Energy Release in Earthquakes

Leon Knopoff

(Received 1957 September 30)

Summary

The notion of applying methods of static elasticity to the study of energy differences in two states of a given structure was extended by Starr to yield the solution to problems involving the propagation of cracks in shear fields. This technique may be modified to include the solution of the problem of the energy release upon the introduction of a tear fault, such as the San Andreas fault, into an otherwise homogeneous medium subjected to a uniform shear stress. The idealized properties of such a fault are a structure elongated compared with its depth and a strike-slip motion along the fault. This configuration, by symmetry, may be imaged in the Earth's surface so that the problem is reducible to that of a strip fault of infinite length in a homogeneous, isotropic, elastic, infinite medium. The medium is subjected to a uniform shear stress at infinity and the shear stress is assumed to vanish upon the strip. This two-dimensional problem has a vector solution rather than a tensor one, and thus it has an analogue in the electrical problem of a perfectly conducting strip placed in a uniform electric field or that of a strip obstacle placed in a uniform hydrodynamic stream field. The stress distribution and the relative motion throughout the medium before and after faulting can be obtained. For the case of the 1906 San Francisco earthquake, this model yields an energy difference of the shear fields before and after faulting of 4×10^{23} ergs. This value must exceed the elastic wave energy.

1. Introduction

The methods of elastostatics have been applied with some success to the problems of rupture in a solid. The general procedure is to compare the solid in two states: the state of the solid in a uniform, flawless condition and the state of the solid after a crack has been introduced. The procedure is convenient since only problems in static elasticity need be solved for the above comparison; no properties of the motion of the solid during the formation or extension of the flaw are required.

Using this general technique, Griffith (1921) obtained a solution for the energy released upon the formation of a tensile crack, that is to say one formed at right angles to the minimum principal stress. From this elastostatically computed energy, Griffith was able to derive conditions for the propagation of the crack.

Starr (1928) found the energy release and the conditions for the propagation of a shear crack, i.e. one formed parallel to the maximum shear stress in a body. Starr's geometry was that of a crack in the shape of a long thin strip, the elongation being in the z-direction of a cartesian coordinate system (Figure 1), the strip having its width dimension parallel to x, and the normal to the plane of the strip pointing in the y-direction. The applied stress was the shear stress τ_{xy}. This stress tends to move the faces of the crack parallel to the width or x-dimension. The crack is thus elongated at right angles to the shear direction and the crack extends itself in the direction of the shear.

In this paper we consider the energy release upon the introduction of another type of shear crack, in this case a crack elongated parallel to the shear direction.

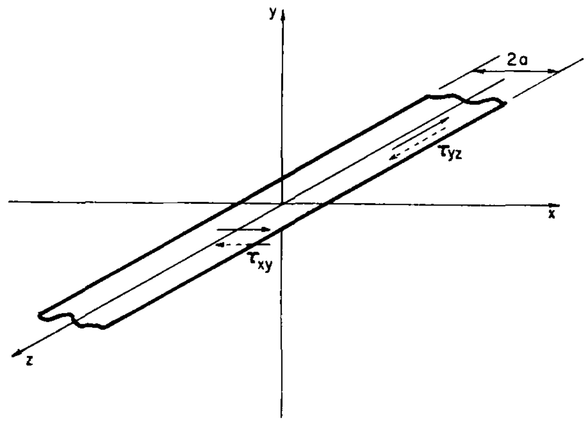

FIG. 1.

The applied stress is the shear stress τ_{yz} for the crack geometry and the coordinate system already considered. In this case all particle motions are parallel to the long dimension of the crack. Since this crack is already of infinite length in the direction of motion, the displacement conditions at the ends of the crack cannot be obtained; hence we have no criteria for the extension of such a crack. We do not attempt to indicate the mechanism of origin of such a crack.

Although originally intended for the problem of the extension of flaws in metals, Starr's solution can be directly applied to faulting in the Earth. The differences are those of the dimensions of the flaws, the energies of the shear fields and the elastic constants of the materials. The model is applicable to normal and thrust faults at depth in the Earth.

The problem considered in this paper is also a model of earth faulting. In this case we solve the problem of the strike-slip fault. As will be seen, the plane $x = 0$ through the centre of the strip and at right angles to it is a plane of zero normal stress. Thus we can solve either the problem of a strike-slip fault at great depth or one which intersects the surface of the Earth.

