identically conserved and again totally dependent on the choice of gauge functions.

Infinitesimal diffeomorphisms are performed on shell by a gauge transformation with gauge function $f^\alpha A_\alpha$ [cf. (59)].

$$\begin{aligned} \delta_f A_\mu &= \partial_\mu (f^\alpha A_\alpha) + [A_\mu, f^\alpha A_\alpha] \\ \delta_f H &= [H, f^\alpha A_\alpha] \end{aligned} \tag{71}$$

Therefore we get an expression for energy by taking in (70) $\theta = f^\alpha A_\alpha$ and $f^\mu = (1, 0)$.

$$4\pi G\, j^\mu_E = \epsilon^{\mu\nu} \partial_\nu \left(\eta_a e^a_0 + \eta_2 \omega_0 + \mathcal{B}\eta_3 a_0\right) \tag{72}$$

$$4\pi G\, E = \left(\eta_a e^a_0 + \eta_2 \omega_0 + \mathcal{B}\eta_3 a_0\right)\Big|_{-\infty}^{+\infty} \tag{73}$$

To see what happens with an explicit solution, let us consider the black hole solution in presence of infalling matter. The inclusion of matter is discussed in Ref. [15] and is described by the interacting action

$$I_m = -\int d\tau\, \left\{ \left(p_a \epsilon^a{}_b e^b{}_\mu + \mathcal{A}\omega_\mu + \mathcal{B} a_\mu\right) \dot{x}^\mu + \frac{1}{2} N (p^2 + m^2) \right\} \tag{74}$$

(a_μ is defined by $\epsilon^{\mu\nu}\partial_\mu a_\nu = \sqrt{-g}$). Notice the additional terms proportional to $\mathcal{A}$ and $\mathcal{B}$. Setting them to zero gives the usual action for a point particle of mass m in a background geometry, $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$. We present a solution to the classical equations of motion coming from $I_g + I_m$. The geometry is given by $\left[x^\mu = (\tau, \sigma)\right]$

$$e^a_\mu = e^{\lambda\sigma} \delta^a_\mu \quad , \quad a_0 = -\frac{1}{2\lambda} e^{2\lambda\sigma} \quad , \quad a_1 = 0 \quad , \quad \omega_0 = -\lambda \quad , \quad \omega_1 = 0 \quad . \tag{75a}$$

with an arbitrary parameter λ. The point particle trajectory is then

$$\sigma(\tau)+\tau=\frac{1}{\lambda}\log\left[-\frac{\sqrt{2}\lambda}{\mathcal{B}}\hat{p}_-+\frac{\left(\frac{\lambda m}{\mathcal{B}}\right)^2}{e^{-\lambda(\sigma(\tau)-\tau)}+\frac{\sqrt{2}\lambda}{\mathcal{B}}\hat{p}_+}\right] \tag{75b}$$

where $\hat{p}_+$ and $\hat{p}_-$ ($\hat{p}^+, -\hat{p}_- \geq 0$ if $\frac{\lambda}{\mathcal{B}}>0$) are the two independent constants of motion of the trajectory related to translation invariance of I_m. Finally the Lagrange multiplier is

$$\begin{aligned}
\eta_0 &= 4\lambda e^{\lambda\sigma}+4\pi G\,\theta(\sigma-\sigma(\tau))\left[\frac{\hat{p}_+}{\sqrt{2}}e^{\lambda\tau}+\frac{\hat{p}_-}{\sqrt{2}}e^{-\lambda\tau}+\frac{\mathcal{B}}{\lambda}e^{\lambda\sigma}\right]\\
\eta_1 &= 4\pi G\,\theta(\sigma-\sigma(\tau))\left[\frac{\hat{p}_+}{\sqrt{2}}e^{\lambda\tau}-\frac{\hat{p}_-}{\sqrt{2}}e^{-\lambda\tau}\right]\\
\eta_2 &= 2e^{2\lambda\sigma}+4\pi G\,\theta(\sigma-\sigma(\tau))\left[\mathcal{A}+\frac{2\hat{p}_+\hat{p}_--m^2}{2\mathcal{B}}+\frac{1}{\lambda}\left(\frac{\hat{p}_+}{\sqrt{2}}e^{\lambda\tau}+\frac{\hat{p}_-}{\sqrt{2}}e^{-\lambda\tau}+\frac{\mathcal{B}}{2\lambda}e^{\lambda\sigma}\right)e^{\lambda\sigma}\right]\\
\eta_3 &= \frac{4\lambda^2}{\mathcal{B}}+4\pi G\,\theta(\sigma-\sigma(\tau))
\end{aligned} \tag{75c}$$

