Monograph:

Lang, Z.Q. and Billings, S.A. (2004) Energy Transfer Properties of Nonlinear Systems in the Frequency Domain. Research Report. ACSE Research Report 862 . Department of Automatic Control and Systems Engineering, University of Sheffield.

Energy Transfer Properties of Nonlinear Systems in the Frequency Domain

Z Q Lang and S A Billings

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street, Sheffield
S1 3JD, UK

Research Report No. 862
June 2004

Energy Transfer Properties of Nonlinear Systems in the Frequency Domain

Z Q Lang and S A Billings

Department of Automatic Control and Systems Engineering
The University of Sheffield, Mappin Street,
Sheffield, S1 3JD, UK

Abstract: In this paper, an analysis of the energy transfer properties of nonlinear systems in the frequency domain is studied based on a new concept known as Nonlinear Output Frequency Response Functions (NOFRFs). The new concept allows the analysis to be implemented in a manner similar to the analysis of linear systems in the frequency domain, and provides great insight into the mechanisms which dominate the nonlinear behaviour. The new analysis is also helpful for the design of nonlinear systems in the frequency domain.

1 Introduction

Linear systems are in general relatively easy to analyse, have provided a basis for the development of the majority of control system design and signal processing methods, and are widely accepted by practitioners in many different fields. However, there are certain types of qualitative behaviour such as the generation of superharmonics, subharmonics, and chaos that linear models cannot exhibit (Pearson 1994). In the cases where these are dominant or significant system behaviours, nonlinear models are required to describe the system, and nonlinear system analysis methods have to be applied to investigate the behaviour. The term superharmonic generation refers to the generation of higher harmonic terms when a system is subject to a sinusoidal input. This phenomenon is the simplest and most common of the three mentioned above and is directly related to the frequency domain energy transfer in nonlinear systems considered in the present study.

It is well known that in linear systems the possible output frequencies at steady state are exactly the same as the frequencies of the input. This concept has been widely applied in various engineering disciplines to address system analysis and design issues. For example, a common feature in the design of many mechanical and civil engineering systems is that the dynamic amplification is minimised by designing the systems' natural frequencies to be outside the range of the main loading frequency spectra. This is based on the assumption that the dominant behaviour of the systems is linear.

In the nonlinear system case, however, a distinct frequency domain property is that the output frequencies at steady state are often richer than the frequencies in the input. The generation of superhamonics is a typical example of this phenomenon. A more complicated example is that when the system input is composed of several frequency components such as, e.g., the frequency components at angular frequencies $\omega_1, \omega_2, \omega_3$, the corresponding output frequencies at steady state could include the original input frequencies: $\omega_1, \omega_2, \omega_3$; harmonics $2\omega_1, 2\omega_2, 2\omega_3, \ldots$; intermodulations: $\omega_1 - \omega_2$, $\omega_3 - \omega_2$, …; and many others. Some energy of the input signal is therefore transferred from the input frequency modes to modes at other frequency locations.

This energy transfer phenomenon has been observed in many practical systems in electronic, mechanical, civil, and materials engineering (Szczepanski and Schaumann 1989 and 1993, Popovic et al. 1995, Neyfeh and Neyfeh 1994, Wilhem, et al. 1999). In offshore structures, for example, the phenomenon is apparent in three different effects; slow drift motion, springing, and ringing (Davies et al 1994, Nesteyard 1993, Jefferys and Raincy 1994, Remery and Hermans 1972, Hsu and Blenkarn 1972). The common feature of these effects is that the energy of the system input (wave movement) is transferred from the frequency modes in the input to the output (motion of the structure) frequency locations which are either below or above the input frequency band. The study of nonlinear phenomena is therefore of considerable practical significance, and not accounting for the nonlinear behaviour at the design stage may prove to be troublesome if not catastrophic (Nayfeh and Nayfeh 1994, Popovic et al 1995).

Previous studies of nonlinear phenomenon can be classified as time domain or frequency domain methods. Time domain methods typically involve the numerical integration of the differential equation description of the nonlinear structures under investigation or finite element simulation techniques (Rainey 1989, Jefferys and Rainey 1994, Stansberg 1993, Farnes, et al. 1994, Lee and Newman 1994, Taylor et al. 1992, Taylor and Natvig 1994). These techniques have been extensively applied to many nonlinear engineering systems including offshore structures. But often the results achieved can only indicate whether the phenomenon takes place under considered loading situations. Many of the phenomena of interest, however, depend on the system mechanisms in the frequency domain. Recently, the Generalised Frequency Response Functions (GFRFs) have been used to explain nonlinear energy transfer phenomena (Billings and Yusof 1996, Boaghe et al. 2000, Boaghe, Billings, and et al. 2002). The GFRFs are the extension of the frequency response function of linear systems to the nonlinear case and provide a description of the characteristics of nonlinear systems in the frequency domain.

In this paper, the frequency domain analysis of nonlinear systems is conducted from a new perspective. The study is based on a new concept known as Nonlinear Output Frequency Response Functions (NOFRFs) and emphasises the combined effects of the input and the system frequency domain characteristics that produce phenomena in the nonlinear case. The concept of NOFRFs allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear system frequency responses. This provides great insight into when, why, and how the particular phenomena of new frequency generation occur in nonlinear systems, and can reveal the real mechanisms which dominate the occurrence of the nonlinear behaviour. The new analysis provides a straightforward and transparent interpretation for nonlinear phenomena in the frequency domain. The results also have application in the design of nonlinear systems where a desired frequency domain behaviour is important for system performance.

2 The frequency response of linear and nonlinear systems

Historically frequency domain methods have dominated the theory and practice of control and signal processing for many years. This is because frequency domain analysis often provides a physically meaningful insight into the behaviour of systems under investigation which complements time domain analysis.

The output frequency response of a stable time-invariant linear system can be expressed by

$$Y(j\omega) = H(j\omega)U(j\omega) \tag{2.1}$$

when the system is subject to an input where the Fourier Transform exists. In (2.1), $Y(j\omega)$ and $U(j\omega)$ are the system input and output spectrum which are the Fourier Transforms of the system time domain input $u(t)$ and output $y(t)$ respectively.

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \tag{2.2}$$

where $h_n(\tau_1,\ldots,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity. In Lang and Billings (1996), an expression for the output frequency response of this class of nonlinear systems to a general input was derived. The result is given by

$$\begin{cases} Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \quad \text{for } \forall\omega \\ Y_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \displaystyle\int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \end{cases} \tag{2.3}$$

This expression reveals how the nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (2.3), $Y_n(j\omega)$ represents the nth order output frequency response of the system,

$$H_n(j\omega_1,\ldots,j\omega_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) e^{-(\omega_1\tau_1+\ldots+\omega_n\tau_n)j} d\tau_1 \ldots d\tau_n \tag{2.4}$$

is the definition of the nth-order Generalised Frequency Response Function (GFRF), and

$$\int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,\ldots,j\omega_n) \prod_{i=1}^{n} U(j\omega_i)$ over the n-dimensional hyperplane $\omega_1 + \cdots + \omega_n = \omega$. Equation (2.3) is a natural extension of the well-known linear relationship (2.1) to the nonlinear case.

For linear systems, equation (2.1) shows that the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation

(2.2), however, the relationship between the input and output frequencies is generally given by

$$f_Y = \bigcup_{n=1}^{N} f_{Y_n} \tag{2.5}$$

where f_Y denotes the non-negative frequency range of the system output, and f_{Y_n} represents the non-negative frequency range produced by the nth-order system nonlinearity. This is much more complicated than in the linear system case.

In 1997, Lang and Billings derived an explicit expression for the output frequency range f_Y of nonlinear systems when subjected to a general input with a spectrum given by

$$U(j\omega) = \begin{cases} U(j\omega) & \text{when } \ |\omega| \in (a,b) \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

where $b > a \geq 0$. The result obtained is

$$\begin{cases} f_Y = f_{Y_N} \bigcup f_{Y_{N-(2p^*-1)}} \\ f_{Y_n} = \begin{cases} \bigcup_{k=0}^{i^*-1} I_k & \text{when } \dfrac{nb}{(a+b)} - \left\lfloor \dfrac{na}{(a+b)} \right\rfloor < 1 \\ \bigcup_{k=0}^{i^*} I_k & \text{when } \dfrac{nb}{(a+b)} - \left\lfloor \dfrac{na}{(a+b)} \right\rfloor \geq 1 \end{cases} \\ i^* = \left\lfloor \dfrac{na}{(a+b)} \right\rfloor + 1 \\ \text{where } \lfloor . \rfloor \text{ means to take the integer part} \\ I_k = \big(na - k(a+b), nb - k(a+b)\big) \ \text{ for } k = 0,\ldots, i^* - 1, \\ \qquad I_{i^*} = \big(0, nb - i^*(a+b)\big) \end{cases} \tag{2.7}$$

In (2.7) p^* could be taken as $1,2,\cdots,\lfloor N/2 \rfloor$, the specific value of which depends on the system nonlinearities. If the system GFRFs $H_{N-(2i-1)}(.) = 0$, for $i = 1,\cdots,q-1$, and $H_{N-(2q-1)}(.) \neq 0$, then $p^* = q$. This is the first analytical description for the output frequencies of nonlinear systems, which extends the well-known relationship between the input and output frequencies of linear systems to the nonlinear case. A similar result was developed one year later by Raz and Veen (1998) from a different perspective.

The evaluation of the output frequency range f_Y of a nonlinear system can be implemented from equation (2.7) given the input frequency range (a,b) and the orders of nonlinearity involved in the system description. Clearly f_Y contains much richer frequency components than the range (a,b) of the input frequencies.

Note that the f_Y, determined by (2.7), is the frequency range where frequency components may exist in the output of a nonlinear system. The result, however, does not mean that the output frequency components are definitely available over the whole range of f_Y.

The above basic results show the composition of the frequency responses of a general class of nonlinear systems and the possibility of the system moving signal energy from the input frequency range to other frequency locations. However, when, why, and how the relocation of the frequency domain response takes place with a nonlinear system cannot be determined from these results. In the next section, a new concept in the frequency domain analysis of nonlinear systems known as Nonlinear Output Frequency Response Functions (NOFRFs) is proposed in order to provide an informative and physically meaningful interpretation of this phenomenon.

