ENGINEERING SYSTEMS AND FREE SEMI-ALGEBRAIC GEOMETRY

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To Scott Joplin and his eternal RAGs

Abstract. This article sketches a few of the developments in the recently emerging area of real algebraic geometry (in short RAG) in a free* algebra, in particular on "noncommutative inequalities". Also we sketch the engineering problems which both motivated them and are expected to provide directions for future developments. The free* algebra is forced on us when we want to manipulate expressions where the unknowns enter naturally as matrices. Conditions requiring positive definite matrices force one to noncommutative inequalities. The theory developed to treat such situations has two main parts, one parallels classical semialgebraic geometry with sums of squares representations (Positivstellensätze) and the other has a new flavor focusing on how noncommutative convexity (similarly, a variety with positive curvature) is very constrained, so few actually exist.

1. Introduction

This article sketches a few of the developments in the recently emerging area of real algebraic geometry in a free* algebra, and the engineering problems which both motivated them and are expected to provide directions for future developments. Most linear control problems with mean square or worst case performance requirements lead directly to matrix inequalities (MIs). Unfortunately, many of these MIs are badly behaved and unsuited to numerics. Thus engineers have spent considerable energy and cleverness doing non-commutative algebra to convert, on an ad hoc basis, various given MIs into equivalent better behaved MIs.

A classical core of engineering problems are expressible as linear matrix inequalities (LMIs). Indeed, LMIs are the gold standard of MIs, since they are evidently convex and they are the subject of many excellent numerical packages. However, for a satisfying theory and successful numerics a convex MI suffices and so it is natural to ask:

How much more restrictive are LMIs than convex MIs?

It turns out that the answer depends upon whether the MI is, as is the case for systems engineering problems, fully characterized by performance criteria based on L^2 and signal flow

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diagrams (as are most textbook classics of control). Such problems have the property we refer to as "*dimension-free*".

Indeed, there are two fundamentally different classes of linear systems problems: dimension free and dimension dependent. A dimension free MI is a MI where the unknowns are g-tuples of matrices which appear in the formulas in a manner which respects matrix multiplication. Dimension dependent MIs have unknowns which are tuples of numbers.

The results presented here suggest the surprising conclusion that for dimension free MIs convexity offers no greater generality than LMIs. Indeed, we conjecture:

Dimension free convex problems are equivalent to an LMI

The key ingredient in passing from convex MIs to LMIs and proving their equivalence lies in the recently blossoming and vigorously developing direction of semi-algebraic in a free $*$ algebra; i.e., semi-algebraic geometry with variables which, like matrices, do not commute. Indeed at this stage there are two main branches of this subject. One includes non-commutative Positivstellensätze which characterize things like one polynomial p being positive where another polynomial q is positive. The other classifies situations with prescribed curvature.

As of today there are numerous versions of the Positivstellensätze for a free $*$- algebra, with typically cleaner statements than in the commutative case. For instance, in the non-commutative setting, positive polynomials are sums of squares. Through the connection between convexity and positivity of a Hessian, non-commutative semi-algebraic dictates a rigid structure for polynomials, and even rational functions, in non-commuting variables. For instance, a noncommutative polynomial p has second derivative p'' which is again a polynomial. Further, if p is matrix convex (as defined below), then p'' is matrix positive (also defined below) and is thus a sum of squares. It is a bizarre twist that p'' can be sum of squares only if p has degree at most two (see §3. The authors suspect that this is a harbinger of a very rigid structure in a free $*$-algebra for "irreducible varieties" whose curvature is either nearly positive or nearly negative; but this is a tale for another day.

A substantial opportunity for noncommutative algebra and symbolic computation lies in numerical computation for problems whose variables are naturally matrices. The goal is to exploit this special structure to accelerate and to increase the allowable size of computation. This is the subject of Section 9.

This survey is not intended to be comprehensive. Rather its purpose is to provide some snippets of results in non-commutative semi-algebraic geometry and their related computer algebra and numerical algorithms, and of motivating engineering problems with the idea of entertaining and even piquing the readers interest in the subject. In particular, this article draws heavily from [HP07] and [HPMV]. Sometimes we shall abbreviate the word noncommutative to NC.

As examples of other important directions and themes, some of which are addressed in other articles in this volume, there is a non-commutative algebraic geometry based on the Weyl algebra and corresponding computer algebra implementations, for example, Gröbner basis generators for the Weyl algebra are in the standard computer algebra packages such as Plural/Singular. A very different and elegant area is that of rings with a polynomial identity, in short PI rings, e.g. $N \times N$ matrices for fixed N. While most PI research concerns identities, there is one line of work on polynomial inequalities, indeed sums of squares, by Procesi-Schacher [PS76]. A Nullstellensätz for PI rings is discussed in [Ami57].

As indicated LMIs play a large role in this paper, so now we describe them precisely.

1.1. **LMIs and Noncommutative LMIS.** Since they play a central role in engineering and the study of convexity in the free $*$ setting, we digress, in the next subjection to define the notion of an LMI.

Given $d \times d$ symmetric (real entry) matrices $\Lambda_0, \Lambda_1, \ldots, \Lambda_g$, the function $L : \mathbb{R}^g \to S_d(\mathbb{R})$ given by

$$L(x) = \sum_{j=0} \Lambda_j x_j$$

is a classical linear pencil; and the inequality $L(x) \succeq 0$ is the classical (commutative) linear matrix inequality. Here $(x_1, \ldots, x_g) \in \mathbb{R}^g$.

In the non-commutative (dimension free) setting it is natural to substitute $X \in \mathbb{S}_n(\mathbb{R}^g)$ for the x above, obtaining the non-commutative version of a linear pencil. Namely, for each n a function $L : \mathbb{S}_n(\mathbb{R}^g) \to \mathbb{S}\mathbb{R}^{n\times n}$

$$L_n(X) = L(X) = \sum \Lambda_j \otimes X_j$$

The inequality $L(X) \succeq 0$ is what we will generally mean by LMI. And, as with polynomials, when we discuss LMIs and linear pencils it will be understood in the non-commutative sense.

Example 1.1. For $x := (x_1, x_2)$ being either commuting or noncommuting variables L written as

$$L(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix} x_2$$

denotes a linear pencil or NC linear pencil. For $X := (X_1, X_2)$ with $X_j \in \mathbb{R}^{n\times n}$

$$L(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} \otimes X_1 + \begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix} \otimes X_2 = \begin{pmatrix} I_n + 2X_1 + 3X_2 & 3X_1 + 5X_2 \\ 3X_1 + 2X_2 & I_n \end{pmatrix}.$$

For $X := (X_1, X_2)$ with $X_j \in \mathbb{R}$, the set of solutions to $L(X) \succeq 0$ is

$$\mathcal{C} := \{(X_1, X_2) : \quad 1 + 2X_1 + 3X_2 - (3X_1 + 5X_2)(3X_1 + 2X_2) \succeq 0.\}, \tag{1.1}$$

□

This last equivalence follows from taking an appropriate Schur complement which we now recall. The *Schur complement* of a matrix (with pivot γ^{-1}) is defined by

$$SchurComp \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} := \alpha - \beta\gamma^{-1}\beta^*.$$

A key fact is: if γ is invertible, then the matrix is positive semi-definite if and only if $\gamma > 0$ and its Schur complement is positive semi-definite.

Example 1.2. Apply this to the LMI in our example to obtain (1.1) for $X := (X_1, X_2)$ with $X_j \in \mathbb{R}^{n\times n}$ and X_j symmetric.

The Schur complement of $L(x)$ using the other pivot is the "rational expression"

$$I_n - (3X_1 + 2X_2)(I_n + 2X_1 + 3X_2)^{-1}(3X_1 + 5X_2).$$

□

1.2. **Outline.** The remainder of the survey is organized as follows. We expand upon the connection between systems engineering problems and dimension free MIs in Section 2. Convexity in the non-commutative (namely equal free $*$) setting is formalized in Section 3. This section also contains a brief glimpse into the *NCAlgebra* package. *NCAlgebra*, and the related *NCGB* (stands for non-commutative Gröbner basis) [HdOSM05] do symbolic computation in a free $*$- algebra and greatly aided the discovery of the results discussed in this survey. *NCAlgebra* and *NCGB* are free, but run under Mathematica which is not. Section 4 describes the engineering necessity for having a theory of matrix-valued non-commutative polynomials whose coefficients are themselves polynomials in non-commuting variables; much of the analysis of Section 3 carries over naturally in this setting. The shockingly rigid structure of convex rational functions is described in Section 5, with a sketch of proofs behind this "curvature oriented" non-commutative semi-algebraic geometry in §6. Section 7 discusses numerics designed to take advantage of matrix variables. Sections 8 gives the solution to the H^∞ control problem stated in §4.

