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Energy quantization for a singular super-Liouville boundary value problem

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Abstract
In this paper, we develop the blow-up analysis and establish the energy quantization for solutions to super-Liouville type equations on Riemann surfaces with conical singularities at the boundary. In other problems in geometric analysis, the blow-up analysis usually strongly utilizes conformal invariance, which yields a Noether current from which strong estimates can be derived. Here, however, the conical singularities destroy conformal invariance. Therefore, we develop another, more general, method that uses the vanishing of the Pohozaev constant for such solutions to deduce the removability of boundary singularities.

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1 Introduction

Many problems with a noncompact symmetry group, like the conformal group, are limit cases where the Palais–Smale condition no longer applies, and therefore, solutions may blow up at isolated singularities, see for instance [31]. Therefore, a blow-up analysis is needed, and this has become one of the fundamental tools in the geometric calculus of variations. This usually depends on the fact that the invariance yields an

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associated Noether current whose algebraic structure can be turned into estimates. In the case of conformal invariance this Noether current is a holomorphic quadratic differential. For harmonic map type problems, finiteness of the energy functional in question implies that that differential is in L^1. This then can be used to obtain fundamental estimates. For other problems, however, like (super-) Liouville equations, finiteness of the energy functional is not sufficient to get the L^1 bound of that differential and hence this is an extra assumption leading to the removability of local singularities (Prop 2.6, [23]).

But for (super-) Liouville equations on surfaces with conical singularities, we do not even have conformal invariance, because the scaling behavior at the singularities is different from that at regular points, see [27]. It turns out, however, that for an important class of two-dimensional geometric variational problems, there is a condition that is weaker than conformal invariance, the vanishing of a so-called Pohozaev constant (i.e. the Pohozaev identity), that is not only sufficient but also necessary for the blow-up analysis. This Pohozaev constant on one hand measures the extent to which the Pohozaev identity fails and on the other hand provides a characterization of the singular behavior of a solution at an isolated singularity. Such kind of quantity appears also in other two dimensional geometric variational problems and can be applied to study the qualitative asymptotic behaviour of solutions defined on degenerating surfaces [20, 40,41]. This vanishing condition is already known to play a crucial role in geometric analysis (see e.g. [34]), but for super-Liouville equations, as mentioned, this identity by itself suffices for the blow-up analysis.

In this paper, we shall apply this strategy to the blow-up analysis of the (super-) Liouville boundary problem on surfaces with conical singularities. To this purpose, let M be a compact Riemann surface with nonempty boundary ∂M and with a spin structure. We also denote this compact Riemann surface as $(M, \mathcal{A}, g)$, where g is its Riemannian metric with the conical singularities of divisor

$$\mathcal{A} = \sum_{j=1}^{m} \alpha_j q_j$$

(for definition of $\mathcal{A}$, see Sect. 2). Associated to the metric g, one can define the gradient ∇ and the Laplace operator Δ in the usual way.

We then have our main object of study, the **super-Liouville functional** that couples a real-valued function u and a spinor ψ on M

$$E_B(u, \psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \left\langle (\not{D} + e^u)\psi, \psi \right\rangle_g - e^{2u} \right\} dv + \int_{\partial M} \{h_g u - c e^u\} d\sigma, \tag{1}$$

where K_g is the Gaussian curvature in M, and h_g is the geodesic curvature of ∂M and c is a given positive constant. The Dirac operator $\not{D}$ is defined by $\not{D}\psi := \sum_{\alpha=1}^{2} e_\alpha \cdot \nabla_{e_\alpha}\psi$, where $\{e_1, e_2\}$ is an orthonormal basis on TM, ∇ is the Levi–Civita connection on M with respect to g and $\cdot$ denotes Clifford multiplication in the spinor bundle ΣM of M. Finally, $\langle \cdot, \cdot \rangle_g$ is the natural Hermitian metric on ΣM induced by g. We also write $|\cdot|_g^2$ as $\langle \cdot, \cdot \rangle_g$. For the geometric background, see [28] or [19].

The Euler-Lagrange system for $E_B(u,\psi)$ with Neumann / chirality boundary conditions is

$$\begin{cases} -\Delta_g u = 2e^{2u} - e^u \langle \psi, \psi \rangle_g - K_g, & \text{in } M^o \setminus \{q_1, q_2, \ldots, q_m\}, \\ \not{D}_g \psi = -e^u \psi, & \text{in } M^o \setminus \{q_1, q_2, \ldots, q_m\}, \\ \dfrac{\partial u}{\partial n} = ce^u - h_g, & \text{on } \partial M \setminus \{q_1, q_2, \ldots, q_m\}, \\ B^{\pm}\psi = 0, & \text{on } \partial M \setminus \{q_1, q_2, \cdots, q_m\}. \end{cases} \tag{2}$$

Here $B^{\pm}$ are the chirality operators (see Sect. 2 for the definition).

When $\psi = 0$ and (M, g) is a closed smooth Riemann surface, we obtain the classical Liouville functional

$$E(u) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u - e^{2u} \right\} dv.$$

The Euler-Lagrange equation for $E(u)$ is the Liouville equation

$$-\Delta_g u = 2e^{2u} - K_g.$$

Liouville [32] studied this equation in the plane, that is, for $K_g = 0$. The Liouville equation comes up in many problems of complex analysis and differential geometry of Riemann surfaces, for instance the prescribing curvature problem. The interplay between the geometric and analytic aspects makes the Liouville equation mathematically very interesting.

When $\psi \neq 0$ and (M, g) again is a closed smooth Riemann surface, we obtain the super-Liouville functional

$$E(u,\psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \left\langle (\not{D} + e^u)\psi, \psi \right\rangle_g - e^{2u} \right\} dv.$$

The Euler-Lagrange system for $E(u, \psi)$ is

$$\begin{cases} -\Delta_g u = 2e^{2u} - e^u \langle \psi, \psi \rangle_g - K_g \\ \not{D}_g \psi = -e^u \psi \end{cases} \text{in } M.$$

The supersymmetric version of the Liouville functional and equation have been studied extensively in the physics literature, see for instance [1,15,33]. As all supersymmetric functionals that arise in elementary particle physics, it needs anticommuting variables.

Motivated by the super-Liouville functional, a mathematical version of this functional that works with commuting variables only, but otherwise preserves the structure and the invariances of it, was introduced in [21]. That model couples the bosonic scalar field to a fermionic spinor field. In particular, the super-Liouville functional is conformally invariant, and it possesses a very interesting mathematical structure.

The analysis of classical Liouville type equations was developed in [2,4,29,30] etc, and the corresponding analysis for super-Liouville equations in [21,23,26]. In particular, the complete blow-up theory for sequences of solutions was established, including the energy identity for the spinor part, the blow-up value at blow-up points and the profile for a sequence of solutions at the blow-up points. For results by physicists about super-Liouville equations, we refer to [1,15,33] etc.

When $(M, \mathcal{A}, g)$ is a closed Riemann surface (without boundary) with conical singularities of divisor $\mathcal{A}$ and with a spin structure, we obtain that

$$E(u, \psi) = \int_M \left\{ \frac{1}{2} |\nabla u|^2 + K_g u + \left\langle (\not{D} + e^u)\psi, \psi \right\rangle_g - e^{2u} \right\} dv_g.$$

The Euler-Lagrange system for $E(u, \psi)$ is

$$\begin{cases} -\Delta_g u = 2e^{2u} - e^u \langle \psi, \psi \rangle_g - K_g \\ \not{D}_g \psi = -e^u \psi \end{cases} \text{in } M \backslash \{q_1, q_2, \ldots, q_m\}. \tag{3}$$

This system is closely related to the classical Liouville equation, or the prescribing curvature equation on M with conical singularites (see [12,36]). In addition, [3,4,6–8, 35] studied the blow-up theory of the following Liouville type equations with singular data:

$$-\Delta_g u = \lambda \frac{K e^u}{\int_M K e^u dg} - 4\pi \left(\sum_{j=1}^m \alpha_j \delta_{q_j} - f \right),$$

where (M, g) is a smooth surface and the singular data appear in the equation, which is the asymptotic behavior associated to the m interior punctures. For system (3), [27] provides an analytic foundation and the blow-up theory.

For Liouville boundary problems on (M, g) with or without conical singularites, there are also lots of results on the blow-up analysis, see [9,16,22,43,44]. For super-Liouville boundary problems on a smooth Riemann surface M, the corresponding results can be found in [24,25].

In this paper, we aim to provide an analytic foundation and to establish the blow-up analysis for the system (2). Our main result is the following energy quantization property for solutions to (2):

Theorem 1.1 *Let (u_n, ψ_n) be a sequence of solutions of (2) with energy conditions:*

$$\int_M e^{2u_n} dg < C, \qquad \int_M |\psi_n|_g^4 dg < C.$$

Define

$$\Sigma_1 = \{x \in M, \textit{ there is a sequence } y_n \to x \textit{ such that } u_n(y_n) \to +\infty\}.$$

If $\Sigma_1 \neq \emptyset$, then the possible values of

$$\lim_{n\to\infty}\left\{\int_M 2e^{2u_n} - e^{u_n}|\psi_n|_g^2 dv_g + \int_{\partial M} ce^{u_n} d\sigma_g\right\}$$

are

$$4\pi m_1 + 2\pi m_2 + \sum_{j\in J_1} 4\pi(1+\alpha_j) + \sum_{j\in J_1} 2\pi(1+\alpha_j),$$

where m_1, m_2 are nonnegative integers, $J_1 \subset \{1, \ldots, l\}$ and $J_2 \subset \{l+1, \ldots, m\}$. Here $\{1, \ldots, l\}$ and $\{l+1, \ldots, m\}$ are the indexing sets of the conical singularities of M. J_1 and J_2 could be empty.

From the energy quantization property, one can deduce the concentration properties of conformal volume and the compactness of solutions. It turns out that understanding of this property is the key step to study existence from a variational point of view by a refined Moser–Trudinger inequality, see e.g. [10,11,18].

If we assume that the points $q_1, q_2, \ldots, q_l$ are in M^o for $1 \leq l < m$ and the points $q_{l+1}, q_{l+2}, \ldots, q_m$ are on ∂M for the surface $(M, \mathcal{A}, g)$ with the divisor $\mathcal{A} = \Sigma_{j=1}^m \alpha_j q_j, \alpha_j > 0$, we have the following Gauss–Bonnet formula

$$\frac{1}{2\pi}\int_M K_g dv_g + \frac{1}{2\pi}\int_{\partial M} h_g d\sigma_g = \mathcal{X}(M) + |\mathcal{A}|,$$

where $\mathcal{X}(M) = 2 - 2g_M$ is the topological Euler characteristic of M itself, g_M is the genus of M and

$$|\mathcal{A}| = \sum_{j=1}^{l} \alpha_j + \sum_{j=l+1}^{m} \frac{\alpha_j}{2}$$

is the degree of $\mathcal{A}$, see [36]. From (2) we obtain that

$$\int_M 2e^{2u_n} - e^{u_n}|\psi_n|_g^2 dv_g + \int_{\partial M} ce^{u_n} d\sum_g = \int_M K_g dv_g + \int_{\partial M} h_g d\sigma_g$$
$$= 2\pi(\mathcal{X}(M) + |\mathcal{A}|).$$

Then we can use Theorem 1.1 to get the following:

Theorem 1.2 *Let $(M, \mathcal{A}, g)$ be as above, and (u_n, ψ_n) be as in Theorem 1.1. Then the value in Theorem 1.1 satisfies*

$$4\pi m_1 + 2\pi m_2 + \sum_{j\in J_1} 4\pi(1+\alpha_j) + \sum_{j\in J_1} 2\pi(1+\alpha_j) = 4\pi(1-g_M) + 2\pi|\mathcal{A}|.$$

In particular,

(i) *If*

$$4\pi(1-g_M)+2\pi|\mathcal{A}|=2\pi,$$

then the blow-up set Σ_1 *contains at most one point.*

(ii) *If*

$$4\pi(1-g_M)+2\pi|\mathcal{A}|<2\pi,$$

then the blow-up set $\Sigma_1=\emptyset$.

To show Theorem 1.1, a key step is to compute the blow-up value

$$m(p)=\lim_{R\to 0}\lim_{n\to\infty}\left\{\int_{B_R^M(p)}(2e^{2u_n}-e^{u_n}|\psi_n|_g^2-K_g)dv_g+\int_{\partial M\cap B_R^M(p)}(ce^{u_n}-h_g)d\sigma_g\right\},$$

at the blow-up point $p\in\Sigma_1$ for a blow-up sequence (u_n,ψ_n). Here $B_R^M(p)$ is a geodesic ball of (M,g) at p. For this purpose, we need to study the following local super-Liouville boundary value problem (see Sect. 3):

$$\begin{cases} -\Delta u(x)=2V^2(x)|x|^{2\alpha}e^{2u(x)}-V(x)|x|^{\alpha}e^{u(x)}|\Psi|^2, & \text{in } D_r^+,\\ \not{D}\Psi=-V(x)|x|^{\alpha}e^{u(x)}\Psi, & \text{in } D_r^+,\\ \frac{\partial u}{\partial n}=cV(x)|x|^{\alpha}e^{u(x)}, & \text{on } L_r,\\ B^{\pm}\Psi=0, & \text{on } L_r.\end{cases}\tag{4}$$

Here $\alpha\geq 0$, $V(x)$ is in $C^1_{loc}(D_r^+\cup L_r)$ and satisfies $0<a\leq V(x)\leq b$. L_r and S_r^+ here and in the sequel are portions of ∂D_r^+, which are defined in section 3. Then we have the following Brezis–Merle type concentration compactness theorem:

Theorem 1.3 *Let* (u_n,Ψ_n) *be a sequence of regular solutions to (4) satisfying*

$$\int_{D_r^+}|x|^{2\alpha}e^{2u_n}+|\Psi_n|^4dx+\int_{L_r}|x|^{\alpha}e^{u_n}ds<C.$$

Define

$$\Sigma_1=\left\{x\in D_r^+\cup L_r,\ \text{there is a sequence } y_n\to x \text{ such that } u_n(y_n)\to+\infty\right\},$$
$$\Sigma_2=\left\{x\in D_r^+\cup L_r,\ \text{there is a sequence } y_n\to x \text{ such that } |\Psi_n(y_n)|\to+\infty\right\}.$$

Then, we have $\Sigma_2\subset\Sigma_1$. *Moreover,* (u_n,Ψ_n) *admits a subsequence, still denoted by* (u_n,Ψ_n), *that satisfies*

a) $|\Psi_n|$ *is bounded in* $L^\infty_{loc}((D_r^+\cup L_r)\backslash\Sigma_2)$.
b) *For* u_n, *one of the following alternatives holds:*

  i) u_n *is bounded in* $L^\infty_{loc}(D_r^+\cup L_r)$.
  ii) $u_n\to-\infty$ *uniformly on compact subsets of* $D_r^+\cup L_r$.

iii) Σ_1 *is finite, nonempty and either*

$$u_n \text{ is bounded in } L^\infty_{loc}((D_r^+ \cup L_r)\backslash\Sigma_1) \tag{5}$$

or

$$u_n \to -\infty \text{ uniformly on compact subsets of } (D_r^+ \cup L_r)\backslash\Sigma_1. \tag{6}$$

To show the quantization property of the blow-up value, we need to rule out (5) in the above theorem. When the spinor field is vanishing, namely, in the case of Liouville-type problems, a technique based on Pohozaev identity was introduced in [7] to prove the vanishing of the mass in the neck region for blow-up sequences of solutions with interior conical singularities. In the case of super-Liouville-type problems, we need to overcome some new difficulties caused by the spinor part. For the case of interior conical singularities, this was achieved in [27]. In the present paper, we shall handle the boundary conical singularities case. To this end, the decay estimates of the spinor part Ψ_n, the Pohozaev identity of the local coupled system (4) and the energy identity of Ψ_n, which means there is no energy contribution on the neck domain, play the essential roles. The corresponding theorem is the following:

Theorem 1.4 *Let* (u_n, Ψ_n) *be a sequence of regular solutions to (4) satisfying*

$$\int_{D_r^+} |x|^{2\alpha} e^{2u_n} + |\Psi_n|^4 dx + \int_{L_r} |x|^\alpha e^{u_n} ds < C.$$

Denote by $\Sigma_1 = \{x_1, x_2, \dots, x_l\}$ *the blow-up set of* u_n*. Then there are finitely many solutions* $(u^{i,k}, \Psi^{i,k})$ *that satisfy*

$$\begin{cases} -\Delta u^{i,k} = 2|x|^\alpha e^{2u^{i,k}} - |x|^\alpha e^{u^{i,k}} \left\langle \Psi^{i,k}, \Psi^{i,k} \right\rangle - 1, & \text{in } S^2, \\ \not{D}\Psi^{i,k} = -|x|^\alpha e^{u^{i,k}} \Psi^{i,k}, & \text{in } S^2, \end{cases} \tag{7}$$

for $i = 1, 2, \dots, I$*, and* $k = 1, 2, \dots, K_i$*, and* $\alpha \geq 0$*, or there are finitely many solutions* $(u^{j,l}, \Psi^{j,l})$ *that satisfy*

$$\begin{cases} -\Delta u^{j,l} = 2|x|^\alpha e^{2u^{j,l}} - |x|^\alpha e^{u^{j,l}} \left\langle \Psi^{j,l}, \Psi^{j,l} \right\rangle - 1, & \text{in } S^2_{c'}, \\ \not{D}\Psi^{j,l} = -|x|^\alpha e^{u^{j,l}} \Psi^{j,l}, & \text{in } S^2_{c'}, \\ \dfrac{\partial u^{j,l}}{\partial n} = c|x|^\alpha e^{u^{j,l}} - c', & \text{on } \partial S^2_{c'}, \\ B^\pm \Psi^{j,l} = 0, & \text{on } \partial S^2_{c'}, \end{cases} \tag{8}$$

for $j = 1, 2, \dots, J$*, and* $l = 1, 2, \cdots, L_j$*, and* $\alpha \geq 0$*. Here* $S^2_{c'}$ *is a portion of the sphere cut out by a 2-plane with constant geodesic curvature* c'*. After selection of a subsequence,* Ψ_n *converges in* C^∞_{loc} *to* Ψ *on* $(B_r^+ \cup L_r)\backslash\Sigma_1$ *and we have the energy identity:*

$$\lim_{n\to\infty}\int_{D_r^+}|\Psi_n|^4dv=\int_{D_r^+}|\Psi|^4dv+\sum_{i=1}^{I}\sum_{k=1}^{K_i}\int_{S^2}|\Psi^{i,k}|^4dv+\sum_{j=1}^{J}\sum_{l=1}^{L_j}\int_{S^2_{c'}}|\Psi^{j,l}|^4dv. \tag{9}$$

A crucial step in proving the above theorem is to show the removability of isolated singularities at the boundary, which is equivalent to the vanishing of the Pohozaev constant (see Theorem 4.5). Once the energy identity for the spinor part (9) is established, we can then obtain

Theorem 1.5 *Let (u_n,Ψ_n) be solutions as in Theorem 1.3. Assume that (u_n,Ψ_n) blows up and the blow-up set $\Sigma_1\neq\emptyset$. Then*

$$u_n\to-\infty\quad\textit{uniformly on compact subsets of } (D_r^+\cup L_r)\setminus\Sigma_1.$$

Furthermore,

$$\int_{D_r^+(0)}[2V(x)|x|^{2\alpha}e^{2u_n}-V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2]\phi dx+\int_{L_r}cV(x)|x|^{\alpha}e^{u_n}$$
$$\to\sum_{x_i\in\Sigma_1}m(x_i)\phi(x_i)$$

for every $\phi\in C_o^{\infty}(D_r^+\cup L_r)$ and $m(x_i)>0$.

In the end, with the help of the Pohozaev identity (see Proposition 4.1) and the Green function at some singular points, we have the following:

Theorem 1.6 *Let (u_n,Ψ_n) be solutions as in Theorem 1.3. Assume that (u_n,Ψ_n) blows up and the blow-up set $\Sigma_1\neq\emptyset$. Let $p\in\Sigma_1$ and assume that p is the only blow-up point in $\overline{D^+_{\delta_0}(p)}$ for some $\delta_0>0$. If there exists a positive constant C such that*

$$\max_{S^+_{\delta_0}(p)}u_n-\min_{S^+_{\delta_0}(p)}u_n\le C,$$

then the blow-up value $m(p)=4\pi$ when $p\notin L_{\delta_0}(p)$, $m(p)=2\pi$ when $p\in L_{\delta_0}(p)\setminus\{0\}$, and $m(p)=2\pi(1+\alpha)$ when $p=0$.

2 Preliminaries

In this section, we will first recall the definition of surfaces with conical singularities, following [36]. Then we shall recall the chirality boundary condition for the Dirac operator $\not{D}$, see e.g. [17]. In particular, we will see that under the chirality boundary conditions $B^{\pm}$, the Dirac operator $\not{D}$ is self-adjoint.

A conformal metric g on a Riemannian surface M (possibly with boundary) has a conical singularity of order α (a real number with $\alpha>-1$) at a point $p\in M\cup\partial M$ if in some neighborhood of p

$$g = e^{2u}|z - z(p)|^{2\alpha}|dz|^2$$

where z is a coordinate of M defined in this neighborhood and u is smooth away from p and continuous at p. The point p is then said to be a conical singularity of *angle* $\theta = 2\pi(\alpha+1)$ if $p \notin \partial M$ and a *corner* of angle $\theta = \pi(\alpha+1)$ if $p \in \partial M$. For example, a (somewhat idealized) American football has two singularities of equal angle, while a teardrop has only one singularity. Both these examples correspond to the case $-1 < \alpha < 0$; in case $\alpha > 0$, the angle is larger than 2π, leading to a different geometric picture. Such singularities also appear in orbifolds and branched coverings. They can also describe the ends of complete Riemann surfaces with finite total curvature. If (M, g) has conical singularities of order $\alpha_1, \alpha_2, \ldots, \alpha_m$ at $q_1, q_2, \ldots, q_m$, then g is said to represent the divisor $\mathcal{A} = \Sigma_{j=1}^m \alpha_j q_j$. Importantly, the presence of such conical singularities destroys conformal invariance, because the conical points are different from the regular ones.

The chirality boundary condition for the Dirac operator $\not{D}$ is a natural boundary condition for spinor part ψ. Let M be a compact Riemann surface with $\partial M \neq \emptyset$ and with a fixed spin structure, admitting a chirality operator G, which is an endomorphism of the spinor bundle ΣM satisfying:

$$G^2 = I, \qquad \langle G\psi, G\varphi\rangle = \langle\psi, \varphi\rangle,$$

and

$$y\nabla_X(G\psi) = G\nabla_X\psi, \qquad X \cdot G\psi = -G(X \cdot \psi),$$

for any $X \in \Gamma(TM)$, $\psi, \varphi \in \Gamma(\Sigma M)$. Here I denotes the identity endomorphism of ΣM and $\Gamma(\cdot)$ denotes the space of sections of a given bundle.

We usually take $G = \gamma(\omega_2)$, which denotes the Clifford multiplication by the complex volume form $\omega_2 = ie_1e_2$, where e_1, e_2 is a local orthonormal frame on M.

Denote by

$$S := \Sigma M|_{\partial M}$$

the restricted spinor bundle with induced Hermitian product.

Let $\vec{n}$ be the outward unit normal vector field on ∂M. One can verify that $\vec{n}G : \Gamma(S) \to \Gamma(S)$ is a self-adjoint endomorphism satisfying

$$(\vec{n}G)^2 = I, \qquad \langle\vec{n}G\psi, \varphi\rangle = \langle\psi, \vec{n}G\varphi\rangle,$$

Hence, we can decompose $S = V^+ \bigoplus V^-$, where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue ± 1. One verifies that the orthogonal projection onto the eigensubbundle $V^\pm$:

$$\begin{aligned} B^\pm : L^2(S) &\to L^2(V^\pm) \\ \psi &\mapsto \frac{1}{2}(I \pm \vec{n}G)\psi, \end{aligned}$$

defines a local elliptic boundary condition for the Dirac operator $\not{D}$, see e.g. [17]. We say that a spinor $\psi \in L^2(\Gamma(\Sigma M))$ satisfies the chirality boundary conditions $B^{\pm}$ if

$$B^{\pm}\psi|_{\partial M} = 0.$$

It is well known (see e.g. [17]) that if $\psi, \phi \in L^2(\Gamma(\Sigma M))$ satisfy the chirality boundary conditions $B^{\pm}$ then

$$\langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0, \quad \text{on } \partial M.$$

In particular,

$$\int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \varphi \rangle = 0. \tag{10}$$

It follows that the Dirac operator $\not{D}$ is self-adjoint under the chirality boundary conditions $B^{\pm}$.