3 Nonlinear Output Frequency Response Functions (NOFRFs)

3.1 The Concept

The concept of the frequency response function in linear systems theory is well-known and has been widely applied. The Generalised Frequency Response Functions (GFRFs) can be considered as the extension of the linear frequency response function concept to the nonlinear case. However, the relationship between the input and output spectra of nonlinear systems is given by equation (2.3), which is much more complicated than in the linear system case. This relationship shows that the GFRFs cannot be used to provide a complete description of the output frequency response of nonlinear systems. The concept of Nonlinear Output Frequency Response Functions is introduced in this study to partly address this problem. The new concept can be regarded as another extension of the linear frequency response function concept to the nonlinear case, which is a complement to the GFRFs.

Consider a comparison between the output frequency response of a static linear system

$$y(t) = ku(t) \tag{3.1}$$

and a static nonlinear system

$$y(t) = ku^2(t) \tag{3.2}$$

The frequency domain representation of (3.1) and (3.2) are

$$Y(j\omega) = kU(j\omega) \tag{3.3}$$

and

$$Y(j\omega) = kU_2(j\omega) \tag{3.4}$$

respectively, where $U_2(j\omega)$ denotes the Fourier transform of $u^2(t)$ and can be expressed in terms of the input spectrum $U(j\omega)$ as

$$U_2(j\omega) = \frac{1/\sqrt{2}}{(2\pi)^{2-1}} \int_{\omega_1+\omega_2=\omega} \prod_{i=1}^{2} U(j\omega_i) d\sigma_{2\omega} \tag{3.5}$$

The comparison of (3.3) with (3.4) indicates that $U_2(j\omega)$ should be regarded as the natural extension of the input spectrum $U(j\omega)$ to the second order nonlinear case.

Because of this, the natural extension of $U(j\omega)$ to the nth order homogeneous nonlinear situation should be

$$U_n(j\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \tag{3.6}$$

which is the Fourier transform of the system input $u(t)$ raised to the nth power. $U_n(j\omega)$ is an important concept for the description of the output frequency response of nonlinear systems. However, as far as we are aware, this is the first time that this concept has been introduced and used for the analysis of nonlinear systems.

The output frequency response of linear systems is given by equation (2.1). This can be rewritten as

$$Y_1(j\omega) = G_1(j\omega)U_1(j\omega) \tag{3.7}$$

where $Y_1(j\omega) = Y(j\omega)$, $G_1(j\omega) = H(j\omega)$, and $U_1(j\omega) = U(j\omega)$.

The nth order output frequency response of nonlinear systems is given by the second equation in (2.3). Given that $U_n(j\omega)$ is the natural extension of $U_1(j\omega)$ to the nth order homogeneous nonlinear situation, rewrite the second equation in (2.3) as

$$\begin{aligned} Y_n(j\omega) &= \frac{\int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}} \frac{1\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \\ &= G_n(j\omega)U_n(j\omega) \end{aligned} \tag{3.8}$$

under the condition

$$\int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \neq 0 \tag{3.9}$$

$Y_n(j\omega)$ is expressed in a manner similar to the description for the output frequency response of linear systems. In (3.8),

$$G_n(j\omega) = \frac{\int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}} \tag{3.10}$$

is defined as the nth order Nonlinear Output Frequency Response Function when ω is in the frequency range where condition (3.9) holds.

3.2 The Properties

The concept of NOFRFs has been proposed by introducing $U_n(j\omega)$ into the second equation in (2.3), the description for the nth order output frequency response $Y_n(j\omega)$. Notice that different definitions can be made by introducing a different term associated with $U(j\omega)$ to yield a similar description for $Y_n(j\omega)$. The justification for the present definition is based on its properties.

The NOFRFs $G_n(j\omega)$, n=1,...,N, have the following three important properties.

(i) $G_n(j\omega)$ allows the system nth order output frequency response $Y_n(j\omega)$ to be described in a manner similar to the description for the output frequency response of linear systems,

(ii) $G_n(j\omega)$ is valid over f_{Y_n}, the output frequency range contributed by the nth order system nonlinearity, provided that condition (3.9) is satisfied.

(iii) $G_n(j\omega)$ is insensitive to a change of the input spectrum by a constant gain, that is,

$$G_n(j\omega)\Big|_{U(j\omega)=\alpha\bar{U}(j\omega)} = G_n(j\omega)\Big|_{U(j\omega)=\bar{U}(j\omega)}$$

The first property is straightforward. From equation (3.8) the total output frequency response of the nonlinear systems can be described as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega) \tag{3.11}$$

This yields a description for $Y(j\omega)$, which is similar to the description for the output frequency response of linear systems.

From (3.8), $G_n(j\omega)$ can be written as

$$G_n(j\omega) = Y_n(j\omega)/U_n(j\omega)$$

Because $Y_n(j\omega) = U_n(j\omega) = 0$ when ω is outside the frequency range of f_{Y_n}, we have the second property of $G_n(j\omega)$. This property implies that $G_n(j\omega)$ can be used to study the behaviour of nth order homogeneous nonlinear systems over the frequency range of f_{Y_n} where the system nth order output frequency response is located.

The third property is also straightforward since

$$G_n(j\omega)\Big|_{U(j\omega)=\alpha\bar{U}(j\omega)} = \frac{\alpha^n \int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}\bar{U}(j\omega_i)d\sigma_{n\omega}}{\alpha^n \int_{\omega_1+\ldots+\omega_n=\omega} \prod_{i=1}^{n}\bar{U}(j\omega_i)d\sigma_{n\omega}} = G_n(j\omega)\Big|_{U(j\omega)=\bar{U}(j\omega)} \tag{3.12}$$

This property reflects the similarity in function between the NOFRF $G_n(j\omega)$ in nth order homogeneous nonlinear systems and the frequency response function in linear systems. This property and equation (3.8) imply that $G_n(j\omega)$ behaves like a dynamic filter and operates on $U_n(j\omega)$ over the frequency range of f_{Y_n} to produce the contribution of nth order nonlinearity on the system output frequency response.

To clearly demonstrate the filtering effect of $G_n(j\omega)$, consider the Hammerstein system which represents a class of nonlinear systems which can be described by a memoryless nonlinear element

$$\bar{u}(t) = f[u(t)] = \alpha_1 u(t) + \alpha_2 u^2(t)+,\cdots,+\alpha_N u^N(t) \tag{3.13}$$

followed by linear dynamics with transfer function $H(s)$. The GFRFs of the Hammerstein system can readily be derived as

$$H_n(j\omega_1,\cdots,j\omega_n)=\alpha_n H[j(\omega_1+\cdots+\omega_n)] \quad n=1,\cdots,N \tag{3.14}$$

Therefore, the NOFRFs of the system are

$$G_n(j\omega)=\frac{\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}$$

$$=\frac{\int_{\omega_1+,\ldots,+\omega_n=\omega}\alpha_n H[j(\omega_1+\cdots+\omega_n)]\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}$$

$$=\alpha_n H(j\omega) \qquad n=1,\ldots,N \tag{3.15}$$

and, in this case, equation (3.8) becomes

$$Y_n(j\omega)=G_n(j\omega)U_n(j\omega)=\alpha_n H(j\omega)U_n(j\omega) \tag{3.16}$$

These results show that in the case of the Hammerstein system $G_n(j\omega)$ happens to be a linear filter, which operates on $U_n(j\omega)$ to produce the nth order output frequency response of the system.

Although it is reasonable to interpret the effect of $G_n(j\omega)$ in terms of filtering, and this is the basis of the analysis in Section 4, it should be emphasised that $G_n(j\omega)$ generally depends on both the frequency domain characteristics of the system nth order nonlinearity and the input spectrum. This is a distinct difference between the linear (n=1) case and the homogeneous nonlinear ($n\geq 2$) situation. It is well known that, unlike linear systems, the behaviour of nonlinear systems generally depend on both the system properties and the input. So the dependence of $G_n(j\omega)$ on the input spectrum is a manifestation of this general nonlinear property in the frequency domain.

The effect of the input spectrum $U(j\omega)$ on $G_n(j\omega)$ is complicated. Rewriting equation (3.10) as

$$G_n(j\omega)=\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\left[\prod_{i=1}^{n}U(j\omega_i)\Big/\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}\right]d\sigma_{n\omega} \tag{3.17}$$

shows that $G_n(j\omega)$ can be regarded as the result of a 'weighted' sum of the GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ over the n dimensional hyper-plane $\omega_1+\cdots+\omega_n=\omega$. The 'weight' at each point of $\{\omega_1,\cdots,\omega_n\}$ is $\prod_{i=1}^{n}U(j\omega_i)$ normalised by $\int_{\omega_1+,\ldots,+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}$ which is the sum of $\prod_{i=1}^{n}U(j\omega_i)$ over the same hyper-plane. This interpretation for $G_n(j\omega)$ emphasises its main dependence on the nth order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$, but also indicates that the effect of the input spectrum $U(j\omega)$ determines how $H_n(j\omega_1,\cdots,j\omega_n)$ over the whole hyper-plane $\omega_1+\cdots+\omega_n=\omega$ is combined together to form the nth order output frequency response function. This observation is the basis of another novel application of the concept of NOFRFs. This application is concerned with fault detection and categorisation using the NOFRF concept for engineering systems and structures and will be introduced in details in later publications.

4 New Frequency Generation Effects in Nonlinear Systems

The introduction of the concept of NOFRFs leads to a new expression for the output frequency response of nonlinear systems given by equations (3.8) and (3.11). In the present study, two objectives are to be achieved from equations (3.8) and (3.11). The first is to provide a qualitative interpretation regarding when and why new frequency generation happens in nonlinear systems so as to reveal the mechanism which dominates the nonlinear behaviour. The second is to describe quantitatively how a specific new frequency generation phenomenon occurs with a nonlinear system when subject to a specific input. This is helpful for engineers trying to understand the mechanisms behind the phenomenon, and informative for system design where the objective is to achieve a desired frequency domain performance.