Sections 9 describes noncommutative semi-algebraic geometry aimed at positivity and Positivstellensäten, this is an analogue of classical semi-algebraic geometry which is elegant though it does not have direct engineering applications.

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2. Dimension free engineering: the map between systems and algebra.

This section illustrates how linear systems problems lead to semi-algebraic geometry over a free or nearly free $*$- algebra and the role of convexity in this setting. The discussion will also inform the necessary further directions in the developing theory of non-commutative semi-algebraic needed to fully treat engineering problems.

In the engineering literature, the action takes place over the real field. Thus in much of this article, and in particular in this section, we restrict to real scalars. However,we do break from the engineering convention in that we will use A^* to denote the transpose of a (real entries) matrix and at the same time the usual involution on matrices with complex entries. Context will evidently determine the meaning.

The inner product of vectors in a real Hilbert space will be denoted $u \cdot v$.

2.1. **Linear systems.** A *linear system* $\mathfrak{F}$ is given by the linear differential equations

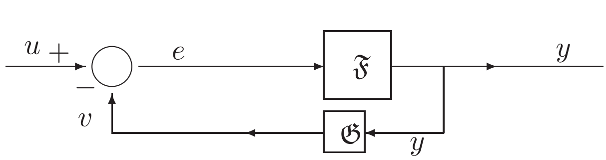
$$\frac{dx}{dt} = Ax + Bu,$$
$$y = Cx,$$

with the vector

- $x(t)$ at each time t being in the vector space $\mathcal{X}$ called the *state space*,
- $u(t)$ at each time t being in the vector space $\mathcal{U}$ called the *input space*,
- $y(t)$ at each time t being in the vector space $\mathcal{Y}$ called the *output space*,

and A, B, C being linear maps on the corresponding vector spaces.

2.2. **Connecting linear systems.** Systems can be connected in incredibly complicated configurations. We describe a simple connection and this goes along way toward illustrating the general idea. Given two linear systems $\mathfrak{F}$, $\mathfrak{G}$, we describe the formulas for connecting them as follows.



Systems $\mathfrak{F}$ and $\mathfrak{G}$ are respectively given by the linear differential equations

$$\frac{dx}{dt} = Ax + Be, \qquad\qquad \frac{d\xi}{dt} = a\,\xi + b\,w,$$
$$y = Cx, \qquad\qquad v = c\,\xi.$$

The connection diagram is equivalent to the algebraic statements

$$w = y \qquad \text{and} \qquad e = u - v.$$

The *closed loop system* is a new system whose differential equations are

$$\begin{aligned} \frac{dx}{dt} &= Ax - Bc\xi + Bu, \\ \frac{d\xi}{dt} &= a\,\xi + b\,y = a\,\xi + b\,Cx, \\ y &= Cx. \end{aligned}$$

In matrix form this is

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} &= \begin{pmatrix} A & -Bc \\ b\,C & a \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u, \\ y &= \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}, \end{aligned} \tag{2.1}$$

where the state space of the closed loop systems is the direct sum '$\mathcal{X} \oplus \mathcal{Y}$' of the state spaces $\mathcal{X}$ of $\mathfrak{F}$ and $\mathcal{Y}$ of $\mathfrak{G}$. The moral of the story is:

System connections produce a new system whose coefficients are matrices with entries which are polynomials or at worst "rational expressions" in the coefficients of the component systems.

Complicated signal flow diagrams give complicated matrices of polynomials or rationals. Note in what was said the dimensions of vector spaces and matrices never entered explicitly; the algebraic form of (2.1) is completely determined by the flow diagram. Thus, such linear systems lead to *dimension free* problems.

2.3. **Energy dissipation.** We have a system $\mathfrak{F}$ and want a condition which checks whether

$$\int_0^\infty |u|^2 dt \geq \int_0^\infty |\mathfrak{F}u|^2 dt, \qquad x(0) = 0,$$

holds for all input functions u, where $\mathfrak{F}u = y$ in the above notation . If this holds $\mathfrak{F}$ is called a *dissipative system*

$$\xrightarrow{\quad L^2[0,\infty] \quad} \boxed{\mathfrak{F}} \xrightarrow{\quad L^2[0,\infty] \quad}$$

The energy dissipative condition is formulated in the language of analysis, but it converts to algebra (or at least an algebraic inequality) because of the following construction, which assumes the existence of a "potential energy" like function V on the state space. A function V which satisfies $V \geq 0$, $V(0) = 0$, and

$$V(x(t_1)) + \int_{t_1}^{t_2} |u(t)|^2 dt \quad \geq \quad V(x(t_2)) + \int_{t_1}^{t_2} |y(t)|^2 dt$$

for all input functions u and initial states x_1 is called a *storage function.* The displayed inequality is interpreted physically as

potential energy now + energy in ≥ potential energy then + energy out.

Assuming enough smoothness of V, we can manipulate this integral condition to obtain first a differential inequality and then an algebraic inequality, as follows:

$$0 \geq \frac{V(x(t_2)) - V(x(t_1))}{t_2 - t_1} \quad + \quad \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |y(t)|^2 - |u(t)|^2 dt,$$

$$0 \geq \nabla V(x(t_1)) \cdot \frac{dx}{dt}(t_1) \quad + \quad |y(t_1)|^2 - |u(t_1)|^2.$$

Substituting $\frac{dx}{dt}(t_1) = \ Ax(t_1) + Bu(t_1)$ and $y = Cx$ gives

$$0 \ \geq \ \nabla V(x(t_1)) \cdot (Ax(t_1) + Bu(t_1)) \quad + \quad |Cx(t_1)|^2 - |u(t_1)|^2.$$

The system is dissipative if this inequality holds for all $u(t_1)$, $x(t_1)$ which can occur when it runs (starting at $x(0) = 0$). Denote $x(t_1)$ by x and $u(t_1)$ by u. With these notations, the inequality becomes

$$0 \ \geq \ \nabla V(x) \cdot (Ax + Bu) \quad + \quad |Cx|^2 - |u|^2, \tag{2.2}$$

All vectors $u(t_1)$ in $\mathcal{U}$ can certainly occur as an input and if all $x(t_1)$ can occur we call the system *reachable* . In the case of linear systems, V can be chosen quadratic of the form $V(x) = \langle Ex, x\rangle$ with $E \succeq 0$ and $\nabla V(x) = 2Ex$.

Theorem 2.1. *The linear system A, B, C is dissipative if inequality (2.2) holds for all $u \in \mathcal{U}, x \in \mathcal{X}$. Conversely, if A, B, C is reachable, then dissipativity implies inequality (2.2) holds for all $u \in \mathcal{U}, x \in \mathcal{X}$.*

2.3.1. *Riccati inequalities.* In the linear case, we may substitute $\nabla V(x) = 2Ex$ in (2.2) to obtain

$$0 \geq 2Ex \cdot (Ax + Bu) + |Cx|^2 - |u|^2,$$

for all u, x. Thus,

$$0 \geq \max_u \left([EA + A^*E + C^*C]x \cdot x + 2B^*Ex \cdot u - |u|^2 \right). \tag{2.3}$$

Since the maximizer in u is $u = B^*Ex$,

$$0 \geq 2Ex \cdot Ax + 2|B^*Ex|^2 + |Cx|^2 - |B^*Ex|^2.$$

This last inequality is conveniently expressed as

$$0 \geq [EA + A^*E + EBB^*E + C^*C]x \cdot x.$$

Thus the classical *Riccati matrix inequality*

$$0 \succeq \; EA + A^*E + EBB^*E + C^*C \qquad \textit{with} \quad E \succeq 0 \tag{2.4}$$

insures dissipativity of the system; and, it turns out, is also implied by dissipativity when the system is reachable.