It may be helpful if we recall that on a surface the (usual) Dirac operator $\not{D}$ can be seen as the (doubled) Cauchy-Riemann operator. Consider $\mathbb{R}^2$ with the Euclidean metric $ds^2 + dt^2$. Let $e_1 = \frac{\partial}{\partial s}$ and $e_2 = \frac{\partial}{\partial t}$ be the standard orthonormal frame. A spinor field is simply a map $\Psi : \mathbb{R}^2 \to \Delta_2 = \mathbf{C}^2$, and the actions of e_1 and e_2 on spinor fields can be identified by multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If $\Psi := \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^2 \to \mathbf{C}^2$ is a spinor field, then the Dirac operator is

$$\not{D}\Psi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial s} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial t} \end{pmatrix} = 2i \begin{pmatrix} \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial \bar{z}} \end{pmatrix},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right).$$

Therefore, the elliptic estimates developed for (anti-) holomorphic functions can be used to study the Dirac equation.

If M be the upper-half Euclidean space $\mathbb{R}^2_+$, then the chirality operator is $G = ie_1e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\overrightarrow{n} = -e_2$, we get that

$$B^{\pm} = \frac{1}{2}(I \pm \overrightarrow{n} \cdot G) = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

By the standard chirality decomposition, we can write $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, and then the boundary condition becomes

$$\psi_+ = \mp\psi_- \quad \text{on } \partial M.$$

Without loss of generality, in the sequel, we shall only consider the chirality boundary condition $B = B^+$.

We have the following geometric property:

Proposition 2.1 *The functional $E_B(u, \psi)$ is invariant under conformal diffeomorphisms $\varphi : M \to M$ preserving the divisor, that is, be $\varphi^*\mathcal{A} = \mathcal{A}$. In other word, if we write that $\varphi^*(g) = \lambda^2 g$, where $\lambda > 0$ is the conformal factor of the conformal map φ, and set*

$$\begin{aligned} \widetilde{u} &= u \circ \varphi - ln\lambda, \\ \widetilde{\psi} &= \lambda^{-\frac{1}{2}} \psi \circ \varphi, \end{aligned} \tag{11}$$

then $E_B(\tilde{u}, \tilde{\psi}) = E_B(u, \psi)$. In particular, if (u, ψ) is a solution of (2), so is $(\tilde{u}, \tilde{\psi})$.

3 The local singular super-Liouville boundary problem

In this section, we shall first derive the local version of the super-Liouville boundary problem. Then we shall analyze the regularity of solutions under the small energy condition.

First we take a point $p \in M^o$, choose a small neighborhood $U(p) \subset M^o$, and define an isothermal coordinate system $x = (x_1, x_2)$ centered at p, such that p corresponds to $x = 0$ and $g = e^{2\phi}|x|^{2\alpha}(dx_1^2 + dx_2^2)$ in $D_r(0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < r^2\}$, where ϕ is smooth away from p, continuous at p and $\phi(p) = 0$. We can choose such a neighborhood small enough so that if p is a conical singular point of g, then $U(p) \cap \mathcal{A} = \{p\}$ and $\alpha > 0$, while, if p is a smooth point of g, then $U(p) \cap \mathcal{A} = \emptyset$ and $\alpha = 0$. Consequently, with respect to the isothermal coordinates, we can obtain the local version of the singular super-Liouville-type equations,

$$\begin{cases} -\Delta u(x) = 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^{\alpha}e^{u(x)}|\Psi|^2 \\ \quad\; D\!\!\!/\Psi = -V(x)|x|^{\alpha}e^{u(x)}\Psi \end{cases} \text{in } D_r(0), \tag{12}$$

which has no boundary condition since p is a interior point of M. Here $\Psi = |x|^{\frac{\alpha}{2}}e^{\frac{\phi(x)}{2}}\psi$, $V(x)$ is a $C^{1,\beta}$ function and satisfies $0 < a \le V(x) \le b$. The detailed arguments can be found in the section 3 of [27]. We also assume that (u, Ψ) satisfy the energy condition:

$$\int_{D_r(0)} |x|^{2\alpha}e^{2u} + |\Psi|^4 dx < +\infty. \tag{13}$$

We put $D_r := D_r(0)$. We say that (u, Ψ) is a weak solution of (12) and (13), if $u \in W^{1,2}(D_r)$ and $\Psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma D_r))$ satisfy

$$\int_{D_r} \nabla u \nabla \phi dx = \int_{D_r} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^{\alpha} e^{u} |\Psi|^2)\phi dx,$$
$$\int_{D_r} \langle \Psi, \not{D}\xi \rangle dx = -\int_{D_r} V(x)|x|^{\alpha} e^{u} \langle \Psi, \xi \rangle dx,$$

for any $\phi \in C_0^\infty(D_r)$ and any spinor $\xi \in C^\infty \cap W_0^{1,\frac{4}{3}}(\Gamma(\Sigma D_r))$. A weak solution is a classical solution by the following:

Proposition 3.1 *Let* (u, Ψ) *be a weak solution of* (12) *and* (13). *Then* $(u, \Psi) \in C^2(D_r) \times C^2(\Gamma(\Sigma D_r))$.

Note that when $\alpha = 0$ this proposition is proved in [21] (see Proposition 4.1). When $\alpha > 0$, this proposition is proved in [27] (see Proposition 3.1).

For $p \in \partial M$, we also can choose a small geodesic ball $U(p) \subset M$ and define an isothermal coordinate system $x = (x_1, x_2)$ centered at p, such that p corresponds to $x = 0$ and $g = e^{2\phi}|x|^{2\alpha}(dx_1^2 + dx_2^2)$ in $\overline{D_r^+}(0) = \{(s,t) \in \mathbb{R}^2 \mid s^2 + t^2 < r^2, t \geq 0\}$, where ϕ is smooth away from p and continuous at p. We can choose such a geodesic ball small enough so that if p is a conical singular point of g, then $U(p) \cap \mathcal{A} = \{p\}$ and $\alpha > 0$, while, if p is a smooth point of g, then $U(p) \cap \mathcal{A} = \emptyset$ and $\alpha = 0$. Set $L_r = \partial D_r^+ \cap \partial \mathbb{R}_+^2$, and $S_r^+ = \partial D_r^+ \cap \mathbb{R}_+^2$. Also in the sequel, we will set $L_r(x_0) = \partial D_r^+(x_0) \cap \partial \mathbb{R}_+^2$, and $S_r^+(x_0) = \partial D_r^+(x_0) \cap \mathbb{R}_+^2$. Consequently, with respect to the isothermal coordinates, (u, ψ) satisfies

$$\begin{cases} -\Delta u(x) = e^{2\phi(x)}|x|^{2\alpha}(2e^{2u(x)} - e^{u(x)}|\psi|^2(x) - K_g) & \text{in } D_r^+, \\ \not{D}\left(e^{\frac{\phi(x)}{2}}|x|^{\frac{\alpha}{2}}\psi\right) = -e^{\phi(x)}|x|^{\alpha} e^{u(x)} \left(e^{\frac{\phi(x)}{2}}|x|^{\frac{\alpha}{2}}\psi\right) & \text{in } D_r^+, \\ \dfrac{\partial u}{\partial n} = e^{\phi(x)}|x|^{\alpha}(ce^{u} - h_g), & \text{on } L_r, \\ B\left(e^{\frac{\phi(x)}{2}}|x|^{\frac{\alpha}{2}}\psi\right) = 0, & \text{on } L_r. \end{cases} \tag{14}$$

Here $\Delta = \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2}$ is the usual Laplacian, and the Dirac operator $\not{D}$ can be seen as doubled Cauchy–Riemann operator, B is the chirality boundary operator of spinors.

Note that the relation between the two Gaussian curvatures and between the two geodesic curvatures are respectively

$$\begin{cases} -\Delta \phi = e^{2\phi}|x|^{2\alpha} K_g, \\ \frac{\partial \phi}{\partial n} = e^{\phi}|x|^{\alpha} h_g. \end{cases}$$

By standard elliptic regularity we conclude that $\phi \in W^{2,p}_{loc}(D_r^+ \cup L_r)$ for some $p > 1$ if $\alpha \geq 0$ and if the curvature K_g and h_g of M is regular enough. Therefore, by Sobolev embedding, $\phi \in C^1_{loc}(D_r^+ \cup L_r)$. If we denote $V(x) = e^{\phi}$, $W_1(x) = e^{2\phi}|x|^{2\alpha} K_g$ and $W_2(x) = e^{\phi}|x|^{\alpha} h_g$, then $0 < a \leq V(x) \leq b$, $W_1(x)$ is in $L^p(D_r^+)$ and $W_2(x)$ is in

$L^p(L_r)$ for all $p > 1$ if the curvature K_g and h_g of M is regular enough. Therefore, the Eq. (14) can be rewritten as:

$$\begin{cases} -\Delta u(x) = 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^{\alpha}e^{u(x)}|\Psi|^2 - W_1(x), & \text{in } D_r^+, \\ \not{D}\Psi = -V(x)|x|^{\alpha}e^{u(x)}\Psi, & \text{in } D_r^+, \\ \frac{\partial u}{\partial n} = cV(x)|x|^{\alpha}e^{u} - W_2, & \text{on } L_r, \\ B(\Psi) = 0, & \text{on } L_r. \end{cases}$$

Furthermore, let $w(x)$ satisfy

$$\begin{cases} -\Delta w(x) = -W_1(x), & \text{in } D_r^+, \\ \frac{\partial u}{\partial n} = -W_2(x), & \text{on } L_r, \\ w(x) = 0, & \text{on } S_r^+. \end{cases}$$

It is easy to see that $w(x)$ is in $C^2(D_r^+) \cap C^1(D_r^+ \cup L_r)$. Setting $v(x) = u(x) - w(x)$, then (v, Ψ) satisfies

$$\begin{cases} -\Delta v(x) = 2V^2(x)|x|^{2\alpha}e^{2v(x)} - V(x)|x|^{\alpha}e^{v(x)}|\Psi|^2, & \text{in } D_r^+, \\ \not{D}\Psi = -V(x)|x|^{\alpha}e^{v(x)}\Psi, & \text{in } D_r^+, \\ \frac{\partial v}{\partial n} = cV(x)|x|^{\alpha}e^{v(x)}, & \text{on } L_r, \\ B(\Psi) = 0, & \text{on } L_r. \end{cases}$$

Here $\alpha \geq 0$, $V(x)$ is in $C^1_{loc}(D_r^+ \cup L_r)$ and satisfies $0 < a \leq V(x) \leq b$. Thus we get the local system (4) of the boundary problem (2).

As the interior case, we can also define (u, Ψ) be a weak solution of (4) if $u \in W^{1,2}(D_r^+)$ and $\Psi \in W_B^{1,\frac{4}{3}}(\Gamma(\Sigma D_r^+))$ satisfy

$$\int_{D_r^+} \nabla u \nabla \phi dx = \int_{D_r^+} (2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^{\alpha}e^{u(x)}|\Psi|^2)\phi dx + \int_{L_r} (cV(x)|x|^{\alpha}e^{v(x)})\phi d\sigma$$

$$\int_{D_r^+} \langle \Psi, \not{D}\xi \rangle dx = -\int_{D_r^+} V(x)|x|^{\alpha}e^{v(x)}\langle \Psi, \xi \rangle dx$$

for any $\phi \in C_0^\infty(D_r^+ \cup L_r)$ and any spinor $\xi \in C_0^\infty(\Gamma(\Sigma(D_r^+ \cup L_r))) \cap W_B^{1,\frac{4}{3}}(\Gamma(\Sigma D_r^+))$. Here

$$W_B^{1,\frac{4}{3}}(\Gamma(\Sigma D_r^+)) = \{\psi | \psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma D_r^+)), B\psi|_{L_r} = 0\}.$$

For weak solutions of (4) we also have the following regularity result.

Proposition 3.2 *Let (u, Ψ) be a weak solution of (4) with the energy condition*

$$\int_{D_r^+} |x|^{2\alpha}e^{2u} + |\Psi|^4 dv + \int_{L_r} |x|^{\alpha}e^{u} d\sigma < \infty. \tag{15}$$

Then $u \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r)$ *and* $\Psi \in C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r)))$.

Note that when $\alpha = 0$ this proposition has been proved in [24]. When $\alpha > 0$, to get the L^1 integral of u^+, we need a trick which was introduced in [7] and also was used in [27]. That is, by using the fact that for some $t > 0$

$$\int_{D_r^+} \frac{1}{|x|^{2t\alpha}} dx \leq C,$$

we can choose $s = \frac{t}{t+1} \in (0,1)$ when $\alpha > 0$ and $s = 1$ when $\alpha = 0$ such that

$$2s\int_{D_r^+} u^+ dx \leq \int_{D_r^+} e^{2su} dx \leq \left(\int_{D_r^+} |x|^{2\alpha} e^{2u} dx\right)^s \left(\int_{D_r^+} |x|^{-2t\alpha} dx\right)^{1-s} < \infty.$$

Once we get the L^1 integral of u^+, we can get the conclusion of Proposition 3.2 by use the same argument in [24]. We omit the proof here.

We call (u, ψ) a *regular* solution to (4) if $u \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r)$ and $\Psi \in C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r)))$.

Next we consider the convergence of a sequence of regular solutions to (4) under a smallness condition for the energy. We assume that (u_n, Ψ_n) satisfy that

$$\begin{cases} -\Delta u_n(x) = 2V^2(x)|x|^{2\alpha} e^{2u_n(x)} - V(x)|x|^{\alpha} e^{u_n(x)} |\Psi_n|^2, & \text{in } D_r^+, \\ \not{D}\Psi_n = -V(x)|x|^{\alpha} e^{u_n(x)} \Psi_n, & \text{in } D_r^+, \\ \frac{\partial u_n}{\partial n} = cV(x)|x|^{\alpha} e^{u_n(x)}, & \text{on } L_r, \\ B(\Psi_n) = 0, & \text{on } L_r, \end{cases} \tag{16}$$

with the energy condition

$$\int_{D_r^+} |x|^{2\alpha} e^{2u_n} + |\Psi_n|^4 dv + \int_{L_r} |x|^{\alpha} e^{u_n} d\sigma < C \tag{17}$$

for some constant $C > 0$. First, we study the small energy regularity, i.e. when the energy $\int_{D_r^+} |x|^{2\alpha} e^{2u_n} dx$ and $\int_{L_r} |x|^{\alpha} e^{u_n} dx$ are small enough, u_n will be uniformly bounded from above. Our Lemma is:

Lemma 3.3 *For* $\varepsilon_1 < \pi$, *and* $\varepsilon_2 < \pi$. *If a sequence of regular solutions* (u_n, Ψ_n) *to* (16) *with*

$$\int_{D_r^+} 2V^2(x)|x|^{2\alpha} e^{2u_n} dx < \varepsilon_1, \quad |c|\int_{L_r} V(x)|x|^{\alpha} e^{u_n} d\sigma < \varepsilon_2, \quad \int_{D_r^+} |\Psi_n|^4 dx < C$$

for some fixed constant $C > 0$, *we have that* $||u_n^+||_{L^\infty(\overline{D}^+_{\frac{r}{4}})}$ *and* $||\Psi_n||_{L^\infty(\overline{D}^+_{\frac{r}{8}})}$ *are uniformly bounded.*

Proof As the same situation as in Proposition 3.2, we can no longer use the inequality $2\int u_n^+ < \int e^{2u_n}$ to get the uniform bound of the L^1-integral of u_n^+ when $\alpha > 0$. But notice that there exists a constant $t > 0$ such that

$$\int_{D_r^+} \frac{1}{|x|^{2t\alpha}} dx \leq C.$$

Setting $s = \frac{t}{t+1} \in (0, 1)$, then we obtain

$$2s \int_{D_r^+} u_n^+ dx \leq \int_{D_r^+} e^{2su_n} dx \leq \left(\int_{D_r^+} |x|^{2\alpha} e^{2u_n} dx \right)^s \left(\int_{D_r^+} |x|^{-2t\alpha} dx \right)^{1-s} < C.$$

Then by a similar argument as in the proof of Lemma 3.5 in [24] we can prove this Lemma. □

When the energy $\int_{D_r^+} 2V^2(x)|x|^{2\alpha} e^{2u_n} + \int_{L_r} V(x)|x|^{\alpha} e^{u_n} ds$ is large, in general, blow-up phenomenon may occur, i.e., Theorem 1.3 holds.

Remark 3.4 Let $v_n = u_n + \alpha \log |x|$, then (v_n, Ψ_n) satisfies

$$\begin{cases} -\Delta v_n(x) = 2V^2(x)e^{2v_n(x)} - V(x)e^{v_n(x)}|\Psi_n|^2, & \text{in } D_r^+, \\ \not{D}\Psi_n = -V(x)e^{v_n(x)}\Psi_n, & \text{in } D_r^+, \\ \frac{\partial v_n}{\partial n} = cV(x)e^{v_n(x)} + \pi\alpha\delta_{p=0}, & \text{on } L_r, \\ B\Psi_n = 0, & \text{on } L_r, \end{cases}$$

with the energy condition

$$\int_{D_r^+} e^{2v_n} + |\Psi_n|^4 dx + \int_{L_r} e^{v_n} ds < C.$$

Then, by using similar arguments as in [7], the two blow-up sets of u_n and v_n are the same. To show this conclusion, it is sufficient to show the point $x = 0$ is a blow-up point for u_n if and only if it is a blow-up point for v_n. In fact, if 0 is the only blow-up point for v_n in a small neighbourhood $D_{\delta_0}^+ \cup L_{\delta_0}$, that is, for any $\delta \in (0, \delta_0)$, $\exists C_\delta > 0$, such that

$$\max_{D_{\delta_0}^+ \setminus D_\delta^+} v_n \leq C_\delta, \quad \text{and} \quad \max_{D_{\delta_0}^+} v_n \to +\infty, \tag{18}$$

then, it is easy to see that 0 is also the only blow-up point for u_n in a small neighbourhood $D_{\delta_0}^+ \cup L_{\delta_0}$, that is, for any $\delta \in (0, \delta_0)$, $\exists C_\delta > 0$, such that

$$\max_{D_{\delta_0}^+ \setminus D_\delta^+} u_n \leq C_\delta, \quad \text{and} \quad \max_{D_{\delta_0}^+} u_n \to +\infty. \tag{19}$$

In converse, we assume that 0 is the only blow-up point for u_n in a small neighbourhood $D_{\delta_0}^+ \cup L_{\delta_0}$ such that (19) is holds. We argue by contradiction and suppose that there exists a uniform constant C, such that $v_n(x) \leq C$ for any $x \in \overline{D}_{\delta_0}^+$. First, we can obtain that there exists a uniform constant C, such that $|\Psi_n|^2(x) \le C$ for any $x \in \overline{D^+_{\frac{\delta_0}{2}}}$. For this purpose, we extend (v_n, Ψ_n) to the lower half disk D_r^-. Assume $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$, and define

$$\begin{aligned} v_n(\bar{x}) &:= v_n(x), \quad \bar{x} \in D_r^-, \\ \Psi_n(\bar{x}) &:= ie_1 \cdot \Psi_n(x), \quad \bar{x} \in D_r^-, \\ A_n(x) &:= \begin{cases} e^{v_n(x)}, & x \in D_r^+, \\ e^{v_n(\bar{x})}, & x \in D_r^-. \end{cases} \end{aligned}$$

Then Ψ_n satisfies

$$\not{D}\Psi_n = -A_n(x)\Psi_n, \quad \text{in } D_r.$$

Since $A_n(x)$ is uniformly bounded in $L^\infty(D_{\delta_0})$ and $\int_{D_{\delta_0}} |\Psi_n|^4 dx < C$, we have Ψ_n is uniformly bounded in $W^{1,\frac{4}{3}}(\Gamma(\Sigma D_{\frac{\delta_0}{2}}))$ and in particular Ψ_n is uniformly bounded $C^\gamma(\Gamma(\Sigma \overline{D^+_{\frac{\delta_0}{2}}}))$ for some $0 < \gamma < 1$. Further, since

$$\begin{aligned} f_n(x) &:= 2V_n^2(x)|x|^{2\alpha}e^{2u_n(x)} - V_n(x)|x|^\alpha e^{u_n(x)}|\Psi_n|^2 \\ &= 2V_n^2(x)e^{2v_n(x)} - V_n(x)e^{v_n(x)}|\Psi_n|^2 \end{aligned}$$

and

$$g_n := -V_n(x)|x|^\alpha e^{u_n(x)}\Psi_n = -V_n(x)e^{v_n(x)}\Psi_n$$

are uniformly bounded in $\overline{D^+_{\frac{\delta_0}{2}}}$. Then by Harnack type inequality of Neumann boundary problem (see Lemma A.2 in [24]), it follows that $\inf_{\overline{D^+_{\frac{\delta_0}{2}}}} u_n \to +\infty$. Thus we get a contradiction since the blow-up set of u_n is finite.

4 Removability of local sigularities

The Pohozaev identity is closely related to the removability of singularities. In this section, we shall first establish the Pohozaev identity for regular solutions to (4). Then for solutions defined on a domain with isolated singularity, we define a constant which is called the Pohozaev constant. The most important is that a necessary and sufficient condition for the removability of local singularities is the vanishing of Pohazaev constant.

Proposition 4.1 (Pohozaev identity) *Let* (u, Ψ) *be a regular solution of (4), that is* (u, Ψ) *satisfies*

$$\begin{cases} -\Delta u(x) = 2V^2(x)|x|^{2\alpha}e^{2u(x)} - V(x)|x|^{\alpha}e^{u(x)}|\Psi|^2, & \text{in } D_R^+, \\ \not{D}\Psi = -V(x)|x|^{\alpha}e^{u(x)}\Psi, & \text{in } D_R^+, \\ \frac{\partial u}{\partial n} = cV(x)|x|^{\alpha}e^{u(x)}, & \text{on } L_R, \\ B\Psi = 0, & \text{on } L_R. \end{cases}$$

Then we have the following Pohozaev identity

$$\begin{aligned} &R\int_{S_R^+}\left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ &= (1+\alpha)\left\{\int_{D_R^+}\left(2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2\right)dv + \int_{L_R} cV(x)|x|^{\alpha}e^{u}ds\right\} \\ &\quad -R\int_{S_R^+}V^2(x)|x|^{2\alpha}e^{2u}d\sigma + \int_{L_R} c\frac{\partial V(s,0)}{\partial s}|s|^{\alpha}se^{u(s,0)}ds \\ &\quad - cV(s,0)|s|^{\alpha}se^{u(s,0)}|_{s=-R}^{s=R} \\ &\quad + \int_{D_R^+} x\cdot\nabla(V^2(x))|x|^{2\alpha}e^{2u}dv - \int_{D_R^+} x\cdot\nabla V(x)|x|^{\alpha}e^{u}|\psi|^2 dv \\ &\quad + \frac{1}{4}\int_{S_R^+}\left\langle\frac{\partial\Psi}{\partial\nu}, (x+\bar{x})\cdot\Psi\right\rangle d\sigma + \frac{1}{4}\int_{S_R^+}\left\langle(x+\bar{x})\cdot\Psi, \frac{\partial\Psi}{\partial\nu}\right\rangle d\sigma, \end{aligned} \tag{20}$$

where ν *is the outward normal vector to* S_R^+*, and* $\bar{x}$ *is the reflection point of* x *about* $\partial\mathbb{R}^2_+$.

Proof The case of $\alpha = 0$ and $V \equiv 1$ has already been treated in [25]. The calculation of the Pohozaev identity is standard. Since in the sequel we will need to calculate the Pohozaev identity for different equations, for reader's convenience, we give the detailed proof for this general case.