2. Electric-elastic analogy

Problems in static elasticity, although tensor problems interrelating dilatations and rotations, have vector solutions in certain special cases. In this paper we shall investigate one circumstance in which tensor elastic problems are reducible to vector elastic problems. This reduction is useful because of the wealth of solutions for problems in electrostatics, whose direct application may be advantageously taken to the problems of elasticity if the analogy is formalized.

Consider a shear field in a homogeneous, isotropic, elastic medium such that the displacement vector $\mathbf{U}$ everywhere points in the z-direction. Let $\mathbf{A}$ be the rotation vector

$$\mathbf{A} = \operatorname{curl}\mathbf{U}. \tag{1}$$

In the absence of dilatations, i.e. in a pure shear field, the differential equation of static elasticity

$$(\lambda+2\mu)\operatorname{grad}\operatorname{div}\mathbf{U}-\mu\operatorname{curl}\ \operatorname{curl}\mathbf{U} = 0, \tag{2}$$

reduces to

$$\operatorname{curl}\mathbf{A} = 0. \tag{3}$$

λ and μ are the Lamé moduli of elasticity. Thus $\mathbf{A}$ is derivable from a scalar potential

$$\mathbf{A} = -\operatorname{grad}\phi, \tag{4}$$

in which the scalar potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0, \tag{5}$$

since

$$\operatorname{div}\mathbf{A} = 0 \tag{6}$$

from equation (1). Hence the rotation vector $\mathbf{A}$ describes both a solenoidal and an irrotational field.

Superficially the vector field $\mathbf{A}$ satisfies the same differential equations as does the electrostatic field in a charge-free situation. If the vector $\mathbf{A}$ can be shown to satisfy the same boundary conditions as the electric field vector $\mathbf{E}$ then the correspondence will be complete and the solutions for certain electrostatic boundary value problems can be applied to the corresponding elastostatic boundary value problems.

We propose to introduce a surface S, in the elastostatic case, which will act as a singular region in the shear field. We shall require that the shear stress vanishes everywhere on this surface,

$$\tau_{yz} = 0 \text{ on } S.$$

Hence, we require the shear strain $e_{yz} = 0$ on S. Now

$$e_{yz} = \tfrac{1}{2}(\partial U_y/\partial z+\partial U_z/\partial y). \tag{7}$$

But by hypothesis we have a displacement field pointing only in the z-direction. Thus

$$\partial U_z/\partial y = 0 \text{ on } S. \tag{8}$$

Now

$$\mathbf{A} = (\partial U_z/\sigma y,\ -\partial U_z/\partial x, 0). \tag{9}$$

If now S is assumed to occupy parts of the planes $y =$ constant, then $\mathbf{A}$ is normal to S on S. This corresponds to the electrostatic boundary condition upon the electric field in the vicinity of a perfect conductor. The analogy is now complete. In order that $\mathbf{A}$ shall never have a z-component, we only solve two-dimensional problems so that z is not a variable in the problem.

It is now an elementary task to determine the remaining properties of the analogy. These are given in Table 1. $\mathbf{n}$ is a unit vector normal to a surface infinite in the z-direction; $\mathbf{z}_1$ is a unit vector in the z-direction.

Table 1

Two-dimensional analogies

	Electric Quantity		*Elastic Quantity*
$\mathbf{E}$	Electric field	$\mathbf{A}$	Rotation vector
ϵ	Dielectric constant	μ	Shear modulus
ϕ	Potential of field	ϕ	Potential of rotation
ψ	Stream function	U_z	Displacement vector (unidirectional)
$\mathbf{D}\times\mathbf{z}_1$	Rotation of Displacement	$\boldsymbol{\tau}$	Stress vector $\boldsymbol{\tau} = \boldsymbol{\tau}_{xz}\mathbf{x}_1 + \boldsymbol{\tau}_{yz}\mathbf{y}_1$
$\mathbf{E}\times\mathbf{z}_1$	Rotation of Electric Field	$\mathbf{e}$	Strain vector $\mathbf{e} = \mathbf{e}_{xz}\mathbf{x}_1 + \mathbf{e}_{yz}\mathbf{y}_1$
$\mathbf{D}$	Displacement	$\mathbf{z}_1\times\boldsymbol{\tau}$	Rotation of stress
σ	Surface charge	$\mathbf{z}_1\times\boldsymbol{\tau}\,.\,\mathbf{n}$	Component of stress
$\mathbf{E}\times\mathbf{n} = 0$	Perfect conductor	$\boldsymbol{\tau}\,.\,\mathbf{n} = 0$	Perfectly weak crack

3. The field around a strip

As an example of the calculation, consider a uniform shear field at infinity and a cut in the finite domain in the shape of a strip of infinite length, of width $2a$ and occupying part of the plane $y = 0$ as shown on Figure 1. Let the strip extend to infinity in the z-direction. At infinity, or in the absence of the strip, let the displacement be

$$\mathbf{U} = (0, 0, A_0 y). \tag{10}$$

The rotation vector in the absence of the strip is

$$\mathbf{A} = (A_0, 0, 0). \tag{11}$$

Thus we have set up the problem of a uniform field perturbed by a strip of infinite length whose short dimension is parallel to the unperturbed field.