2.3.2. *Schur Complements and Linear Matrix Inequalities.* Using Schur complements, the Riccati inequality of equation (2.4) is equivalent to the inequality

$$L(E) := \begin{pmatrix} EA + A^*E + C^*C & EB \\ B^*E & -I \end{pmatrix} \preceq 0. \tag{2.5}$$

Here A, B, C describe the system and E is an unknown matrix. If the system is reachable, then A, B, C is dissipative if and only if $L(E) \preceq 0$ and $E \succeq 0$.

The key feature in this reformulation of the Ricatti inequality is that $L(E)$ is linear in E, so inequality $L(E) \preceq 0$ is a Linear Matrix Inequality (LMI) in E. This is more general than was introduced in §1.1 since the coefficients A, B, C are themselves matrices rather than scalars.

2.4. **The basic questions.** What we have seen so far are the basic components of how one produces matrix inequalities (MI's) from engineering problems. It is in fact a very mechanical procedure. The trouble is that one typically produces messy MI's having no apparent good properties (see §4.1 for an example). We would like for them to be convex or to transform to a convex matrix inequality, justifying the claim that major issues in linear systems theory are:

1. *Are convex matrix inequalities more general than LMIs?*

2. *Which problems convert to a convex matrix inequality? How does one do the conversion?*
3. *Find numerics which will solve large convex problems. How do you use special structure, such as most unknowns are matrices and the formulas are all built of noncommutative rational functions?*

The mathematics here aims toward helping an engineer who writes a toolbox which other engineers will use for designing systems, like control systems. What goes in such toolboxes is algebraic formulas with matrices A, B, C unspecified and reliable numerics for solving them when a user does specify A, B, C as matrices. A user who designs a controller for a helicopter puts in the mathematical systems model for his helicopter and puts in matrices, for example, A is a particular $R^{8\times 8}$ matrix etc. Another user who designs a satellite controller might have a 50 dimensional state space and of course would pick completely different A, B, C. Essentially any matrices of any compatible dimensions can occur and our claim that our algebraic formulas are convex in the ranges we specify must be true.

The toolbox designer faces two completely different tasks. One is manipulation of algebraic inequalities; the other is numerical solutions. Often the first is far more daunting since the numerics is handled by some standard package although for numerics problem size is a demon. Thus there is a great need for algebraic theory.

Much of this paper bears on the first question when the unknowns are matrices, which though not fully solved has already motivated the construction of a considerable amount of noncommutative semi-algebraic geometry outlined in this survey. Section 7 bears on Question 3.

3. Convexity in a free algebra

Convexity of functions, domains and their close relative, positive curvature of varieties, are very natural notions in a $*$-free algebra. Shockling, convex polynomials and rational functions have a structure so rigid as to be nearly trivial. Indeed, a polynomial whose zero variety has positive curvature on a Zaraski open set is itself convex. In this section we survey what is known about convex polynomials in a free $*$-algebra. In later sections we treat convex rational functions, varieties with some positive curvature, and polynomials whose coefficients are themselves formally letters as in equation (2.5).

Let $\mathbb{R}\langle x\rangle$ denotes the free $*$-algebra in indeterminates $x = (x_1, ..., x_g)$, over the real field. Elements of $\mathbb{R}\langle x\rangle$ are non-commutative polynomials. There is a natural involution on $\mathbb{R}\langle x\rangle$ which reverses the order of multiplication $(fp)^* = p^* f^*$. In particular $x_j^* = x_j$ and for this reason the variables are *symmetric*. It is also possible to allow for non-symmetric variables by introducing the g additional variables x_j^*, but in the literature we are summarizing typically x_j can be taken either free or symmetric with no change in the conclusion. Thus for expositional purposes we will stick with symmetric variables in this survey, except for Section 9.

Let $\mathbb{S}_n(\mathbb{R}^g)$ denote g-tuples $X = (X_1, \ldots, X_g)$ of symmetric $n \times n$ matrices. Noncommutative polynomials are naturally evaluated at an $X \in \mathbb{S}_n(\mathbb{R}^g)$ by substitution. The involution on $\mathbb{R}\langle x \rangle$ is compatible with transpose of matrices in that $p(X)^* = p^*(X)$. A polynomial p is *symmetric* if $p = p^*$. Thus, if p is symmetric, then $p(X)^* = p(X)$.

A symmetric polynomial p is *matrix convex*, or simply *convex* for short, if for each positive integer n, each pair of tuples $X \in \mathbb{S}_n(\mathbb{R}^g)$ and $Y \in \mathbb{S}_n(\mathbb{R}^g)$, and each $0 \le t \le 1$,

$$p(tX + (1-t)Y) \preceq tp(X) + (1-t)p(Y). \tag{3.1}$$

Even in one-variable, convexity in the noncommutative setting differs from convexity in the commuting case because here Y need not commute with X. For example, to see that the polynomial $p = x^4$ is not matrix convex, let

$$X = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and compute

$$\frac{1}{2}X^4 + \frac{1}{2}Y^4 - \left(\frac{1}{2}X + \frac{1}{2}Y\right)^4 = \begin{pmatrix} 164 & 120 \\ 120 & 84 \end{pmatrix}$$

which is not positive semi-definite. On the other hand, to verify that x^2 is a matrix convex polynomial, observe that

$$\begin{aligned} &tX^2 + (1-t)Y^2 - (tX + (1-t)Y)^2 \\ &\qquad = t(1-t)(X^2 - XY - YX + Y^2) = t(1-t)(X-Y)^2 \succeq 0. \end{aligned}$$

It is possible to automate checking for convexity, rather than depending upon lucky choices of X and Y as was done above. The theory described in [CHSY03], sketched later in §6, leads to and validates a symbolic algorithm for determining regions of convexity of noncommutative rational functions (noncommutative rationals are formally introduced in §5) which is currently implemented in NCAlgebra.

We introduce now a sample NCAlgebra command, leaving a more detailed discussion for later (see §6.2). Noncommutative multiplication will be denoted $**$. The command is

NCConvexityRegion[Function F, Variable x].

Let us illustrate it on the example $p(x) = x^4$ with $x = x^*$.

```
In[1]:=  SetNonCommutative[x]};
In[2]:=   NCConvexityRegion[ x**x**x**x, x ]}
Out[2]:= { {2, 0, 0},  {0, 2 }, {-2, 0} }
```

which we interpret as saying that $p(x) = x^4$ is convex on the set of matrices X for which the the 3×3 block matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \tag{3.2}$$

is positive semi-definite. Thus, we conclude that p is *nowhere* convex.

This is a simple special case of the following theorem.

Theorem 3.1. [HM03] *Every convex symmetric polynomial in the free algebra $\mathbb{R}\langle x\rangle$ has degree two or less.*

A symmetric polynomial q is *matrix positive* or *positive* for short if for each n and $X \in \mathbb{S}_n(\mathbb{R}^g)$, $q(X) \succeq 0$. As we shall see convexity of p is equivalent to its "second directional derivative" being a positive polynomial. More generally, a symmetric polynomial whose k^{th} derivative is nonnegative has degree at most k (see Theorem 3.6 below).

It turns out that even if p is convex only on an open non-commutative domain, then in fact it is convex everywhere. To state the result we need to introduce some notation and terminology.

Let $\mathcal{P}$ denote a subset of $\mathbb{R}\langle x\rangle$ consisting of symmetric polynomials. Define the matrix nonnegativity domain $\mathcal{D}(\mathcal{P})$ of $\mathcal{P}$ to be the sequence of sets $(\mathcal{D}(\mathcal{P})_n)_{n=1}^{\infty}$ where

$$\mathcal{D}(\mathcal{P}) = \{X \in \mathbb{S}_n(\mathbb{R}^g) : p(X) \succeq 0, \quad q \in \mathcal{P}\}.$$

Theorem 3.2. [HM03] *Suppose $\mathcal{P}$ is a set of symmetric polynomials whose matrix nonnegativity domain $\mathcal{D}(\mathcal{P})$ contains open sets in all large enough dimensions; i.e., there is an n_0 so that for each $n \geq n_0$ the set $\mathcal{D}(\mathcal{P})_n$ contains an open set. If $p \in \mathbb{R}\langle x\rangle$ is symmetric and convex on $\mathcal{D}(\mathcal{P})$, then p has degree at most two.*

The proofs will be sketched shortly.

