Energy Transfer Analysis of Turbulent Plane Couette Flow

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Abstract. An energy transfer analysis of turbulent plane Couette flow is performed. It is found that nonlinear interaction between the $[0, \pm 1]$ modes is principally responsible for maintaining the mean streamwise turbulent velocity profile. The $[0, \pm 1]$ modes extract energy from the laminar flow by linear non-modal growth mechanisms and transfer it directly to the mean flow mode. The connection of this work to linear/nonlinear models of transition is discussed.

1 Introduction

In recent work a model for maintenance of turbulence in wall-bounded flows has been proposed [11,15]. It is argued that the backbone of the turbulent state is a quasi-periodic, self-sustaining cycle consisting of streak generation by linear advection, streak breakdown by a spanwise-inflectional instability, and vortex regeneration by nonlinear interaction of the products of the streak breakdown. This work was carried out by direct simulation of turbulent flow by reducing the size of the computational box to a minimal flow unit for sustaining turbulence and by analysis of low-dimensional dynamical systems.

In this paper we investigate movement of energy in wavenumber space in the turbulent Couette flow considered in [11] using energy transfer analysis. We find that it is interaction between modes $[0, 1]$ and $[0, -1]$ that is principally responsible for maintaining mean streamwise turbulent velocity profile. These two modes extract energy from the laminar flow by linear, non-modal mechanisms and transfer it directly to the mean flow mode. The relationship of these findings to linear/nonlinear models of sustained turbulence is discussed.

2 Energy Transfer Analysis

In the plane channel geometry the flow is between two infinite flat horizontal plates at $y = \pm 1$. All distances are non-dimensionalized by h, the channel half-height, and all velocities by V, the difference between the centerline and wall velocities. The Reynolds number is defined by $R = hV/\nu$, where ν is

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the kinematic viscosity. We let x, y, and z denote the streamwise, normal, and spanwise directions, respectively. For Couette flow the laminar velocity is $\mathbf{U}(y) = (y, 0, 0)$.

We assume that disturbances are periodic in the horizontal directions and have the form:

$$\mathbf{u}(x,y,z,t) = \text{Real}\{ \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{\mathbf{u}}_{mn}(y,t) e^{im\alpha_f x + in\beta_f z} \}, \tag{1}$$

where $\hat{\mathbf{u}}_{mn} = (\hat{u}_{mn}, \hat{v}_{mn}, \hat{w}_{mn})$ and α_f and β_f are the fundamental wavenumbers. Let $[m,n]$ denote the mode with integer streamwise and spanwise wavenumbers m and n, respectively.

The disturbance energy density in mode $[m,n]$ is

$$E_{mn}(t) = \frac{1}{2} \int_{-1}^{1} (|\hat{u}_{mn}(y,t)|^2 + |\hat{v}_{mn}(y,t)|^2 + |\hat{w}_{mn}(y,t)|^2) dy. \tag{2}$$

It can be shown that [13]

$$\frac{dE_{mn}}{dt} = \mathcal{L}_{mn}(t) + \mathcal{M}_{mn}(t) + \sum \Gamma_{mn}^{[pq][k\ell]}(t), \tag{3}$$

where the sum is over all modes satisfying $m = p + k$ and $n = q + \ell$. The term $\mathcal{L}_{mn}$ is the contribution due to linear terms in the disturbance equations, such as dissipation and interaction between mode $[m,n]$ and the laminar flow, $\mathcal{M}_{mn}$ is the contribution due to nonlinear interaction between mode $[m,n]$ and the modified mean flow $\hat{\mathbf{u}}_{00}$, and $\Gamma_{mn}^{[pq][k\ell]}$ is the contribution due to nonlinear interaction between modes $[p,q]$ and $[k,\ell]$. For the $[0,0]$ mode the linear term is non-positive, as it represents dissipation, $\mathcal{M}_{00}(t) = 0$, and the nonlinear interaction term consists of integrals involving Reynolds stress quantities like $\hat{u}_{pq}\hat{v}_{k\ell}$ and $\hat{w}_{pq}\hat{v}_{k\ell}$, where $p + k = q + \ell = 0$.