4.1 Qualitative analysis

In Section 3.1, It has been shown that $U_n(j\omega)$ is the Fourier Transform of the system time domain input raised to the nth power. A comparison of the second equation in (2.3) with equation (3.6) indicates that $U_n(j\omega)$ can also be interpreted as the output spectrum of the nth order homogeneous nonlinear system whose GFRF is a constant of value one. This means that the frequency range of $U_n(j\omega)$ is f_{Y_n} given by (2.7), which is also the real output frequency range of the system if condition (3.9) is satisfied. By the real output frequency range, we mean the frequency range where the amplitude of the system output frequency components is not zero or not negligible. Consequently equation (3.8), that is, $Y_n(j\omega)=G_n(j\omega)U_n(j\omega)$ where $G_n(j\omega)$ is well defined over f_{Y_n} implies that the real output frequency range $f_{Y_{nR}}$ of an nth order homogeneous nonlinear system satisfies the relationship

$$f_{Y_{nR}}\in f_{Y_n} \tag{4.1}$$

For an nth ($n\geq 2$) order homogeneous nonlinear system, assume that the input frequency components are within the range of (a,b). When the nonlinearities of the system generate power at new frequency locations a part of or the whole $f_{Y_{nR}}$ must have no overlap with the input frequency range (a,b). This can be explained from (3.8) in terms of the composition of $U_n(j\omega)$ and the filtering effect of $G_n(j\omega)$. The nonlinear composition of $U_n(j\omega)$ from $U(j\omega)$ generates the frequency components in f_{Y_n} which can be determined using (2.7) and which are often much richer than the frequency components in the input. The filtering effect of $G_n(j\omega)$ attenuates some of the frequency components in f_{Y_n} and produces the output frequency components $f_{Y_{nR}}$ some or all of which are not available in the input. In such cases, energy is generated at new frequency locations.

Fig. 4.1 schematically illustrates a specific case where the combination of the two mechanisms generate frequencies at a new location. First, the nth order homogeneous nonlinear system with GFRF $H_n(j\omega_1,\cdots,j\omega_n)=1$ operates upon the input spectrum $U(j\omega)$ and produces $U_n(j\omega)$ the frequency range f_{U_n} of which is greater than the frequency range (a,b) of the input. Then $G_n(j\omega)$ operates on $U_n(j\omega)$ like a filter within the frequency range of $f_{Y_n}=f_{U_n}$ resulting in an output spectrum $Y_n(j\omega)$ whose real frequency range $f_{Y_{nR}}$ is within f_{Y_n} but does not overlap with (a,b), resulting in a power transfer to a new frequency range.

For nonlinear systems with nonlinearities up to Nth order, the output spectrum is given by (3.11). The real frequency range f_{Y_R} of the system output is therefore within the union of $\bigcup_{n=1}^{N} f_{Y_{nR}}$, that is,

$$f_{Y_R}\in\bigcup_{n=1}^{N} f_{Y_{nR}} \tag{4.2}$$

Power can be transferred to new frequency locations in this general situation when a part of or the whole f_{Y_R} has no overlap with (a,b). This can happen in two different scenarios. Denote the output frequency range which is contributed by the nth ($n\geq 2$) order nonlinearity and which is outside the input frequency range (a,b) as $f_{Y_R}^{n}$. Then a power transfer to a new frequency range happens when

(a) $f_{Y_R}^{n}$, n=2,...,N, have no overlap at all with each other, or

(b) $f_{Y_R}^{n}$, n=2,...,N, have overlap but the output spectrum $Y(j\omega)$ is not zero or negligible over the whole range of $\bigcup_{n=2}^{N} f_{Y_R}^{n}$.

In case (a), some input signal energy will be transferred from (a,b) to $\bigcup_{n=2}^{N} f_{Y_R}^{n}$. In case (b), some input signal energy will be transferred from (a,b) to a frequency range within $\bigcup_{n=2}^{N} f_{Y_R}^{n}$.

Based on the above analysis the generation of power at new frequency locations in nonlinear systems is produced due to the combined effects of

(a) The nonlinear composition of $U_n(j\omega)$ from the input spectrum $U(j\omega)$, which often generates much richer frequency components than the frequency components in the input.

(b) The filtering effect of the NOFRF $G_n(j\omega)$, which determines the real output frequency range contributed by nth order homogeneous nonlinear systems. When an energy transfer phenomenon happens with the system, the whole or part of the real output frequency range is outside the input frequency band.

(c) The inter-kernel interference between the different orders of homogeneous nonlinear systems involved in the system. This eventually determines whether a frequency component outside the input frequency band produced by the homogeneous nonlinear systems can be observed in the system output.

Given the homogeneous nonlinear system components involved in a system and a specific input, the filtering effect of the NOFRF $G_n(j\omega)$ is an important factor which determines where the output power components are located in the frequency domain. A considerable gain of $G_n(j\omega)$ at a frequency over the frequency range $f_{Y_R}^n$ implies that the input signal energy could be transferred by the system to this frequency unless the inter-kernel interference is counteracting and produces a negligible response at this frequency. An inspection of the gain of the NOFRFs $G_n(j\omega)$ for n=2, ... , N can therefore provide a significant insight into how the phenomenon takes place.

4.2 Quantitative Interpretation

The mechanism which dominates the occurrence of a frequency domain energy transfer phenomenon in a nonlinear system can generally be explained based on the qualitative analysis discussed in Section 4.1. However, the behaviour of nonlinear systems is input dependent. In order to understand exactly how the phenomenon happens with a system, it is necessary to investigate quantitatively how the combined effects of the system and a specific input produce the phenomenon. The new concept of the NOFRFs reflects the combined contribution of the system and its input to the system frequency domain output behaviour. The procedure for a quantitative interpretation therefore includes

(i) Evaluate the NOFRFs under a considered specific input,

(ii)Inspect how the NOFRF $G_n(j\omega)$ operates, like a filter, on $U_n(j\omega)$ to produce the nth order output frequency response $Y_n(j\omega) = G_n(j\omega)U_n(j\omega)$, and

(iii) Inspect how the inter-kernel interference effect of

$$\sum_{i=l}^{N} Y_n(j\omega) = \sum_{i=l}^{N} G_n(j\omega)U_n(j\omega)$$

works to generate the output frequency response $Y(j\omega)$ at a frequency or over a frequency range.

4.2.1 Evaluation of $G_n(j\omega)$

From the definition of $G_n(j\omega)$ in (3.10), a direct evaluation of $G_n(j\omega)$ requires a knowledge of the GFRF $H_n(j\omega_l,\cdots j\omega_n)$ of the system and the spectrum $U(j\omega)$ of the considered input. In the following, however, an effective algorithm is proposed which allows the evaluation of the NOFRFs to be implemented directly using the system input output data.

From equation (3.11), it is known that

$$
\begin{aligned}
Y(j\omega) &= \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega) = \sum_{n=1}^{N}\left(G_n^R + jG_n^I\right)\left(U_n^R + jU_n^I\right) \\
&= \sum_{n=1}^{N}\left(G_n^R U_n^R - G_n^I U_n^I\right) + \left(G_n^R U_n^I + G_n^I U_n^R\right)j
\end{aligned} \tag{4.3}
$$

where G_n^R and G_n^I are the real and imaginary part of $G_n(j\omega)$, and U_n^R and U_n^I are the real and imaginary part of $U_n(j\omega)$. Therefore

$$
\begin{bmatrix} \operatorname{Re} Y(j\omega) \\ \operatorname{Im} Y(j\omega) \end{bmatrix} = \begin{bmatrix} U_1^R, \cdots, U_N^R, -U_1^I, \cdots, -U_N^I \\ U_1^I, \cdots, U_N^I, U_1^R, \cdots, U_N^R \end{bmatrix} \begin{bmatrix} G^R \\ G^I \end{bmatrix} \tag{4.4}
$$

where $G^R = [G_1^R, \cdots, G_N^R]^T$ and $G^I = [G_1^I, \cdots, G_N^I]^T$.

Consider the case of $u(t) = \alpha u^*(t)$ where α is a constant and $u^*(t)$ is the input signal under which the NOFRFs of the system are to be evaluated.

$$
\begin{aligned}
U_n(j\omega) &= \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \\
&= \alpha^n \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U^*(j\omega_i) d\sigma_{n\omega} = \alpha^n U_n^*(j\omega)
\end{aligned} \tag{4.5}
$$

where $U^*(j\omega)$ is the Fourier Transform of $u^*(t)$ and

$$
U_n^*(j\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U^*(j\omega_i) d\sigma_{n\omega}
$$

Equation (4.4) can be written as

$$
\begin{bmatrix} \operatorname{Re} Y(j\omega) \\ \operatorname{Im} Y(j\omega) \end{bmatrix} = \begin{bmatrix} \alpha U_1^{*R}, \cdots, \alpha^N U_N^{*R}, -\alpha U_1^{*I}, \cdots, -\alpha^N U_N^{*I} \\ \alpha U_1^{*I}, \cdots, \alpha^N U_N^{*I}, \alpha U_1^{*R}, \cdots, \alpha^N U_N^{*R} \end{bmatrix} \begin{bmatrix} G^{*R} \\ G^{*I} \end{bmatrix} \tag{4.6}
$$

where U_n^{*R} and U_n^{*I} are the real and imaginary part of $U_n^*(j\omega)$, and G^{*R} and G^{*I} are the real and imaginary part of $G^* = [G_1^*, \cdots, G_N^*]^T$ which are the NOFRFs to evaluate.