First, we multiply the first equation by $x\cdot\nabla u$ and integrate over D_R^+ to obtain

$$-\int_{D_R^+}\triangle u x\cdot\nabla u dv = \int_{D_R^+}2V^2(x)|x|^{2\alpha}e^{2u}x\cdot\nabla u dv - \int_{D_R^+}V(x)|x|^{\alpha}e^{u}|\Psi|^2 x\cdot\nabla u dv.$$

It follows from direct computations that

$$\begin{aligned} &\int_{D_R^+}\triangle u x\cdot\nabla u dv \\ &= R\int_{S_R^+}\left|\frac{\partial u}{\partial\nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma + \int_{L_R}\frac{\partial u}{\partial n}(x\cdot\nabla u)ds \\ &= R\int_{S_R^+}\left|\frac{\partial u}{\partial\nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma + \int_{L_R}cV(x)|x|^{\alpha}e^{u}(x\cdot\nabla u)ds \end{aligned}$$

$$
\begin{aligned}
&= R\int_{S_R^+}\left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma - (\alpha+1)\int_{L_R} cV(x)|x|^{\alpha}e^u ds \\
&\quad - \int_{L_R} c\frac{\partial V(s,0)}{\partial s}|s|^{\alpha}se^{u(s,0)}ds + cV(s,0)|s|^{\alpha}se^{u(s,0)}|_{s=-R}^{s=R},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{D_R^+} 2V^2(x)|x|^{2\alpha}e^{2u}x\cdot\nabla u dv \\
&= R\int_{S_R^+} V^2(x)|x|^{2\alpha}e^{2u}d\sigma - (2+2\alpha)\int_{D_R^+} V^2(x)|x|^{2\alpha}e^{2u}dv \\
&\quad - \int_{D_R^+} x\cdot\nabla(V^2(x))|x|^{2\alpha}e^{2u}dv,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{D_R^+} V(x)|x|^{\alpha}e^u|\Psi|^2 x\cdot\nabla u dv \\
&= R\int_{S_R^+} V(x)|x|^{\alpha}e^u|\Psi|^2 d\sigma - \int_{D_R^+}|x|^{\alpha}e^u x\cdot\nabla(V(x)|\Psi|^2)dv \\
&\quad -(2+\alpha)\int_{D_R^+} V(x)|x|^{\alpha}e^u|\Psi|^2 dv.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&R\int_{S_R^+}\left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\
&= (1+\alpha)\int_{D_R^+} 2V^2(x)|x|^{2\alpha}e^{2u}dv - (2+\alpha)\int_{D_R^+} V(x)|x|^{\alpha}e^u|\Psi|^2 dv \\
&\quad +(\alpha+1)\int_{L_R} cV(x)|x|^{\alpha}e^u ds \\
&\quad -R\int_{S_R^+} V^2(x)|x|^{2\alpha}e^{2u}d\sigma + R\int_{S_R^+} V(x)|x|^{\alpha}e^u|\Psi|^2 d\sigma \\
&\quad +\int_{L_R} c\frac{\partial V(s,0)}{\partial s}|s|^{\alpha}se^{u(s,0)}ds - cV(s,0)|s|^{\alpha}se^{u(s,0)}|_{s=-R}^{s=R} \\
&\quad +\int_{D_R^+} x\cdot\nabla(V^2(x))|x|^{2\alpha}e^{2u}dv - \int_{D_R^+}|x|^{\alpha}e^u x\cdot\nabla(V(x)|\Psi|^2)dv \qquad (21)
\end{aligned}
$$

On the other hand, for $x\in\mathbb{R}_+^2$, we denote $x = x_1e_1 + x_2e_2$ under the local orthonormal basis $\{e_1, e_2\}$ on $\mathbb{R}_+^2$. Using the Clifford multiplication relation

$$
e_i\cdot e_j + e_j\cdot e_i = -2\delta_{ij},\ \text{for } 1\le i, j\le 2
$$

and

$$\langle\psi, \varphi\rangle = \langle e_i \cdot \psi, e_i \cdot \varphi\rangle$$

for any spinors $\psi, \varphi \in \Gamma(\Sigma M)$. We know that

$$\langle\psi, e_i \cdot \psi\rangle + \langle e_i \cdot \psi, \psi\rangle = 0 \tag{22}$$

for any $i = 1, 2$. Using the chirality boundary condition of Ψ, we extend (u, Ψ) to the lower half disk D_R^-. Assume $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$, and define

$$u(\bar{x}) := u(x), \quad \bar{x} \in D_R^-, \tag{23}$$

$$\Psi(\bar{x}) := ie_1 \cdot \Psi(x), \quad \bar{x} \in D_R^-. \tag{24}$$

Then it follows from the argument in Lemma 3.4 of [24] that we obtain

$$\not{D}\psi = -A(x)\psi \quad \text{in } D_R.$$

Here

$$A(x) = \begin{cases} V(x)|x|^\alpha e^{u(x)}, & x \in D_R^+, \\ V(\bar{x})|\bar{x}|^\alpha e^{u(\bar{x})}, & x \in D_R^-. \end{cases}$$

Using the Schrödinger–Lichnerowicz formula $\not{D}^2 = -\triangle + \frac{1}{2}K_g$, we have

$$-\triangle\Psi = -dA(x) \cdot \psi + A^2(x)\Psi \quad \text{in } D_R. \tag{25}$$

Then we multiply (25) by $x \cdot \Psi$ (where $\cdot$ denotes the Clifford multiplication) and integrate over D_R to obtain

$$\int_{D_R} \langle\triangle\Psi, x \cdot \Psi\rangle dv = \int_{B_R} \langle dA(x) \cdot \Psi, x \cdot \Psi\rangle dv - \int_{D_R} A^2(x)\langle\Psi, x \cdot \Psi\rangle dv,$$

and

$$\int_{D_R} \langle x \cdot \Psi, \triangle\Psi\rangle dv = \int_{D_R} \langle x \cdot \Psi, dA(x) \cdot \Psi\rangle dv - \int_{D_R} A^2(x)\langle x \cdot \Psi, \Psi\rangle dv.$$

On the other hand, by partial integration,

$$\begin{aligned}
&\int_{D_R} \langle\triangle\Psi, x \cdot \Psi\rangle dv \\
&= \int_{D_R} div\langle\nabla\Psi, x \cdot \Psi\rangle dv - \int_{D_R} \sum_{\alpha=1}^{2} \langle\nabla_{e_\alpha}\Psi, e_\alpha \cdot \Psi\rangle dv - \int_{D_R} \langle\nabla\Psi, x \cdot \nabla\Psi\rangle \\
&= \int_{\partial D_R} \left\langle \frac{\partial\Psi}{\partial\nu}, x \cdot \Psi \right\rangle d\sigma + \int_{D_R} \langle\not{D}\Psi, \Psi\rangle dv - \int_{D_R} \langle\nabla\Psi, x \cdot \nabla\Psi\rangle \\
&= \int_{\partial D_R} \left\langle \frac{\partial\Psi}{\partial\nu}, x \cdot \Psi \right\rangle d\sigma - \int_{D_R} A(x)|\Psi|^2 dv - \int_{D_R} \langle\nabla\Psi, x \cdot \nabla\Psi\rangle,
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial D_R^+ \cap \mathbb{R}^2_+} \left\langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \right\rangle d\sigma - 2 \int_{D_R^+} V(x)|x|^\alpha e^u |\Psi|^2 dv \\
&\quad - \int_{D_R} \langle \nabla \Psi, x \cdot \nabla \Psi \rangle,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int_{D_R} \langle x \cdot \Psi, \triangle \Psi \rangle &= \int_{\partial D_R^+ \cap \mathbb{R}^2_+} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma - 2 \int_{D_R^+} V(x)|x|^\alpha e^u |\Psi|^2 dv \\
&\quad - \int_{D_R} \langle x \cdot \nabla \Psi, \nabla \Psi \rangle.
\end{aligned}
$$

Furthermore we also have

$$
\begin{aligned}
&\int_{D_R} \langle dA(x) \cdot \Psi, x \cdot \Psi \rangle dv + \int_{D_R} \langle x \cdot \Psi, dA(x) \cdot \Psi \rangle dv \\
&\quad = \int_{D_R} \sum_{\alpha,\beta=1}^{2} \langle \nabla_{e_\alpha} A(x) e_\alpha \cdot \Psi, e_\beta \cdot \Psi \rangle x_\beta dv \\
&\qquad + \int_{D_R} \sum_{\alpha,\beta=1}^{2} \langle e_\beta \cdot \Psi, \nabla_{e_\alpha} A(x) e_\alpha \cdot \Psi \rangle x_\beta dv \\
&\quad = 2 \int_{D_R} \sum_{\alpha=1}^{2} \langle \nabla_{e_\alpha} A(x) e_\alpha \cdot \Psi, e_\alpha \cdot \Psi \rangle x_\alpha dv \\
&\quad = 2 \int_{D_R} x \cdot \nabla(A(x)) |\Psi|^2 dv \\
&\quad = -2 \int_{D_R} A(x) x \cdot \nabla(|\Psi|^2) dv - 4 \int_{D_R} A(x) |\Psi|^2 dv + 2R \int_{\partial D_R} A(x) |\Psi|^2 dv \\
&\quad = -4 \int_{D_R^+} V(x)|x|^\alpha e^u x \cdot \nabla(|\Psi|^2) dv - 8 \int_{D_R^+} V(x)|x|^\alpha e^u |\Psi|^2 dv \\
&\qquad + 4R \int_{\partial D_R^+ \cap \mathbb{R}^2_+} V(x)|x|^\alpha e^u |\Psi|^2 dv.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&R \int_{\partial D_R^+ \cap \mathbb{R}^2_+} V(x)|x|^\alpha e^u |\Psi|^2 d\sigma - \int_{D_R^+} V(x)|x|^\alpha e^u x \cdot \nabla(|\Psi|^2) dv \\
&\quad = \frac{1}{4} \int_{\partial D_R^+ \cap \mathbb{R}^2_+} \left\langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \right\rangle d\sigma + \frac{1}{4} \int_{\partial D_R^+ \cap \mathbb{R}^2_+} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma \\
&\qquad + \int_{D_R^+} V(x)|x|^\alpha e^u |\Psi|^2 dv. \qquad (26)
\end{aligned}
$$

Putting (21) and (26) together, we obtain our Pohozaev type identity (20). □

Pohozaev type identity is shown to be closely related to the removability of local singularities of solutions. For a solution of (12) and (13), we defined in [27] the following Pohozaev constant:

Definition 4.2 ([27]). Let $(u, \Psi) \in C^2(D_r\backslash\{0\}) \times C^2(\Gamma(\Sigma(D_r\backslash\{0\})))$ be a solution of (12) and (13). For $0 < R < r$, we define the *Pohozaev constant* with respect to the Eqs. (12) with the constraint (13) as follows:

$$\begin{aligned} C(u,\Psi) := R\int_{\partial D_R(0)} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ - (1+\alpha)\int_{D_R(0)} (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2)dx \\ + R\int_{\partial D_R(0)} V^2(x)|x|^{2\alpha}e^{2u}d\sigma - \frac{1}{2}\int_{\partial D_R(0)} \left\langle \frac{\partial \Psi}{\partial \nu}, x\cdot\Psi\right\rangle + \left\langle x\cdot\Psi, \frac{\partial \Psi}{\partial \nu}\right\rangle d\sigma \\ - \int_{D_R(0)} (|x|^{2\alpha}e^{2u}x\cdot\nabla(V^2(x)) - |x|^{\alpha}e^{u}|\Psi|^2 x\cdot\nabla V(x))dx \end{aligned}$$

where ν is the outward normal vector of $\partial D_R(0)$.

It is clear that $C(u, \Psi)$ is independent of R for $0 < R < r$. Thus, the vanishing of the Pohozaev constant $C(u, \Psi)$ is equivalent to the *Pohozaev identity*

$$\begin{aligned} R\int_{\partial D_R(0)} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ = (1+\alpha)\int_{D_R(0)} (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2)dx \\ -R\int_{\partial D_R(0)} V^2(x)|x|^{2\alpha}e^{2u}d\sigma + \frac{1}{2}\int_{\partial D_R(0)} \left(\left\langle \frac{\partial \Psi}{\partial \nu}, x\cdot\Psi\right\rangle + \left\langle x\cdot\Psi, \frac{\partial \Psi}{\partial \nu}\right\rangle\right) d\sigma \\ + \int_{D_R(0)} (|x|^{2\alpha}e^{2u}x\cdot\nabla(V^2(x)) - |x|^{\alpha}e^{u}|\Psi|^2 x\cdot\nabla V(x))dx \end{aligned} \tag{27}$$

for a solution $(u, \Psi) \in C^2(D_r(0)) \times C^2(\Gamma(\Sigma D_r(0)))$ of (12) and (13).

We also proved in [27] that a local singularity is removable if the Pohozaev identity (27) holds, that is, iff the Pohozaev constant vanishes.

Theorem 4.3 [27]. *Let $(u, \Psi) \in C^2(D_r \setminus \{0\}) \times C^2(\Gamma(\Sigma(D_r \setminus \{0\})))$ be a solution of (12) and (13). Then there is a constant $\gamma < 2\pi(1+\alpha)$ such that*

$$u(x) = -\frac{\gamma}{2\pi}\log|x| + h, \quad \text{near } 0,$$

where h is bounded near 0. The Pohozaev constant $C(u, \Psi)$ and γ satisfy:

$$C(u, \Psi) = \frac{\gamma^2}{4\pi}.$$

In particular, $(u, \Psi) \in C^2(D_r) \times C^2(\Gamma(\Sigma D_r))$, *i.e. the local singularity of* (u, Ψ) *is removable, iff* $C(u, \Psi) = 0$.

For the singular boundary problem (4), we can define the Pohozaev constant in a similar way:

Definition 4.4 Let $(u, \Psi) \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r \backslash \{0\}) \times C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r \backslash \{0\})))$ be a solution of (4) and (15). For $0 < R < r$, we define the *Pohozaev constant* with respect to the Eqs. (4) with the constraint (15) as follows:

$$
\begin{aligned}
C_B(u, \Psi) &:= R \int_{\partial D_R^+ \cap \mathbb{R}_+^2} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\
&\quad -(1+\alpha) \int_{D_R^+} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dv \\
&\quad -(\alpha+1) \int_{\partial D_R^+ \cap \partial \mathbb{R}_+^2} cV(x)|x|^\alpha e^u ds \\
&\quad +R \int_{\partial D_R^+ \cap \mathbb{R}_+^2} V^2(x)|x|^{2\alpha} e^{2u} d\sigma - \int_{\partial D_R^+ \cap \partial \mathbb{R}_+^2} c \frac{\partial V(s,0)}{\partial s} |s|^\alpha s e^u ds \\
&\quad +cV(s,0)|s|^\alpha s e^u |_{s=-R}^{s=R} \\
&\quad - \int_{D_R^+} x \cdot \nabla(V^2(x))|x|^{2\alpha} e^{2u} dv + \int_{D_R^+} x \cdot \nabla V(x)|x|^\alpha e^u |\psi|^2 dv \\
&\quad -\frac{1}{4} \int_{\partial D_R^+ \cap \mathbb{R}_+^2} \left\langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \right\rangle d\sigma - \frac{1}{4} \int_{\partial D_R^+ \cap \mathbb{R}_+^2} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma.
\end{aligned}
$$

The removability theorem of a local singularity at the boundary is following:

Theorem 4.5 (Removability of a local boundary singularity). *Let* $(u, \Psi) \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r \backslash \{0\}) \times C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r \backslash \{0\})))$ *be a solution of* (4) *and* (15)*, then there is a constant* $\gamma < \pi(1+\alpha)$ *such that*

$$u(x) = -\frac{\gamma}{2\pi} \log|x| + h, \quad \text{near } 0,$$

where h is bounded near 0. *The Pohozaev constant* $C(u, \Psi)$ *and* γ *satisfy:*

$$C(u, \Psi) = \frac{\gamma^2}{2\pi}.$$

In particular, $(u, \Psi) \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r) \times C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r)))$, *i.e. the local singularity of* (u, Ψ) *is removable, iff* $C(u, \Psi) = 0$.

To prove Theorem 4.5, we need to derive the decay of spinor part Ψ near the singular point. For the case of $\alpha = 0$ and $V(x) = 1$, this is shown in [25]. By using similar arguments, we can also get the following lemma for the general case:

Lemma 4.6 *There are* $0 < \varepsilon_1 < 2\pi$ *and* $0 < \varepsilon_2 < \pi$ *such that* (v, ϕ) *satisfy*

$$\begin{cases} -\Delta v = 2V^2(x)|x|^{2\alpha}e^{2v} - V(x)|x|^{\alpha}e^{v}\langle\phi,\phi\rangle, & \text{in } B^+_{r_0},\\ \not{D}\phi = -V(x)|x|^{\alpha}e^{v}\phi, & \text{in } B^+_{r_0},\\ \frac{\partial v}{\partial n} = cV(x)|x|^{\alpha}e^{v}, & \text{on } L_{r_0}\setminus\{0\},\\ B\phi = 0, & \text{on } L_{r_0}\setminus\{0\}, \end{cases}$$

with energy conditions

$$\int_{B^+_{r_0}} |x|^{2\alpha}e^{2v}dx \le \varepsilon_1 < 2\pi, \quad \int_{B^+_{r_0}} |\phi|^4 dx \le C, \quad |c|\int_{L_{r_0}} |x|^{\alpha}e^{v}ds \le \varepsilon_2 < \pi.$$

Then for any $x \in \overline{B}^+_{\frac{r_0}{2}}$ *we have*

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \le C\left(\int_{B^+_{2|x|}} |\phi|^4 dx\right)^{\frac{1}{4}}. \tag{28}$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2(1+\alpha)-\varepsilon}})$*, then, for any* $x \in \overline{B}^+_{\frac{r_0}{2}}$*, we have*

$$|\phi(x)||x|^{\frac{1}{2}} + |\nabla\phi(x)||x|^{\frac{3}{2}} \le C|x|^{\frac{1}{4C}}\left(\int_{B^+_{r_0}} |\phi|^4 dx\right)^{\frac{1}{4}}, \tag{29}$$

for some positive constant C*. Here* ε *is any sufficiently small positive number.*

Proof of Theorem 4.5: By the conformal invariance, we assume without loss of generality that $\int_{B^+_r} |x|^{2\alpha}e^{2v}dx \le \varepsilon_1$ and $|c|\int_{L_r} |x|^{\alpha}e^{v}ds \le \varepsilon_2$ where ε_1 and ε_2 are as in Lemma 4.6. By standard potential analysis, it follows that there is a constant γ such that

$$\lim_{|x|\to 0} \frac{u}{-\log|x|} = \frac{\gamma}{\pi}.$$

By $\int_{D^+_r} |x|^{2\alpha}e^{2u}dx < C$, we obtain that $\gamma \le \pi(1+\alpha)$. Furthermore, by using Lemma 4.6 and by a similar argument as in the proof of Proposition 5.4 of [23], we can improve this to the strict inequality $\gamma < \pi(1+\alpha)$. Next we set

$$\begin{aligned} v(x) = &-\frac{1}{\pi}\int_{B^+_r} \log|x-y|(2V^2(y)|y|^{2\alpha}e^{2u} - V(y)|y|^{\alpha}e^{u}|\Psi|^2)dy\\ &-\frac{1}{\pi}\int_{L_r} \log|x-y|(cV(y)|y|^{\alpha}e^{u})d\sigma \end{aligned}$$

and set $w = u - v$. Notice that v satisfies that

$$\begin{cases} -\Delta v = 2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^\alpha e^u|\Psi|^2, & \text{in } D_r^+, \\ \frac{\partial v}{\partial n} = cV(x)|x|^\alpha e^u, & \text{on } L_r, \end{cases}$$

and w satisfies that

$$\begin{cases} -\Delta w = 0, & \text{in } D_r^+, \\ \frac{\partial v}{\partial n} = 0, & \text{on } L_r\backslash\{0\}. \end{cases}$$

We can check that

$$\lim_{|x|\to 0} \frac{v(x)}{-\log|x|} = 0.$$

Since we can extend w to $B_r\backslash\{0\}$ evenly to get a harmonic function w in $B_r\backslash\{0\}$, then we obtain that

$$\lim_{|x|\to 0} \frac{w(x)}{-\log|x|} = \lim_{|x|\to 0} \frac{u-v}{-\log|x|} = \frac{\gamma}{\pi}.$$

Duo to w is harmonic in $B_1\backslash\{0\}$ we have

$$w = -\frac{\gamma}{\pi}\log|x| + w_0$$

with a smooth harmonic function w_0 in B_r. Therefore we have

$$u = -\frac{\gamma}{\pi}\log|x| + v + w_0 \quad \text{near } 0.$$

To compute the Pohozaev constant of (u, Ψ) we need the decay of the gradient of u near the singular point. We denote that $f_1(x) := 2V^2(x)|x|^{2\alpha}e^{2u(x)}$, $f_2(x) := -V(x)|x|^\alpha e^{u(x)}|\Psi|^2(x)$ and $f_3(x) := cV(x)|x|^\alpha e^u$. Since each f_i is L^1 integrable, we can obtain $e^{|v(x)|} \in L^p(D_r^+)$ for any $p \geq 1$ and $e^{|v(x)|} \in L^p(L_r)$ for any $p \geq 1$. Since

$$f_1(x) = |x|^{-\frac{2\gamma}{\pi}+2\alpha}\left(2V^2(x)e^{2w_0(x)+2v(x)}\right),$$
$$f_2(x) = -|x|^{-\frac{\gamma}{\pi}+\alpha-1}\left(V(x)e^{w_0(x)+v(x)}|x||\Psi|^2(x)\right),$$

and

$$f_3(x) = |x|^{-\frac{\gamma}{\pi}+\alpha}(cV(x)e^{w_0(x)+v(x)}),$$

we set $s_1 = \frac{2\gamma}{\pi} - 2\alpha$ and $s_2 = \frac{\gamma}{\pi} - \alpha + 1$. Then $\max\{s_1, s_2\} = s_2 < 2$. Since $|\Psi| \leq C|x|^{-\frac{1}{2}}$ near 0 and $w_0(x)$ is smooth in B_r, we have by Hölder's inequality that $f_1 \in L^t(D_r^+)$ for any $t \in (1, \frac{2}{s_1})$ if $s_1 > 0$, and $f_1 \in L^t(D_r^+)$ for any $t > 1$ if $s_1 \leq 0$. For f_2, we have $f_2 \in L^t(D_r^+)$ for any $t \in (1, \frac{2}{s_2})$ if $s_2 > 0$, and $f_2 \in L^t(D_r^+)$ for any $t > 1$ if $s_2 \le 0$. For f_3, we have $f_3 \in L^t(L_r)$ for any $t \in (1, \frac{2}{s_1})$ if $s_1 > 0$, and $f_3 \in L^t(L_r)$ for any $t > 1$ if $s_1 \le 0$. Putting all together and by standard elliptic theory, we have $v(x)$ is in $L^\infty(\overline{D_r^+})$. On the other hand, since $v(x)$ is in $L^\infty(\overline{D_r^+})$, it follows from Lemma 4.6 that there exists a small $\delta_0 > 0$ such that

$$|\Psi| \le C|x|^{\delta_0 - \frac{1}{2}}, \quad \text{near } 0,$$

and

$$|\nabla\Psi| \le C|x|^{\delta_0 - \frac{3}{2}}, \quad \text{near } 0.$$

Next we estimate $\nabla v(x)$. If $s_1 < 0$ and $s_2 < 0$, then $v(x)$ is in $C^1(\overline{B_r^+})$. If $s_1 > 0$ or $s_2 > 0$, $\nabla v(x)$ will have a decay when $|x| \to 0$. Without loss of generality, we assume that $0 < s_1 < 2$ and $0 < s_2 < 2$. For any $x \in D_r^+$ we hanve

$$\begin{aligned}
|\nabla v(x)| &\le \frac{1}{\pi}\int_{D_r^+} \frac{1}{|x-y|}(|f_1(y)| + |f_2(y)|)dy + \frac{1}{\pi}\int_{L_r} \frac{1}{|x-y|}|f_3(y)|dy \\
&= \frac{1}{\pi}\int_{\{|x-y|\ge\frac{|x|}{2}\}\cap D_r^+} \frac{1}{|x-y|}(|f_1(y)| + |f_2(y)|)dy \\
&\quad + \frac{1}{\pi}\int_{\{|x-y|\le\frac{|x|}{2}\}\cap D_r^+} \frac{1}{|x-y|}(|f_1(y)| + |f_2(y)|)dy \\
&\quad + \frac{1}{\pi}\int_{\{|x-y|\ge\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|}|f_3(y)|dy + \frac{1}{\pi}\int_{\{|x-y|\le\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|}|f_3(y)|dy \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Fix $t \in (1, \frac{2}{s_2})$ and choose $0 < \tau_1 < 1$ such that $\frac{\tau_1 t}{t-1} < 2$. Hence, we have $0 < \tau_1 < 2 - s_2$. Then by Hölder's inequality we obtain

$$\begin{aligned}
I_1 &\le \left(\int_{\{|x-y|\ge\frac{|x|}{2}\}\cap D_r^+} \frac{1}{|x-y|^{\frac{\tau_1 t}{t-1}}}dy\right)^{\frac{t-1}{t}} \\
&\quad \times \left(\int_{\{|x-y|\ge\frac{|x|}{2}\}\cap D_r^+} \frac{1}{|x-y|^{(1-\tau_1)t}}(|f_1| + |f_2|)^t dy\right)^{\frac{1}{t}} \le \frac{C}{|x|^{1-\tau_1}}.
\end{aligned}$$

For I_2, since $y \in \{y||x-y| \le \frac{|x|}{2}\}$ implies that $|y| \ge \frac{|x|}{2}$, we can get that

$$I_2 \le C\int_{\{|x-y|\le\frac{|x|}{2}\}\cap D_r^+} \frac{1}{|x-y||y|^{s_2}}dy \le C|x|^{1-s_2}.$$

Similarly, for I_3, we fix $t \in (1, \frac{2}{s_1})$ and choose $\tau_2 > 0$ such that $\frac{\tau_2 t}{t-1} < 1$, and hence we have $0 < \tau_2 < 1 - \frac{s_1}{2}$. By Holder's inequality we obtain,

$$
\begin{aligned}
|I_3| &\leq \frac{1}{\pi}\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|^{\frac{\tau_2 t}{t-1}}}dy\right)^{\frac{t-1}{t}}\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|^{t(1-\tau_2)}}|f_3(y)|^t dy\right)^{\frac{1}{t}} \\
&\leq \frac{C}{|x|^{1-\tau_2}}.
\end{aligned}
$$

For I_4 we have

$$
|I_4| \leq C\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|}\frac{1}{|y|^{\frac{s_1}{2}}}dy \leq \frac{C}{|x|^{\frac{s_1}{2}}}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap L_r} \frac{1}{|x-y|}dy \leq \frac{C}{|x|^{\tau_3}},
$$

for some τ_3 with $0 < \tau_3 < 1$. In conclusion, for all $x \in B_r^+(0)$ we have

$$
|\nabla v(x)| \leq \frac{C}{|x|^{1-\tau_1}} + \frac{C}{|x|^{1-\tau_2}} + \frac{C}{|x|^{\tau_3}} \tag{30}
$$

for suitable constants $0 < \tau_1 < 2 - s_2$, $0 < \tau_2 < 1 - \frac{s_1}{2}$ and $0 < \tau_3 < 1$.