The evolution of the total disturbance energy, E, in all modes is given by the Reynolds-Orr equation [5]. Assuming periodicity of the disturbance in the horizontal directions, we have

$$\frac{dE}{dt} = -\int (U'(y)uv + \frac{1}{R}\mathcal{D}) d\mathbf{x}, \tag{4}$$

where integration is over one periodic unit, $\mathcal{D}$, which denotes dissipation, is positive, $U(y)$ is the laminar flow, and u and v are the disturbance streamwise and normal velocities [5]. Both terms in (4) originate from linear terms in the disturbance equations. Nonlinear terms in the disturbance equations do not appear explicitly in (4). Hence, only linear mechanisms are directly responsible for changes in total disturbance energy. The nonlinear terms in the disturbance equations are energy conserving. They do not contribute directly to total changes in energy, but do move energy in wavenumber space.

3 Results and Discussion

The simulations are performed on a Cray T90 using a code that solves the three-dimensional N-S equations in the channel [12] and the results are analyzed on a workstation using Matlab. The flow parameters are the same as in [11]: $\alpha_f = 1.14$ and $\beta_f = 1.66$, $R = 400$, and the number of modes in horizontal directions in (1) is $M = N = 7$.

We generate a turbulent flow starting with a relatively large amplitude ($E(0) \approx 2.5 \times 10^{-3}$) optimal streamwise vortex in modes $[0, \pm 1]$ [4]. The flow evolves and we find that disturbances are maintained to at least $t = 1500$, where time is scaled by h/V. Flow quantities are analyzed for the time interval $[1000, 1150]$. The mean streamwise velocity averaged over the time interval satisfies the wall law in the viscous sub-layer and the Reynolds based on friction velocity has an average value of ≈ 33.5, well above the value 20 for laminar flow.

We have computed disturbance energy in various wavenumbers. The disturbance energy is greatest in modes with streamwise wavenumber $m = 0$, primarily $[0, 0]$ and $[0, \pm 1]$. We will focus our attention on energy transfer into these modes.

Figure 1 plots dE_{00}/dt and various terms on the right hand side of (3) for this mode. The principal positive energy transfer is from the interaction of the $[0, 1]$ and $[0, -1]$ modes. The linear term makes a large negative contribution. Interactions involving other modes with streamwise wavenumber 0 or modes with non-zero streamwise wavenumber make a smaller net contribution to the energy transfer. We have checked the magnitude of various interactions that are not plotted and it does not appear that relevant contributions to the transfer into $[0, 0]$ are being neglected. In physical space, the $[0, \pm 1]$ interaction works to maintain mean streamwise shear; the key term is Reynolds stress product of the streamwise and normal velocities.

Figure 2 plots dE_{01}/dt and various terms on the right-hand side of (3) for this mode. The principal positive contribution is linear. The mean flow contribution makes a large negative contribution. Other nonlinear interactions make relatively small contributions to the net transfer. By symmetry the analogous plot for mode $[0, -1]$ is the same as Figure 2. Note that energy is continually extracted from the laminar flow.

As mentioned in the Introduction, the process by which vortices generate streaks is essentially linear. During this process there can be substantial growth in disturbance energy as energy is extracted from the laminar flow. This is due to non-modal (or transient) growth mechanisms [8,14], as Couette flow is linearly stable for all Reynolds numbers. In inviscid flow the growth is algebraic in time [6]. In the viscous case there can be substantial transient growth in energy before the eventual decay in linear calculations. Streamwise vortices are among the most effective structures in extracting energy from the laminar flow. The mode $[0, 1]$ corresponds to streamwise and spanwise wavenumbers $\alpha = 0$ and $\beta = 1.66$, respectively. Among streamwise-