3.1. **Some History of Convex Polynomials.** The earliest related results we know of are due to Karl Löwner who studied a class of real analytic functions in one real variable called matrix monotone functions, which we shall not define here. Löwner gave integral representations and these have developed beautifully over the years. The impact on our story comes a few years later when Löwner's student Klaus [Kra36] introduced matrix convex functions f in one variable. Such a function f on $[0, \infty] \subset \mathbb{R}$ can be represented as $f(t) = tg(t)$ with g matrix monotone, so the representations for g produce representations for f. It seems to be a folk theorem that the one variable version of Theorem 3.1 was known as a consequence of more general results on these matrix convex functions. Modern references are [OST07], [Uch05]. Frank Hansen has extensive deep work on matrix convex and monotone functions whose definition in several variables is different than the one we use here, see[HT07]; and for a more recent reference see [Han97].

3.2. **The Proof of Theorem 3.1 and its Ingredients.** Just as in the commutative case, convexity of a symmetric $p \in \mathbb{R}\langle x \rangle$ is equivalent to positivity of its Hessian. Unlike the commutative case, positive non-commutative polynomials are sums of squares. Combinatorial considerations say that a Hessian which is also a sum of squares must come from a polynomial of degree two. In the remainder of this section we flesh out this argument, introducing the needed techniques and results.

The proof of Theorem 3.2 requires different, though certainly related, machinery which is discussed in §6.

3.2.1. *Noncommutative Derivatives.* For a polynomial $p \in \mathbb{R}\langle x \rangle$ define the k^{th}*-directional derivative* : by

$$p^{(k)}(x)[h] = \frac{d^k}{dt^k} p(x + th)\Big|_{t=0}.$$

Note that $p^{(k)}(x)[h]$ is homogeneous of degree k in h.

More formally, we regard the directional derivative $p'(x)[h] \in \mathbb{R}\langle x, h \rangle$ as a polynomial in $2g$ free symmetric (i.e. invariant under *) variables $(x_1, \dots, x_g, h_1, \dots, h_g)$; In the case of a word $w = x_{j_1} x_{j_2} \cdots x_{j_n}$ the derivative is:

$$w'[h] = h_{j_1} x_{j_2} \cdots x_{j_n} + x_{j_1} h_{j_2} x_{j_3} \cdots x_{j_n} + \ \dots \ + x_{j_1} \cdots x_{j_{n-1}} h_{j_n}$$

and for a polynomial $p = p'(x)[h] = \sum p_w w$ the derivative is

$$p'(x)[h] = \sum p_w w'[h].$$

If p is symmetric, then so is p'.

For $X, H \in \mathbb{S}_n(\mathbb{R}^g)$ observe that

$$p'(X)[H] = \lim_{t \to 0} \frac{p(X + tH) - p(X)}{t}.$$

Alternately, with $q(t) = p(X + tH)$,

$$p'(X)[H] = q'(0).$$

Likewise for a polynomial $p \in \mathbb{R}\langle x\rangle$, the *Hessian* $p''(x)[h]$ of $p(x)$ can be thought of as the formal second directional derivative of p in the "direction" h. Equivalently, the Hessian of $p(x)$ can also be defined as the part of the polynomial

$$r(x)[h] := p(x + h) - p(x)$$

in the free algebra in the symmetric variables that is homogeneous of degree two in h.

If $p'' \neq 0$, that is, if degree $p \geq 2$, then the degree of $p''(x)[h]$ as a polynomial in the $2g$ variables $x_1, \dots, x_g, h_1 \dots, h_g$ is equal to the degree of $p(x)$ as a polynomial in $x_1, \dots, x_g$. Likewise for k^{th} derivatives.

Example 3.3. The first (non-commutative) derivative of $p(x) = x_2x_1x_2$ is

$$p'(x)[h] = \frac{d}{dt}[(x_2 + th_2)(x_1 + th_1)(x_2 + th_2)]\Big|_{t=0} = h_2x_1x_2 + x_2h_1x_2 + x_2x_1h_2.$$

□

Example 3.4. The one variable $p(x) = x^4$ has first derivative

$$p'(x)[h] = hxxx + xhxx + xxhx + xxxh.$$

Note each term is linear in h and h replaces each occurrence of x once and only once.

The Hessian, or second derivative, of p is

$$\begin{aligned} p''(x)[h] = hhxx + &hhxx + hxhx + hxxh \\ &+ hxhx + xhhx + xhhx + xhxh + hxxh + xhxh + xxhh + xxhh, \end{aligned}$$

which simplifies to

$$p''(x)[h] = 2hhxx + 2hxhx + 2hxxh + 2xhhx + 2xhxh + 2xxhh.$$

Note each term is degree two in h and h replaces each pair of x's exactly once. Likewise $p^{(3)}(x)[h] = 6(hhhx + hhxh + hxhh + xhhh)$ and $p^{(4)}(x)[h] = 24hhhh$ and $p^{(5)}(x)[h] = 0$. □

Example 3.5. The Hessian of $p = x_1^2x_2$ is $p''(x)[h] = h_1^2x_2 + h_1x_1h_2 + x_1h_1h_2$. □

Theorem 3.1 is the $k = 2$ case of the following result.

Theorem 3.6 ([HP07])**.** *Every symmetric polynomial $p \in \mathbb{R}\langle x\rangle$ whose k^{th} derivative is a matrix positive polynomial has degree k or less.*

Proof See [HP07] for the full proof or [HM03] for case of $k = 2$. The very intuitive proof based upon a little non-commutative semi-algebraic geometry is sketched in the next subsubsection. ■

3.2.2. *A Little NonCommutative Semi-Algebraic Geometry.* A central theme of semi-algebraic geometry are positivstellensätz which, in the simplest forms, represent polynomials which are positive, or positive on a domain. It turns out positivstellensätzae in the free $*$ setting generally have cleaner statements than in the commutative case. Proofs of Theorems 3.1 and 3.2 require a non-commutative positivstellensätz.

Recall, a symmetric polynomial p is **matrix positive polynomial** or simply **positive** provided $p(X_1, \cdots, X_g)$ is positive semidefinite for every $X \in \mathbb{S}_n(\mathbb{R}^g)$ (and every n). An example of a matrix positive polynomial is a **Sum of Squares** of polynomials, meaning an expression of the form

$$p(x) = \sum_{j=1}^{c} h_j(x)^* h_j(x).$$

Substituting $X \in \mathbb{S}_n(\mathbb{R}^g)$ gives $p(X) = \sum_{j=1}^{c} h_j(X)^* h_j(X) \succeq 0$. Thus p is positive. Remarkably these are the only positive non-commutative polynomials.

Theorem 3.7. *Every matrix positive polynomial is a sum of squares.*

As noted above, this non-commutative behavior is much cleaner than that of conventional "commutative" semi-algebraic geometry. See [Par00, Las01] for a beautiful treatment of applications of commutative semialgebraic geometry. This theorem is just a sample of the structure of noncommutative semialgebraic geometry, the topic of §9.

Suppoe $p \in \mathbb{R}\langle x \rangle$ is (symmetric and) convex and $Z, H \in \mathbb{S}_n(\mathbb{R}^g)$ and $t \in \mathbb{R}$ are given. In the definition of convex, choosing $X = Z + tH$ and $Y = Z - tH$, it follows that

$$0 \preceq p(Z + tH) + p(Z - tH) - 2p(Z),$$

and therefore

$$0 \preceq \lim_{t \to 0} \frac{p(X + tH) + p(X - tH) - 2p(X)}{t^2} \to p''(X)[H].$$

Thus the Hessian of p is matrix positive and since, in the noncommutative setting, positive polynomials are sums of squares we obtain the following theorem.

Proposition 3.8. *If p is matrix convex, then its Hessian $p''(x)[h]$ is a sum of squares.*

3.2.3. *Proof of Theorem 3.2 by example.* Example 3.4 serves to illustrate the proof of Theorem 3.2 in the case $k = 2$.