At this point we are ready to compute the Pohozaev constant $C(u, \Psi)$. We denote

$$
\nabla u = -\frac{\gamma}{\pi}\frac{x}{|x|^2} + \nabla(w_0 + v(x)) = -\frac{\gamma}{\pi}\frac{x}{|x|^2} + \nabla\eta(x).
$$

By (30), we have

$$
\begin{aligned}
& r\int_{S_r^+}\left(\frac{1}{2}|\nabla u|^2 - |\frac{\partial u}{\partial \nu}|^2\right)ds \\
&= r\int_{S_r^+}\frac{1}{2}\left[\left(\frac{\gamma}{\pi}\right)^2\frac{1}{|x|^2} - 2\frac{\gamma}{\pi}\frac{x\cdot\nabla\eta}{|x|^2} + |\nabla\eta|^2\right]ds - r\int_{S_r^+}\left(-\frac{\gamma}{\pi}\frac{1}{|x|} + \frac{x\cdot\nabla\eta}{|x|}\right)^2 ds \\
&= r\int_{S_r^+}\left[-\frac{1}{2}\left(\frac{\gamma}{\pi}\right)^2\frac{1}{|x|^2} + \frac{\gamma}{\pi}\frac{x\cdot\nabla\eta}{|x|^2} + \frac{1}{2}|\nabla\eta|^2 - \left(\frac{x\cdot\nabla\eta}{|x|}\right)^2\right]d\sigma \\
&= -\frac{1}{2}\left(\frac{\gamma}{\pi}\right)^2\pi + \frac{\gamma}{\pi}r\int_{S_r^+}\frac{x\cdot\nabla\eta}{|x|^2} + \frac{r}{2}\int_{S_r^+}|\nabla\eta|^2 - r\int_{S_r^+}\left(\frac{x\cdot\nabla\eta}{|x|}\right)^2 \\
&= -\frac{\gamma^2}{2\pi} + o_r(1),
\end{aligned}
$$

where $o_r(1) \to 0$ as $r \to 0$. We also have

$$(1+\alpha)\int_{D_r^+} 2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2 dx = o_r(1),$$

and

$$r\int_{S_r^+} V^2(x)|x|^{2\alpha}e^{2u}d\sigma = o_r(1),$$

and

$$\int_{D_r^+} (|x|^{2\alpha}e^{2u}x\cdot\nabla(V^2(x)) - |x|^{\alpha}e^{u}|\Psi|^2 x\cdot\nabla V(x))dx = o_r(1),$$

and

$$(\alpha+1)\int_{S_r^+} cV(x)|x|^{\alpha}e^{u}d\sigma - \int_{L_r} c\frac{\partial V(s,0)}{\partial s}|s|^{\alpha}se^{u}ds + cV(s,0)|s|^{\alpha}se^{u}|_{s=-r}^{s=r} = o_r(1),$$

and

$$\int_{S_r^+}\left\langle \frac{\partial\Psi}{\partial\nu}, (x+\bar{x})\cdot\nabla\Psi\right\rangle d\sigma + \int_{S_r^+}\left\langle (x+\bar{x})\cdot\nabla\Psi, \frac{\partial\Psi}{\partial\nu}\right\rangle d\sigma = o_r(1).$$

Putting all together and letting $r \to 0$, we get

$$C(u,\Psi) = \lim_{r\to 0} C(u,\Psi,r) = \frac{\gamma^2}{2\pi}.$$

Since $C(u,\Psi) = 0$ for (u,Ψ), therefore we get $\gamma = 0$. This implies that the local singularity of (u,Ψ) is removable. □

5 Bubble energy

After a suitable rescaling at a boundary blow-up point, we will obtain a bubble, i.e. an entire solution on the upper half-plane $\mathbb{R}^2_+$ with finite energy. In this section, we will investigate such entire solutions. We will first show the asymptotic behavior of an entire solution and compute the bubble energy, and then show that an entire solution can be conformally extended to a spherical cap, i.e., the singularity at infinity is removable.

The considered equations are

$$\begin{cases} -\Delta u = 2|x|^{2\alpha}e^{2u} - |x|^{\alpha}e^{u}\langle\psi,\psi\rangle, & \text{in } \mathbb{R}^2_+, \\ \not{D}\psi = -|x|^{\alpha}e^{u}\psi, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial n} = c|x|^{\alpha}e^{u}, & \text{on } \partial\mathbb{R}^2_+, \\ B\psi = 0, & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{31}$$

The energy condition is

$$I(u,\psi)=\int_{\mathbb{R}^2_+}(|x|^{2\alpha}e^{2u}+|\psi|^4)dx+\int_{\partial\mathbb{R}^2_+}|x|^{\alpha}e^{u}ds<\infty. \tag{32}$$

First, let us notice that if (u,ψ) is a weak solution of (31) and (32) with $u\in H^{1,2}_{loc}(\mathbb{R}^2_+)$ and $\psi\in W^{1,\frac{4}{3}}_{loc}(\Gamma(\Sigma\mathbb{R}^2_+))$, by using similar arguments as in the proof of Proposition 3.2, we have $u^+\in L^\infty(\overline{\mathbb{R}}^2_+)$. Consequently, it follows that $u\in C^2_{loc}(\mathbb{R}^2_+)\cap C^1_{loc}(\overline{\mathbb{R}}^2_+)$ and $\psi\in C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+))\cap C^1_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+))$.

We call (u,ψ) a *regular* solution of (31) and (32), if $u\in C^2_{loc}(\mathbb{R}^2_+)\cap C^1_{loc}(\overline{\mathbb{R}}^2_+)$ and $\psi\in C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+))\cap C^1_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+))$.

Next, we denote by (v,ϕ) the Kelvin transformation of (u,ψ), i.e.

$$\begin{aligned} v(x)&=u\left(\frac{x}{|x|^2}\right)-2(1+\alpha)\ln|x|,\\ \phi(x)&=|x|^{-1}\psi\left(\frac{x}{|x|^2}\right). \end{aligned}$$

Then (v,ϕ) satisfies

$$\begin{cases} -\Delta v=2|x|^{2\alpha}e^{2v}-|x|^{\alpha}e^{v}\langle\phi,\phi\rangle, & \text{in } \mathbb{R}^2_+,\\ \not{D}\phi=-|x|^{\alpha}e^{v}\phi, & \text{in } \mathbb{R}^2_+,\\ \frac{\partial v}{\partial n}=c|x|^{\alpha}e^{v}, & \text{on } \partial\mathbb{R}^2_+\backslash\{0\},\\ B\phi=0, & \text{on } \partial\mathbb{R}^2_+\backslash\{0\}. \end{cases} \tag{33}$$

And, by change of variable, we can choose r_0 small enough such that (v,ϕ) satisfies

$$\int_{|x|\le r_0}|x|^{2\alpha}e^{2v}dx\le\varepsilon_1<2\pi,\quad \int_{|x|\le r_0}|\phi|^4dx\le C,\quad |c|\int_{|s|\le r_0}|x|^{\alpha}e^{v}ds\le\varepsilon_2<\pi. \tag{34}$$

Applying Lemma 4.6 to (33) and (34), and by the Kelvin transformation, we obtain the asymptotic estimate of the spinor $\psi(x)$

$$|\psi(x)|\le C|x|^{-\frac{1}{2}-\delta_0}\qquad \text{for } |x| \text{ near } \infty, \tag{35}$$

and

$$|\nabla\psi(x)|\le C|x|^{-\frac{3}{2}-\delta_0}\qquad \text{for } |x| \text{ near } \infty, \tag{36}$$

for some positive number δ_0 provided that $e^{2v}=O(\frac{1}{|x|^{2(1+\alpha)-\varepsilon}})$, where ε is any small positive number.

Denote

$$d=\int_{\mathbb{R}^2_+}2|x|^{2\alpha}e^{2u}-|x|^{\alpha}e^{u}|\psi|^2dx+\int_{\partial R^2_+}c|x|^{\alpha}e^{u}ds,$$

and

$$\xi_0 = \int_{\mathbb{R}^2_+} e^u \psi dx.$$

Next, we will show that $d = 2(1+\alpha)\pi$ and ξ_0 is a well-defined constant spinor.

Proposition 5.1 *Let (u, ψ) be a regular solution of (31) and (32) and let c be a non-negative constant. Then we have*

$$u(x) = -\frac{d}{\pi} \ln|x| + C + O(|x|^{-1}) \quad \text{for} \quad |x| \quad \text{near} \quad \infty, \tag{37}$$

$$\psi(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} (I + ie_1) \cdot \xi_0 + o(|x|^{-1}) \quad \text{for} \quad |x| \quad \text{near} \quad \infty, \tag{38}$$

where $\cdot$ is the Clifford multiplication, C is a positive universal constant, and I is the identity. In particular we have $d = 2(1+\alpha)\pi$ and ξ_0 is well defined.

Proof We shall apply standard potential analysis to prove this proposition. Similar arguments can be found in [13,21,22] and the references therein. The essential facts used in this case are the Pohozaev identity and the decay estimate for the spinor. For readers' convenience, we sketch the proof here.

Step 1. $\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = -\frac{d}{\pi}$ and $d > \pi(1+\alpha)$.

Let

$$w(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2_+} (\log|x-y| + \log|\overline{x} - y| - 2\log|y|) \left(2|y|^{2\alpha} e^{2u(y)} - |y|^\alpha e^{u(y)} |\psi(y)|^2\right) dy$$
$$+ \frac{1}{2\pi} \int_{\partial\mathbb{R}^2_+} (\log|x-y| + \log|\overline{x} - y| - 2\log|y|) c|y|^\alpha e^{u(y)} dy.$$

where $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$. It is easy to check that $w(x)$ satisfies

$$\begin{cases} \Delta w = 2|x|^{2\alpha} e^{2u} - |x|^\alpha e^u |\psi|^2, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial w}{\partial n} = -c|x|^\alpha e^u, & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

and

$$\lim_{|x|\to\infty} \frac{w(x)}{\ln|x|} = \frac{d}{\pi}.$$

Consider $v(x) = u + w$. Then $v(x)$ satisfies

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial n} = 0, & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

We extend $v(x)$ to $\mathbb{R}^2$ by even reflection such that $v(x)$ is harmonic in $\mathbb{R}^2$. From Lemma 5.1 we know $v(x) \le C(1 + \ln(|x|+1))$ for some positive constant C. Thus

$v(x)$ is a constant. This completes the proof of Step 1. Since $\int_{\mathbb{R}^2_+} |x|^{2\alpha} e^{2u} dx < \infty$, we get that $d \geq \pi(1+\alpha)$. Furthermore, similarly as in the case of the usual Liouville or super-Liouville equation, we can show that $d > \pi(1+\alpha)$.

Step 2. The proof of (37) and $d = 2\pi(1+\alpha)$.

Notice that we have shown $d > \pi(1+\alpha)$ in Step 2, we then can improve the estimates of e^{2u} to

$$e^{2u} \leq C|x|^{-2(1+\alpha)-\varepsilon} \quad \text{for } |x| \text{ near } \infty.$$

Therefore the asymptotic estimates (35) and (36) of the spinor $\psi(x)$ hold. By using the standard potential analysis we can obtain that

$$u(x) = -\frac{d}{\pi} \ln |x| + C + O(|x|^{-1}) \qquad \text{for} \quad |x| \quad \text{near} \quad \infty$$

for some constant $C > 0$. Thus we get the proof of (37).

Furthermore, we can show that $d = 2\pi(1+\alpha)$. For sufficiently large $R > 0$, the Pohozaev identity for the solution (u, ψ) gives

$$\begin{aligned} &R \int_{S_R^+} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ &= (1+\alpha) \int_{D_R^+} 2|x|^{2\alpha} e^{2u} - |x|^\alpha e^u |\Psi|^2 dv + (\alpha+1) \int_{L_R} c|x|^\alpha e^u ds \\ &\quad -R \int_{S_R^+} |x|^{2\alpha} e^{2u} d\sigma - c|s|^\alpha s e^u |_{s=-R}^{s=R} \\ &\quad +\frac{1}{4} \int_{S_R^+} \langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \rangle d\sigma + \frac{1}{4} \int_{S_R^+} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma. \end{aligned} \tag{39}$$

By the asymptotic estimates (35), (36) and (37) of (u, ψ) we have

$$\lim_{R\to+\infty} R \int_{S_R^+} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma = \frac{d^2}{2\pi},$$

and

$$\lim_{R\to+\infty} R \int_{S_R^+} |x|^{2\alpha} e^{2u} d\sigma + c|s|^\alpha s e^u |_{s=-R}^{s=R} = 0,$$

and

$$\lim_{R\to+\infty} \int_{S_R^+} \left\langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \right\rangle d\sigma + \int_{S_R^+} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma = 0.$$

Let $R \to +\infty$ in (39), we get that

$$\frac{d^2}{2\pi} = (1+\alpha)d.$$

It follows that $d = 2\pi(1+\alpha)$.

Step 3. The proof of (38).

Since $d = 2\pi(1+\alpha)$ by Step 2, we can improve the estimate for e^{2u} to

$$e^{2u} \le C|x|^{-4(1+\pi)} \qquad \text{for } |x| \text{ near } \infty. \tag{40}$$

This implies that the constant spinor ξ_0 is well defined. By using the chirality boundary condition of spinor, we extend (u, ψ) to the lower half plane $\mathbb{R}^2_+$(see (23) and (24)) to get

$$\not{D}\psi = -A(x)\psi, \quad \text{in } \mathbb{R}^2.$$

Here $A(x)$ is defined by

$$A(x) = \begin{cases} |x|^\alpha e^{u(x)}, & x \in \mathbb{R}^2_+, \\ |\bar{x}|^\alpha e^{u(\bar{x})}, & x \in \mathbb{R}^2_-. \end{cases}$$

Define

$$\xi_1 = \int_{\mathbb{R}^2} A(x)\psi dx.$$

The constant spinor ξ_1 is also well defined. From the asymptotic estimates (35) and (40) and a similar argument in [27] we obtain

$$\psi(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}\cdot \xi_1 + o(|x|^{-1}) \qquad \text{for } |x| \text{ near } \infty. \tag{41}$$

Since

$$\begin{aligned}
\xi_1 &= \int_{\mathbb{R}^2_+} A(x)\psi dx + \int_{\mathbb{R}^2_-} A(x)\psi dx \\
&= \int_{\mathbb{R}^2_+} |x|^\alpha e^u \psi dx + \int_{\mathbb{R}^2_-} |\bar{x}|^\alpha e^{u(\bar{x})} i e_1 \cdot \psi(\bar{x}) dx \\
&= \int_{\mathbb{R}^2_+} |x|^\alpha e^u \psi dx + \int_{\mathbb{R}^2_+} |y|^\alpha e^{u(y)} i e_1 \cdot \psi(y) dy \\
&= (I + ie_1)\cdot \int_{\mathbb{R}^2_+} |x|^\alpha e^u \psi dx \\
&= (I + ie_1)\cdot \xi_0.
\end{aligned}$$

Hence we obtain from (41)

$$\psi(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}(I + ie_1)\cdot\xi_0 + o(|x|^{-1}) \qquad \text{for} \quad |x| \quad \text{near} \quad \infty.$$

Thus we finish the proof of Step 3 and we complete the proof of the Proposition. □

Proposition 5.1 indicates that the singularity at infinity of regular solutions for (31) and (32) can be removed as in many other conformally invariant problems.

Theorem 5.2 *Let (u, ψ) be a regular solution of (31) and (32). Then (u, ψ) extends conformally to a regular solution on a spherical cap $\mathbb{S}^2_{c'}$, where c' is the geodesic curvature of $\partial\mathbb{S}^2_{c'}$.*

Proof Let (v, ϕ) be the Kelvin transformation of (u, ψ) as before. Then (v, ϕ) satisfies the system (33). To prove the theorem, by conformal invariance, it is sufficient to show that (v, ϕ) is regular on $\overline{\mathbb{R}}^2_+$. Applying Proposition 5.1, we get

$$v(x) = \left(\frac{d}{\pi} - 2(1+\alpha)\right)\ln|x| + O(1) \qquad \text{for} \quad |x| \quad \text{near} \quad 0. \tag{42}$$

Since $\alpha = 2\pi(1+\alpha)$, it follows that v is bounded near the singularity 0. Recall that ϕ is also bounded near 0, we can apply elliptic theory to obtain that (v, ϕ) is regular on $\overline{\mathbb{R}}^2_+$. □

6 Energy identity for spinors

The energy identity for spinor part of solutions to the super-Liouville equations on closed Riemann surfaces was derived in [23,27]. In this section, we shall prove an analogue for the singular super-Liouville boundary problem, i.e. Theorem 1.4. For harmonic maps in dimension two and J-holomorphic curves as well as for solutions of certain nonlinear Dirac type equations, similar results are derived in [14,38,39,42] and the references therein.

To prove Theorem 1.4, we shall derive the local estimate for the spinor part on an upper half annulus. Since we can extend (u, Ψ) to the lower half disk D_r^- by the chirality boundary condition of Ψ, the proof of this local estimate can be established by using the result of Lemma 3.1 of [23]. Here we just state the Lemma and omit the proof.

Lemma 6.1 *Let (u, Ψ) satisfies (4) and*

$$\int_{D_r^+} |x|^{2\alpha}e^{2u} + |\Psi|^4 dx + \int_{L_r} |x|^{\alpha}e^u ds < C.$$

For $0 < r_1 < 2r_1 < \frac{r_2}{2} < r_2 < r$*, consider the annulus* $A_{r_1,r_2} = \{x \in \mathbb{R}^2 | r_1 \le |x| \le r_2\}$ *and the upper half annulus* $A^+_{r_1,r_2} = A_{r_1,r_2} \cap \mathbb{R}^2_+$*. Then we have*

$$\begin{aligned}&\left(\int_{A^+_{2r_1,\frac{r_2}{2}}} |D\Psi|^{\frac{4}{3}}\right)^{\frac{3}{4}} + \left(\int_{A^+_{2r_1,\frac{r_2}{2}}} |\Psi|^4\right)^{\frac{1}{4}}\\ &\le C_0 \left(\int_{A^+_{r_1,r_2}} |x|^{2\alpha} e^{2u}\right)^{\frac{1}{2}} \left(\int_{A^+_{r_1,r_2}} |\Psi|^4\right)^{\frac{1}{4}} + C\left(\int_{A^+_{r_1,2r_1}} |\Psi|^4\right)^{\frac{1}{4}}\\ &\quad + C\left(\int_{A^+_{\frac{r_2}{2},r_2}} |\Psi|^4\right)^{\frac{1}{4}} \end{aligned} \tag{43}$$

for a positive constant C_0 *and some universal positive constant* C*.*

Proof of Theorem 1.4 We will follow closely the argument for the energy identity of harmonic maps, see [14], or for super-Liouville equations, see [7,23,25,27] . Since the blow-up set Σ_1 is finite, we can find small disk $D^+_{\delta_i}(x_i)$, which is centered at each blow-up point x_i, such that $D^+_{\delta_i}(x_i) \cap D^+_{\delta_j}(x_j) = \emptyset$ for $i \ne j, i, j = 1, 2, \ldots, P$, and on $(D^+_r \cup Ł_r) \backslash \bigcup_{i=1}^{P} (D^+_{\delta_i}(x_i) \cup L_{\delta_i}(x_i))$, Ψ_n converges strongly to Ψ in L^4. So, we need to prove that there are $(u^{i,k}, \xi^{i,k})$, which are solutions of (7), $i = 1, 2, \ldots, I; k = 1, 2, \ldots, K_i$, such that

$$\lim_{\delta_i \to 0} \lim_{n\to\infty} \int_{D^+_{\delta_i}(x_i)} |\Psi_n|^4 dv = \sum_{k=1}^{L_i} \int_{S^2} |\xi^{i,k}|^4 dv, \text{ for } i = 1, 2, \ldots, I; \tag{44}$$

or, we need to prove that there are $(u^{j,l}, \xi^{j,l})$, which are solutions of (8), $j = 1, 2, \ldots, J; l = 1, 2, \ldots, L_j$, such that

$$\lim_{\delta_j \to 0} \lim_{n\to\infty} \int_{D^+_{\delta_j}(x_i)} |\Psi_n|^4 dv = \sum_{l=1}^{L_j} \int_{S^2_{c'}} |\xi^{j,l}|^4 dv, \text{ for } j = 1, 2, \ldots, J; \tag{45}$$

When $p \in (D^+_r)^o$, from [27], we know that (44) holds. So, without loss of generality, we assume that $p \in L_r$ and there is only one bubble at each blow-up point p. Furthermore, we may assume that $p = 0$. The case of $p \ne 0$ can be handled in an analogous way and in fact this case is simpler, as $|x|^\alpha$ is a smooth function near p. Then what we need to prove is that there exists a bubble (u, ξ) as (7), such that

$$\lim_{\delta \to 0} \lim_{n\to\infty} \int_{D^+_\delta} |\Psi_n|^4 dv = \int_{S^2} |\xi|^4 dv, \tag{46}$$

or there exists a bubble (u, ξ) as (8) such that such that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{D_\delta^+}|\Psi_n|^4 dv = \int_{S^2_{c'}}|\xi|^4 dv. \tag{47}$$

Next we rescale functions (u_n, Ψ_n) at the blow-up point $p = 0$ and then try to get the bubble of (u_n, Ψ_n). To this purpose, we let $x_n \in \overline{D}_\delta^+$ such that $u_n(x_n) = \max_{\overline{D}_\delta^+} u_n(x)$. Write $x_n = (s_n, t_n)$. It is clear that $x_n \to p$ and $u_n(x_n) \to +\infty$. Define $\lambda_n = e^{-\frac{u_n(x_n)}{\alpha+1}}$. We know λ_n, $|x_n|$ and t_n converge to 0 as $n \to 0$, but their rates of converging to 0 may be different. Next we will distinguish three cases.

Case I. $\frac{|x_n|}{\lambda_n} = O(1)$ as $n \to +\infty$.

In this case, we define the rescaling functions

$$\begin{cases} \widetilde{u}_n(x) = u_n(\lambda_n x) + (1+\alpha)\ln\lambda_n \\ \widetilde{\Psi}_n(x) = \lambda_n^{\frac{1}{2}}\Psi_n(\lambda_n x) \end{cases}$$

for any $x \in \overline{D}^+_{\frac{\delta}{2\lambda_n}}$. Then $(\widetilde{u}_n(x), \widetilde{\Psi}_n(x))$ satisfies

$$\begin{cases} -\Delta\widetilde{u}_n(x) = 2V^2(\lambda_n x)|x|^{2\alpha}e^{2\widetilde{u}_n(x)} - V(\lambda_n x)|x|^\alpha e^{\widetilde{u}_n(x)}|\widetilde{\Psi}_n(x)|^2, & \text{in } D^+_{\frac{\delta}{2\lambda_n}}, \\ \not{D}\widetilde{\Psi}_n(x) = -V(\lambda_n x)|x|^\alpha e^{\widetilde{u}_n(x)}\widetilde{\Psi}_n(x), & \text{in } D^+_{\frac{\delta}{2\lambda_n}}, \\ \frac{\partial\widetilde{u}_n(x)}{\partial n} = cV(\lambda_n x)|x|^\alpha e^{\widetilde{u}_n(x)}, & \text{on } L_{\frac{\delta}{2\lambda_n}}, \\ B\widetilde{\Psi}_n(x) = 0, & \text{on } L_{\frac{\delta}{2\lambda_n}}, \end{cases}$$

with the energy condition

$$\int_{D^+_{\frac{\delta}{2\lambda_n}}}|x|^{2\alpha}e^{2\widetilde{u}_n(x)} + |\widetilde{\Psi}_n(x)|^4 dv + \int_{L_{\frac{\delta}{2\lambda_n}}}|x|^\alpha e^{\widetilde{u}_n(x)}d\sigma < C.$$

We know that

$$\max_{\bar{D}^+_{\frac{\delta}{2\lambda_n}}}\widetilde{u}_n(x) = \widetilde{u}_n(\frac{x_n}{\lambda_n}) = u_n(x_n) + (\alpha_n + 1)\ln\lambda_n = 0.$$

Notice that the maximum point of $\widetilde{u}_n(x)$, i.e. $\frac{x_n}{\lambda_n}$, is bounded, namely $|\frac{x_n}{t_n}| \le C$. So by taking a subsequence, we can assume that $\frac{x_n}{t_n} \to x_0 \in \bar{\mathbb{R}}^2_+$ with $|x_0| \le C$. Therefore it follows from Theorem 1.3 that, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+)) \cap C^1_{loc}(\Gamma(\Sigma\bar{\mathbb{R}}^2_+))$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$\begin{cases} -\Delta\widetilde{u} = 2V^2(0)|x|^{2\alpha}e^{2\widetilde{u}} - V(0)|x|^\alpha e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2_+, \\ \not{D}\widetilde{\Psi} = -V(0)|x|^\alpha e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial\widetilde{u}}{\partial n} = cV(0)|x|^\alpha e^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_+, \\ B\Psi = 0, & \text{on } \partial\mathbb{R}^2_+ \end{cases} \tag{48}$$

with the energy condition $\int_{\mathbb{R}^2_+}(|x|^{2\alpha}e^{2\widetilde{u}}+|\widetilde{\Psi}|^4)dx+\int_{\partial\mathbb{R}^2_+}|x|^{\alpha}e^{\widetilde{u}}d\sigma<\infty$. By Proposition 5.1, there holds

$$\int_{\mathbb{R}^2_+}(2V^2(0)|x|^{2\alpha}e^{2\widetilde{u}}-V(0)|x|^{\alpha}e^{\widetilde{u}}|\widetilde{\Psi}|^2)dx+\int_{\partial\mathbb{R}^2_+}cV(0)|x|^{\alpha}e^{\widetilde{u}}d\sigma=2\pi(1+\alpha).$$

By the removability of a global singularity (Theorem 5.2), we get a bubbling solution on $S^2_{c'}$.