This problem, as is well known, is solved in electrostatics or in hydrodynamics by a conformal transformation. The potential is (Smythe 1950, p. 92)

$$\phi = -A_0 \,\mathrm{Re}(w^2 - a^2)^{\frac{1}{2}}, \tag{12}$$

where $w = x + iy$. This is seen to satisfy the condition at infinity, since for $|w| \gg a$, $\phi = -A_0 x$ leading to $\mathbf{A} = (A_0, 0, 0)$ at infinity. On the strip, $w = x$ where $|x| < a$. Hence, $\phi = 0$ on the strip. Thus the strip is an equipotential surface and the $\mathbf{A}$ lines are everywhere normal to it.

We are now in a position to solve for several features of the field in the presence of the strip.

3.1 *The displacement field.*

From Equation (4)

$$\mathbf{A} = (-\partial\phi/\partial x, -\partial\phi/\partial y, 0). \tag{13}$$

Comparing equation (13) with equation (9), we see that the Cauchy-Riemann conditions are satisfied. Hence U_z is orthogonal to ϕ in the complex w-plane. Thus, if ϕ is given by equation (12), the displacement anywhere is

$$U_z = A_0 \,\mathrm{Im}(w^2 - a^2)^{\frac{1}{2}}. \tag{14}$$

In the plane $y = 0$,

$$U_z = 0, \qquad |x| > a$$

$$U_z = A_0\,(a^2 - x^2)^{\frac{1}{2}} \qquad |x| < a.$$

At the origin, $U_z = A_0 a$. Of course, on the opposite side of the strip, $U_z = -A_0 a$.

In the plane $x = 0$,

$$U_z = \pm A_0(y^2+a^2)^{\frac{1}{2}} \qquad y \gtrless 0.$$

This, logically, approaches the value $\pm A_0 y$ as $|y| \to \infty$.

We may inquire into the distance from the strip in the plane $x = 0$, that the relative displacement falls to half the value at the crack. The displacement in the absence of the strip is $A_0 y$. In the presence of the strip it is $A_0(y^2+a^2)^{\frac{1}{2}}$. The relative displacement is

$$A_0[(y^2+a^2)^{\frac{1}{2}}-y]. \tag{15}$$

At the strip this has the value $A_0 a$. Solving the equation

$$(y_0{}^2+a^2)^{\frac{1}{2}}-y_0 = a/2, \tag{16}$$

we obtain a distance $y_0 = 3a/4$ for the "half-displacement" point.

3.2 *The strain field*

The only strain components are

$$e_{zy} = \tfrac{1}{2}\partial U_z/\partial y = -\tfrac{1}{2}A_0 \operatorname{Re} w(w^2-a^2)^{-\frac{1}{2}} \tag{17}$$

$$e_{zx} = \tfrac{1}{2}\partial U_z/\partial x = \tfrac{1}{2}A_0 \operatorname{Im} w(w^2-a^2)^{-\frac{1}{2}}, \tag{18}$$

all other terms in the strain tensor are zero either because of the nature of the displacement vector or because of the two-dimensionality of the problem.

On the plane $y = 0$,

$$\begin{aligned} e_{zy} &= 0 & |x| &< a \\ &= -\tfrac{1}{2}A_0 x(x^2-a^2)^{\frac{1}{2}} & |x| &> a. \end{aligned}$$

The first of these two expressions is consistent with the boundary condition (8).

On the plane $x = 0$, $e_{zx} = 0$. Thus the surface $x = 0$ is a free surface. If we wish to consider the present problem as a model of displacement due to faulting, the surface $x = 0$ may be taken as the free surface of the Earth, and the crack $y = 0$, $0 < x < a$, may be taken as the fault. The total relative displacement between the two sides of the fault is $2A_0 a$ at the surface of the crack.