Example 3.9. The one-variable polynomial $p = x^4$ is not matrix convex.

Here is a sketch of the proof based upon Proposition 3.8.

If $p(x) = x^4$ is matrix convex, then $p''(x)[h]$ is matrix positive and therefore, by Proposition 3.8, there exists a ℓ and polynomials $f_1(x,h), \dots, f_\ell(x,h)$ such that

$$\begin{aligned} p''(x)[h] =& hhxx + hxhx + hxxh + xhhx + xhxh + xxhh \\ =& f_1(x,h)^* f_1(x,h) + \cdots + f_\ell(x,h)^* f_\ell(x,h). \end{aligned}$$

One can show that each $f_j(x,h)$ is linear in h. On the other hand, some term $f_i^* f_i$ contains $hhxx$ and thus f_i contains hx^2. Let m denote the largest ℓ such that some f_j contains the term hx^ℓ. Then $m \geq 1$ and for such j, the product $f_j^* f_j$ contains the term $hx^{2m}h$ which can't be cancelled out, a contradiction. ■ □

The proof of the more general, order k derivative, is similar, see [HP07].

4. A bit of engineering reality: coefficients in an algebra

A level of generality which most linear systems problems require is polynomials p or noncommutative rational functions (to be discussed later) in two classes of variables, say a and x, rather than x alone. As the example in Subsection 4.1 below illustrates, the x play the role of unknowns and the a the role of systems parameters and we are interested in matrix convexity in x over ranges of the variable(s) a. Describing this setup fully takes a while, as one can see in [CHSY03] where it is worked out. An engineer might look at [CHS06], especially the first part which describe a computational noncommutative algebra attack on convexity, it seems to be the most intuitive read on the subject at hand.

In this section some sample results from [HHLM] and a motivating engineering example are presented.

In [HHLM] one shows that second derivatives of a symmetric polynomial $p(a,x)$ in x determine convexity in x and that convexity in the x variable on some "open set" of a, x implies that p has degree 2 or less in x.

Theorem 4.1. *If $P(a,x)$ is a symmetric $d \times d$ matrix with polynomial entries $p_{ij}(a,x)$, then convexity in x for all X and all A satisfying some strict algebraic inequality of the form $g(A) \succ 0$, implies each p_{ij} has degree 2 or less.*

Proof See [HP07] survey combined with [HHLM]. ■

Assume a $d \times d$ matrix of polynomials $P(a, x)$ has degree 2 in x. There are tests (not perfect) to see where in the a variable $P(X, A)$ is negative semi-definite for all X. Equivalently, to see where P is convex in x, see [HP07].

The following is a further example of results from [HHLM].

Theorem 4.2. *A symmetric $p(a, x)$ is convex in x and concave in a if and only if*

$$p(a, x) = L(a, x) + R(x)^* R(x) - S(a)^* S(a),$$

where $L(a, x)$ has degree at most one in each of x and a and R and S are vectors which are linear in x and a respectively.

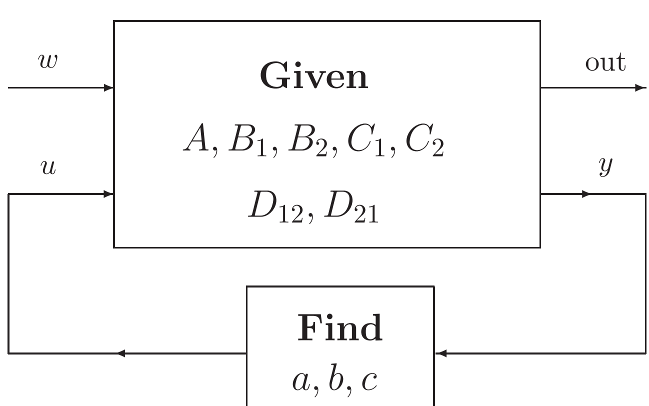
Note that $R(x)^* R(x)$ is a homogeneous of degree two sum of squares.

The subsection below gives a flavor of how two types of variables a, x as well as matrices with noncommutative polynomial entries arise naturally in engineering applications. It continues the discussions of Section 2.

4.1. **A sample engineering messy algebra problem.** Here is a basic engineering problem, the *standard problem of H^∞ control*:

Make a given system dissipative by designing a feedback law.

To be more specific, we are given a signal flow diagram:



where the given system is

$$\frac{ds}{dt} = As + B_1 w + B_2 u,$$

$$\text{out} = C_1 s + D_{12} u + D_{11} w,$$

$$y = C_2 s + D_{21} w,$$

$$D_{21} = I, \qquad D_{12}D'_{12} = I, \qquad D'_{12}D_{12} = I, \qquad D_{11} = 0.$$

The assumptions on D are to simplify calculations. In practice one needs something messier.

We want to find an unknown system

$$\frac{d\xi}{dt} = a\,\xi + b\,y, \qquad u = c\,\xi,$$

called the *controller*, which makes the system dissipative over every finite horizon. Namely:

$$\int_0^T |w(t)|^2 dt \geq \int_0^T |out(t)|^2 dt, \qquad s(0) = 0.$$

So a, b, c are the critical unknowns.

4.1.1. *Conversion to algebra.* The dynamics of the "closed loop" system has the form

$$\frac{d}{dt}\begin{pmatrix} s \\ \xi \end{pmatrix} = \mathcal{A}\begin{pmatrix} s \\ \xi \end{pmatrix} + \mathcal{B}w,$$

$$\text{out} = \mathcal{C}\begin{pmatrix} s \\ \xi \end{pmatrix} + \mathcal{D}w,$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are "2×2 block matrices" whose entries are polynomials in the $A's, B's, \cdots, a, b, c$ etc. The storage function inequality which corresponds to energy dissipation (see Subsection 2.3) has the form

$$H := \mathcal{A}^* E + E\mathcal{A} + E\mathcal{B}\mathcal{B}^* E + \mathcal{C}^*\mathcal{C} \preceq 0. \tag{4.1}$$

Expressing E and H as 2×2 block matrices,

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \succeq 0, \qquad E_{12} = E_{21}^*,$$

$$H = \begin{pmatrix} H_{ss} & H_{sy} \\ H_{ys} & E_{yy} \end{pmatrix} \preceq 0, \qquad H_{sy} = H_{ys}^*,$$

the challenge of the algebraic inequality of equation (4.1) is to find $H \preceq 0$ where the entries of H are the polynomials:

$$
\begin{aligned}
H_{ss} &= E_{11}\, A + A^*\, E_{11} + C_1^*\, C_1 + E_{12}^*\, b\, C_2 + C_2^*\, b^*\, E_{12}^* + E_{11}\, B_1\, b^*\, E_{12}^* + \\
&\qquad E_{11}\, B_1\, B_1^*\, E_{11} + E_{12}\, b\, b^*\, E_{12}^* + E_{12}\, b\, B_1^*\, E_{11}, \\
H_{sz} &= E_{21}\, A + \frac{1}{2} a^*\, (E_{21} + E_{12}^*) + c^*\, C_1 + E_{22}\, b\, C_2 + c^*\, B_2^*\, E_{11}^* + \\
&\qquad \frac{1}{2} E_{21}\, B_1\, b^*\, (E_{21} + E_{12}^*) + E_{21}\, B_1\, B_1^*\, E_{11}^* + \frac{1}{2} E_{22}\, b\, b^*\, (E_{21} + E_{12}^*) + E_{22}\, b\, B_1^*\, E_{11}^*, \\
H_{zs} &= A^*\, E_{21}^* + C_1^*\, c + \frac{1}{2}(E_{12} + E_{21}^*)\, a + E_{11}\, B_2\, c + C_2^*\, b^*\, E_{22}^* + E_{11}\, B_1\, b^*\, E_{22}^* + \\
&\qquad E_{11}\, B_1\, B_1^*\, E_{21}^* + \frac{1}{2}(E_{12} + E_{21}^*)\, b\, b^*\, E_{22}^* + \frac{1}{2}(E_{12} + E_{21}^*)\, b\, B_1^*\, E_{21}^*, \\
H_{zz} &= E_{22}\, a + a^*\, E_{22}^* + c^*\, c + E_{21}\, B_2\, c + c^*\, B_2^*\, E_{21}^* + E_{21}\, B_1\, b^*\, E_{22}^* + \\
&\qquad E_{21}\, B_1\, B_1^*\, E_{21}^* + E_{22}\, b\, b^*\, E_{22}^* + E_{22}\, b\, B_1^*\, E_{21}^*.
\end{aligned}
$$

Here A, B_1, B_2, C_1, C_2 are known and the unknowns are a, b, c and for E_{11}, E_{12}, E_{21} and E_{22}. If one can find E, then it turns out that there are explicit formulas for a, b, c in terms of E.