Excite the system under study $\overline{N}$ times by the input signals

$\alpha_i u^*(t)$, $i = 1, \ldots, \overline{N}$

where $\overline{N} \geq N$ and $\alpha_{\overline{N}}, \alpha_{\overline{N}-l}, \cdots, \alpha_l$ are constants which satisfy the condition

$\alpha_{\overline{N}} > \alpha_{\overline{N}-l} > \cdots > \alpha_l > 0$

to generate $\overline{N}$ output frequency responses $Y^i(j\omega)$, $i = 1, \ldots, \overline{N}$. From equation (4.6), it is known that the output frequency responses can be related to the NOFRFs to be evaluated as below.

$$
\mathbf{Y}^{1,\ldots,\overline{N}}(j\omega) = \mathbf{AU}^{1,\ldots,\overline{N}}(j\omega) \begin{bmatrix} G^{*R} \\ G^{*I} \end{bmatrix} \tag{4.7}
$$

where

$$\mathbf{Y}^{1,\ldots,\overline{N}}(j\omega)=\begin{bmatrix}\operatorname{Re}Y^{1}(j\omega)\\ \operatorname{Im}Y^{1}(j\omega)\\ \vdots\\ \vdots\\ \operatorname{Re}Y^{\overline{N}}(j\omega)\\ \operatorname{Im}Y^{\overline{N}}(j\omega)\end{bmatrix} \tag{4.8}$$

and

$$\mathbf{AU}^{1,\ldots,\overline{N}}(j\omega)=\begin{bmatrix}\alpha_1U_1^{*R},\cdots,\alpha_1{}^{N}U_N^{*R},-\alpha_1U_1^{*I},\cdots,-\alpha_1{}^{N}U_N^{*I}\\ \alpha_1U_1^{*I},\cdots,\alpha_1{}^{N}U_N^{*I},\alpha_1U_1^{*R},\cdots,\alpha_1{}^{N}U_N^{*R}\\ \vdots\\ \vdots\\ \alpha_{\overline{N}}U_1^{*R},\cdots,\alpha_{\overline{N}}{}^{N}U_N^{*R},-\alpha_{\overline{N}}U_1^{*I},\cdots,-\alpha_{\overline{N}}{}^{N}U_N^{*I}\\ \alpha_{\overline{N}}U_1^{*I},\cdots,\alpha_{\overline{N}}{}^{N}U_N^{*I},\alpha_{\overline{N}}U_1^{*R},\cdots,\alpha_{\overline{N}}{}^{N}U_N^{*R}\end{bmatrix} \tag{4.9}$$

Consequently the values of the NOFRFs, $G_1^*(j\omega),\cdots,G_N^*(j\omega)$, can be determined using an Least Square based approach as

$$\begin{aligned}\begin{bmatrix}G^{*R}\\ G^{*I}\end{bmatrix}&=[G_1^{*R}(\omega),\cdots,G_N^{*R}(\omega),G_1^{*I}(\omega),\cdots,G_N^{*I}(\omega)]^T\\ &=\left[\left(\mathbf{AU}^{1,\ldots,\overline{N}}(j\omega)\right)^T\left(\mathbf{AU}^{1,\ldots,\overline{N}}(j\omega)\right)\right]^{-1}\left(\mathbf{AU}^{1,\ldots,\overline{N}}(j\omega)\right)^T\mathbf{Y}^{1,\ldots,\overline{N}}(j\omega)\end{aligned} \tag{4.10}$$

This algorithm for the determination of the NOFRFs requires experimental or simulation results for the system under $\overline{N}$ different input signal excitations which are $\alpha_i u^*(t)$, $i=1,\ldots,\overline{N}$. The approach can be applied when either a simulation model such as a mathematical or finite element model is available or practical experiments can be performed on the system as required.

4.2.2 Inspection of the filtering effect of $G_n(j\omega)$ on $U_n(j\omega)$

It is well known in linear system frequency response analysis that an inspection of the magnitude of the frequency response function is an important means of studying the system behaviour. The frequency locations where peaks can be found in the magnitude characteristic of the frequency response function indicate resonance of the system. In mechanical engineering systems, for example, this is called the analysis of transmissibility, indicating that the results reflect how the system transmits the input signal energy to the output at any frequency.

The inspection of the filtering effect of $G_n(j\omega)$ can be regarded as an extension of the well-known linear analysis to the nth order homogeneous nonlinear system case. The implementation of this suggests a graphical display of $|G_n(j\omega)|$ over the interested frequency range. Because

$$Y_n(j\omega)=G_n(j\omega)U_n(j\omega)$$

$|G_n(j\omega)|$ can be explained to be the transmissibility of the nth order homogeneous nonlinear system for the specific input under consideration. The analysis of $|G_n(j\omega)|$ can therefore be used to reveal the mechanism, which induces new frequency generation. A peak in $|G_n(j\omega)|$ at a frequency implies that the contribution of the system input to the nth order output, which is measured by $|U_n(j\omega)|$, is greatly enhanced at this frequency. Therefore, when the peak appears outside the input frequency band new frequency components are generated. For n>1 $G_n(j\omega)$ is generally dependent on the input. So the analysis result is input dependent.

4.2.3 Study of the effects of inter-kernal interference

The output frequency response $Y(j\omega)$ is determined by equation (3.11), the summation of the output frequency responses of all homogeneous nonlinear systems involved in the system. If some input signal energy is transferred by the nonlinear system to a frequency location outside the input frequency band, the NOFRF analysis above indicates the combined contribution of the system and the input at different nonlinear orders to the occurrence of the phenomenon. But to unravel the phenomenon a study of the composition of $Y(j\omega)$ by $Y_n(j\omega) = G_n(j\omega)U_n(j\omega)$, for n=1,...,N, at this frequency is also needed to reveal the effects of inter-kernal interference.

It has been shown in Section 4.1 that there are two scenarios where the energy transfer phenomenon may take place. In scenario (a), there is no overlap between the frequency ranges which are outside the input frequency band and which are produced by different order homogeneous nonlinear systems. In this case, there is no interkernal interference effect on the new frequency generation phenomenon. In scenario (b), however, the phenomenon needs to be interpreted by taking the influences of all involved system nonlinearities into account. Therefore, the NOFRF $G_n(j\omega)$ based analyses for all involved nonlinear orders are needed to determine exactly how the phenomenon arises.

Given a nonlinear system and a frequency domain energy transfer phenomenon with the system, the above three step procedure can be used to quantitatively reveal the mechanisms behind the phenomenon. The significance of this study is in two respects. First, the interpretation of the new frequency generation can help engineers to fully understand the phenomena which is totally different from the linear system case. Secondly, given a specific input, a comparison of the results of this study under different values of system parameters will show the effects of the parameters on the energy transfer phenomenon. This could be an effective means for the frequency domain design of nonlinear systems and is worth further study.

The transmissibility concept in linear system frequency domain analysis has been mentioned above. Transmissibility is normally defined as the magnitude of the ratio between the system output and input spectrum. Although this definition is theoretically not correct for nonlinear systems, it is still widely applied in industry for nonlinear frequency domain analysis due to its simplicity. The three step analysis can be a powerful complement for the transmissibility analysis for nonlinear systems. The results achieved are more accurate regarding the system frequency domain behaviour and informative for system design.

5 Energy transfer phenomena of a nonlinear system

In this section, the quantitative analysis procedure in Section 4.2 is applied to investigate the generation of new frequency components in a nonlinear system.

5.1 The nonlinear system

The differential equation of the considered nonlinear system is given by

$$m\frac{d^2y(t)}{dt^2}+c\frac{dy(t)}{dt}+k_1y(t)+k_2y^2(t)+k_3y^3(t)=u(t) \tag{5.1}$$

where y(t) and u(t) are the input and output of the system, m, c, k_1, k_2, and k_3 are the system parameters. An example of a practical system which can be described by the differential equation is the mechanical oscillator shown in Fig.5.1 where u(t) is the force imposed on the mass m, y(t) is the displacement of the mass, and c, k_1, k_2, and k_3 are characteristic parameters of the damper and spring respectively.

In the following, consider the case where (Worden et al. 1994)

$$m=1,\ c=20,\ k_1=10^4,\ \ k_2=10^7,\ \ k_3=5\times10^9$$

to study the frequency domain behaviour of the system when subject to two specific inputs using the analysis procedure in Section 4.2.

5.2 Analysis of the system output frequency response to the first input

The first input is given by

$$u(t)=\frac{3}{2\pi}\frac{\sin(2\times55\times\pi\times t)-\sin(2\times30\times\pi\times t)}{t} \tag{5.2}$$

$$t=-511\times0.005\,\text{sec},\cdots,512\times0.005\text{sec}$$

and shown in Fig. 5.2(a). The output response of the system to this input is obtained using a fifth order Runge-Kutta simulation method, and the result is shown in Fig. 5.2(b). Fig. 5.3 shows the spectra of the input and output and indicates that some of the input energy is transferred by the system from the input frequency band (30,55) Hz to the lower frequency range (0,30) Hz.

The NOFRFs of system (5.1) when subjected to input (5.2) are evaluated up to the fourth order over the frequency range (10,20) Hz to yield $G_n(j2\pi f)$, n=1,2,3,4, as shown in Fig.5.4. The results indicate that over this frequency range a maximum gain appears near 16Hz for $G_n(j2\pi f)$, n=2,3,4. The nth order output frequency response $Y_n(j2\pi f)$ and the contribution of the system input to the response $U_n(j2\pi f)$ are shown in Figs 5.5 and 5.6 respectively. How the frequency domain behaviour of the four homogeneous nonlinear systems are produced over the frequency range (10,20) Hz can be clearly explained from Figs 5.4-5.6. $G_n(j2\pi f)$ behaves like a filter that shapes $U_n(j2\pi f)$ to yield $Y_n(j\omega)=G_n(j\omega)U_n(j\omega)$. Over the frequency range

(10,20) Hz, the maximum gain of $G_n(j2\pi f)$ near 16Hz produces a maximum nth order output frequency response at this frequency for n=2,3,4.

The system output frequency response is the effect of a combination of the output frequency responses of the involved homogeneous nonlinear systems. This can be interpreted, to a certain extent, from the above NOFRF based analysis under the assumption that

$$Y(j2\pi f) \approx \sum_{n=1}^{4} G_n(j2\pi f)U_n(j2\pi f) \tag{5.3}$$

But an analysis of the inter-kernel interference may also be needed to determine exactly how a particular phenomenon arises.