Case II. $\frac{|x_n|}{\lambda_n}\to+\infty$ as $n\to+\infty$.

In this case, we must have

$$\overline{u}_n(y_n):=u_n(x_n)+(\alpha+1)\ln|x_n|=(\alpha+1)\ln|x_n|-(\alpha+1)\ln\lambda_n\to+\infty. \quad (49)$$

Therefore we can rescale twice to get the bubble. First, we define the rescaling functions

$$\begin{cases}\overline{u}_n(x)=u_n(|x_n|x)+(\alpha+1)\ln|x_n|\\ \overline{\Psi}_n(x)=|x_n|^{\frac{1}{2}}\Psi_n(|x_n|x)\end{cases}$$

for any $x\in\overline{D}^+_{\frac{\delta}{2|x_n|}}$. Then $(\overline{u}_n(x),\overline{\Psi}_n(x))$ satisfies that

$$\begin{cases}-\Delta\overline{u}_n(x)=2V^2(|x_n|x)|x|^{2\alpha}e^{2\overline{u}_n(x)}-V(|x_n|x)|x|^{\alpha}e^{\overline{u}_n(x)}|\overline{\Psi}_n(x)|^2, & \text{in } D^+_{\frac{\delta}{2|x_n|}},\\ \not{D}\overline{\Psi}_n(x)=-V(|x_n|x)|x|^{\alpha}e^{\overline{u}_n(x)}\overline{\Psi}_n(x), & \text{in } D^+_{\frac{\delta}{2|x_n|}},\\ \frac{\partial\overline{u}_n(x)}{\partial\nu}=cV(|x_n|x)|x|^{\alpha}e^{\overline{u}_n(x)}, & \text{on } L_{\frac{\delta}{2|x_n|}},\\ B\overline{\Psi}_n(x)=0, & \text{on } L_{\frac{\delta}{2|x_n|}}.\end{cases}$$

Set that $y_n=\frac{x_n}{|x_n|}$. We assume that $y_0=\lim_{n\to\infty}\frac{x_n}{|x_n|}$. By (49), we know y_0 is a blow-up point of $(\overline{u}_n,\overline{\Psi}_n)$. We can set $\delta_n=e^{-\overline{u}_n(y_n)}$, and $\rho_n=\frac{e^{-u_n(x_n)}}{|x_n|^{\alpha}}=\lambda_n(\frac{\lambda_n}{|x_n|})^{\alpha}$. It is clear that $\delta_n\to0$, $\rho_n\to0$ and $\frac{|x_n|}{\rho_n}\to+\infty$ as $n\to\infty$. We define the rescaling functions

$$\begin{cases}\widetilde{u}_n(x)=\overline{u}_n(\delta_nx+y_n)+\ln\delta_n=u_n(x_n+\rho_nx)-u_n(x_n)\\ \widetilde{\Psi}_n(x)==\delta_n^{\frac{1}{2}}\overline{\Psi}_n(\delta_nx+y_n)=\rho_n^{\frac{1}{2}}\Psi_n(x_n+\rho_nx)\end{cases}$$

for any x such that $y_n+\delta_nx\in\overline{D}^+_R(y_n)$ with any $R>1$. By a direct computation, we have

$$\Omega_n=\{x\in\mathbb{R}^2|y_n+\delta_nx\in\overline{D}^+_R(y_n)\}=\{x\in\mathbb{R}^2|x_n+\rho_nx\in\overline{D}^+_{R|x_n|}(x_n)\}.$$

We set $L_n = \partial\Omega_n \cap \{x \in \mathbb{R}^2 | t = -\frac{t_n}{\rho_n}\}$. Then $(\widetilde{u}_n(x), \widetilde{\Psi}_n(x))$ satisfies

$$\begin{cases} -\Delta\widetilde{u}_n(x) = 2V^2(x_n+\rho_n x)|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x|^{2\alpha}e^{2\widetilde{u}_n(x)} \\ \qquad\qquad -V(x_n+\rho_n x)|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x|^{\alpha}e^{\widetilde{u}_n(x)}|\widetilde{\Psi}_n(x)|^2, & \text{in } \Omega_n, \\ \not{D}\widetilde{\Psi}_n(x) = -V(x_n+\rho_n x)|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x|^{\alpha}e^{\widetilde{u}_n(x)}\widetilde{\Psi}_n(x), & \text{in } \Omega_n, \\ \frac{\partial\widetilde{u}_n(x)}{\partial n} = cV(x_n+\rho_n x)|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x|^{\alpha}e^{\widetilde{u}_n(x)}, & \text{on } L_n, \\ B\widetilde{\Psi}_n(x) = 0, & \text{on } L_n, \end{cases}$$

with the energy condition

$$\int_{\Omega_n}\left|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x\right|^{2\alpha}e^{2\widetilde{u}_n(x)}+\left|\widetilde{\Psi}_n(x)|^4dv+\int_{L_n}|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x\right|^{\alpha}e^{\widetilde{u}_n(x)}d\sigma < C.$$

It is clear that

$$\widetilde{u}_n(x) \le \max_{\Omega_n}\widetilde{u}_n(x) = \widetilde{u}_n(0) = 0.$$

Now we proceed by distinguishing two subcases.

Case II.1 $\frac{t_n}{\rho_n} \to +\infty$ as $n \to \infty$.

Notice that $|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x| \to 1$ as $n \to \infty$ in $C^0_{loc}(\mathbb{R}^2)$. It follows from Theorem 1.3 that, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2))$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$\begin{cases} -\Delta\widetilde{u} = 2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2, \\ \not{D}\widetilde{\Psi} = -V(0)e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2, \end{cases} \tag{50}$$

with the energy condition $\int_{\mathbb{R}^2} e^{2\widetilde{u}} + |\widetilde{\Psi}|^4 dx < \infty$. By Proposition 6.4 in [21], there holds

$$\int_{\mathbb{R}^2}(2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2)dx = 4\pi.$$

By the removability of a global singularity (Theorem 6.5 in [21]), we get a bubbling solution on S^2.

Case II.2 $\frac{t_n}{\rho_n} \to \Lambda$ as $n \to \infty$.

Similar in the Case II.1, we have from Theorem 1.3 that, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2_{-\Lambda}) \cap C^1_{loc}(\bar{\mathbb{R}}^2_{-\Lambda}) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_{-\Lambda})) \cap C^1_{loc}(\Sigma\bar{\mathbb{R}}^2_{-\Lambda})$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$\begin{cases} -\Delta\widetilde{u} = 2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2_{-\Lambda}, \\ \not{D}\widetilde{\Psi} = -V(0)e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2_{-\Lambda}, \\ \frac{\partial\widetilde{u}}{\partial n} = cV(0)e^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_{-\Lambda}, \\ B\widetilde{\Psi} = 0, & \text{on } \partial\mathbb{R}^2_{-\Lambda}, \end{cases} \tag{51}$$

with the energy condition $\int_{\mathbb{R}^2_{-\Lambda}} e^{2\widetilde{u}} + |\widetilde{\Psi}|^4 dx + \int_{\partial\mathbb{R}^2_{-\Lambda}} e^{\widetilde{u}} d\sigma < \infty$. By Proposition 6.4 in [21], there holds

$$\int_{\mathbb{R}^2_{-\Lambda}} (2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2)dx + \int_{\partial\mathbb{R}^2_{-\Lambda}} cV(0)e^{\widetilde{u}} d\sigma = 2\pi.$$

By the removability of a global singularity (Theorem 6.5 in [25]), we get a bubbling solution on $S^2_{c'}$.

It is well know, in order to prove (46) or (47), we need to prove that there is no any energy of Ψ_n in the neck domain, i.e.

$$\lim_{\delta\to 0}\lim_{R\to+\infty}\lim_{n\to\infty}\int_{A^+_{\delta,R,n}} |\Psi_n|^4 dv = 0, \tag{52}$$

where $A^+_{\delta,R,n}$ is the neck domain which is defined latter. To this purpose, we shall proceed separately for Case I, Case II.1 and Case II.2.

For **Case I**, we define the neck domain is

$$A^+_{\delta,R,n} = \{x \in \mathbb{R}^2_+ | \lambda_n R \le |x| \le \delta\}.$$

We have two claims.

Claim 1 For any $\varepsilon > 0$, there is an $N > 1$ such that for any $n \ge N$, we have

$$\int_{D^+_r\setminus D^+_{e^{-1}r}} (|x|^{2\alpha}e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_r\setminus D^+_{e^{-1}r})\cap\partial\mathbb{R}^2_+} |x|^\alpha e^{u_n} < \varepsilon; \quad \forall r \in [e\lambda_n R, \delta].$$

To prove this claim, we note two facts. The first fact is: for any $T > 0$, there exists some $N(T)$ such that for any $n \ge N(T)$, we have

$$\int_{D^+_\delta\setminus D^+_{\delta e^{-T}}} (|x|^{2\alpha}e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_\delta\setminus D^+_{\delta e^{-T}})\cap\partial\mathbb{R}^2_+} |x|^\alpha e^{u_n} < \varepsilon. \tag{53}$$

Actually, since (u_n, Ψ_n) has no blow-up point in $\overline{D^+_\delta}\setminus\{p\}$, then $|\Psi_n|$ is uniformly bounded in $\overline{D^+_\delta\setminus D^+_{\delta e^{-T}}}$, and u_n will either be uniformly bounded in $\overline{D^+_\delta\setminus D^+_{\delta e^{-T}}}$ or uniformly tend to $-\infty$ in $\overline{D^+_\delta\setminus D^+_{\delta e^{-T}}}$. So if u_n uniformly tends to $-\infty$ in $\overline{D_\delta\setminus D_{\delta e^{-T}}}$, it is clear that, for any given $T > 0$, we have an $N(T)$ big enough such that when $n \ge N(T)$

$$\int_{D^+_\delta\setminus D^+_{\delta e^{-T}}} (|x|^{2\alpha}e^{2u_n} + \int_{\partial(D^+_\delta\setminus D^+_{\delta e^{-T}})\cap\partial\mathbb{R}^2_+} |x|^\alpha e^{u_n} < \frac{\varepsilon}{2}.$$

Moreover, since Ψ_n converges to Ψ in $L^4_{loc}((D_r^+ \cap L_r) \setminus \Sigma_1)$ and hence

$$\int_{D_\delta^+ \setminus D_{\delta e^{-T}}^+} |\Psi_n|^4 \to \int_{D_\delta^+ \setminus D_{\delta e^{-T}}^+} |\Psi|^4.$$

For any small $\varepsilon > 0$, we may choose $\delta > 0$ small enough such that $\int_{D_\delta^+} |\Psi|^4 < \frac{\varepsilon}{4}$, then for any given $T > 0$, we have an $N(T)$ big enough such that when $n \geq N(T)$

$$\int_{D_\delta^+ \setminus D_{\delta e^{-T}}^+} |\Psi_n|^4 < \frac{\varepsilon}{2}.$$

Consequently, we get (53).

If (u_n, Ψ_n) is uniformly bounded in $\overline{D_\delta^+ \setminus D_{\delta e^{-T}}^+}$, then we know (u_n, Ψ_n) converges to a weak solution (u, Ψ) strongly on compact sets of $D_\delta^+ \setminus \{p\}$. Therefore, we can also choose $\delta > 0$ small enough such that, for any given $T > 0$, there exists an $N(T)$ big enough, when $n \geq N(T)$, (53) holds.

The second fact is: For any small $\varepsilon > 0$, and $T > 0$, we may choose an $N(T)$ such that when $n \geq N(T)$

$$\begin{aligned} &\int_{D^+_{\lambda_n R e^T} \setminus D^+_{\lambda_n R}} (|x|^{2\alpha} e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_{\lambda_n R e^T} \setminus D^+_{\lambda_n R}) \cap \partial\mathbb{R}^2_+} |x|^\alpha e^{u_n} \\ &= \int_{D^+_{R e^T} \setminus D^+_R} (|x|^{2\alpha} e^{2\widetilde{u}_n} + |\widetilde{\Psi}_n|^4) + \int_{\partial(D^+_{R e^T} \setminus D^+_R) \cap \partial\mathbb{R}^2_+} |x|^\alpha e^{\widetilde{u}_n} < \varepsilon, \end{aligned}$$

if R is big enough.

Now we can prove the claim. We argue by contradiction by using the above two facts. If there exists $\varepsilon_0 > 0$ and a sequence r_n, $r_n \in [e\lambda_n R, \delta]$, such that

$$\int_{D^+_{r_n} \setminus D^+_{e^{-1} r_n}} (|x|^{2\alpha} e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_{r_n} \setminus D^+_{e^{-1} r_n}) \cap \partial\mathbb{R}^2_+} |x|^\alpha e^{u_n} \geq \varepsilon_0.$$

Then, by the above two facts, we know that $\frac{\delta}{r_n} \to +\infty$ and $\frac{\lambda_n R}{r_n} \to 0$, in particular, $r_n \to 0$ as $n \to +\infty$. Rescaling again, we set

$$\begin{cases} v_n(x) = u_n(r_n x) + (1+\alpha) \ln r_n, \\ \varphi_n(x) = r_n^{\frac{1}{2}} \Psi(r_n x) \end{cases}$$

for any $x \in D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}}$.

It is clear that

$$\int_{D_1^+ \setminus D^+_{e^{-1}}} (|x|^{2\alpha} e^{2v_n} + |\varphi_n|^4) + \int_{\partial(D_1^+ \setminus D^+_{e^{-1}}) \cap \partial\mathbb{R}^2_+} |x|^\alpha e^{v_n} \geq \varepsilon_0. \tag{54}$$

And (v_n, φ_n) satisfies for any $R > 0$

$$\begin{cases} -\Delta v_n(x) = 2V^2(r_n x)|x|^{2\alpha}e^{2v_n(x)} - V(r_n x)|x|^{\alpha}e^{v_n(x)}|\varphi_n(x)|^2, & \text{in } \left(D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}}\right), \\ \not{D}\varphi_n(x) = -V(r_n x)|x|^{\alpha}e^{v_n(x)}\varphi_n(x), & \text{in } \left(D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}}\right), \\ \frac{\partial v_n(x)}{\partial n} = cV(r_n x)|x|^{\alpha}e^{v_n(x)}, & \text{on } \partial\left(D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}}\right) \cap \partial\mathbb{R}^2_+, \\ B\varphi_n(x) = 0, & \text{on } \partial\left(D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}}\right) \cap \partial\mathbb{R}^2_+. \end{cases}$$

According to Theorem 1.3, there are three possible cases:

1. There exists some $q \in Q_n = (D^+_{\frac{\delta}{r_n}} \setminus D^+_{\frac{\lambda_n R}{r_n}})$ and energy concentration occurs near the point q, namely along some subsequence we have

$$\lim_{n\to\infty} \int_{D_r(q)\cap Q_n} (|x|^{2\alpha}e^{2v_n} + |\varphi_n|^4) + \int_{D_r(q)\cap\partial Q_n\cap\{t=0\}} |x|^{\alpha}e^{v_n} \geq \varepsilon_0 > 0$$

   for any small $r > 0$. In such a case, we still obtain the second "bubble" by the rescaling argument. Thus we get a contradiction.
2. For any $R > 0$, there is no blow-up point in $D^+_R \setminus D^+_{\frac{1}{R}}$ and $v_n \to -\infty$ uniformly in $\overline{D^+_R \setminus D^+_{\frac{1}{R}}}$. Then, it is clear that φ_n converges to a spinor φ in $L^4_{loc}(\overline{\mathbb{R}^2_+} \setminus \{0\})$ which satisfies

$$\begin{cases} \not{D}\varphi = 0, & \text{in } \mathbb{R}^2_+, \\ B\varphi = 0, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \end{cases}$$

   We translate φ to be a harmonic spinor on $\mathbb{R}^2_+ \setminus \{0\}$ satisfying the corresponding chirality boundary condition and then extend it as in (24) to a harmonic spinor $\overline{\varphi}$ on $\mathbb{R}^2 \setminus \{0\}$ with bounded energy, i.e., $||\overline{\varphi}||_{L^4(\mathbb{R}^2)} < \infty$. As discussed in [23], $\overline{\varphi}$ conformally extends to a harmonic spinor on S^2. By the well known fact that there is no nontrivial harmonic spinor on S^2, we have that $\overline{\varphi} \equiv 0$ and hence φ_n converges to 0 in $L^4_{loc}(\mathbb{R}^2_+ \setminus \{0\})$. This will contradict (54)
3. For any $R > 0$, there is no blow-up point in $(D^+_R \setminus D^+_{\frac{1}{R}})$ and (v_n, φ_n) is uniformly bounded in $(D^+_R \setminus D^+_{\frac{1}{R}})$. In such a case (v_n, φ_n) will converge to (v, φ) strongly on $(D^+_R \setminus D^+_{\frac{1}{R}})$ and (v, φ) satisfying

$$\begin{cases} -\Delta v = 2V^2(0)|x|^{2\alpha}e^{2v} - V(0)|x|^{\alpha}e^{v}|\varphi|^2, & \text{in } \mathbb{R}^2_+, \\ \not{D}\varphi = -V(0)|x|^{\alpha}e^{v}\varphi, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial n} = cV(0)|x|^{\alpha}e^{v}, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \\ B\varphi = 0, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\} \end{cases}$$

   with finite energy. It is clear that (v, φ) is regular.

Next we need to remove the singularities of (v, φ) and then obtain the second bubble of the system. Consequently we get a contradiction. To this purpose, let us use the

Pohozaev identity of (u_n, Ψ_n) in D_δ^+, it follows for any ρ with $r_n\rho < \delta$

$$
\begin{aligned}
& r_n\rho \int_{S_{r_n\rho}^+} \left|\frac{\partial u_n}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u_n|^2 d\sigma \\
&= (1+\alpha)\int_{D_{r_n\rho}^+} 2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2 dv \\
&\quad +(\alpha+1)\int_{L_{r_n\rho}} cV(x)|x|^{\alpha}e^{u_n} ds \\
&\quad -r_n\rho\int_{S_{r_n\rho}^+} V^2(x)|x|^{2\alpha}e^{2u_n} d\sigma \\
&\quad +\int_{L_{r_n\rho}} c\frac{\partial V(s,0)}{\partial s}|s|^{\alpha}se^{u_n} ds - cV((s,0))|s|^{\alpha}se^{u_n}\Big|_{s=-r_n\rho}^{s=r_n\rho} \\
&\quad +\int_{D_{r_n\rho}^+} x\cdot\nabla(V^2(x))|x|^{2\alpha}e^{2u_n} dv - \int_{D_{r_n\rho}^+} x\cdot\nabla V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2 dv \\
&\quad +\frac{1}{4}\int_{S_{r_n\rho}^+}\left\langle\frac{\partial\Psi_n}{\partial\nu}, (x+\bar{x})\cdot\Psi_n\right\rangle d\sigma + \frac{1}{4}\int_{S_{r_n\rho}^+}\left\langle(x+\bar{x})\cdot\Psi_n, \frac{\partial\Psi_n}{\partial\nu}\right\rangle d\sigma.
\end{aligned}
$$

Hence for rescaling functions (v_n, φ_n) we have

$$
\begin{aligned}
& \rho\int_{S_\rho^+}\left|\frac{\partial v_n}{\partial\nu}|^2 - \frac{1}{2}|\nabla v_n\right|^2 d\sigma \\
&= (1+\alpha)\int_{D_\rho^+} 2V^2(r_nx)|x|^{2\alpha}e^{2v_n} - V(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2 dv \\
&\quad +(\alpha+1)\int_{L_\rho} cV(r_nx)|x|^{\alpha}e^{v_n} ds \\
&\quad -\rho\int_{S_\rho^+} V^2(r_nx)|x|^{2\alpha}e^{2v_n} d\sigma + \int_{L_\rho} c\frac{\partial V((r_ns,0))}{\partial s}|s|^{\alpha}se^{v_n} ds \\
&\quad -cV((r_ns,0))|s|^{\alpha}se^{v_n}\Big|_{s=-\rho}^{s=\rho} \\
&\quad +\int_{D_\rho^+} x\cdot(\nabla V^2)(r_nx)|x|^{2\alpha}e^{2v_n} dv - \int_{D_\rho^+} x\cdot(\nabla V)(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2 dv \\
&\quad +\frac{1}{4}\int_{S_\rho^+}\left\langle\frac{\partial\varphi_n}{\partial\nu}, (x+\bar{x})\cdot\varphi_n\right\rangle d\sigma + \frac{1}{4}\int_{S_\rho^+}\left\langle(x+\bar{x})\cdot\varphi_n, \frac{\partial\varphi_n}{\partial\nu}\right\rangle d\sigma.
\end{aligned}
$$

This implies that the associated Pohozaev constant of (v_n, φ_n) satisfies

$$
\begin{aligned}
& C(v_n, \varphi_n) = C(v_n, \varphi_n, \rho) \\
&= \rho\int_{S_\rho^+}\left|\frac{\partial v_n}{\partial\nu}\right|^2 - \frac{1}{2}|\nabla v_n|^2 d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&-(1+\alpha)\int_{D_\rho^+} 2V^2(r_nx)|x|^{2\alpha}e^{2v_n} - V(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2 dv \\
&-(\alpha+1)\int_{L_\rho} cV(r_nx)|x|^{\alpha}e^{v_n}ds \\
&+\rho\int_{S_\rho^+} V^2(r_nx)|x|^{2\alpha}e^{2v_n}d\sigma - \int_{L_\rho} c\frac{\partial V((r_ns,0))}{\partial s}|s|^{\alpha}se^{v_n}ds \\
&+cV((r_ns,0))|s|^{\alpha}se^{v_n}|_{s=-\rho}^{s=\rho} \\
&-\int_{D_\rho^+} x\cdot(\nabla V^2)(r_nx)|x|^{2\alpha}e^{2v_n}dv + \int_{D_\rho^+} x\cdot(\nabla V)(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2 dv \\
&-\frac{1}{4}\int_{S_\rho^+}\left\langle\frac{\partial\varphi_n}{\partial\nu},(x+\bar{x})\cdot\varphi_n\right\rangle d\sigma - \frac{1}{4}\int_{S_\rho^+}\left\langle(x+\bar{x})\cdot\varphi_n,\frac{\partial\varphi_n}{\partial\nu}\right\rangle d\sigma \\
&=0.
\end{aligned}
$$

Since, for any $\rho > 0$, $\int_{D_\rho^+}|x|^{2\alpha}e^{2v_n}+|\varphi_n|^4dv+\int_{L_\rho}|x|^{\alpha}e^{v_n}ds < C$, it is easy to check that

$$
\lim_{\rho\to 0}\lim_{n\to\infty}\int_{D_\rho^+} x\cdot(\nabla V^2)(r_nx)|x|^{2\alpha}e^{2v_n}dv + \int_{D_\rho^+} x\cdot(\nabla V)(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2dv = 0,
$$

and

$$
\lim_{\rho\to 0}\lim_{n\to\infty}\int_{L_\rho} c\frac{\partial V((r_ns,0))}{\partial s}|s|^{\alpha}se^{v_n}ds = 0.
$$

This implies that

$$
\begin{aligned}
0 &= \lim_{\rho\to 0}\lim_{n\to\infty} C(v_n,\varphi_n,\rho) \\
&= \lim_{\rho\to 0} C(v,\varphi,\rho) - (1+\alpha)\lim_{r\to 0}\lim_{n\to\infty}\int_{D_r^+} 2V^2(r_nx)|x|^{2\alpha}e^{2v_n} - V(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2dv \\
&\quad -(1+\alpha)\lim_{r\to 0}\lim_{n\to\infty}\int_{L_r} cV(r_nx)|x|^{\alpha}e^{v_n}ds \\
&= C(v,\varphi) - (1+\alpha)\beta.
\end{aligned}
$$