We very much wish that these inequalities (4.1) are convex in the unknowns (so that numerical solutions will be reliable). But our key inequality above is not convex in the unknowns.

The key question: *Is there is a set of noncommutative convex inequalities whose set of solutions is equivalent to those of* (4.1)*?*

This is a question in algebra not in numerics and we leave it as a nearly impossible challenge to the reader. Later in §8 we give the answer. There are many ways to derive the solution for this as well as a broad class of related problems, but all appeal to very special structure. An issue driving our development of $*-$ free semi-algebraic geometry is how to give a general theory which solves this and many other examples as a special case. It is clear that such a theory must includes the possibility to change of variables, identifying which non-convex problems can be converted to convex problems, and automating the conversion when possible.

4.2. **What is needed for engineering.** Many linear systems problems which are "dimension free" readily convert to non-commutative inequalities on $d \times d$ matrices of polynomials of the form $P(a, x) \preceq 0$ as the example in Section 4.1 illustrates. Often the inequality $P(a, x) \preceq 0$ can be simplified by various means such as solving for some variables and substituting to get other hopefully simpler inequality $R(a, x) \preceq 0$. In fact what one gets by standard manipulations in all circumstances (to our knowledge) is matrices $R(a, x)$ with noncommutative rational expressions as entries. Thus there is the need to generalize Theorem 4.1 from polynomials to rational expressions. The notion of a noncommutative rational function is given in the following section where Theorem 5.3 gives solid support for our conjecture that convex noncommutative rational functions $R(a,x)$ have a surprisingly simple structure in x.

This very strong conclusion is bad news for engineers because it says convexity for dimension free problems is much rarer than for dimension dependent problems. We emphasize that the result does not preclude transformation, by change of variable say, to achieve convexity and understanding such transformation is a challenging mostly open area.

5. Convexity for Noncommutative Rationals

This section describes the extension of the convex noncommutative polynomial theorem, Theorem 3.1, to symmetric noncommutative rational functions, $r = r(x)$, of the x variable alone which are convex near the origin, see [HMV06]. The results provide further evidence for the rigidity of convexity in the noncommutative (dimension free) setting.

5.1. Noncommutative Rational Functions.

We shall discuss the notion of a noncommutative rational function in terms of rational expressions. We refer to [HMV06, Section 2 and Section 16] for details. In what follows, the casual reader can ignore the technical condition, "analytic at 0", which we include for the sake of precision.

A **noncommutative rational expression analytic at** 0 is defined recursively. Noncommutative polynomials are noncommutative rational expressions as are all sums and products of noncommutative rational expressions. If r is a noncommutative rational expression and $r(0) \neq 0$, then the inverse of r is a rational expression analytic at 0.

The notion of the **formal domain of a rational expression** r, denoted $\mathcal{F}_{r,\mathrm{for}}$, and the evaluation $r(X)$ of the rational expression at a tuple $X \in \mathbb{S}_n(\mathbb{R}^g) \cap \mathcal{F}_{r,\mathrm{for}}$ are also defined recursively[1]. Example (5.1) below is illustrative.

An example of a noncommutative rational expression is the Riccati expression for discrete-time systems:

$$r = a^*xa - x + c^*c + (a^*xb + c^*d)(I - d^*d - b^*xb)^{-1}(b^*xa + d^*c)$$

Here some variables are symmetric some are not. A difficulty is two different expressions, such as

$$r_1 = x_1(1 - x_2x_1)^{-1} \quad \text{and} \quad r_2 = (1 - x_1x_2)^{-1}x_1,$$

[1]The formal domain of a polynomial p is all of $\mathbb{S}_n(\mathbb{R}^g)$ and $p(X)$ is defined just as before. The formal domain of sums and products of rational expressions is the intersection of their respective formal domains. If r is an invertible rational expression analytic at 0 and $r(X)$ is invertible, then X is in the formal domain of r^{-1}.

that can be converted into each other with algebraic manipulation represent the same rational function. Thus it is necessary to specify an equivalence relation on rational expressions to arrive at what are typically called **noncommutative rational functions**. (This is standard and simple for commutative (ordinary) rational functions.) There are many alternate ways to describe the noncommutative rational functions and they go back 50 years or so in the algebra literature. The simplest one for our purposes is **evaluation equivalence** — two rational expressions r_1 and r_2 are evaluation equivalent if $r_1(X) = r_2(X)$ for all $X \in \mathcal{F}_{r_1,\text{for}} \cap \mathcal{F}_{r_2,\text{for}}$. For engineering purposes one need not be too concerned, since what happens is that two expressions r_1 and r_2 are equivalent whenever the usual manipulations you are accustomed to with matrix expressions convert r_1 to r_2.

For $\mathfrak{r}$ a rational function, that is, an "equivalence class of rational expressions r", define its **domain** by

$$\mathcal{F}_{\mathfrak{r},\text{for}} := \cup_{\{r \textit{ represents } \mathfrak{r}\}} \mathcal{F}_{r,\text{for}}.$$

Let $\mathcal{F}^0_{\mathfrak{r},\text{for}}$ denote the arcwise connected component of $\mathcal{F}_{\mathfrak{r},\text{for}}$ containing 0 (and similarly for $\mathcal{F}^0_{r,\text{for}}$). We call $\mathcal{F}^0_{\mathfrak{r},\text{for}}$ the **principal component** of $\mathcal{F}_{\mathfrak{r},\text{for}}$. Henceforth we do not distinguish between the rational functions $\mathfrak{r}$ and rational expressions r, since this causes no confusion. We give several examples.

Example 5.1.

$$r(x_1, x_2) = (1 + x_1 - (3 + x_2)^{-1})^{-1},$$

where we take $x_1 = x_1^*, x_2 = x_2^*$ is a symmetric noncommutative rational expression. The domain $\mathcal{F}_{r,\text{for}}$ is

$$\cup_{n>0}\{(X_1, X_2) \in \mathbb{SR}^{n\times n}(\mathbb{R}^2) : \quad 1 + X_1 - (3 + X_2)^{-1} \text{ and } 3 + X_2 \text{ are invertible}\}.$$

Its principal component $\mathcal{F}^0_{r,\text{for}}$ is

$$\cup_{n>0}\{(X_1, X_2) \in \mathbb{SR}^{n\times n}(\mathbb{R}^2) : \quad 1 + X_1 - (3 + X_2)^{-1} \succ 0 \text{ and } 3 + X_2 \succ 0\}.$$

□

Example 5.2. We return to convexity checker command and illustrate it on

$$F((a, b, r), (x, y)) := -(y + a^*xb)(r + b^*xb)^{-1}(y + b^*xa) + a^*xa. \tag{5.1}$$

where $x = x^*, y = y^*$. Here we are viewing F as a function of two classes of variables (see Section 4). An application of the command **NCConvexityRegion**$[F, \{x, y\}]$ outputs the list

$$\{-2\,(r + b^*xb)^{-1},\ 0,\ 0,\ 0\}.$$

This output has the meaning that whenever A, B, R are fixed matrices, the function F is "x, y-matrix concave" on the domain of matrices X, and Y

$$\mathcal{G}_{A,B,R} := \{(X,Y) : (R + B^*XB)^{-1} \succ 0\}.$$

The command NCConvexityRegion also has an important feature which, for this problem, assures us no domain bigger than

$$\bar{\mathcal{G}}_{A,B,R} := \{(X,Y) : R + B^*XB \succeq 0\}$$

is a "domain of concavity" for F. The algorithm is discussed briefly in §6. For details and proof of the last assertion, see [CHSY03]. □

5.2. **Convexity vs LMIs.** Now we restrict from functions $r(a, x)$ in two types of variables to $r(x)$ of only one type. The following theorem characterizes symmetric noncommutative rational functions (in x) which are convex near the origin in terms of an LMI. The more general $r(a, x)$ has not been worked out.