3.3 *Energy loss by insertion of the crack*

We can now inquire into the energy loss by reduction of the stress in the vicinity of the crack. The energy in a medium having only shear strain is

$$W = \tfrac{1}{2}\mu \int_V (\text{curl } \mathbf{U})^2 dv$$

integrated over all space where dv is an element of volume and μ is the shear modulus. Thus the energy difference between the two static states of strain is

$$W = \tfrac{1}{2}\mu \int_V [A_0{}^2-|\mathbf{A}|^2] dv, \tag{19}$$

where $\mathbf{A}$ is the rotation in the case of the medium in the presence of the crack. It is at this stage that the value of the electrical analogy is fully appreciated. If the value of **A** is inserted in equation (19) from the expressions for the potential given in equations (4) and (12), the integral to be evaluated diverges. That is to say, it leads to an infinite result when integrated over one coordinate and to a zero result when integrated over the other coordinate. To avoid this singular circumstance two procedures are available and both draw heavily upon experience obtained in the corresponding electrical problem. Stratton (1941, p. 118) shows that the integral of equation (19) may be modified and written in the form

$$\Delta W = \tfrac{1}{2}\mu \int_{V_0} A_0^2 dv + \tfrac{1}{2}\mu \int_{V_1} (A_0 - A_1)^2 dv, \tag{20}$$

where V_0 is the volume of a conductor inserted into the field and where V_1 is the volume remaining after this insertion has taken place. A_0 is the field distribution before the insertion of the conductor; A_1 is the field distribution after insertion. In the present case $A_0 = (A_0, 0, 0)$ and $V_0 = 0$ since the strip encloses no volume. V_1 now occupies all space. Thus if dS is an element of area in the x–y plane, the energy difference per unit length* is

$$\Delta E = \tfrac{1}{2}\mu \int_S [(\Delta A_x)^2 + (\Delta A_y)^2] dS, \tag{21}$$

where

$$\Delta A_x = A_0(1 - \operatorname{Re} w(w^2 - a^2)^{-\frac{1}{2}}, \tag{22}$$

and

$$\Delta A_y = A_0 \operatorname{Im} w(w^2 - a^2)^{-\frac{1}{2}}. \tag{23}$$

It is now convenient to solve this problem in elliptical coordinates (Stratton, pp. 53–54). We let

$$x = a\xi\eta \tag{24}$$

$$y = a(\xi^2 - 1)^{\frac{1}{2}}(1 - \eta^2)^{\frac{1}{2}}. \tag{25}$$

The element of area is

$$dS = \frac{a^2(\xi^2 - \eta^2) d\xi d\eta}{(\xi^2 - 1)^{\frac{1}{2}}(1 - \eta^2)^{\frac{1}{2}}}.$$

After much algebra

$$\Delta A_x = A_0 \left\{ \frac{\xi^2 - \eta^2 - \xi(\xi^2 - 1)^{\frac{1}{2}}}{\xi^2 - 1} \right\} \tag{26}$$

$$\Delta A_y = -A_0 \frac{\eta(1 - \eta^2)^{\frac{1}{2}}}{\xi^2 - \eta^2}. \tag{27}$$

Substituting these quantities into equation (21),

$$\Delta E = 2\mu A_0^2 a^2 \int_0^1 \frac{d\eta}{(1 - \eta^2)^{\frac{1}{2}}} \int_1^\infty \frac{2\xi^2 - 1 - 2\xi^2(\xi^2 - 1)^{\frac{1}{2}}}{(\xi^2 - 1)^{\frac{1}{2}}} d\xi. \tag{28}$$

* We use the notation W to represent total energy and E to represent energy per unit length.

This integral is convergent. The result of the integration is

$$\Delta E = \tfrac{1}{2}\pi\mu A_0{}^2 a^2 \tag{29}$$

per unit length.

The second technique of obtaining this result is to use the form of equation (19) where it is converted to an integral only over the volume of the inserted conductor. This expression is (Stratton, p. 113)

$$\Delta W = \tfrac{1}{2}\int_{V_0}(\mu'-\mu)\mathbf{A}'\cdot\mathbf{A}_0 dv \tag{30}$$

where instead of inserting a conducting strip, we insert an elliptical cylinder into the region where the new cylinder has a shear modulus μ' differing from that of the surrounding medium μ. The field inside the cylinder, $\mathbf{A}'$ is given by Smythe (p. 97) as

$$\begin{cases} A_x' = A_0(a+b)(a+b\mu'/\mu)^{-1} \\ A_y' = A_2' = 0. \end{cases} \tag{31}$$