Here

$$
\begin{aligned}
\beta = \lim_{r\to 0}\lim_{n\to\infty}\Bigg[&\int_{D_r^+} 2V^2(r_nx)|x|^{2\alpha}e^{2v_n} - V(r_nx)|x|^{\alpha}e^{v_n}|\varphi_n|^2dv \\
&+\int_{L_r} cV(r_nx)|x|^{\alpha}e^{v_n}ds\Bigg],
\end{aligned}
$$

and $C(v,\varphi)=C(v,\varphi,\rho)$ is the Pohozaev constant of (v,φ), i.e.

$$\begin{aligned}C(v,\varphi)=\rho\int_{S_\rho^+}\left|\frac{\partial v}{\partial\nu}|^2-\frac{1}{2}|\nabla v\right|^2 d\sigma\\ -(1+\alpha)\left[\int_{D_\rho^+}2V^2(0)|x|^{2\alpha}e^{2v}-V(0)|x|^\alpha e^v|\varphi|^2dv+\int_{L_\rho}cV(0)|x|^\alpha e^v ds\right]\\ +\rho\int_{S_\rho^+}V^2(0)|x|^{2\alpha}e^{2v}d\sigma+cV(0)|s|^\alpha se^v|_{s=-\rho}^{s=\rho}\\ -\frac{1}{4}\int_{S_\rho^+}\left\langle\frac{\partial\varphi}{\partial\nu},(x+\bar{x})\cdot\varphi\right\rangle d\sigma-\frac{1}{4}\int_{S_\rho^+}\left\langle(x+\bar{x})\cdot\varphi,\frac{\partial\varphi}{\partial\nu}\right\rangle d\sigma.\end{aligned}$$

On the other hand, we use the fact that (v_n,φ_n) converges to (v,φ) in $C^2_{loc}(\mathbb{R}^2_+)\cap C^1_{loc}(\overline{\mathbb{R}}^2_+\backslash\{0\})\times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+))\cap C^1_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+\backslash\{0\}))$ again to get

$$\begin{aligned}&\int_{D_\rho^+}2V^2(r_nx)|x|^{2\alpha}e^{2v_n}-V(r_nx)|x|^\alpha e^{v_n}|\varphi_n|^2dv+\int_{L_\rho}cV(r_nx)|x|^\alpha e^{v_n}ds\\ &\quad\to\int_{D_\rho^+}2V^2(0)|x|^{2\alpha}e^{2v}-V(0)|x|^\alpha e^v|\varphi|^2dv+\int_{L_\rho}cV(0)|x|^\alpha e^v ds+\beta\end{aligned}$$

as $n\to\infty$. By using Green's representation formula for u_n in D_ρ^+ and then take $n\to\infty$, we have

$$v(x)=\frac{\beta}{\pi}\ln\frac{1}{|x|}+\phi(x)+\gamma(x),$$

where

$$\begin{aligned}\phi(x)=\frac{1}{\pi}\int_{D_\rho^+}\ln\frac{1}{|x-y|}(2V^2(0)|y|^{2\alpha}e^{2v(y)}-V(0)|y|^\alpha e^{v(y)}|\varphi|^2(y))dy\\ +\frac{1}{\pi}\int_{L_r}\ln\frac{1}{|x-y|}(cV(0)|y|^\alpha e^{v(y)})dy,\end{aligned}$$

and

$$\gamma(x)=\frac{1}{\pi}\int_{S_\rho^+}\ln\frac{1}{|x-y|}\frac{\partial v}{\partial\nu}+\frac{(x-y)\cdot\nu}{|x-y|^2}v(y)dy.$$

It is clear that $\gamma(x)$ is in $C^1(\overline{D_\rho^+})$ and ϕ satisfies

$$\begin{cases}-\Delta\phi=2V^2(0)|x|^{2\alpha}e^{2v}-V(0)|x|^\alpha e^v|\varphi|^2, & \text{in } D_\rho^+,\\ \frac{\partial\phi}{\partial\nu}=cV(0)|x|^\alpha e^v, & \text{on } L_\rho.\end{cases}$$

By similar arguments as the proof of Proposition 4.5, we can obtain that

$$C(v,\varphi)=\frac{\beta^2}{2\pi},$$

This implies that

$$(1+\alpha)\beta=\frac{\beta^2}{2\pi}.$$

Noticing that $\int_{D_\rho^+}|x|^{2\alpha}e^{2v}dx<\infty$, we have $\beta\le(1+\alpha)\pi$. Therefore we obtain that $\beta=0$, i.e. $C(v,\varphi)=0$, and the singularity at 0 of (v,φ) is removed by Proposition 4.5. Furthermore, the singularity at ∞ of (v,φ) is also removed by Theorem 5.2. Thus we get another bubble on $S^2_{c'}$, and we get a contradiction to the assumption that $m=1$. Concequently we complete the proof of the claim 1.

Claim 2 We can separate $A^+_{\delta,R,n}$ into finitely many parts

$$A^+_{\delta,R,n}=\bigcup_{k=1}^{N_k}A^+_k$$

such that on each part

$$\int_{A^+_k}|x|^{2\alpha}e^{2u_n}\le\frac{1}{4\Lambda^2},\quad k=1,2,\dots,N_k.$$

where $N_k\le N_0$ for N_0 is a uniform integer for all n large enough, $A^+_k=D^+_{r^{k-1}}\setminus D^+_{r^k}$, $r^0=\delta$, $r^{N_k}=\lambda_nR$, $r^k<r^{k-1}$ for $k=1,2,\dots,N_k$, and C_0 is a constant as in Lemma 6.1.

The proof of this claim is very similar to those in [23,25,41] and the argument is now standard, so we omit it.

Now we apply **Claim 1** and **Claim 2** to prove (52). Let $\varepsilon>0$ be small, and let δ be small enough, and let R and n be big enough. We apply Lemma 6.1 to each part A^+_k to obtain

$$\begin{aligned}\left(\int_{A^+_l}|\Psi_n|^4\right)^{\frac14}&\le C_0\left(\int_{D^+_{er^{l-1}}\setminus D^+_{e^{-1}r^l}}|x|^{2\alpha}e^{2u_n}\right)^{\frac12}\left(\int_{D^+_{er^{l-1}}\setminus D^+_{e^{-1}r^l}}|\Psi_n|^4\right)^{\frac14}\\&\quad+C\left(\int_{D^+_{er^{l-1}}\setminus D^+_{r^{l-1}}}|\Psi_n|^4\right)^{\frac14}+C\left(\int_{D^+_{r^l}\setminus D^+_{e^{-1}r^l}}|\Psi_n|^4\right)^{\frac14}\\&\le C_0\left(\left(\int_{A^+_l}|x|^{2\alpha}e^{2u_n}\right)^{\frac12}+\varepsilon^{\frac12}+\varepsilon^{\frac12}\right)\left(\left(\int_{A^+_l}|\Psi_n|^4\right)^{\frac14}+\varepsilon^{\frac14}+\varepsilon^{\frac14}\right)+C\varepsilon^{\frac14}\\&\le C_0\left(\int_{A^+_l}|x|^{2\alpha}e^{2u_n}\right)^{\frac12}\left(\int_{A^+_l}|\Psi_n|^4\right)^{\frac14}+C\left(\varepsilon^{\frac14}+\varepsilon^{\frac12}+\varepsilon^{\frac34}\right)\end{aligned}$$

$$\leq \frac{1}{2}\left(\int_{A_l^+}|\Psi_n|^4\right)^{\frac{1}{4}} + C\left(\varepsilon^{\frac{1}{4}}+\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{3}{4}}\right).$$

Therefore we have

$$\left(\int_{A_l^+}|\Psi_n|^4\right)^{\frac{1}{4}} \leq C\left(\varepsilon^{\frac{1}{4}}+\varepsilon^{\frac{1}{2}}+\varepsilon^{\frac{3}{4}}\right).$$

Since ε is small, we may assume $\varepsilon \leq 1$. Then we get

$$\left(\int_{A_l^+}|\Psi_n|^4\right)^{\frac{1}{4}} \leq C\varepsilon^{\frac{1}{4}}. \tag{55}$$

With similar arguments, and using (55), we have

$$\left(\int_{A_l^+}|\nabla\psi_n|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq C\varepsilon^{\frac{1}{4}}. \tag{56}$$

Summing up (55) and (56) on A_l^+ we get

$$\int_{A^+_{\delta,R,n}}|\Psi_n|^4 + \int_{A^+_{\delta,R,n}}|\nabla\psi_n|^{\frac{4}{3}} = \sum_{l=1}^{N_0}\int_{A_l^+}|\Psi_n|^4 + |\nabla\psi_n|^{\frac{4}{3}} \leq C\varepsilon^{\frac{1}{3}}. \tag{57}$$

Thus we have shown (52) in the first case.

For **Case II**, according the blow-up process, we define the neck domain is

$$A^+_{S,R,n} = \{x \in \mathbb{R}^2_+ | \rho_n R \leq |x - x_n| \leq |x_n| S\}.$$

Notice that

$$\begin{aligned}
\int_{D_\delta^+}|\Psi_n|^4 dv &= \int_{D^+_{\frac{\delta}{|x_n|}}}|\overline{\Psi}_n|^4 dv \\
&= \int_{D^+_{\frac{\delta}{|x_n|}}\setminus D^+_{R_1}(y_n)}|\overline{\Psi}_n|^4 dv + \int_{D^+_{R_1}(y_n)\setminus D^+_{\delta_n R_2}(y_n)}|\overline{\Psi}_n|^4 dv + \int_{D^+_{\delta_n R_2}(y_n)}|\overline{\Psi}_n|^4 dv \\
&= \int_{D^+_{\frac{\delta}{|x_n|}}\setminus D^+_{R_1}(y_n)}|\overline{\Psi}_n|^4 dv + \int_{D^+_{|x_n|R_1}(x_n)\setminus D^+_{|x_n|\delta_n R_2}(x_n)}|\Psi_n|^4 dv + \int_{D^+_{\delta_n R_2}(y_n)}|\overline{\Psi}_n|^4 dv.
\end{aligned}$$

Duo to the assumption that (u_n, Ψ_n) has only one bubble at the blow-up point $p = 0$, $(\overline{u}_n, \overline{\Psi}_n)$ also has only one bubble at its blow-up point y_0. Therefore, we have

$$\lim_{\delta\to 0}\lim_{R_1\to\infty}\lim_{n\to\infty}\int_{D^+_{\frac{\delta}{|x_n|}}\setminus D^+_{R_1}(y_n)}|\overline{\Psi}_n|^4 dv = 0.$$

While $D^+_{\delta_n R_2}(y_n)$ is a bubble domain, we know to prove (52) it is sufficient to prove that

$$\lim_{S\to\infty}\lim_{R\to\infty}\lim_{n\to\infty}\int_{A^+_{S,R,n}}|\Psi_n|^4 dv = 0. \tag{58}$$

To prove (58), by using the similar argument as the case 1, we have the following facts:

Fact II.1 For any small $\varepsilon > 0$, and $T > 0$, there exists some $N(T)$ such that for any $n \geq N(T)$ we have

$$\int_{D^+_{|x_n|S}(x_n)\setminus D^+_{|x_n|Se^{-T}}(x_n)}\left(|x|^{2\alpha}e^{2u_n}+|\Psi_n|^4\right)+\int_{\partial(D^+_{|x_n|S}(x_n)\setminus D^+_{|x_n|Se^{-T}}(x_n))\cap\partial\mathbb{R}^2_+}|x|^{\alpha}e^{u_n} < \varepsilon,$$

for sufficiently large S.

Fact II.2 For any small $\varepsilon > 0$, and $T > 0$, we may choose an $N(T)$ such that when $n \geq N(T)$

$$\begin{aligned}
&\int_{D^+_{\rho_n Re^T}(x_n)\setminus D^+_{\rho_n R}(x_n)}\left(|x|^{2\alpha}e^{2u_n}+|\Psi_n|^4\right)+\int_{\partial(D^+_{\rho_n Re^T}(x_n)\setminus D^+_{\rho_n R}(x_n))\cap\partial\mathbb{R}^2_+}|x|^{\alpha}e^{u_n}\\
&=\int_{(D_{Re^T}\setminus D_R)\cap\{t>-\frac{t_n}{\rho_n}\}}\left(\left|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x\right|^{2\alpha}e^{2\widetilde{u}_n}+|\widetilde{\Psi}_n|^4\right)\\
&\quad+\int_{(D_{Re^T}\setminus D_R)\cap\{t=-\frac{t_n}{\rho_n}\}}\left|\frac{x_n}{|x_n|}+\frac{\rho_n}{|x_n|}x\right|^{\alpha}e^{\widetilde{u}_n}\\
&<\varepsilon,
\end{aligned}$$

if R is large enough.

Buy using the above two facts, we need to prove the following claim:

Claim II.1 For any $\varepsilon > 0$, there is an $N > 1$ such that for any $n \geq N$, we have

$$\begin{aligned}
&\int_{D^+_r(x_n)\setminus D^+_{e^{-1}r}(x_n)}(|x|^{2\alpha}e^{2u_n}+|\Psi_n|^4)\\
&\quad+\int_{\partial(D^+_r(x_n)\setminus D^+_{e^{-1}r}(x_n))\cap\partial\mathbb{R}^2_+}|x|^{\alpha}e^{u_n} < \varepsilon;\quad \forall r\in[e\rho_n R, |x_n|S].
\end{aligned}$$

Proof of Claim II.1 We assume by a contradiction that there exists $\varepsilon_0 > 0$ and a sequence r_n, $r_n \in [e\rho_n R, |x_n|S]$, such that

$$\int_{D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n)}(|x|^{2\alpha}e^{2u_n}+|\Psi_n|^4)+\int_{\partial(D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n))\cap\partial\mathbb{R}^2_+}|x|^{\alpha}e^{u_n}\geq\varepsilon_0.$$

Then, by Facts II.1 and II.2, we know that $\frac{|x_n|S}{r_n}\to+\infty$ and $\frac{\rho_n R}{r_n}\to 0$, in particular, $r_n \to 0$ as $n \to +\infty$. We assume that $\Lambda = \lim_{n\to\infty}\frac{t_n}{r_n}$. Here Λ is either a nonnegative real number or $+\infty$. Next we proceed by distinguishing two cases.

Case II.1 $\Lambda > 0$.

In this case, we note that $D_{r_n\rho}(x_n)$ is in $\mathbb{R}^2_+$ when n is sufficient small and $0 < \rho < \Lambda$. We define the rescaling functions again

$$\begin{cases} v_n(x) = u_n(r_nx + x_n) + \ln(r_n|x_n|^\alpha), \\ \varphi_n(x) = r_n^{\frac{1}{2}}\Psi(r_nx + x_n) \end{cases}$$

for any $r_nx + x_n \in D^+_{|x_n|S}(x_n) \setminus D^+_{\rho_n R}(x_n)$. Then $(v_n(x), \varphi_n(x))$ satisfies that

$$\int_{(D_1\setminus D_{e^{-1}})\cap\{t>-\frac{t_n}{r_n}\}} \left(\left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x \right|^{2\alpha} e^{2v_n} + |\varphi_n|^4 \right) + \int_{(D_1\setminus D_{e^{-1}})\cap\left\{t=-\frac{t_n}{r_n}\right\}} \left| \frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x \right|^{\alpha} e^{v_n} \geq \varepsilon_0. \tag{59}$$

Note that (v_n, φ_n) satisfies for any $R > 0$ and $S > 0$

$$\begin{cases} -\triangle v_n(x) = 2V^2(r_nx + x_n)|\frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x|^{2\alpha}e^{2v_n(x)} \\ \qquad -V(r_nx + x_n)|\frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x|^{\alpha}e^{v_n(x)}|\varphi_n(x)|^2, & \text{in } \left(D_{\frac{|x_n|S}{r_n}} \setminus D_{\frac{\rho_n R}{r_n}}\right) \cap \left\{t > -\frac{t_n}{r_n}\right\}, \\ \not{D}\varphi_n(x) = -V(r_nx + x_n)|\frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x|^{\alpha}e^{v_n(x)}\varphi_n(x), & \text{in } \left(D_{\frac{|x_n|S}{r_n}} \setminus D_{\frac{\rho_n R}{r_n}}\right) \cap \left\{t > -\frac{t_n}{r_n}\right\}, \\ \frac{\partial v_n(x)}{\partial n} = cV(r_nx + x_n)|\frac{x_n}{|x_n|} + \frac{r_n}{|x_n|}x|^{\alpha}e^{v_n(x)}, & \text{on } \left(D_{\frac{|x_n|S}{r_n}} \setminus D_{\frac{\rho_n R}{r_n}}\right) \cap \left\{t = -\frac{t_n}{r_n}\right\}, \\ B\varphi_n(x) = 0, & \text{on } \left(D_{\frac{|x_n|S}{r_n}} \setminus D_{\frac{\rho_n R}{r_n}}\right) \cap \left\{t = -\frac{t_n}{r_n}\right\}. \end{cases}$$

According to Theorem 1.3, there are three possible cases. Similar to the Case I, we can rule out the first and the second possible cases. If the third case happens, then there is no blow-up point in $(D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -b\}$ for any $R > 0$ and any $b < \Lambda$. Furthermore (v_n, φ_n) will converge to (v, φ) strongly on $(D_R \setminus D_{\frac{1}{R}}) \cap \{t \geq -b\}$. If $\Lambda > 0$, then (v, φ) satisfies

$$\begin{cases} -\triangle v = 2V^2(0)e^{2v} - V(0)e^v|\varphi|^2, & \text{in } \mathbb{R}^2_\Lambda \setminus \{0\}, \\ \not{D}\varphi = -V(0)e^v\varphi, & \text{in } \mathbb{R}^2_\Lambda \setminus \{0\}, \\ \frac{\partial v}{\partial n} = cV(0)e^v, & \text{on } \partial\mathbb{R}^2_\Lambda \text{ (in the case of } \Lambda < +\infty), \\ B\varphi = 0, & \text{on } \partial\mathbb{R}^2_\Lambda \text{ (in the case of } \Lambda < +\infty) \end{cases} \tag{60}$$

with finite energy.

Since $D_{r_n\rho}(x_n)$ contains completely in $\mathbb{R}^2_+$ when n is sufficient small and $0 < \rho < \Lambda$, we know that the origin is actually an interior singular point of (v, φ) to (60). Then this local singular can be removed by using the similar arguments in the case II of [27]. After removing the local singularity 0, we can remove the singularity at ∞ of (v, φ) to (60) by Theorem 5.2. Thus we get another bubble on $S^2_{c'}$, and we get a contradiction to the assumption that $m = 1$. Concequently we complete the proof of the claim II.1.

Case II.2 $\Lambda = 0$.

In this case, noticing that $x_n = (s_n, t_n)$ and $\lim_{n\to\infty}\frac{|x_n|}{r_n} = +\infty$, we have $\lim_{n\to\infty}\frac{|s_n|}{t_n} = +\infty$ and $\lim_{n\to\infty}\frac{|s_n|}{r_n} = +\infty$. We set $x_n' = (s_n, 0)$. Then we define the rescaling functions in this case

$$\begin{cases} v_n(x) = u_n(r_n x + x_n') + \ln(r_n|s_n|^{\alpha}), \\ \varphi_n(x) = r_n^{\frac{1}{2}}\Psi(r_n x + x_n') \end{cases}$$

for any $r_n x + x_n' \in D^+_{|x_n|S}(x_n') \setminus D^+_{\rho_n R}(x_n')$. Since that

$$\begin{aligned} &\int_{D^+_{\frac{3}{2}r_n}(x_n')\setminus D^+_{\frac{1}{2}e^{-1}r_n}(x_n')} (|x|^{2\alpha}e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_{\frac{1}{2}r_n}(x_n')\setminus D^+_{\frac{3}{2}e^{-1}r_n}(x_n'))\cap\partial\mathbb{R}^2_+} |x|^{\alpha}e^{u_n} \\ &\geq \int_{D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n)} (|x|^{2\alpha}e^{2u_n} + |\Psi_n|^4) + \int_{\partial(D^+_{r_n}(x_n)\setminus D^+_{e^{-1}r_n}(x_n))\cap\partial\mathbb{R}^2_+} |x|^{\alpha}e^{u_n} \\ &\geq \varepsilon_0, \end{aligned}$$

we have that $(v_n(x), \varphi_n(x))$ satisfies that

$$\begin{aligned} &\int_{D^+_{\frac{3}{2}}\setminus D^+_{\frac{e^{-1}}{2}}} \left(\left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{2\alpha} e^{2v_n} + |\varphi_n|^4\right) \\ &+ \int_{\partial\left(D^+_{\frac{3}{2}}\setminus D^+_{\frac{e^{-1}}{2}}\right)\cap\{t=0\}} \left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{\alpha} e^{v_n} \geq \varepsilon_0. \end{aligned} \tag{61}$$

Note that (v_n, φ_n) satisfies for any $R > 0$ and $S > 0$

$$\begin{cases} -\Delta v_n(x) = 2V^2(r_n x + x_n')|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x|^{2\alpha}e^{2v_n(x)} \\ \qquad\qquad -V(r_n x + x_n')|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x|^{\alpha}e^{v_n(x)}|\varphi_n(x)|^2, & \text{in } \left(D^+_{\frac{|x_n|S}{r_n}} \setminus D^+_{\frac{\rho_n R}{r_n}}\right), \\ \not{D}\varphi_n(x) = -V(r_n x + x_n')|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x|^{\alpha}e^{v_n(x)}\varphi_n(x), & \text{in } \left(D^+_{\frac{|x_n|S}{r_n}} \setminus D^+_{\frac{\rho_n R}{r_n}}\right), \\ \frac{\partial v_n(x)}{\partial n} = cV(r_n x + x_n')|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x|^{\alpha}e^{v_n(x)}, & \text{on } \partial\left(D^+_{\frac{|x_n|S}{r_n}} \setminus D^+_{\frac{\rho_n R}{r_n}}\right)\cap\partial\mathbb{R}^2_+, \\ B\varphi_n(x) = 0, & \text{on } \partial\left(D^+_{\frac{|x_n|S}{r_n}} \setminus D^+_{\frac{\rho_n R}{r_n}}\right)\cap\partial\mathbb{R}^2_+. \end{cases}$$

According to Theorem 1.3, there are three possible cases. From (61), we can rule out the first and the second possible cases by using the similar arguments of Case I. Next we assume that the third case happens, i.e. there is no blow-up point in $\overline{D^+_R \setminus D^+_{\frac{1}{R}}}$ for any $R > 0$. Furthermore (v_n, φ_n) will converge to (v, φ) strongly on $\overline{D^+_R \setminus D^+_{\frac{1}{R}}}$, and (v, φ) satisfies

$$\begin{cases} -\Delta v = 2V^2(0)e^{2v} - V(0)e^v|\varphi|^2, & \text{in } \mathbb{R}^2_+, \\ \not{D}\varphi = -V(0)e^v\varphi, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial n} = cV(0)e^v, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \\ B\varphi = 0, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\} \end{cases} \tag{62}$$

with finite energy.