Theorem 5.3. [HMV06] *Suppose $r = r(x)$ is a noncommutative symmetric rational function which is convex (in x) near the origin. Then*

(1) r has a representation

$$r(x) = r_0 + r_1(x) + \ell(x)\ell(x)^* + \Lambda(x)(I - L(x))^{-1}\Lambda(x)^*, \tag{5.2}$$

where

$$L(x), \quad \ell(x), \quad \Lambda(x), \quad r_0 + r_1(x)$$

are linear pencils in $x_1, \cdots, x_g$ satisfying

$$L(0) = 0, \quad \ell(0) = 0, \quad \Lambda(0) = 0, \quad r_1(0) = 0.$$

In addition L and r_1 are symmetric, for example, $L(x)$ has the form $L(x) = A_1x_1 + \cdots + A_gx_g$ for symmetric matrices A_j.

Thus for γ any real number $r - \gamma$ is a Schur complement of the noncommutative linear pencil

$$\mathcal{L}_\gamma(x) := \begin{pmatrix} -1 & 0 & \ell(x)^* \\ 0 & -(I - L(x)) & \Lambda(x)^* \\ \ell(x) & \Lambda(x) & r_0 - \gamma + r_1(x) \end{pmatrix}.$$

(2) The principal component of the domain of r is a convex set, indeed it is the positivity set of the pencil $I - L(x)$. Indeed this holds for any r of the form (5.2), subject to a minimality type condition on $\mathcal{L}_\gamma$.

This correspondence between properties of the pencil and properties of r yields

Corollary 5.4. *For any $\gamma \in \mathbb{R}$, the principal component, $\mathcal{G}_\gamma^0$, of the set of solutions X to the NCMI*

$$r(X) \prec \gamma I$$

equals the set of solutions to a NCLMI based on a certain linear pencil $\mathcal{L}_\gamma(x)$.

That is, **numerically solving matrix inequalities based on r is equivalent to numerically solving a NCLMI associated to r.**

5.3. **Proof of Corollary 5.4.** By item (2) of Theorem 4.2 the upper 2×2 block of $\mathcal{L}_\gamma(X)$ is negative definite if and only if $I - L(X) \succ 0$ if and only if X is in the component of 0 of the domain of r. Given that the upper 2×2 block of $\mathcal{L}_\gamma(X)$ is negative definite, by the LDL^* (Cholesky) factorization , $0 \succ \mathcal{L}_\gamma(X)$ is negative definite if and only if $\gamma I \succ r(X)$.■

6. Ideas behind some proofs and the convexity checker algorithm

The proofs of Theorem 3.2, the results from Section 4 on polynomials in two classes of variables, and many of the results on rational functions exposited in the previous section begin, just as in the case of everywhere convex polynomials, with the observation that matrix convexity of a noncommutative rational function on a *noncommutative convex domain* is equivalent to its noncommutative second directional derivative being matrix positive. This link between convexity and positivity remains in the noncommutative setting. While we will not define carefully the notion of a noncommutative convex domain, a special example is the $\epsilon > 0$ neighborhood of 0 which is the sequence of sets $(N_{\epsilon,n})_n$ where

$$N_{\epsilon,n} = \{X \in \mathbb{S}_n(\mathbb{R}^g) : \sum X_j^2 \preceq \epsilon^2 I_n\}.$$

The phrase, *for X near zero*, is shorthand for *in some noncommutative ϵ neighborhood of* 0.

Dealing with polynomials in variables (a, x) which are convex (on a domain) in the variable x only, requires noncommutative partial derivatives. The informal definition of the k^{th} *partial derivative* of a noncommutative rational function $r(x)$ with respect to x in the direction h is defined by

$$\frac{\partial^k}{\partial^k x} r(a, x)[h] = \frac{d^k}{dt^k} r(a, x + th)\Big|_{t=0} \tag{6.1}$$

When there are no a variables, we write, as one would expect, $r'(x)[h]$ and $r''(x)[h]$ instead of $\frac{\partial}{\partial x} r(x)$ and $\frac{\partial^k}{\partial^k x} r(x)$.

6.1. **The Middle Matrix.** There is a canonical representation of rational functions $q(b)[h]$ which are homogeneous of degree 2 in h as a matrix product,

$$q(b)[h] = V(b)[h]^* Z(b) V(b)[h]. \tag{6.2}$$

In the case that $r(a,x)$ is a polynomial of degree d and $q(a,x)[h] := \frac{\partial^k}{\partial^k x} r(a,x)[h]$, then (a,x) constitutes b and $V(a,x)[h]$ is a (column) vector whose entries are monomials of the form $h_j m(a,x)$ where $m(a,x)$ is a monomial in the variables (a,x) of degree at most $d-1$ (each such monomial appearing exactly once), where $Z(a,x)$ is a matrix whose entries are polynomials in (a,x). The matrix Z is unique, up to the order determined by the ordering of the $h_j m(a,x)$ in $V(a,x)[h]$. The matrix $Z(a,x)$ is the called **middle matrix** and $V(a,x)[h]$ is the **border** or **tautological vector**. The following basic noncommutative principle, which we state very informally, is key to many of the proofs many of the results presented in this survey.

Principle 6.1. A variety of very weak hypotheses on positivity of $q(a,x)[h]$ imply positivity of the middle matrix ([CHSY03] or [HMV06]).

Indeed, for a polynomial $p(a,x)$, the condition $\frac{\partial^k}{\partial^k x} p(A,X)[H] \succeq 0$ for X near 0 and all A and H is far more than needed to imply $Z(A,X) \succeq 0$ for X near 0.

A key ingredient of the principle is the CHSY-Lemma.

Lemma 6.2 (CHSY)**.** *Let ℓ be given and let $\nu = g\sum_0^\ell g^j$. There is a κ so that if $(X,v) \in \mathbb{S}_n(\mathbb{R}^g) \times \mathbb{R}^n$ and the set $\{m(X)v : m \in \mathbb{R}\langle x\rangle$ is a monomial of degree $\ell\}$ is linearly independent, then the codimension of $\{V(X)[H] : H \in \mathbb{S}_n(\mathbb{R}^g)\}$ in $\mathbb{R}^{n\nu}$ is at most κ (independent of n).*

Consider the perpetually reoccurring example, $p(x) = x^4$. The decomposition of equation $p''(x)[h]$ is given by

$$p''(x)[h] = 2\begin{pmatrix} h & xh & x^2h \end{pmatrix} \begin{pmatrix} x^2 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ hx \\ hx^2 \end{pmatrix}. \tag{6.3}$$

It is evident that the middle matrix for the polynomial $p(x) = x^4$ is not positive semi-definite (for any X) its Hessian is not positive semi-definite near 0 and hence, in view of Principle 6.1 p is not convex. This illustrates the idea behind the proof of Theorem 3.2 and the idea also applies to Theorem 4.1 in (a,x).

6.2. **Automated Convexity Checking.** The example above of $p(x) = x^4$ foreshadows the layout of our convexity checking algorithm.

Convexity Checker Algorithm for an noncommutative rational r:

(1) Compute symbolically the Hessian $q(a,x)[h] := \frac{\partial^2 r}{\partial x^2}(a,x)[h]$.
(2) Represent $q(a,x)[h]$ as $q(a,x)[h] = V(a,x)[h]^* Z(a,x) V(a,x)[h]$.
(3) Apply the noncommutative LDL decomposition to the matrix $Z(a,x) = LDL^*$. The diagonal matrix $D(a,x)$ has the form $D = diag\ \{\rho_1(a,x), \ldots, \rho_c(a,x)\}$.
(4) If $D(A,X) \succeq 0$ (that is, each $\rho_j(A,X) \succeq 0$), then the Hessian $q(A,X)[H]$ is positive semidefinite for all H. Thus a set $\mathcal{D}$ where r is matrix convex is given by

$$\mathcal{D} = \{(A,X) : \quad \rho_j(A,X) \succeq 0,\ j = 1, \ldots, c\}. \tag{6.4}$$

(In the example (6.3) $D(X)$ equals (3.2) which is not diagonal. In particular, convexity fails.)