Inserting the above solution into (73), we get the following value for the energy.

$$E=\lambda\left(\frac{m^2-2\hat{p}_+\hat{p}_-}{2\mathcal{B}}-\mathcal{A}\right) \tag{76}$$

Let us compare this result with the ADM definition of energy. To get an ADM energy, we rewrite the Lagrange density for the gravity sector as

$$\begin{aligned}
4\pi G\,\mathcal{L} &= \eta_a\dot{e}_1^a+\eta_2\dot{\omega}_1+\mathcal{B}\eta_3\dot{a}_1+e_0^a\left(\mathcal{B}\eta_3\epsilon_{ab}e_1^b+\omega_1\epsilon_a{}^b\eta_b+\eta_a'\right)\\
&+\ \omega_0\left(\epsilon^a{}_b\eta_a e_1^b+\eta_2'\right)+\mathcal{B}a_0\eta_3'-\left(\eta_a e_0^a+\eta_2\omega_0+\mathcal{B}\eta_3 a_0\right)'
\end{aligned} \tag{77}$$

where prime and dot denote derivatives on σ and τ respectively.

Note that the variation of $\mathcal{L}$ is

$$\delta\mathcal{L}=\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\phi}\right)\right]\delta\phi-\frac{1}{4\pi G}\left(\delta\eta_a e_0^a+\delta\eta_2\omega_0+\mathcal{B}\delta\eta_3 a_0\right)' \tag{78}$$

where ϕ denotes all the fields. To eliminate boundary contributions, we have to introduce in the action an appropriate boundary term and a boundary condition such that boundary variations cancel. The required boundary term is then identified as the energy. Let us take the boundary condition to be

$$A_0\to A_0\Big|_{\text{free}}\qquad \text{as}\ \ \sigma\to\pm\infty \tag{79}$$

where subscript 'free' denotes an empty space solution [(75a)]. The necessary boundary contributes to the Lagrange density a total derivative.

$$4\pi G\,\mathcal{L}_B=\left(\eta_a e_0^a\Big|_{\text{free}}+\eta_2\omega_0\Big|_{\text{free}}+\mathcal{B}\eta_3 a_0\Big|_{\text{free}}\right)' \tag{80}$$

Therefore the ADM energy is

$$4\pi G\,E_{ADM}=\left(\eta_a e^a\Big|_{\rm free}+\eta_2\omega_0\Big|_{\rm free}+\mathcal{B}\eta_3 a_0\Big|_{\rm free}\right)\Bigg|_{-\infty}^{+\infty} \tag{81}$$

which coincides with the expression (73) with the help of (79). Thus E agrees with E_{ADM}.

It can be shown that if we use the gauge invariance[15] of I_m, we get by the Noether procedure the constant of motion $C_m = p_a\epsilon^a{}_b\theta^b + \mathcal{A}\alpha + \beta$ and the corresponding energy on the solution (75), $E_m = p_a\epsilon^a{}_b e^b_0 + \mathcal{A}\omega_0 + \mathcal{B}a_0$, again takes the value E (76). The equality of these two expressions reflects energy balance: the same total energy can be computed either from the field or from the source. We remark that without matter there is no step function in the solution (75b) and expressions (73) and (81) vanish, thus giving no mass to the 'pure' black hole configuration contrary to what is usually argued.****

Finally note that if one takes θ to be a constant along the P^0 direction for defining energy [as we did in the 2+1 dimensional gauge gravity theory] and calculates the energy using the solution (75), one gets a diverging result.

VI. DISCUSSION

Noether's procedure for constructing conserved symmetry currents provides a universal method, which in particular may be used to derive energy-momentum (pseudo) tensors in various gravity theories. This allows for an *a priori*, symmetry-motivated approach to the problem, in contrast to the conventional construction, which relies on manipulating equations of motion, and which is motivated *a posteriori*, even while a variety of results emerges, reflecting the variety of possible manipulations on the equations of motion.[8,10] In gravity theories that are also gauge theories, as is possible for low dimensionality, the Noether method yields a gauge current from which energy and angular momentum can be reconstructed.