Next we will remove two singular points at 0 and at ∞, and consequently we get the second bubble of the considered system. Thus we get a contradiction. To this purpose, let us computate the Pohozaev constant of (v, φ). Let start with the Pohozaev identity of (u_n, Φ_n). We multiply all terms in (4) by $(x - x_n') \cdot \nabla u_n$ and integrate over $D^+_{r_n\rho}(x_n')$. It follows for any sufficient small ρ that

$$\begin{aligned}
& r_n\rho \int_{S^+_{r_n\rho}(x_n')} \left|\frac{\partial u_n}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u_n|^2 d\sigma \\
&= \int_{D^+_{r_n\rho}(x_n')} (2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dv + \int_{L_{r_n\rho}(x_n')} cV(x)|x|^{\alpha}e^{u_n}ds \\
&\quad -r_n\rho \int_{S^+_{r_n\rho}(x_n')} V^2(x)|x|^{2\alpha}e^{2u_n}d\sigma + \int_{L_{r_n\rho}(x_n')} c\frac{\partial(V(s,0)|s|^{\alpha})}{\partial s}(s - s_n)e^{u_n}ds \\
&\quad -cV(s,0)|s|^{\alpha}(s - s_n)e^{u_n}\Big|^{s=x_n'+r_n\rho}_{s=x_n'-r_n\rho} \\
&\quad + \int_{D^+_{r_n\rho}(x_n')} (x - x_n') \cdot \nabla(V^2(x)|x|^{2\alpha})e^{2u_n}dv \\
&\quad - \int_{D^+_{r_n\rho}(x_n')} (x - x_n') \cdot \nabla(V(x)|x|^{\alpha})e^{u_n}|\Psi_n|^2 dv \\
&\quad + \frac{1}{4}\int_{S^+_{r_n\rho}(x_n')} \left\langle \frac{\partial\Psi}{\partial\nu}, (x + \bar{x} - 2x_n') \cdot \Psi \right\rangle d\sigma + \frac{1}{4}\int_{S^+_{r_n\rho}(x_n')} \left\langle (x + \bar{x} - 2x_n') \cdot \Psi, \frac{\partial\Psi}{\partial\nu} \right\rangle d\sigma
\end{aligned}$$

Hence for rescaling functions (v_n, φ_n) we have

$$\begin{aligned}
& \rho \int_{S^+_\rho} \left|\frac{\partial v_n}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla v_n|^2 d\sigma \\
&= \int_{D^+_\rho} 2V^2(r_n x + x_n') \left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{2\alpha} e^{2v_n} \\
&\quad -V(r_n x + x_n') \left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{\alpha} e^{v_n}|\varphi_n|^2 dv \\
&\quad + \int_{L_\rho} cV(r_n x + x_n') \left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{\alpha} e^{v_n} ds \\
&\quad -\rho \int_{S^+_\rho} V^2(r_n x + x_n') \left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{2\alpha} e^{2v_n} d\sigma \\
&\quad + \int_{L_\rho} c\frac{\partial(V((r_n s + s_n, 0)) \left|\frac{r_n}{|s_n|}s + \frac{s_n}{|s_n|}\right|^{\alpha})}{\partial s} s e^{v_n} ds
\end{aligned}$$

$$
\begin{aligned}
&-cV((r_n s+s_n,0))\left|\frac{r_n}{|s_n|}s+\frac{s_n}{|s_n|}\right|^{\alpha} s e^{v_n}\Big|_{s=-\rho}^{s=\rho} \\
&+\int_{D_\rho^+} x\cdot\nabla\left(V^2(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}\right)e^{2v_n}dv \\
&-\int_{D_\rho^+} x\cdot\nabla(V(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha})e^{v_n}|\varphi_n|^2dv \\
&+\frac{1}{4}\int_{S_\rho^+}\left\langle\frac{\partial\varphi_n}{\partial\nu},(x+\bar{x})\cdot\varphi_n\right\rangle d\sigma+\frac{1}{4}\int_{S_\rho^+}\left\langle(x+\bar{x})\cdot\varphi_n,\frac{\partial\varphi_n}{\partial\nu}\right\rangle d\sigma. \qquad (63)
\end{aligned}
$$

Since the associated Pohozaev constant of (v_n,φ_n) is

$$
\begin{aligned}
&C(v_n,\varphi_n)=C(v_n,\varphi_n,\rho) \\
&=\rho\int_{S_\rho^+}\left|\frac{\partial v_n}{\partial\nu}\right|^2-\frac{1}{2}|\nabla v_n|^2d\sigma \\
&-\int_{D_\rho^+}2V^2(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}e^{2v_n} \\
&-V(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}|\varphi_n|^2dv \\
&-\int_{L_\rho}cV(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}ds \\
&+\rho\int_{S_\rho^+}V^2(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}e^{2v_n}d\sigma \\
&-\int_{L_\rho}c\frac{\partial\left(V((r_n s+s_n,0))\left|\frac{r_n}{|s_n|}s+\frac{s_n}{|s_n|}\right|^{\alpha}\right)}{\partial s}se^{v_n}ds \\
&+cV((r_n s+s_n,0))\left|\frac{r_n}{|s_n|}s+\frac{s_n}{|s_n|}\right|^{\alpha}se^{v_n}\Big|_{s=-\rho}^{s=\rho} \\
&-\int_{D_\rho^+}x\cdot\nabla\left(V^2(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}\right)e^{2v_n}dv \\
&+\int_{D_\rho^+}x\cdot\nabla\left(V(r_n x+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}\right)e^{v_n}|\varphi_n|^2dv \\
&-\frac{1}{4}\int_{S_\rho^+}\left\langle\frac{\partial\varphi_n}{\partial\nu},(x+\bar{x})\cdot\varphi_n\right\rangle d\sigma-\frac{1}{4}\int_{S_\rho^+}\left\langle(x+\bar{x})\cdot\varphi_n,\frac{\partial\varphi_n}{\partial\nu}\right\rangle d\sigma,
\end{aligned}
$$

we have from (63) that

$$
C(v_n,\varphi_n)=C(v_n,\varphi_n,\rho)=0.
$$

Since $|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x|^{2\alpha}$ is a smooth function in $\overline{D_\rho^+}$, by the energy condition,

$$\int_{D_\rho^+}\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}e^{2v_n}+|\varphi_n|^4dv+\int_{L_\rho}\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}ds<C,$$

we can easily to check that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{D_\rho^+}x\cdot\nabla\left(V^2(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}\right)e^{2v_n}dv$$
$$+\int_{D_\rho^+}x\cdot\nabla\left(V(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}\right)e^{v_n}|\varphi_n|^2dv=0,$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{L_\rho}c\frac{\partial\left(V((r_ns+s_n,0))\left|\frac{r_n}{|s_n|}s+\frac{s_n}{|s_n|}\right|^{\alpha}\right)}{\partial s}se^{v_n}ds=0.$$

This implies that

$$\begin{aligned}0&=\lim_{\rho\to 0}\lim_{n\to\infty}C(v_n,\varphi_n,\rho)=\lim_{\rho\to 0}C(v,\varphi,\rho)\\&\quad-\lim_{r\to 0}\lim_{n\to\infty}\int_{D_r^+}2V^2(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}e^{2v_n}\\&\quad-V(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}|\varphi_n|^2dv\\&\quad-\lim_{r\to 0}\lim_{n\to\infty}\int_{L_r}cV(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}ds\\&=C(v,\varphi)-\beta.\end{aligned}$$

Here

$$\begin{aligned}\beta=\lim_{r\to 0}\lim_{n\to\infty}&\left[\int_{D_r^+}2V^2(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{2\alpha}e^{2v_n}\right.\\&\left.-V(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}|\varphi_n|^2dv\right]\\&+\lim_{r\to 0}\lim_{n\to\infty}\int_{L_r}cV(r_nx+x_n')\left|\frac{x_n'}{|s_n|}+\frac{r_n}{|s_n|}x\right|^{\alpha}e^{v_n}ds.\end{aligned}$$

and $C(v,\varphi)=C(v,\varphi,\rho)$ is the Pohozaev constant of (v,φ), i.e.

$$C(v,\varphi)=\rho\int_{S_\rho^+}\left|\frac{\partial v}{\partial \nu}|^2-\frac{1}{2}|\nabla v\right|^2d\sigma$$

$$
\begin{aligned}
&-\int_{D_\rho^+} 2V^2(0)e^{2v} - V(0)e^v|\varphi|^2 dv + \int_{L_\rho} cV(0)e^v ds \\
&+\rho \int_{S_\rho^+} V^2(0)e^{2v} d\sigma + cV((0,0))se^v|_{s=-\rho}^{s=\rho} \\
&-\frac{1}{4}\int_{S_\rho^+}\left\langle \frac{\partial\varphi}{\partial\nu}, (x+\bar{x})\cdot\varphi\right\rangle d\sigma - \frac{1}{4}\int_{S_\rho^+}\left\langle (x+\bar{x})\cdot\varphi, \frac{\partial\varphi}{\partial\nu}\right\rangle d\sigma.
\end{aligned}
$$

On the other hand, we use the fact that (v_n, φ_n) converges to (v, φ) in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\overline{\mathbb{R}}^2_+ \backslash\{0\}) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+)) \cap C^1_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+\backslash\{0\}))$ again to get

$$
\begin{aligned}
&\int_{D_r^+} 2V^2(r_n x + x_n')\left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{2\alpha} e^{2v_n} - V(r_n x + x_n')\left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{\alpha} e^{v_n}|\varphi_n|^2 dv \\
&\quad + \int_{L_r} cV(r_n x + x_n')\left|\frac{x_n'}{|s_n|} + \frac{r_n}{|s_n|}x\right|^{\alpha} e^{v_n} ds \\
&\quad \to \int_{D_\rho^+} 2V^2(0)e^{2v} - V(0)e^v|\varphi|^2 dv + \int_{L_\rho} cV(0)e^v ds + \beta
\end{aligned}
$$

as $n \to \infty$. By using Green's representation formula for u_n in D_ρ^+ and then take $n \to \infty$, we have

$$
v(x) = \frac{\beta}{\pi}\ln\frac{1}{|x|} + \phi(x) + \gamma(x),
$$

with ϕ being a bounded term and $\gamma(x)$ being a regular term. Consequently, we can obtain that

$$
C(v, \varphi) = \frac{\beta^2}{2\pi}.
$$

This implies that

$$
\beta = \frac{\beta^2}{2\pi}.
$$

Noticing that $\int_{D_\rho^+} e^{2v}dx < \infty$, we have $\beta \le \pi$. Therefore we obtain that $\beta = 0$, i.e. $C(v, \varphi) = 0$, and the singularity at 0 of (v, φ) is removed by Propostion 4.5. Furthermore, the singularity at ∞ of (v, φ) is also removed by Theorem 5.2. Thus we get another bubble on $S^2_{c'}$, and we get a contradiction to the assumption that $m = 1$. Consequently we complete the proof of **Claim II.1**.

Next , similarly to **Case I**. we can prove the following:

Claim II.2 We can separate $A^+_{S,R,n}(x_n)$ into finitely many parts

$$A^+_{S,R,n} = \bigcup_{k=1}^{N_k} A^+_k$$

such that on each part

$$\int_{A^+_k} |x|^{2\alpha} e^{2u_n} \le \frac{1}{4\Lambda^2}, \quad k = 1, 2, \dots, N_k.$$

where $N_k \le N_0$ for N_0 is a uniform integer for all n large enough, $A^+_k = D^+_{r^{k-1}}(x_n) \setminus D^+_{r^k}(x_n)$, $r^0 = \delta$, $r^{N_k} = \lambda_n R$, $r^k < r^{k-1}$ for $k = 1, 2, \dots, N_k$, and C_0 is a constant as in Lemma 6.1.

Then, by using **Claim II.1** and **Claim II.2** we can complete the proof of the result. □

7 Blow-up behavior

In this section, we will show that $u_n \to -\infty$ uniformly on compact subset of $(D^+_r \cup L_r) \setminus \Sigma_1$ in means of the energy identity for spinors. Thus we rule out the possibility that u_n is uniformly bounded in $L^\infty_{loc}((D^+_r \cup L_r) \setminus \Sigma_1)$ in Theorem 1.3. The following is the proof of Theorem 1.5.

Proof of Theorem 1.5: We prove the results by contradiction. Assume that the conclusion of the theorem is false. Then by Theorem 1.3, u_n is uniformly bounded in $L^\infty_{loc}((D^+_r \cup L_r) \setminus \Sigma_1)$. Thus we know that (u_n, Ψ_n) converges in C^2 on any compact subset of $(D^+_r \cup L_r) \setminus \Sigma_1$ to (u, Ψ), which satisfies that

$$\begin{cases} -\Delta u(x) = 2u^2(x)|x|^{2\alpha} e^{2u(x)} - V(x)|x|^\alpha e^{u(x)}|\Psi|^2, & \text{in } D^+_r \setminus \Sigma_1, \\ \not{D}\Psi = -V(x)|x|^\alpha e^{u(x)}\Psi, & \text{in } D^+_r \setminus \Sigma_1, \\ \frac{\partial u}{\partial n} = cV(x)|x|^\alpha e^{u(x)}, & \text{on } L_r \setminus \Sigma_1, \\ B(\Psi) = 0, & \text{on } L_r \setminus \Sigma_1. \end{cases} \tag{64}$$

with bounded energy

$$\int_{D^+_r} (|x|^{2\alpha} e^{2u} + |\Psi|^4) dx + \int_{L_r} |x|^\alpha e^u ds < +\infty.$$

Since the blow-up set Σ_1 is not empty, we can take a point $p \in \Sigma_1$. Choose a small $\delta_0 > 0$ such that p is the only point of Σ_1 in $\overline{D}_{2\delta_0}(p) \cap (D^+_r \cup L_r) = \{p\}$. If p is the interior point of D^+_r, then we can choose δ_0 sufficiently small such that $D_{2\delta_0}(p) \subset (D^+_r \cup L_r)$. Hence by Theorem 1.3 in [27] we can get a contradiction.

Next we assume that p is on L_r. Without loss of generality, we assume that $p = 0$. The case of $p \neq 0$ can be dealed with in an analogous way.

We shall first show that the limit (u, Ψ) is regular at the isolated singularity $p = 0$, i.e. $u \in C^2(D_r^+) \cap C^1(D_r^+ \cup L_r)$ and $\Psi \in C^2(\Gamma(\Sigma D_r^+)) \cap C^1(\Gamma(\Sigma(D_r^+ \cup L_r)))$ for some small $r > 0$. To this end, we shall using Theorem 4.5 for removability of a local singularity to remove the singularity. We know that the Phohozaev constant, denote $C_B(u, \Psi)$, of (u, Ψ) at $p = 0$ is

$$
\begin{aligned}
C_B(u, \Psi) := C_B(u, \Psi, \rho) = \rho \int_{S_\rho^+} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\
& - (1+\alpha) \int_{D_\rho^+} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dv \\
& - (\alpha+1) \int_{L_\rho} cV(x)|x|^\alpha e^u ds \\
& + \rho \int_{S_\rho^+} V^2(x)|x|^{2\alpha} e^{2u} d\sigma - \int_{L_\rho} c\frac{\partial V(s,0)}{\partial s}|s|^\alpha s e^u ds \\
& + cV(s,0)|s|^\alpha s e^u|_{s=-\rho}^{s=\rho} \\
& - \int_{D_\rho^+} x \cdot \nabla(V^2(x))|x|^{2\alpha} e^{2u} dv + \int_{D_\rho^+} x \cdot \nabla V(x)|x|^\alpha e^u |\psi|^2 dv \\
& - \frac{1}{4}\int_{S_\rho^+} \left\langle \frac{\partial \Psi}{\partial \nu}, (x+\bar{x}) \cdot \Psi \right\rangle d\sigma - \frac{1}{4}\int_{S_\rho^+} \left\langle (x+\bar{x}) \cdot \Psi, \frac{\partial \Psi}{\partial \nu} \right\rangle d\sigma
\end{aligned}
$$

for any $0 < \rho < \delta_0$. On the other hand, since (u_n, Ψ_n) are the regular solution, the Pohozaev constant $C_B(u_n, \Psi_n) = C_B(u_n, \Psi_n, \rho)$ satisfies

$$
\begin{aligned}
0 = C(u_n, \Psi_n) &= C(u_n, \Psi_n, \rho) \\
&= \rho \int_{S_\rho^+} \left|\frac{\partial u_n}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla u_n|^2 d\sigma \\
&\quad -(1+\alpha) \int_{D_\rho^+} 2V^2(x)|x|^{2\alpha} e^{2u_n} - V(x)|x|^\alpha e^{u_n} |\Psi_n|^2 dv \\
&\quad -(\alpha+1) \int_{L_\rho} cV(x)|x|^\alpha e^{u_n} ds \\
&\quad +\rho \int_{S_\rho^+} V^2(x)|x|^{2\alpha} e^{2u_n} d\sigma - \int_{L_\rho} c\frac{\partial V(s,0)}{\partial s}|s|^\alpha s e^{u_n} ds \\
&\quad +cV((s,0))|s|^\alpha s e^{u_n}|_{s=-\rho}^{s=\rho} \\
&\quad -\int_{D_\rho^+} x \cdot \nabla(V^2(x))|x|^{2\alpha} e^{2u_n} dv + \int_{D_\rho^+} x \cdot \nabla V(x)|x|^\alpha e^{u_n} |\Psi_n|^2 dv \\
&\quad -\frac{1}{4}\int_{S_\rho^+} \left\langle \frac{\partial \Psi_n}{\partial \nu}, (x+\bar{x}) \cdot \Psi_n \right\rangle d\sigma - \frac{1}{4}\int_{S_\rho^+} \left\langle (x+\bar{x}) \cdot \Psi_n, \frac{\partial \Psi_n}{\partial \nu} \right\rangle d\sigma.
\end{aligned}
$$

Let $n \to \infty$ and $\rho \to 0$, by using that (u_n, Ψ_n) converges to (u, Ψ) regularly on any compact subset of $\overline{D}^+_{2\delta_0} \setminus \{0\}$ and that the energy condition (17), to get

$$
\begin{aligned}
0 &= \lim_{\rho\to 0}\lim_{n\to\infty} C(u_n, \Psi_n, \rho) = \lim_{\rho\to 0} C(u, \Psi, \rho) \\
&\quad -(1+\alpha)\lim_{\delta\to 0}\lim_{n\to\infty}\left\{\int_{D^+_\delta}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx \right.\\
&\qquad\qquad \left. + \int_{L_\delta} cV(x)|x|^{\alpha}e^{u_n}ds\right\} \\
&= C(u, \Psi) - (1+\alpha)\beta,
\end{aligned}
$$

where

$$
\beta = \lim_{\delta\to 0}\lim_{n\to\infty}\left\{\int_{D^+_\delta}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx + \int_{L_\delta} cV(x)|x|^{\alpha}e^{u_n}ds\right\}.
$$

Moreover, we can also assume that

$$
\begin{aligned}
&(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx + cV(x)|x|^{\alpha}e^{u_n}ds \\
&\quad \to \nu = (2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2)dx + cV(x)|x|^{\alpha}e^{u}ds + \beta\delta_{p=0}
\end{aligned}
$$

in the sense of distributions in $D^+_\delta \cup L_\delta$ for any small $\delta > 0$. Then, applying similar arguments as in the proof of the local singularity removability in Claim I.1, Theorem 1.4, we can show that $C_B(u, \Psi) = 0$, $\beta = 0$ and hence (u, Ψ) is a regular solution of (4) on $D^+_{2\delta_0}$ with bounded energy

$$
\int_{D^+_{2\delta_0}}(|x|^{2\alpha}e^{2u} + |\Psi|^4)dx + \int_{L_{2\delta}}|x|^{\alpha}e^{u}ds < +\infty.
$$

Hence, we can choose some small $\delta_1 \in (0, \delta_0)$ such that for any $\delta \in (0, \delta_1)$,

$$
\int_{D^+_\delta}(2V^2(x)|x|^{2\alpha}e^{2u} - V(x)|x|^{\alpha}e^{u}|\Psi|^2)dx + \int_{L_\delta} cV(x)|x|^{\alpha}e^{u}ds < \min\left\{\frac{1+\alpha}{10}, \frac{1}{10}\right\}. \tag{65}
$$

Next, as in the proof of Theorem 1.4, we rescale (u_n, Ψ_n) near $p = 0$. We let $x_n \in \overline{D}^+_{\delta_1}$ such that $u_n(x_n) = \max_{\overline{D}^+_{\delta_1}} u_n(x)$. Write $x_n = (s_n, t_n)$. Then $x_n \to p = 0$ and $u_n(x_n) \to +\infty$. Define $\lambda_n = e^{-\frac{u_n(x_n)}{\alpha+1}}$. It is clear that λ_n, $|x_n|$ and t_n converge to 0 as $n \to 0$. we will proceed by distinguishing the following three cases:

Case I. $\frac{|x_n|}{\lambda_n} = O(1)$ as $n \to +\infty$.

In this case, the rescaling functions are

$$
\begin{cases}
\widetilde{u}_n(x) = u_n(\lambda_n x) + (1+\alpha)\ln\lambda_n \\
\widetilde{\Psi}_n(x) = \lambda_n^{\frac{1}{2}}\Psi_n(\lambda_n x)
\end{cases}
$$

for any $x \in \overline{D}^+_{\frac{\delta_1}{2\lambda_n}}$. Moreover, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_+)) \cap C^1_{loc}(\Gamma(\Sigma\bar{\mathbb{R}}^2_+))$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$\begin{cases} -\Delta\widetilde{u} = 2V^2(0)|x|^{2\alpha}e^{2\widetilde{u}} - V(0)|x|^{\alpha}e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2_+, \\ \not{D}\widetilde{\Psi} = -V(0)|x|^{\alpha}e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2_+, \\ \frac{\partial\widetilde{u}}{\partial n} = cV(0)|x|^{\alpha}e^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_+, \\ B\widetilde{\Psi} = 0, & \text{on } \partial\mathbb{R}^2_+ \end{cases}$$

and

$$\int_{\mathbb{R}^2_+}(2V^2(0)|x|^{2\alpha}e^{2\widetilde{u}} - V(0)|x|^{\alpha}e^{\widetilde{u}}|\widetilde{\Psi}|^2)dx + \int_{\partial\mathbb{R}^2_+} cV(0)|x|^{\alpha}e^{\widetilde{u}}d\sigma = 2\pi(1+\alpha).$$

Then for $\delta \in (0, \delta_1)$ small enough, $R > 0$ large enough and n large enough, we have

$$\begin{aligned} &\int_{D^+_\delta}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx + \int_{L_\delta} cV(x)|x|^{\alpha}e^{u_n}ds \\ &= \int_{D_{\lambda_n R}}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx + \int_{L_{\lambda_n R}} cV(x)|x|^{\alpha}e^{u_n}ds \\ &\quad + \int_{D^+_\delta\setminus D^+_{\lambda_n R}}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx + \int_{L_\delta\setminus L_{\lambda_n R}} cV(x)|x|^{\alpha}e^{u_n}ds \\ &\geq \int_{D^+_R}(2V^2(\lambda_n x)|x|^{2\alpha}e^{2\widetilde{u}_n} - V(\lambda_n x)|x|^{\alpha}e^{\widetilde{u}_n}|\widetilde{\Psi}_n|^2) + \int_{L_R} cV(\lambda_n x)|x|^{\alpha}e^{\widetilde{u}_n}ds \\ &\quad - \int_{D^+_\delta\setminus D^+_{t_n R}} V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2 \\ &\geq 2\pi(1+\alpha) - \frac{1+\alpha}{10}. \end{aligned} \tag{66}$$

Here in the last step, we use the fact from Theorem 1.4 that the neck energy of the spinor field Ψ_n is converging to zero. We remark that in the above estimate, if there are multiple bubbles then we need to decompose $D^+_\delta\setminus D^+_{\lambda_n R}$ further into bubble domains and neck domains and then apply the no neck energy result in Theorem 1.4 to each of these neck domains.

On the other hand, we fix some $\delta \in (0, \delta_1)$ small such that (66) holds and then let $n \to \infty$ to conclude that

$$\begin{aligned} 2\pi(1+\alpha) - \frac{1+\alpha}{10} &\leq \int_{D^+_\delta}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^{\alpha}e^{u_n}|\Psi_n|^2)dx \\ &\quad + \int_{L_\delta} cV(x)|x|^{\alpha}e^{u_n}ds \\ &= -\int_{D^+_\delta}\Delta u_n = -\int_{\partial B_\delta}\frac{\partial u_n}{\partial n} \\ &\to -\int_{\partial D^+_\delta}\frac{\partial u}{\partial n} = -\int_{D^+_\delta}\Delta u \end{aligned}$$

$$= \int_{D_\delta^+} (2V^2(x)|x|^{2\alpha} e^{2u} - V(x)|x|^\alpha e^u |\Psi|^2) dx + \int_{L_\delta} cV(x)|x|^\alpha e^u ds < \frac{1+\alpha}{10}$$

Here in the last step, we have used (65). Thus we get a contradiction and finish the proof of the Theorem in this case.

Case II. $\frac{|x_n|}{\lambda_n} \to +\infty$ as $n \to +\infty$.

In this case, as in the arguments in Theorem 1.4, we can rescale twice to get the bubble. First, we define the rescaling functions

$$\begin{cases} \bar{u}_n(x) = u_n(|x_n|x) + (\alpha+1)\ln|x_n| \\ \bar{\Psi}_n(x) = |x_n|^{\frac{1}{2}} \Psi_n(|x_n|x) \end{cases}$$

for any $x \in \overline{D^+}_{\frac{\delta_1}{2|x_n|}}$. Set $y_n := \frac{x_n}{|x_n|}$. Due to $\bar{u}_n(y_n) \to +\infty$, we set that $\delta_n = e^{-\bar{u}_n(y_n)}$ and define the rescaling function

$$\begin{cases} \widetilde{u}_n(x) = \bar{u}_n(\delta_n x + y_n) + \ln \delta_n \\ \widetilde{\Psi}_n(x) = \delta_n^{\frac{1}{2}} \bar{\Psi}_n(\delta_n x + y_n) \end{cases}$$

for any $\delta_n x + y_n \in \overline{D^+}_{\frac{\delta_1}{2|x_n|}}$. Denote that $\rho_n = \frac{e^{-u_n(x_n)}}{|x_n|^\alpha} = \lambda_n (\frac{\lambda_n}{|x_n|})^\alpha$ and $x_n = (s_n, t_n)$.

Case II.1 $\frac{t_n}{\rho_n} \to +\infty$ as $n \to \infty$.