In order to show how the inter-kernal interference works to produce the system output frequency response, the frequency at which a maximum output frequency response takes place in the range of (10,20)Hz is evaluated. The result is $f_{r1} = 15.82$ Hz. The contribution of each of the four homogeneous nonlinear systems to the output response at this frequency can then be determined, and the results are shown in Table 1.

Table 1 The contribution of different homogeneous nonlinear effects to the output response at frequency $f_{r1} = 15.82\text{Hz}$ when the system is subject to the first input

n	nth order output frequency response $G_n(j2\pi f_{r1})U_n(j2\pi f_{r1})\big\|_{f_{r1}=15.82Hz}$	Magnitude of the nth order output frequency response $\left\|G_n(j2\pi f_{r1})U_n(j2\pi f_{r1})\right\|_{f_{r1}=15.82Hz}$
1	0	0
2	$(0.2778 - 0.0513j) \times 10^{-3}$	0.2336×10^{-3}
3	$(-0.2821 + 0.0636j) \times 10^{-3}$	0.28012×10^{-3}
4	$(0.0948 - 0.0214j) \times 10^{-3}$	0.0927×10^{-3}

Table 1 shows that under the condition of (5.3) the system response at frequency $f_{r1} = 15.82Hz$ is

$$Y(j2\pi f_{r1}) \approx \sum_{n=1}^{4} G_n(j2\pi f_{r1})U_n(j2\pi f_{r1}) = 4.05495\times 10^{-5} - 9.13720\times 10^{-6} j$$

the amplitude of which is

$$\left|Y(j2\pi f_{r1})\right| \approx \left|4.05495\times 10^{-5} - 9.13720\times 10^{-6} j\right| = 4.15662\times 10^{-5}$$

The result shows how a combination of the intra and inter kernel interference generates the system output frequency response. The frequency domain behaviour of the system at other frequencies can be quantitatively analysed in the same way to explain how the behaviour occurs.

To confirm the effectiveness of the above analysis where the maximum nonlinear order N is assumed to be 4, the system output spectrum evaluated using the direct FFT at $f_{r1} = 15.82$ Hz was obtained as

$$|Y(j2\pi f_{r1})| = 3.93462 \times 10^{-5}$$

This is close to the result 4.15662×10^{-5} obtained from the above quantitative analysis, and indicates that the analysis based on N=4 is a reasonable approximation.

Note that the small value of the magnitude of the output spectrum at $f_{r1} = 15.82$ Hz is due to the small displacement y(t) of the system under the given input. This is a quite common phenomenon in practical systems. The scale of the system input and output amplitude can be clearly observed from Fig 5.2.

5.3 Analysis of the system frequency response to a second input

A second input was generated from an uncorrelated random sequence uniformly distributed in the interval (0,100) and band limited within the frequency range (30,50)Hz under the sampling interval $T_s = 0.005$ sec. Fig.5.7 (a) and (b) show the input and the output response of the system to this input obtained using a fifth order Runge-Kutta simulation method. The spectra of the input and output signals are shown in Fig.5.8 indicating the occurrence of a frequency domain energy transfer phenomenon. Over the frequency range of (10,20) Hz, a maximum frequency response outside the input frequency band can be observed at about 15 Hz.

To reveal the composition of the system output frequency response over the frequency range (10,20) Hz so as to explain how the energy transfer happens, the NOFRFs of the system under the second input are evaluated up to the fourth order to yield $G_n(j2\pi f)$, n=1,2,3,4, as shown in Fig.5.9. $U_n(j2\pi f)$ and $Y_n(j2\pi f) = G_n(j2\pi f)U_n(j2\pi f)$, n=1,2,3,4, are also determined over this frequency range to yield the results shown in Figs. 5.10 and 5.11. A maximum gain of $G_n(j2\pi f)$ for n=2,3,and 4 at a frequency location a bit below 15 Hz can be observed, and a maximum nth order output frequency response appears between 15Hz and 16Hz for n=2,3,and 4.

Under the assumption of (5.3), Figs.5.9, 5.10, and 5.11 can explain, to a certain extent, the frequency domain behaviour of the system output over the frequency range (10,20) Hz as shown in Fig.5.8, especially why a maximum output frequency response appears at a frequency of about 15 Hz. To determine exactly how the maximum output frequency response is generated, however, an analysis for the effect of the inter-kernel interference at this frequency is also needed.

The frequency where the maximum output frequency response occurs is evaluated. The result is $f_{r2} = 14.65Hz$. The contribution of each of the four homogeneous nonlinear systems to the system response at this frequency can then be determined, and the results are shown in Table 2.

Table 2 The contribution of different homogeneous nonlinear effects to the output response at frequency $f_{r2} = 14.65\text{Hz}$ when the system is subject to the second input.

n	nth order output frequency response $G_n(j2\pi f_r)U_n(j2\pi f_r)\big|_{f_r=14.65Hz}$	Magnitude of the nth order output frequency response $\left|G_n(j2\pi f_r)U_n(j2\pi f_r)\right|_{f_r=14.65Hz}$
1	0	0
2	$(0.0725-0.0838j)\times 10^{-3}$	0.1108×10^{-3}
3	$(-0.0898-0.1038j)\times 10^{-3}$	0.1372×10^{-3}
4	$(0.0302+0.0349j)\times 10^{-3}$	0.0461×10^{-3}

Table 2 shows that under the condition of (5.3) the system response at frequency $f_{r2} = 14.65Hz$ is

$$Y(j2\pi f_r) \approx \sum_{n=1}^{4} G_n(j2\pi f_r)U_n(j2\pi f_r) = 1.2902\times 10^{-5} + 1.4918\times 10^{-5} j$$

the amplitude of which is

$$\left|Y(j2\pi f_r)\right| \approx 1.9723\times 10^{-5}$$

How the intra and inter kernel interference effects work at frequency $f_{r2} = 14.65Hz$ to generate the maximum output frequency response outside the input frequency band is clearly reflected by the quantitative analysis. To confirm this result, the system output spectrum was evaluated using a direct FFT at $f_{r2} = 14.65Hz$. The result was

$$\left|Y(j2\pi f_{r2})\right| = 1.86701\times 10^{-5}$$

which is close to the above obtained result 1.9723×10^{-5} and therefore indicates that the N=4 based analysis is a reasonable appoximation.

Figs 5.4 and 5.9 show the NOFRFs $G_n(j2\pi f)$ n=1,2,3,4, of the nonlinear system under the first and second input respectively. Apart from $G_1(j2\pi f)$, the NOFRFs under the two inputs are obviously different. This reflects the input dependent property of the NOFRFs, and implies that some frequency domain behaviour of nonlinear systems can only be interpreted under a specific input. This is why a quantitative analysis is necessary to show exactly how the nonlinear phenomenon happens. However, some similarities between the NOFRFs under the two different inputs can still be observed. The most obvious similarity is that $\left|G_n(j2\pi f)\right|$ under the two inputs all reach a maximum at a frequency near 15Hz for n=2,3,4. This is because the resonant frequency of the nonlinear system is $\sqrt{\frac{k_1}{m}} = 99\,rad/s = 15.76Hz$, and this important system property is reflected by the NOFRFs evaluated under the two different inputs. Therefore, although the NOFRFs generally describe a combined effect of the system characteristics and the input on the system output frequency response, the concept does reflect some important dynamic characteristics of the system under study.

The above qualitative analysis of the generation of new frequencies indicates why and how this phenomenon can happen with a nonlinear system. As far as the system design in the frequency domain is concerned, the same analysis for a typical input can be conducted under different values of the system parameters that can be changed to show the effects of these parameters on the system performance and to determine the parameter values which correspond to a desired system output frequency response. This is a procedure which can be regarded as an extension of the transmissibility based linear frequency domain design to the nonlinear case.

6 Conclusions

The frequency domain energy transfer in nonlinear systems is a phenomenon which has been observed in systems in many different engineering areas. If this nonlinear behaviour is not accounted for at the design stage unexpected and even dangerous results may occur (Nayfeh and Nayfeh 1994). This indicates the considerable significance of an effective analysis procedure which can reveal the real mechanisms that dominate the occurrence of the nonlinear phenomenon and which can therefore be more informative for system design. Previous approaches to this analysis include techniques in both the time and the frequency domain. But the question regarding when, why, and how such phenomenon happens with a general nonlinear system cannot be properly answered using existing approaches. In the present study, a new concept known as Nonlinear Output Frequency Response Functions has been proposed and used to conduct a new form of analysis. This allows the analysis to be implemented in a manner similar to the analysis of linear system frequency responses. The analysis results provide great insight into the mechanisms behind the nonlinear behaviour which has been shown to depend on a combined effect of the input spectrum and the system frequency domain characteristics. The new analysis should also be of benefit in the system design in the frequency domain. This is a more significant topic and is currently under study.

Acknowledgements

The authors gratefully acknowledge the support of the Engineering and Physical Science Research Council, UK, for this work.

References

Billings, S.A., and Yusof, M.I., 1996, Decomposition of generalised frequency response functions for nonlinear systems using symbolic computation. Int. J. Control, Vol.65, pp589-618.

Boughe, O.M., Balikhim, M.A., Billings, S.A., and Alleyne, H., 2000, Global nonlinear model for the evolution of the Dst index. Research Report No.776, Dept. of AC&SE, Sheffield University.

Boughe, O.M., Billings, S.A., Li, L.M., Fleming, P.J., and Liu, J., 2002, Time and frequency domain identification and analysis of a gas turbine engine. Control Engineering Practice, Vol.10, pp1347-1356.

Davis, K.B., Leverette, S.J., and Spillane, M.W., 1994, Ringing response of TLP and GBS platforms. BOSS 94, Vol.3, pp569-585.

Farnes, K., Skjastad, O., and Hoen, C., 1994, Time domain analysis of transient resonant response of a monotower platform. BOSS 94, Vol.3, pp401-423.

Jefferys, E.R. and Rainey, R.C.T., 1994, Slender body models of TLP and GBS 'ringing'. BOSS 94, Cambridge, MA.