However, the Noether procedure is not without ambiguity. Since a local symmetry is operating, the symmetry current is a divergence of an antisymmetric tensor.[6] But the Noether method, which requires recognizing that the symmetry variation of a Lagrange density is a total derivative, $\delta\mathcal{L} = \partial_\mu X^\mu$, leaves an undetermined contribution to X^μ, which also is the divergence of an antisymmetric tensor. Moreover, as we have seen, a variety of conserved currents may be derived, depending on whether one uses Einstein-Hilbert or Palatini formulations, whether the coordinate invariance is viewed as diffeomorphisms of geometrical variables or a gauge transformations on gauge connections. Finally observe that our expressions are neither diffeomorphism nor gauge invariant. At the same time, in all instances one Noether tensor gives the "correct" integrated expressions.

**** See for example, de Alwis in Ref. [5].

APPENDIX

The Noether current associated with a local symmetry can always be brought in a form that is identically conserved. This theorem is proved in Ref. [6], but the form of the current in terms of the Lagrange density is not given. Here we present the current explicitly for the case that the Lagrange density contains at most first derivatives of fields and the symmetry variation of the fields does not depend on second or higher derivatives of the parameter functions.

Let us consider an action for a field multiplet ϕ,

$$I = \int_\Omega dx \mathcal{L}(\phi, \partial\phi) \tag{A1}$$

which is invariant under a local transformation,

$$\delta\phi = \Delta_A \theta^A + \Delta^\mu_A \partial_\mu \theta^A \tag{A2}$$

where Δ_A may depend on ϕ, and θ^A is a gauge parameter function. First we note the Noether identity

$$\frac{\delta I}{\delta\phi}\Delta_A - \partial_\mu\left(\frac{\delta I}{\delta\phi}\Delta^\mu_A\right) = 0 \tag{A3}$$

where $\frac{\delta I}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi}\right)$. Eq. (A3) is a consequence of the invariance of the action against the transformation (A2) with arbitrary θ^A. $\delta\mathcal{L}$ for arbitrary θ^A reads

$$\delta\mathcal{L} = M_A\theta^A + M^\mu_A\partial_\mu\theta^A + M^{\mu\nu}_A\partial_\mu\partial_\nu\theta^A \tag{A4}$$

where

$$\begin{aligned}
M_A &= \frac{\partial\mathcal{L}}{\partial\phi}\Delta_A + \frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi}\partial_\alpha\Delta_A \\
M^\mu_A &= \frac{\partial\mathcal{L}}{\partial\phi}\Delta^\mu_A + \frac{\partial\mathcal{L}}{\partial\partial_\alpha\phi}\left(\partial_\alpha\Delta^\mu_A + \delta_\alpha{}^\mu\Delta_A\right) \\
M^{\mu\nu}_A &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Delta^\nu_A \equiv M^{(\mu\nu)}_A - \mathcal{F}^{\mu\nu}_A
\end{aligned} \tag{A5}$$

In the last equation $M^{\mu\nu}_A$ is decomposed into its symmetric $[M^{(\mu\nu)}_A]$ and antisymmetric $[-\mathcal{F}^{\mu\nu}_A]$ parts. Using the Noether identity, one easily finds the relation,

$$M_A = \partial_\mu M^\mu_A - \partial_\mu\partial_\nu M^{(\mu\nu)}_A \tag{A6}$$

which shows that $\delta\mathcal{L}$ can be presented as a total derivative, since the transformation (A2) *is* a symmetry.

$$\begin{aligned}
\delta\mathcal{L} &= \partial_\mu(M^\mu_A\theta^A) + M^{(\mu\nu)}_A\partial_\mu\partial_\nu\theta^A - \partial_\mu\partial_\nu M^{(\mu\nu)}_A\,\theta^A \\
&= \partial_\mu\left[M^\mu_A\theta^A + M^{(\mu\nu)}_A\partial_\nu\theta^A - \partial_\nu\,M^{(\mu\nu)}_A\,\theta^A\right]
\end{aligned} \tag{A7}$$

Therefore, the Noether current is

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial \partial_{\mu}\phi}\delta\phi - \left[M_A^{\mu}\theta^A + M_A^{(\mu\nu)}\partial_{\nu}\theta^A - \partial_{\nu}\, M_A^{(\mu\nu)}\,\theta^A\right] \tag{A8}$$

Inserting $\delta\phi$ in (A2) into (A8), and after a little algebra, wherein use is made of the equation of motion $\frac{\delta I}{\delta\phi} = 0$, we get the desired expression for j^{μ}.

$$j^{\mu} = \partial_{\nu}\left(\mathcal{F}_A^{\nu\mu}\theta^A\right) \tag{A9}$$

For the Maxwell Lagrangian, (A9) reproduces (7).

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