Then, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2))$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$\begin{cases} -\Delta \widetilde{u} = 2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2, \\ \not{D}\widetilde{\Psi} = -V(0)e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2, \end{cases}$$

with the bubble energy

$$\int_{\mathbb{R}^2} (2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2) dx = 4\pi.$$

Therefore, for $\delta \in (0, \delta_1)$ small enough, $S, R > 0$ large enough and n large enough, the fact that the neck energy of the spinor field Ψ_n is converging to zero, we have

$$\int_{D_\delta^+} (2V^2(x)|x|^{2\alpha} e^{2u_n} - V(x)|x|^\alpha e^{u_n} |\Psi_n|^2) dx + \int_{L_\delta} cV(x)|x|^\alpha e^{u_n} ds$$
$$= \int_{D^+_{\frac{\delta}{|x_n|}}} (2V^2(|x_n|x)|x|^{2\alpha} e^{2\bar{u}_n} - V(|x_n|x)|x|^\alpha e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx + \int_{L_{\frac{\delta}{|x_n|}}} cV(|x_n|x)|x|^\alpha e^{\bar{u}_n} ds$$

$$
\begin{aligned}
&= \int_{D^+_{\frac{\delta}{|x_n|}} \setminus D^+_S(y_n)} (2V^2(|x_n|x)|x|^{2\alpha} e^{2\bar{u}_n} - V(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\
&\quad + \int_{D^+_S(y_n) \setminus D^+_{\frac{\rho_n}{|x_n|}R}(y_n)} (2V^2(|x_n|x)|x|^{2\alpha} e^{2\bar{u}_n} - V(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\
&\quad + \int_{D^+_{\frac{\rho_n}{|x_n|}R}(y_n)} (2V^2(|x_n|x)|x|^{2\alpha} e^{2\bar{u}_n} - V(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} |\bar{\Psi}_n|^2) dx \\
&\quad + \int_{L_{\frac{\rho_n R}{|x_n|}}(y_n)} cV(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} ds \\
&\quad + \int_{L_S(y_n) \setminus L_{\frac{\rho_n R}{|x_n|}}(y_n)} cV(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} ds + \int_{L_{\frac{\delta}{|x_n|}} \setminus L_S(y_n)} cV(|x_n|x)|x|^{\alpha} e^{\bar{u}_n} ds \\
&\geq \int_{D_R \cap \left\{t > -\frac{t_n}{\rho_n}\right\}} \Bigg(2V^2(x_n + \rho_n x) \left| \frac{x_n}{|x_n|} + \frac{\rho_n}{|x_n|} x \right|^{2\alpha} e^{2\widetilde{u}_n(x)} \\
&\qquad\qquad - V(x_n + \rho_n x) \left| \frac{x_n}{|x_n|} + \frac{\rho_n}{|x_n|} x |^{\alpha} e^{\widetilde{u}_n(x)} \right| \widetilde{\Psi}_n|^2 \Bigg) dx \\
&\quad + \int_{D_R \cap \left\{t = -\frac{t_n}{\rho_n}\right\}} (cV(x_n + \rho_n x) \left| \frac{x_n}{|x_n|} + \frac{\rho_n}{|x_n|} x \right|^{\alpha} e^{\widetilde{u}_n(x)} \\
&\quad - \int_{D^+_{|x_n|S}(x_n) \setminus D^+_{\rho_n R}(x_n)} V(x)|x|^{\alpha} e^{u_n} |\Psi_n|^2 - \int_{D^+_{\frac{\delta}{|x_n|}} \setminus D^+_S(y_n)} V(t_n x)|x|^{\alpha} e^{\bar{u}_n} |\bar{\Psi}_n|^2 \\
&\geq 4\pi - \frac{1}{10}.
\end{aligned}
$$

Then, applying similar arguments as in **Case I**, we get a contradiction, and finish the proof of the Theorem in this case.

Case II.2 $\frac{t_n}{\rho_n} \to \Lambda$ as $n \to \infty$.

Then, by passing to a subsequence, $(\widetilde{u}_n, \widetilde{\Psi}_n)$ converges in $C^2_{loc}(\mathbb{R}^2_{-\Lambda}) \cap C^1_{loc}(\bar{\mathbb{R}}^2_{-\Lambda}) \times C^2_{loc}(\Gamma(\Sigma\mathbb{R}^2_{-\Lambda})) \cap C^1_{loc}(\Sigma\bar{\mathbb{R}}^2_{-\Lambda})$ to some $(\widetilde{u}, \widetilde{\Psi})$ satisfying

$$
\begin{cases}
-\Delta\widetilde{u} = 2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2, & \text{in } \mathbb{R}^2_{-\Lambda}, \\
\not{D}\widetilde{\Psi} = -V(0)e^{\widetilde{u}}\widetilde{\Psi}, & \text{in } \mathbb{R}^2_{-\Lambda}, \\
\frac{\partial\widetilde{u}}{\partial n} = cV(0)e^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_{-\Lambda}, \\
B\widetilde{\Psi} = 0, & \text{on } \partial\mathbb{R}^2_{-\Lambda},
\end{cases}
$$

with the bubble energy

$$
\int_{\mathbb{R}^2_{-\Lambda}} (2V^2(0)e^{2\widetilde{u}} - V(0)e^{\widetilde{u}}|\widetilde{\Psi}|^2) dx + \int_{\partial\mathbb{R}^2_{-\Lambda}} cV(0)e^{\widetilde{u}} d\sigma = 2\pi.
$$

Then, applying similar arguments as in **Case II.1**, we can get a contradiction, and finish the proof of the Theorem. □

8 Blow-up Value

By means of Theorem 1.5, we can further compute the blow-up value at the blow-up point p, which is defined as

$$m(p) = \lim_{\delta\to 0}\lim_{n\to\infty}\left\{\int_{D_\delta^+(p)}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^\alpha e^{u_n}|\Psi_n|^2)dx \right. \\ \left. + \int_{L_\delta(p)} cV(x)|x|^\alpha e^{u_n} ds\right\}.$$

We know from Theorem 1.5 that $m(p) > 0$. Now we shall determine the precise value of $m(p)$ under a boundary condition.

Proof of Theorem 1.6: When $p \notin L_{\delta_0}(p)$, It is clear that we can choose δ_0 sufficiently small such that $\overline{D_{\delta_0}^+(p)} = \overline{D_{\delta_0}(p)}$. Then we have $m(p) = 4\pi$ according to the arguments in [27]. Next we assume that $p \in L_{\delta_0}(p)$. Without loss of generality, we assume $p = 0$. The case of $p \neq 0$ can be handled analogously.

By using the boundary condition, it follows that

$$0 \le u_n - \min_{S_{\delta_0}^+} u_n \le C$$

on $S_{\delta_0}^+$. Let w_n be the unique solution of the following problem

$$\begin{cases} -\Delta w_n = 0, & \text{in } D_{\delta_0}^+, \\ \frac{\partial w_n}{\partial n} = 0, & \text{on } L_{\delta_0}, \\ w_n = u_n - \min_{S_{\delta_0}^+} u_n, & \text{on } S_{\delta_0}^+. \end{cases}$$

It follows from the maximum principle and the Hopf Lemma that w_n is uniformly bounded in $\overline{D_{\delta_0}^+}$, and consequently w_n is $C^2(D_{\delta_0}^+) \cap C^1(D_{\delta_0}^+ \cup L_{\delta_0})$. Now we set $v_n = u_n - \min_{S_{\delta_0}^+} u_n - w_n$. Then v_n satisfies that

$$\begin{cases} -\Delta v_n = 2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^\alpha e^{u_n}|\Psi_n|^2, & \text{in } D_{\delta_0}^+, \\ \frac{\partial v_n}{\partial n} = cV(x)|x|^\alpha e^{u_n}, & \text{on } L_{\delta_0}, \\ v_n = 0, & \text{on } S_{\delta_0}^+, \end{cases}$$

with the energy condition

$$\int_{D_{\delta_0}^+}(2V^2(x)|x|^{2\alpha}e^{2u_n} - V(x)|x|^\alpha e^{u_n}|\Psi_n|^2)dx + \int_{L_{\delta_0}} cV(x)|x|^\alpha e^{u_n} ds \le C. \quad (67)$$

By Green's representation formula, we have

$$v_n(x) = \frac{1}{\pi}\int_{D^+_{\delta_0}} \log\frac{1}{|x-y|}(2V^2(y)|y|^{2\alpha}e^{2u_n} - V(y)|y|^{\alpha}e^{u_n}|\Psi_n|^2)dy \\ +\frac{1}{\pi}\int_{L_{\delta_0}} \log\frac{1}{|x-y|}cV(y)|y|^{\alpha}e^{u_n}dy + R_n(x) \tag{68}$$

where $R_n(x) \in C^1(D^+_{\delta_0} \cup L_{\delta_0})$ is a regular term. By using Theorem 1.5, we know

$$v_n(x) \to \frac{m(p)}{\pi}\ln\frac{1}{|x|} + R(x), \text{ in } C^1_{loc}((D^+_{\delta_0} \cup L_{\delta_0}) \setminus \{0\}) \tag{69}$$

for $R(x) \in C^1(D^+_{\delta_0} \cup L_{\delta_0})$. On the other hand, we observe that (v_n, Ψ_n) satisfies

$$\begin{cases} -\Delta v_n = 2K_n^2(x)|x|^{2\alpha}e^{2v_n} - K_n(x)|x|^{\alpha}e^{v_n}|\Psi_n|^2, & \text{in } D^+_{\delta_0}, \\ \not{D}\Psi_n = -K_n(x)e^{v_n}\Psi_n, & \text{in } D^+_{\delta_0}, \\ \frac{\partial v_n}{\partial n} = cK_n(x)|x|^{\alpha}e^{u_n}, & \text{on } L_{\delta_0}, \\ B(\Psi_n) = 0, & \text{on } L_{\delta_0}, \end{cases}$$

where $K_n = V(x)e^{\min_{S^+_{\delta_0}} u_n + w_n}$. Noticing the Pohozaev identity of (v_n, Ψ_n) in $D^+_{\delta_0}$ for $0 < \delta < \delta_0$ is

$$\begin{aligned} &\delta\int_{S^+_\delta}\left|\frac{\partial v_n}{\partial \nu}\right|^2 - \frac{1}{2}|\nabla v_n|^2 d\sigma \\ &= (1+\alpha)\left\{\int_{D^+_\delta}(2K_n^2(x)|x|^{2\alpha}e^{2v_n} - K_n(x)|x|^{\alpha}e^{v_n}|\Psi_n|^2)dv + \int_{L_\delta} cK_n(x)|x|^{\alpha}e^{v_n}ds\right\} \\ &\quad -\delta\int_{S^+_\delta} K_n^2(x)|x|^{2\alpha}e^{2v_n}d\sigma \\ &\quad +\int_{L_\delta} c\frac{\partial K_n(s,0)}{\partial s}|s|^{\alpha}se^{v_n(s,0)}ds - cK_n(s,0)|s|^{\alpha}se^{v_n(s,0)}|_{s=-\delta}^{s=\delta} \\ &\quad +\int_{D^+_\delta} x\cdot\nabla(K_n^2(x))|x|^{2\alpha}e^{2v_n}dv - \int_{D^+_\delta} x\cdot\nabla K_n(x)|x|^{\alpha}e^{v_n}|\Psi_n|^2 dv \\ &\quad +\frac{1}{4}\int_{S^+_\delta}\left\langle\frac{\partial\Psi_n}{\partial\nu}, (x+\bar{x})\cdot\Psi_n\right\rangle d\sigma + \frac{1}{4}\int_{S^+_\delta}\left\langle(x+\bar{x})\cdot\Psi_n, \frac{\partial\Psi_n}{\partial\nu}\right\rangle d\sigma. \end{aligned} \tag{70}$$

We will take $n \to \infty$ first and then $\delta \to 0$ in (70). By using (69) we get

$$\lim_{\delta\to 0}\lim_{n\to\infty}\delta\int_{S_\delta^+}\left|\frac{\partial v_n}{\partial \nu}\right|^2-\frac{1}{2}|\nabla v_n|^2 d\sigma=\lim_{\delta\to 0}\delta\int_{S_\delta^+}\frac{1}{2}\left|\frac{\partial\left(\frac{m(p)}{\pi}\ln\frac{1}{|x|}\right)}{\partial\nu}\right|^2 d\sigma=\frac{1}{2\pi}m^2(p).$$

By using $u_n \to -\infty$ uniformly on S_δ^+, we also have

$$\lim_{\delta\to 0}\lim_{n\to\infty}\delta\int_{S_\delta^+}K_n^2(x)|x|^{2\alpha}e^{2v_n}d\sigma=\lim_{\delta\to 0}\lim_{n\to\infty}\delta\int_{S_\delta^+}V^2(x)|x|^{2\alpha}e^{2u_n}d\sigma=0,$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}cK_n(s,0)|s|^\alpha s e^{v_n(s,0)}|_{s=-\delta}^{s=\delta}=0.$$

By using the energy condition (67), we have

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{D_\delta^+}\left(|x|^{2\alpha}e^{2v_n}x\cdot\nabla(K_n^2(x))\right)-|x|^\alpha e^{v_n}|\Psi_n|^2 x\cdot\nabla K_n(x))dx=0,$$

and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{L_\delta}c\frac{\partial K_n(s,0)}{\partial s}|s|^\alpha s e^{v_n(s,0)}ds=0.$$

Since $u_n \to -\infty$ uniformly in any compact subset of $(D_{\delta_0}^+\cup L_{\delta_0})\backslash\{0\}$, and $|\Psi_n|$ is uniformly bounded in any compact subset of $(D_{\delta_0}^+\cup L_{\delta_0})\backslash\{0\}$, we know

$$\begin{cases} \not{D}\Psi=0, & \text{in } D_{\delta_0}^+,\\ B\Psi=0, & \text{on } L_{\delta_0}\setminus\{0\}. \end{cases}$$

We extend Ψ a harmonic spinor $\overline{\Psi}$ on $D_{\delta_0}\backslash\{0\}$ with bounded energy, i.e., $||\overline{\Psi}||_{L^4(D_{\delta_0})}<\infty$. Since the local singularity of a harmonic spinor with finite energy is removable, we have $\overline{\Psi}$ is smooth in D_{δ_0}. It follows that Ψ is smooth in $D_{\delta_0}^+\cup L_{\delta_0}$. Therefore we obtain that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\left(\frac{1}{4}\int_{S_\delta^+}\left\langle\frac{\partial\Psi_n}{\partial\nu},(x+\bar{x})\cdot\Psi_n\right\rangle d\sigma+\frac{1}{4}\int_{S_\delta^+}\left\langle(x+\bar{x})\cdot\Psi_n,\frac{\partial\Psi_n}{\partial\nu}\right\rangle d\sigma\right)=0.$$

Putting all together, we obtain that

$$\frac{1}{2\pi}m^2(p)=(1+\alpha)m(p).$$

It follows that $m(p)=2\pi(1+\alpha)$. Thus we finish the proof of Theorem 1.6. □

9 Energy quantization for the global super-Liouville boundary problem

In this section, we will show the quantization of energy for a sequence of blowing-up solutions to the global super-Liouville boundary problem on a singular Riemann surface. Let $(M, \mathcal{A}, g)$ be a compact Riemann surface with conical singularities represented by the divisor $\mathcal{A} = \Sigma_{j=1}^{m} \alpha_j q_j, \alpha_j > 0$ and with a spin structure. We assume that ∂M is not empty and (M, g) has conical singular points $q_1, q_2, \ldots, q_m$ such that $q_1, q_2, \ldots, q_l$ are in M^o for $1 \le l < m$ and $q_{l+1}, q_{l+2}, \ldots, q_m$ are on ∂M. Writing $g = e^{2\phi} g_0$, where g_0 is a smooth metric on M, we can deduce from the results for the local super-Liouville equations:

Proof of Theorem 1.1: Since $g = e^{2\phi} g_0$ with g_0 being smooth, then by the well known properties of ϕ (see e.g. [36] or [5], p. 5639), we know that (u_n, ψ_n) satisfies

$$\begin{cases} -\Delta_{g_0}(u_n + \phi) = 2e^{2(u_n+\phi)} - e^{u_n+\phi} \left\langle e^{\frac{\phi}{2}} \psi_n, e^{\frac{\phi}{2}} \psi_n \right\rangle_{g_0} - K_{g_0} - \sum_{j=1}^{l} 2\pi \alpha_j \delta_{q_j} & \text{in } M^o, \\ \not{D}_{g_0}(e^{\frac{\phi}{2}} \psi_n) = -e^{u_n+\phi}(e^{\frac{\phi}{2}} \psi_n) & \text{in } M^o, \\ \frac{\partial(u_n+\phi)}{\partial n} = ce^{u_n+\phi} - h_{g_0} + \sum_{j=l+1}^{m} \pi \alpha_j \delta_{q_j}, & \text{on } \partial M, \\ B^{\pm}(e^{\frac{\phi}{2}} \psi_n) = 0, & \text{on } \partial M, \end{cases}$$

with the energy conditions:

$$\int_M e^{2(u_n+\phi)} dg_0 + |e^{\frac{\phi}{2}} \psi_n|_{g_0}^4 dv_{g_0} + \int_{\partial M} e^{u_n+\phi} d\sigma_{g_0} < C.$$

If we define the blow-up set of $u_n + \phi$ as

$$\Sigma_1' = \{x \in M, \text{ there is a sequence } y_n \to x \text{ such that } (u_n + \phi)(y_n) \to +\infty\},$$

then by Remark 3.4 and Remark 3.3 in [27], we have $\Sigma_1 = \Sigma_1'$. By the blow-up results of the local system, it follows that one of the following alternatives holds:

(i) u_n is bounded in $L^\infty(M)$.
(ii) $u_n \to -\infty$ uniformly on M.
(iii) Σ_1 is finite, nonempty and

$$u_n \to -\infty \text{ uniformly on compact subsets of } M \backslash \Sigma_1.$$

Furthermore,

$$\int_M (2e^{2(u_n+\phi)} - e^{u_n+\phi} |e^{\frac{\phi}{2}} \psi_n|_{g_0}^2) \varphi dv_{g_0} + \int_{\partial M} ce^{u_n+\phi} \varphi d\sigma_{g_0} \to \sum_{p_i \in \Sigma_1} m(p_i) \varphi(p_i)$$

for any smooth function φ on M.

Next let $p = \frac{q}{q-1} > 2$. Notice that

$$
\begin{aligned}
&||\nabla(u_n+\phi)||_{L^q(M,g_0)}\\
&\le \sup\left\{|\int_M \nabla(u_n+\phi)\nabla\varphi dv_{g_0}||\varphi\in W^{1,p}(M,g_0),\right.\\
&\qquad\qquad \left.\int_M \varphi dv_{g_0}=0, ||\varphi||_{W^{1,p}(M,g_0)}=1\right\}.
\end{aligned}
$$

Due to $||\varphi||_{L^\infty(M,g_0)\le C}$ for any $\varphi\in W^{1,p}(M,g_0)$ with $\int_M \varphi dv_{g_0}=0$ and $||\varphi||_{W^{1,p}(M,g_0)}=1$ by the Sobolev embedding theorem, we get that

$$
\begin{aligned}
&\left|\int_M \nabla(u_n+\phi)\nabla\varphi dv_{g_0}\right|\\
&=\left|-\int_M \Delta_{g_0}(u_n+\phi)\varphi dv_{g_0}+\int_{\partial M}\frac{\partial(u_n+\phi)}{\partial n}\varphi d\sigma_{g_0}\right|\\
&\le \int_M (2e^{2(u_n+\phi)}+e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2_{g_0}+|K_{g_0}|)|\varphi|dg_0+\int_{\partial M}(ce^{u_n+\phi}+|h_{g_0}|)|\varphi|d\sigma_{g_0}\\
&\quad+\sum_{j=1}^{l}\left|\int_M 2\pi\alpha_j\delta_{q_j}\varphi dv_{g_0}\right|+\sum_{j=1+1}^{m}\left|\int_{\partial M}\pi\alpha_j\delta_{q_j}\varphi d\sigma_{g_0}\right|\\
&\le C.
\end{aligned}
$$

This means that $u_n+\phi-\frac{1}{|M|}\int_M(u_n+\phi)dv_{g_0}$ is uniformly bounded in $W^{1,q}(M,g_0)$.

We define the Green function G by

$$
\begin{cases}
-\Delta_{g_0}G=\sum_{p\in M^o\cap\Sigma_1} m(p)\delta_p-K_{g_0}-\sum_{j=1}^{l}2\pi\alpha_j\delta_{q_j},\\
\frac{\partial G}{\partial n}=\sum_{p\in\partial M\cap\Sigma_1} m(p)\delta_p-h_{g_0}+\sum_{j=l+1}^{m}\pi\alpha_j\delta_{q_j},\\
\int_M G dv_{g_0}=0.
\end{cases}
$$

It is clear that $G\in W^{1,q}(M,g_0)\cap C^2_{loc}(M\backslash\Sigma_1)$ with $\int_M Gdg_0=0$ for $1<q<2$.

Now we take $R>0$ small such that, at each blow-up point $p\in\Sigma_1$, the geodesic ball of M, $B^M_R(p)$, satisfies $B^M_R(p)\cap(\Sigma_1\cup\{q_1,q_2,\dots,q_m\})=\{p\}$. Then we also have

$$
G(x)=\begin{cases}
-\frac{1}{2\pi}m(p)\log d(x,p)+g(x), & \text{if } p\in M^o\cap(\Sigma_1\setminus\{q_1,q_2,\dots,q_m\}),\\
-\left(\frac{1}{2\pi}m(p)-\alpha_j\right)\log d(x,p)+g(x), & \text{if } p=q_j\in M^o\cap\Sigma_1\cap\{q_1,q_2,\dots,q_l\},\\
-\left(\frac{1}{\pi}m(p)\right)\log d(x,p)+g(x), & \text{if } p\in\partial M\cap(\Sigma_1\setminus\{q_{l+1},q_{l+2},\dots,q_m\}),\\
-\left(\frac{1}{\pi}m(p)+a_j\right)\log d(x,p)+g(x), & \text{if } p=q_j\in\partial M\cap\Sigma_1\cap\{q_{l+1},q_{l+2},\dots,q_m\},
\end{cases}
$$

for $x\in B^M_R(p)\backslash\{p\}$ with $g\in C^2(B^M_R(p))$, where $d(x,p)$ denotes the Riemannian distance between x and p with respect to g_0 and

$$
\begin{aligned}
m(p)=\lim_{R\to 0}\lim_{n\to\infty}&\left\{\int_{B^M_R(p)}(2e^{2(u_n+\phi)}-e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2_{g_0}-K_{g_0})dv_{g_0}\right.\\
&\left.+\int_{\partial M\cap B^M_R(p)}(ce^{u_n+\varphi}-h_{g_0})d\sigma_{g_0}\right\}.
\end{aligned}
$$

On the other hand, since for any $\varphi \in C^{\infty}(M)$

$$\begin{aligned}
&\int_M \nabla(u_n+\phi-G)\nabla\varphi dv_{g_0}\\
&= -\int_M \Delta_{g_0}(u_n+\phi-G)\varphi dv_{g_0} + \int_{\partial M}\frac{\partial(u_n+\phi-G)}{\partial n}\varphi d\sigma_{g_0}\\
&= \int_M (2e^{2(u_n+\phi)} - e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2_{g_0} - \sum_{p\in M^o\cap\Sigma_1} m(p)\delta_p)\varphi dv_{g_0}\\
&\quad + \int_{\partial M}\left(ce^{u_n+\phi} - \sum_{p\in\partial M\cap\Sigma_1} m(p)\delta_p\right)\varphi d\sigma_{g_0}\\
&\to 0, \quad \text{as } n\to\infty,
\end{aligned}$$

by using the fact that $u_n+\phi-\frac{1}{|M|}\int_M(u_n+\phi)dg_0$ is uniformly bounded in $W^{1,q}(M,g_0)$, we get

$$u_n+\phi-\frac{1}{|M|}\int_M(u_n+\phi)dg_0 \to G$$

strongly in $C^2_{loc}(M\backslash\Sigma_1)$ and weakly in $W^{1,q}(M,g_0)$. Consequently we have

$$\max_{M^o\cap\partial B^M_R(p)} u_n - \min_{M^o\cap\partial B^M_R(p)} u_n \le C.$$

Therefore we get the blow-up value $m(p) = 4\pi$ when $p \in M^o \cap (\Sigma_1 \setminus \{q_1, q_2, \ldots, q_m\})$, $m(p) = 4\pi(1+\alpha_j)$ when $p = q_j \in M^o\cap\Sigma_1\cap\{q_1,q_2,\ldots,q_l\}$, $m(p) = 2\pi$ when $p\in\partial M\cap(\Sigma_1\setminus\{q_{l+1},q_{l+2},\ldots,q_m\})$, and $m(p) = 2\pi(1+\alpha_j)$ when $p = q_j\in\partial M\cap\Sigma_1\cap\{q_{l+1},q_{l+2},\ldots,q_m\}$. By using that

$$\begin{aligned}
&\int_M 2e^{2u_n} - e^{u_n}|\psi_n|^2_g dv_g + \int_{\partial M} ce^{u_n}d\sigma_g = \int_M 2e^{2(u_n+\phi)} - e^{u_n+\phi}|e^{\frac{\phi}{2}}\psi_n|^2_{g_0}dv_{g_0}\\
&\quad + \int_{\partial M} ce^{u_n}d\sigma_{g_0},
\end{aligned}$$

we get the conclusion of the Theorem. □

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