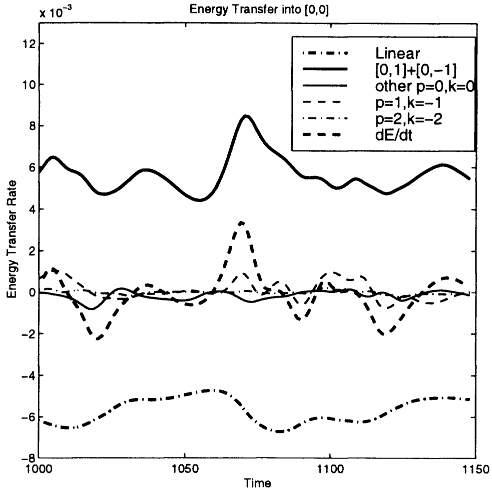

Fig. 1. Energy transfer into $[0,0]$. The curve labeled *other* $p=0, k=0$, corresponds to the sum of all contributions from all allowable interactions of modes $[p=0,q]$ and $[k=0,\ell]$ other than $[0,1]+[0,-1]$. The curve labeled $p=1, k=-1$ corresponds to the sum of all contributions from allowable interactions of modes $[p=1,q]$ and $[k=-1,\ell]$.

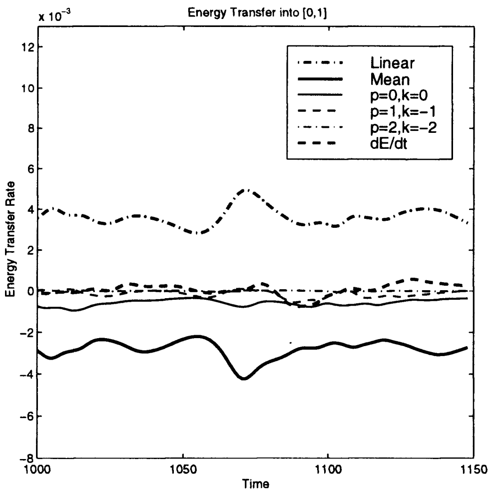

Fig. 2. Energy transfer into $[0,1]$. The curve labeled *Mean* corresponds to the contribution from the modified mean flow.

independent modes, this wavenumber combination has the greatest potential for non-modal growth in Couette flow [4].

Energy transfer among a triad of modes satisfies a conservation law. It can be shown that [13]

$$\Gamma_{00}^{[0,1][0,-1]}(t) = -\mathcal{M}_{0,1}(t) - \mathcal{M}_{0,-1}(t). \tag{5}$$

This equality means that the contribution to the transfer into $[0,0]$ due to the $[0,1] + [0,-1]$ interaction is balanced by the transfer out of $[0,1]$ and $[0,-1]$ due to the mean flow interaction. In light of the discussion above, we are led to the following interpretation. Energy is extracted from the laminar flow by $[0,\pm1]$ and other modes by linear mechanisms. Part of this energy is transferred instantaneously to $[0,0]$, where it maintains the turbulent mean shear. The $[0,0]$ cannot extract energy directly from the laminar flow.

Streamwise streaks are not effective in extracting energy from the laminar flow. In the self-sustaining cycle mentioned above, streamwise streaks break down and regenerate streamwise vorticity in the spanwise modes, which can effectively extract energy from the laminar flow. Although the results will not be plotted here, we have verified that streamwise vorticity is periodically regenerated in the $[0,\pm1]$ modes; the key interactions involve modes with streamwise wavenumbers $m = \pm1$, consistent with results in [11].

We have repeated the above calculations for the time interval $[1150, 1300]$ and the results are qualitatively similar. Energy is continuously extracted from the laminar flow by the $[0,\pm1]$ modes, which then transfer this energy directly to $[0,0]$.

Our findings on turbulent Couette flow are consistent with recent work on heuristic models of transition and turbulence [1–3,7,9,10]. In the models considered in these papers, structures called *inputs* extract energy from the laminar flow by linear non-modal growth mechanisms, converting inputs to *ouputs* – streaks to vortices, for example. The output structures cannot extract energy from the laminar flow. The nonlinear terms (or random scattering terms) are energy conserving. Their role is to convert outputs back to inputs allowing the process to repeat.

In the present work, computations are performed in a small computational box to isolate the self-sustaining cycle. In future work we plan on performing energy transfer analysis of a more realistic turbulent channel flow.

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