Lang, Zi-Qiang and Billings, S. A., 1996, Output frequency characteristics of nonlinear systems, Int. J. Control, Vol.64, pp1049-1067.

Lang, Zi-Qiang and Billings, S. A., 1997, Output frequencies of nonlinear systems, Int. J. Control, Vol.67, pp713-730.

Lee, C.H. and Newman, J.N. , 1994, Second order wave effects on offshore structures. BOSS, Vol.2, pp133-145.

Nestegard, A., 1993, High order wave load effects on TLPs. Tensioned Buoyant Platform Seminar, London, UK, 25-26 May 1993.

Nayfeh S.A. and Nayfeh A.H., 1994, Energy transfer from high- to low- frequency modes in flexible structure via modulation. Journal of Vibration and Acoustics, Vol.116, pp203-207.

Pearson, R.K., 1994, Discrete time dynamic models. Oxford University Press.

Popovic, Pavol Nayfeh, Ali, H., Kyoyal, H., and Nayfeh, Samir A., 1995, An experimental investigation of energy transfer from a high frequency mode to a low frequency mode in a flexible structure. Journal of Vibration and Control, Vol.1, pp115-128.

Remery, G.F.M. and Hermans, A.J., 1972, The slow drift oscillations of a moored object in random seas. Society of Petroleum Engineers Journal, June, pp191-198.

Rainey, R.C., 1989, A new equation for calculating wave loads on offshore structures. J. Fluid Mech., Vol.204, pp295-324.

Hsu, F.H. and Blenkarn, K.A., 1972, Analysis of peak mooring forces caused by slow vessel drift oscillations in random seas. Society of Petroleum Engineers Journal, August, pp329-344.

Raz, Gil M. and Veen, Barry D. Van, 1998, Baseband Volterra filters for implementing carrier based nonlinearities. IEEE Trans. Signal Processing, Vol.46, No.1, pp.103-114.

Stansberg, C.T., 1993, Statistical properties of nonlinear wave-induced high frequency responses in random waves. Proceedings of the Seminar on Tensions Buoyant Platforms.

Szczepanski, Stanislaw and Schaumann, Rolf, 1989, Effects of weak nonlinearties in transconductance capacitor filters. In Proc. IEEE Int. Symp. Circuits and Systems, pp. 1055-1058.

Szczepanski, Stanislaw and Schaumann, Rolf, 1993, Nonlinearity-induced distortion of the transfer function shape in high-order OTA-C filters. Analogue Integrated Circuits and Signal Processing, 3, pp.143-151.

Taylor, E.R. and Dai, D.N., 1992, Nonlinear hydrodynamic analysis of TLP's in extreme waves: Slender body and diffraction theories compared. BOSS 92', pp569-583.

Taylor, E.R. and Natvig, B.J., 1994, Effects of wave drift damping on TLP horizontal offsets. BOSS 94', Vol.2, pp31-34.

Wilhem, M et al., 1999, Fourier rheological materials. Rheol. Acta, Vol. 38, pp349-356.

Worden, K., Stansby, P.K, Tomlionson, G.R., and Billings, S.A., 1994, Identification of nonlinear wave forces. Journal of Fluid and Structures, Vol.8, pp19-71.

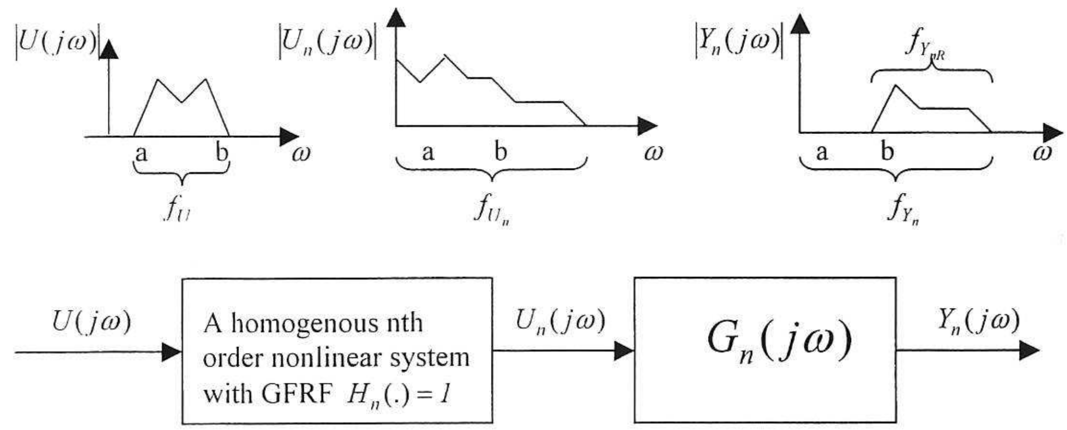

Figure 4.1 An illustration of the energy transfer produced by an nth order homogeneous nonlinear system

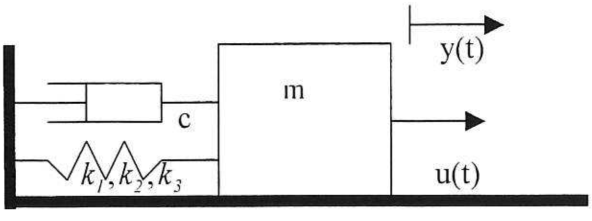

Figure 5.1 A mechanical oscillator which can be described by the nonlinear system in Section 5

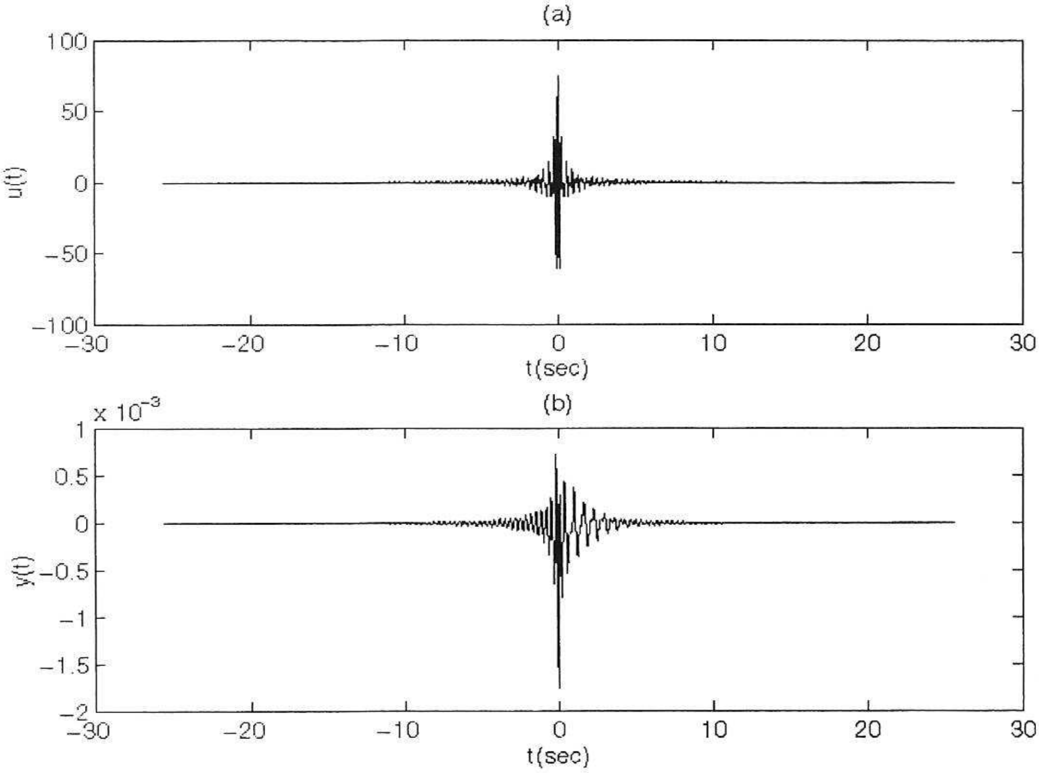

Figure 5.2 The first input and the output response of the studied nonlinear system to this input

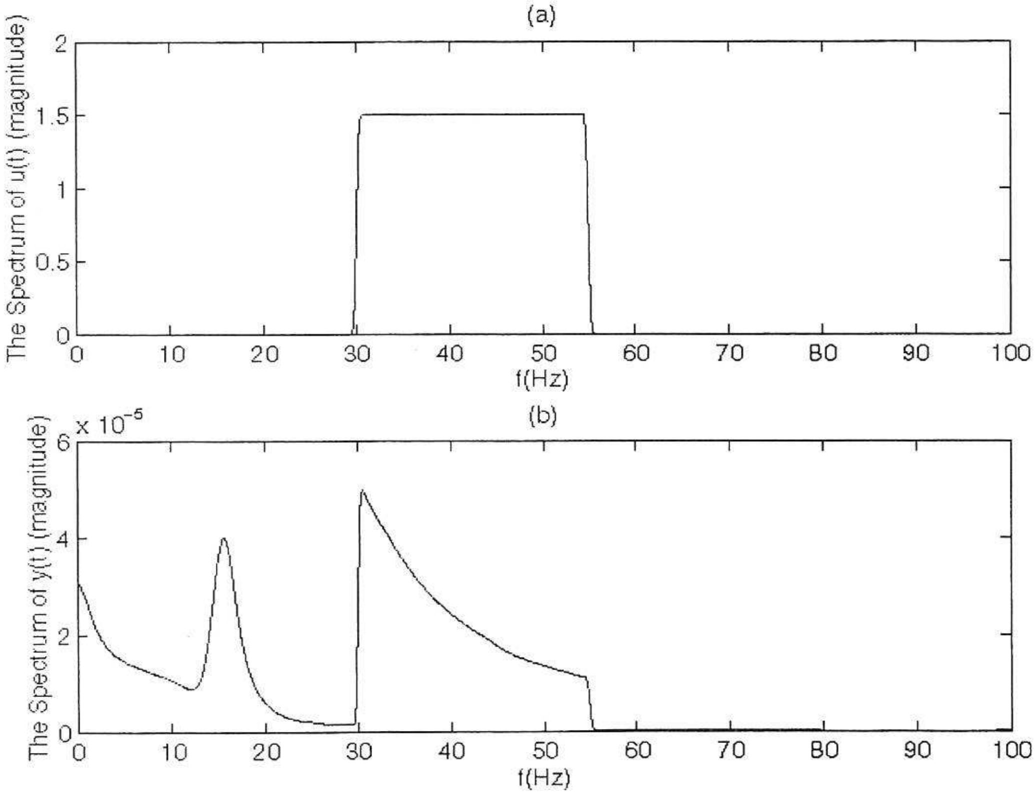

Figure 5.3 The spectra of the first input and the corresponding output of the studied nonlinear system