The energy difference is

$$\Delta E = \frac{1}{2}\cdot\frac{\mu'-\mu}{a+b\mu'/\mu}(a+b)A_0{}^2 ab\pi, \tag{32}$$

since the field on the inside of the elliptic cylinder is uniform and parallel to the external field $\mathbf{A}_0$. a and b are the semi-major and semi-minor axes of the cylinder. Now allowing the cylinder to become perfectly conducting by allowing the shear modulus μ' to become infinite,

$$\Delta E = \tfrac{1}{2}\pi(a+b)aA_0{}^2\mu$$

and finally allowing the cylinder to become infinitely thin, the energy necessary to insert the strip of width $2a$ is

$$\Delta E = \tfrac{1}{2}\pi a^2 A_0{}^2\mu \tag{33}$$

per unit length in agreement with the result of equation (29).

4. Application to the San Francisco earthquake, 1906

Let the San Andreas fault be represented by a plane vertical crack of length L, of infinitesimal width, and of depth to the bottom of the crack a. Before faulting let the region be in a uniform shear field $\mathbf{U} = (0, 0, A_0 y)$ and after faulting let the crack now represent a surface of zero shear stress. The surface of the Earth is the plane $x = 0$. The surface trace of the fault shows an offset $s = 2A_0 a$. The energy release is $\frac{1}{4}\pi\mu A_0{}^2 a^2 L = (1/16)\pi\mu s^2 L$. We neglect end effects due to the termination of the fault. For the values (Reid 1910, p. 22) $\mu = 3\times 10^{11}$dyn/cm^2, $s = 4$m, $L = 435$ km, we obtain the energy $\Delta W = 4\times 10^{23}$ ergs. This figure must include both the seismic wave energy and the energy connected with non-elastic effects such as heat and plastic deformation.

The displacement on the crack is a maximum at the surface of the Earth and falls to zero at the lower edge. Thus there is no discontinuity in the shear strain at the lower edge.

The depth of faulting (i.e. the half-width a of the crack) can be obtained by fitting the triangulation observations (Hayford & Baldwin 1908, pp. 133–134) to the curve given in equation 15. Assuming, as Reid does, that the motion of the Farallon Islands represented a regional offset of the portion of the Earth's crust lying to the west of the fault, the 33 points to which the curve is to be fitted are tabulated in Table 2. We obtain, by a least squares technique, the curve

$$U_y = \pm 0{\cdot}72[(y^2+3{\cdot}2^2)^{\frac{1}{2}} - |y|]$$

Table 2

Triangulation data for San Francisco Earthquake, 1906

Number of Points	Distance of Stations from fault	Average Displacement	Displacement corrected for regional offset
10	1·5 km.	1·54 m.	1·54 m.
12	2·0	2·95	1·17
3	4·2	0·86	0·86
7	5·8	2·38	0·60
1	6·4	0·58	0·58
1	37·	1·78	0 (Farallon I. station)

where U_y is the offset in metres and y is the distance from the fault in kilometres. The plus and minus signs refer to the relative displacements on the two sides of the fault. To these values, the regional offset must be added for the stations west of the fault. The relative displacement at the fault is thus seen to be 4·5m compared to Reid's value of 4m referred to above. The depth to the bottom of the crack is now seen to be 3·2 km, a value considerably less than depths usually assumed for California earthquakes and for the San Francisco earthquake in particular.

It should be noted that the model used here assumes a uniform shear modulus everywhere, including the material at great depth in the Earth, and including the region very close to the fault.

Publication No. 90.
Institute of Geophysics,
University of California,
U.S.A.:
1957 *September.*

References

Griffith, A. A., 1921. The Phenomena of Rupture and Flow in Solids, *Phil. Trans. Roy. Soc.* A, **221,** 163–198.

Hayford, J. F. & Baldwin, A. L. Geodetic Measurements of Earth Movements, *California State Earthquake Commission Report.* (The California Earthquake of April 18, 1906). **1,** 114–146. (Carnegie Institution of Washington, 1908).

Reid, H. F. The Mechanics of the Earthquake, *California State Earthquake Commission Report* (The California Earthquake of April 18, 1906), **2** (Carnegie Institution of Washington, 1910).

Smythe, W. R., 1950. *Static and Dynamic Electricity.* (McGraw-Hill Book Co., New York, 2nd Ed.).

Starr, A. T., 1928. Slip in a Crystal and Rupture in a Solid due to Shear, *Proc. Camb. Phil. Soc.*, **24**, 489–500.

Stratton, J. A., 1941. *Electromagnetic Theory* (McGraw-Hill Book Co., New York).