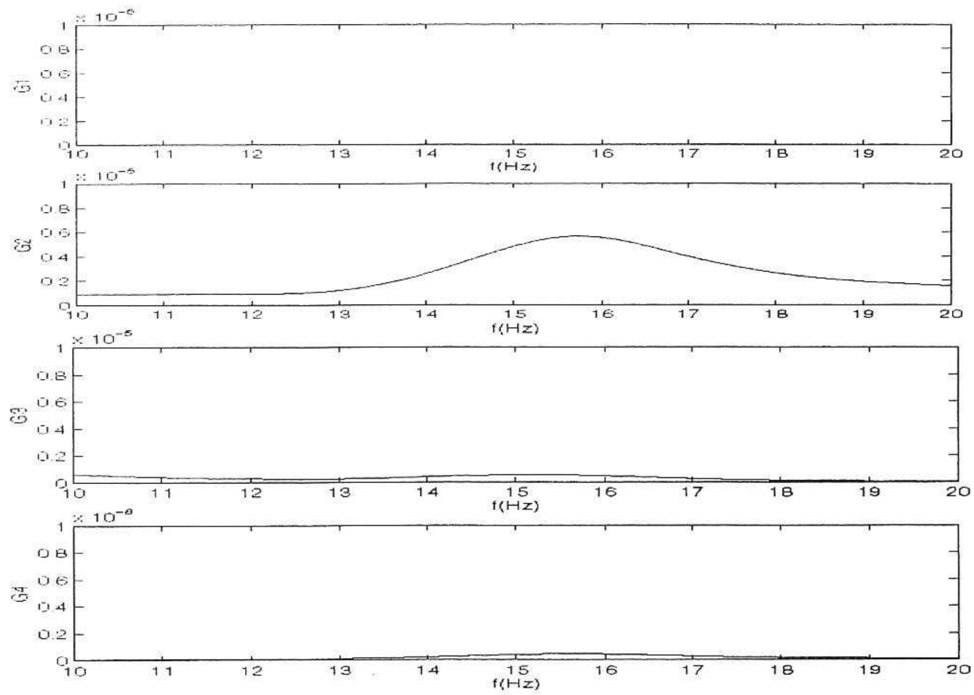

Figure 5.4 The Nonlinear Output Transfer Functions of the nonlinear system under the first input over the frequency range of (10,20) Hz

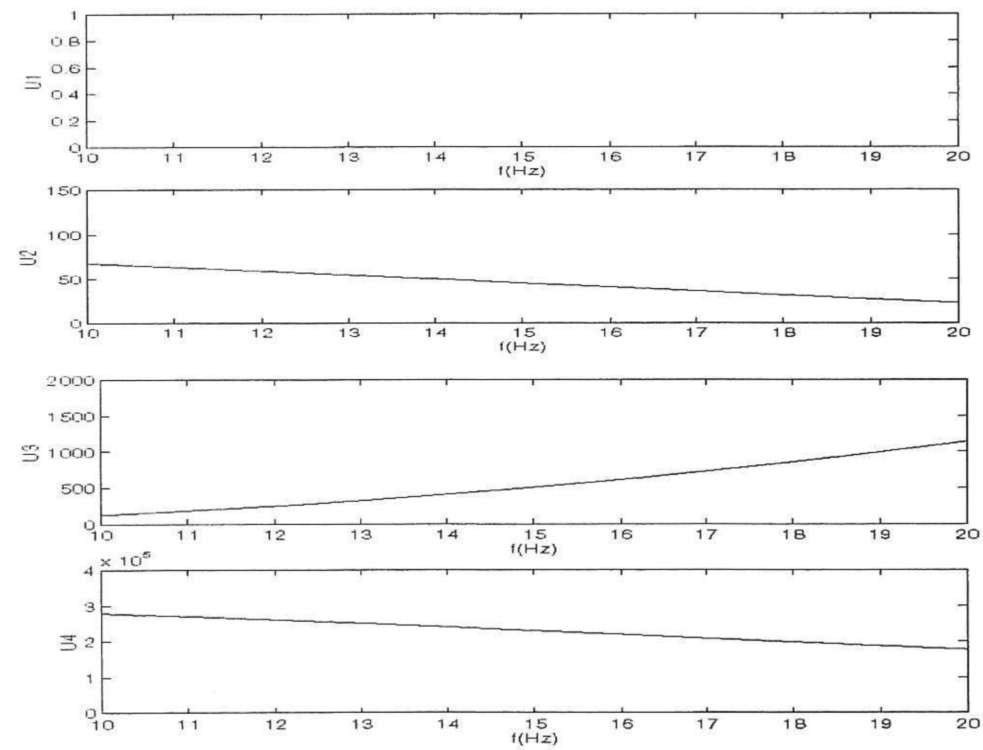

Figure 5.5 The contributions from the first input to the first, second, third, and fourth order output frequency responses of the nonlinear system over the frequency range of (10,20) Hz

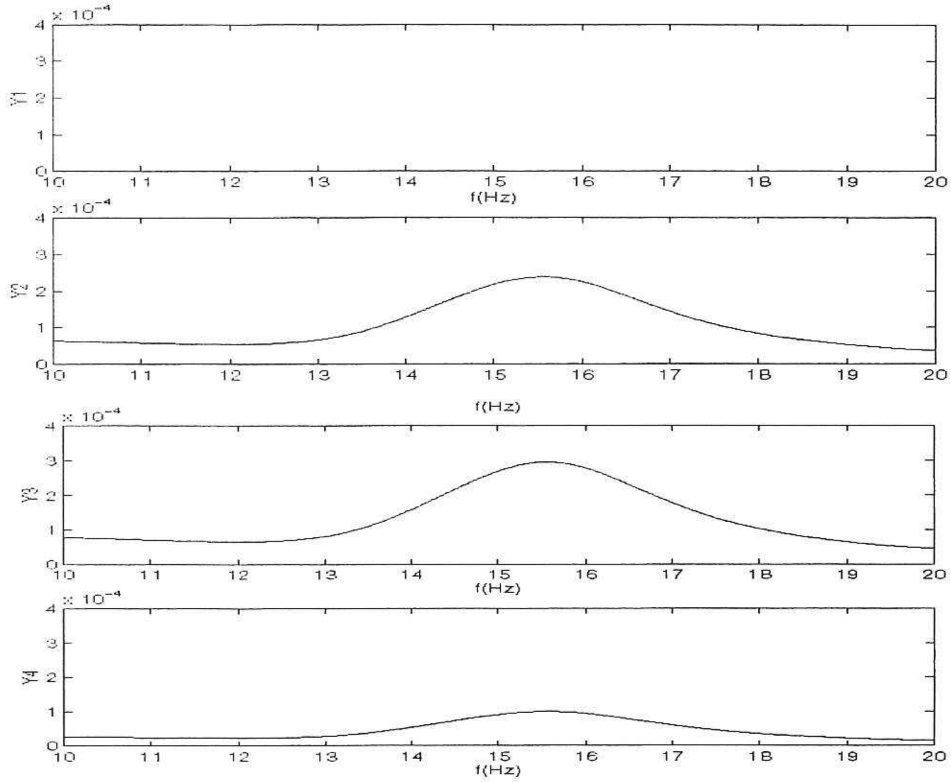

Figure 5.6 The first, second, third, and fourth order output frequency responses of the nonlinear system to the first input over the frequency range of (10,20) Hz

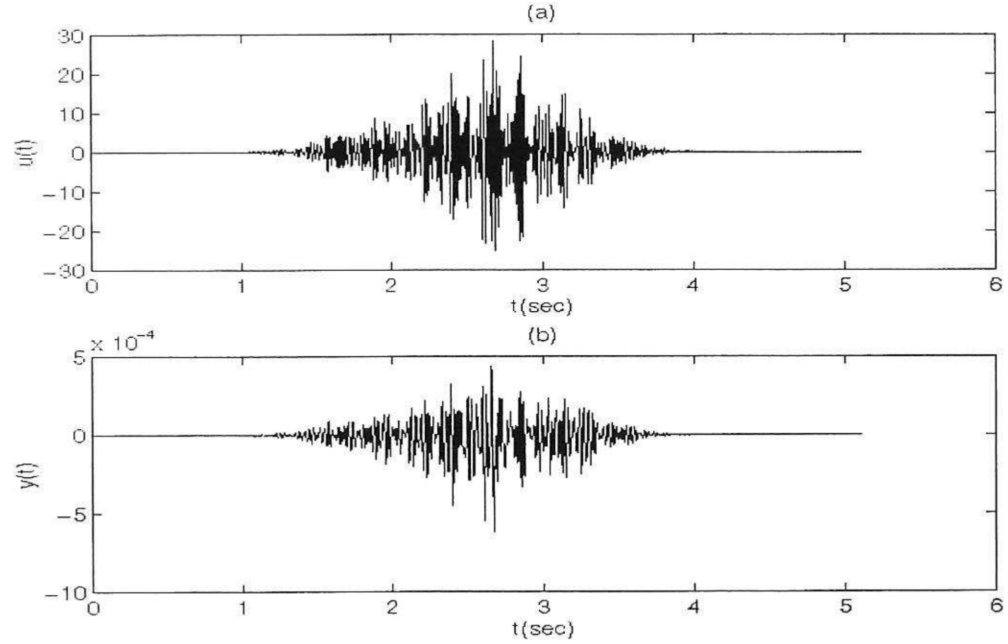

Figure 5.7 The second input and the output response of the nonlinear system to this input

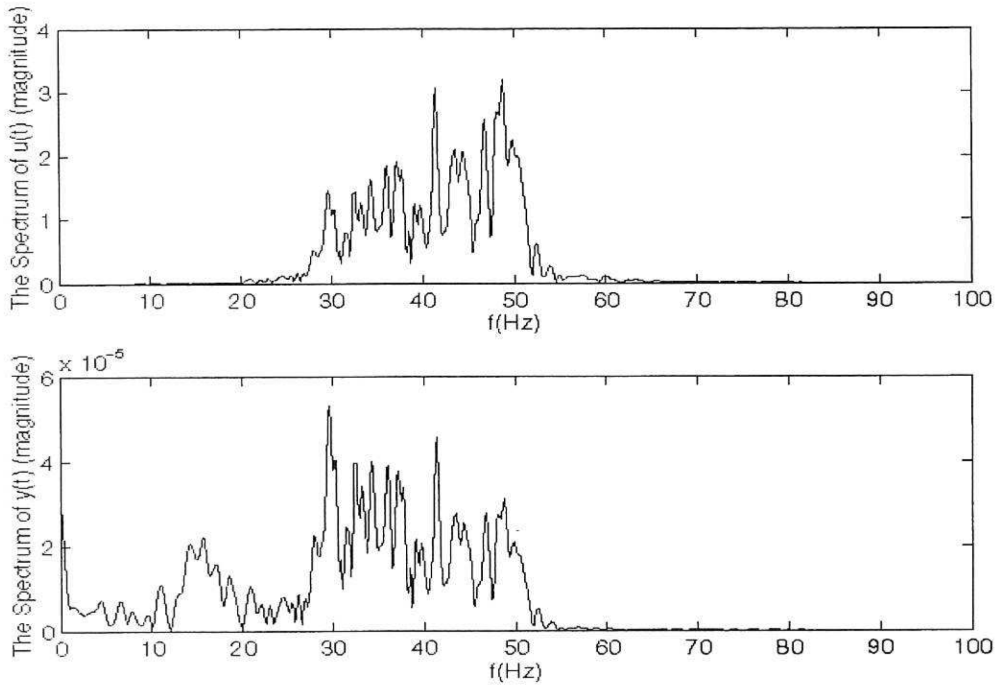

Figure 5.8 The spectra of the second input and the corresponding output of the nonlinear system

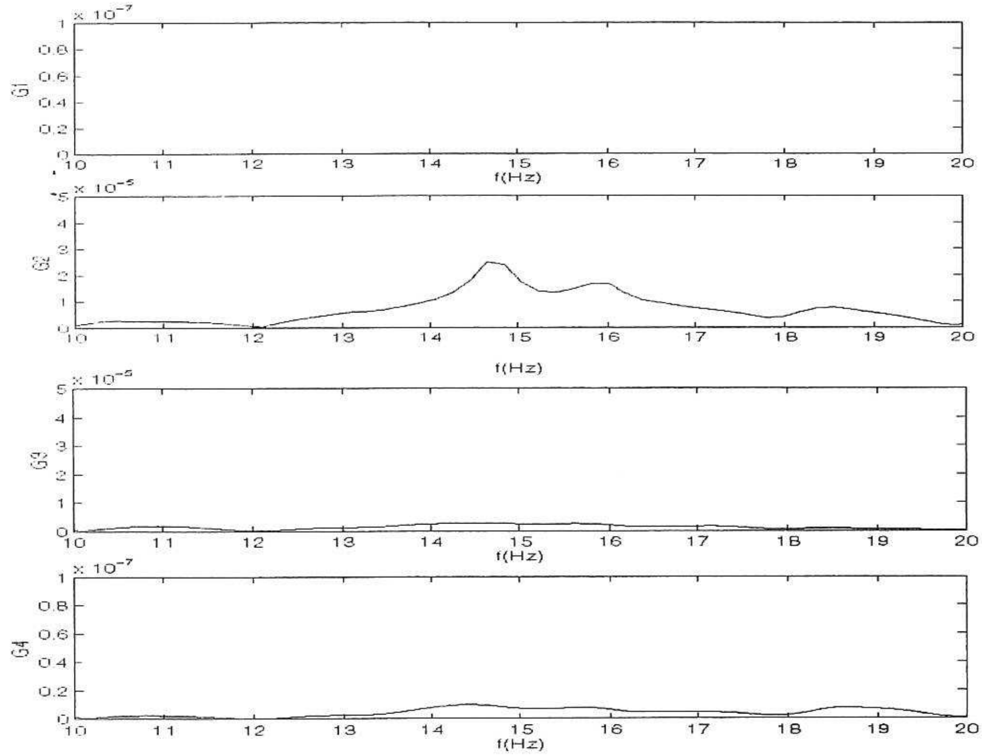

Figure 5.9 The Nonlinear Output Transfer Functions of the nonlinear system under the second input over the frequency range of (10,20) Hz

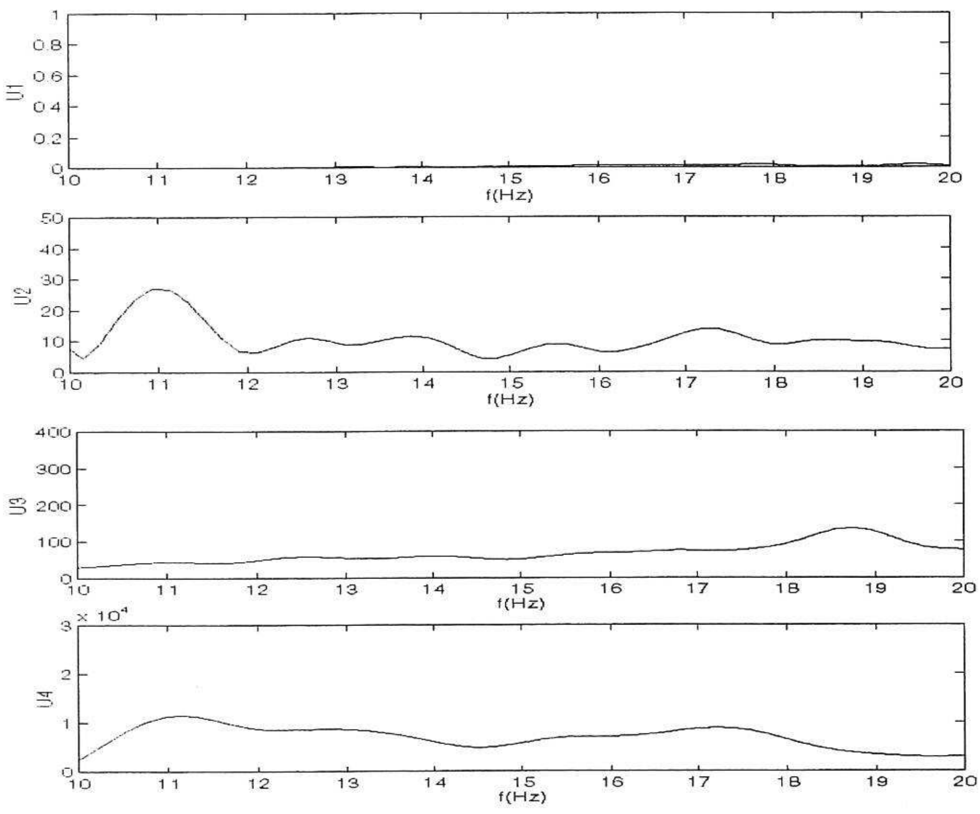

Figure 5.10 The contributions from the second input to the first, second, third, and fourth order output frequency responses of the nonlinear system over the frequency range of (10,20) Hz

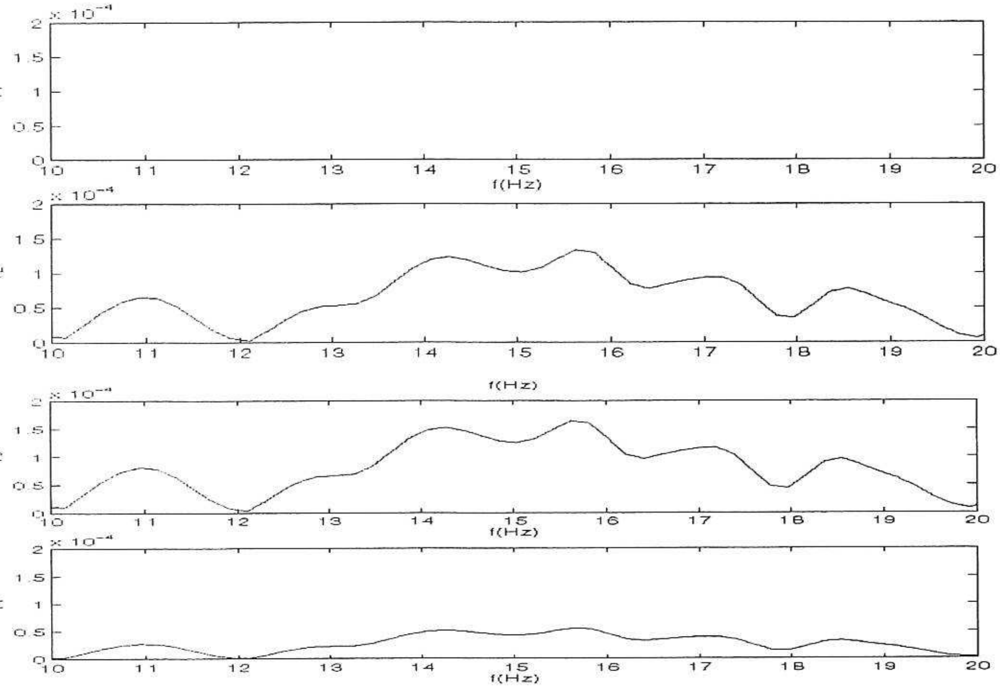

Figure 5.11 The first, second, third, and fourth order output frequency responses of the nonlinear system to the second input over the frequency range of (10,20) Hz