The surprising and deep fact is that (under very weak hypotheses) the closure of $\mathcal{D}$ is the largest possible domain of convexity. See [CHSY03] for the proof. Extensions of the result and streamlined proofs can be found in [HMV06]. See also [KVV07]

It is hard to imagine a precise "convexity region algorithm" not based on noncommutative calculations, the problem being that matrices of practical size often have thousands of entries and so would lead to calculations with huge numbers of polynomials in thousands of variables.

6.3. **Proof of Theorem 5.3.** The proof consists of several stages. It is interesting that the technique for the first stage, which yields an initial representation for r as a Schur Complement of a linear pencil, is classical. In fact, the following representation of any symmetric noncommutative rational function r is the symmetric version of the one due originally to Kleene, Schützenberger, and Fliess (who were motivated by automata and formal languages, and bilinear systems; see [BR84] for a good survey), and further studied recently by Beck [Bec01], see also [BDG96, LZD96], and by Ball–Groenewald–Malakorn [BGM06a, BGM06b, BGM05].

Theorem 6.3. *If r is a noncommutative symmetric rational function which is analytic at the origin, then r has a symmetric minimal* **Descriptor, or Recognizable Series, Realization**. *Namely,*

$$r(x) = r_0 + C(J - \sum_{j=1}^{g} \mathcal{A}_j x_j)^{-1} C^*, \tag{6.5}$$

where and $\mathcal{A}_j \in \mathbb{R}^{n \times n}$ are symmetric and J a signature matrix; i.e., J is symmetric and $J^2 = I$.

Here minimality means that $CJ\mathcal{A}_{i_1} \cdots J\mathcal{A}_{i_k} v = 0$ for all words $i_1 \cdots i_k$ in the indices $1, \ldots, g$ implies $v = 0$.

Of course in general the above symmetric realization is not monic, i.e., $J \neq I$. The second stage of the proof uses the convexity of r near the origin, more precisely, the positivity of the noncommutative Hessian, to force J to be within rank one of I. For notational ease, let $L_{\mathcal{A}}(x) = \sum \mathcal{A}_j x_j$. The Hessian of $r(x)$ is then,

$$r''(x)[h] = 2\Gamma(x)^* L_{\mathcal{A}}[h](J - L_{\mathcal{A}}(x))^{-1} L_{\mathcal{A}}[h]\Gamma(x)$$

where $\Gamma(x) = (J - L_{\mathcal{A}}(x))^{-1}C^*$. The heuristic argument is that there is an $X \in \mathbb{S}_n(\mathbb{R}^g)$ (with n as large as necessary) close to 0 and a vector v so that $\Gamma(X)v$ has components $z_1, \dots, z_d \in \mathbb{R}^n$ which are independent. A minimality hypothesis on the descriptor realization allows for an argument similar to that of the CHSY-Lemma to prevail with the conclusion that $\{L_A[H]\Gamma(X)v : H \in \mathbb{S}_n(\mathbb{R}^g)\}$ has small codimension. Indeed, with n large enough, the restriction on this codimension implies that J can have at most one negative eigenvalue; i.e., is nearly positive definite. From here, algebraic manipulations give Theorem 5.3 item (1).

The third stage of the proof — establishing item (2) — was quite gruelling in [HMV06], but it is subsumed now under the following fairly general singularities theorem for various species of minimal noncommutative realizations.

Theorem 6.4 ([KVV07]). *Suppose*

$$r(x) = d(x) + C(x)(I - \sum_{j=1}^{g} A_j x_j)^{-1} B(x), \tag{6.6}$$

where $d(x)$ is a noncommutative polynomial, $A_j \in \mathbb{R}^{n\times n}$, and $B(x) = \sum B_{j_1\dots j_r} x_{j_1} \cdots x_{j_r}$ and $C(x) = \sum C_{j_1\dots j_l} x_{j_1} \cdots x_{j_l}$ are $n \times 1$ and $1 \times n$ matrix valued noncommutative polynomials, homogeneous of degrees r and l, respectively. Assume the "minimality type" conditions:

$$\operatorname*{span}_{k\geq 0;\, 1\leq i_1,\dots,i_k,\, j_1,\dots,j_r\leq g} \operatorname{ran} A_{i_1} \cdots A_{i_k} B_{j_1\dots j_r} = \mathbb{R}^n,$$

$$\bigcap_{k\geq 0;\, 1\leq i_1,\dots,i_k\, j_1,\dots,j_l\leq g} \ker C_{j_1\dots j_l} A_{i_1} \cdots A_{i_k} = 0.$$

Then

$$\mathcal{F}^0_{r,for} = \{(X_1, \dots, X_g) : \quad \det(I - A_1 \otimes X_1 - \cdots A_g \otimes X_g) \neq 0\}.$$

The proof is based on the formalism of noncommutative backward shifts and thus the theorem applies more generally to matrix-valued noncommutative rational functions.

7. Numerics and symbolics for matrix unknowns

In this section we discuss some ideas for combining symbolic and numerical computations to solve equations and optimization problems involving matrix unknowns. We focus on the big picture rather than on the details.

7.1. **Unconstrained Zero Finding.** The problem is, given a NC rational function $f(a, x)$ and A, find X such that $f(A, X) = 0$. A conceptual algorithm proceeds as follows.

Algorithm 7.1 (Newton-Rapson with line search)**.** Let X_0 and $\epsilon > 0$ be given and set $k = 0$.

(1) Compute $\frac{\partial}{\partial x} f(a, x)[h]$ symbolically.
(2) Find H_k satisfying the linear equation

$$f(A, X_k) + \frac{\partial}{\partial x} f(A, X_k)[H_k] = 0, \tag{7.1}$$

(3) Find $\alpha_k \in R$ such that $X_{k+1} = X_k + \alpha_k\, H_{xk}$ satisfies $\|f(A, X_{k+1})\| < \|f(A, X_k)\|$.
(4) Stop if $\|f(A, X_{k+1})\| \leq \epsilon$. Otherwise increment k and go to (1).

The step (3) is called a **line search** and is often performed on a nonnegative real valued function ϕ which has the key property that $\phi(A, X) = 0$, if $f(A, X) = 0$. In (3) we took $\phi(A, X) = \|f(A, X_{k+1})\|$. Typically a line search selects α_k to approximately achieve

$$\min_{\alpha \in R} \phi(A, X_k + \alpha H_k).$$

Under certain conditions the above algorithm will converge to X^* satisfying $f(A, X^*) = 0$. For example, once very near a minimizer of ϕ, this is rapidly convergent even with α set to 1. The analysis of these convergence conditions is standard (see [GM82], for instance) and will not be pursued any further in here. For very large problems (many unknown variables) there are two main show stoppers and they both relate to the **linear subproblem**, (7.1). The first (widely known) one is the numerical solution of (7.1). The second is the actual construction of the linear subproblem; this can consume large amounts of time and memory. *We feel this second issue is an excellent opportunity for the subject of computer algebra* and which is the emphasis of this section. When f is a NC rational function it is possible to take advantage of noncommutative algebra and organize the problem as a problem in g noncommuting variables rather than a problem in $g\, n(n+1)/2$ commuting variables. Indeed, the structure of problems like (7.1) is revealed in the next theorem. See [CHS06, dOH06b] for more details.

Theorem 7.2. *Let $f(a, x)$ be a NC rational function of (a, x). Equation* (7.1) *is a* Generalized Sylvester Equation*, that is, it can be represented in the form*

$$f(a, x_k) + \sum_i^{Syl} r_i(a, x_k)\, h_k\, s_i(a, x_k) + \sum_j^{SylT} t_j(a, x_k)\, h_k^*\, u_j(a, x_k) = 0 \tag{7.2}$$

where the coefficient r_i, s_i, t_j, and u_j are rational functions of a and x and the sums are both finite.

This representation is not unique as is well illustrated by the following examples. Also these examples illustrate noncommutative symbolic computation which we believe is essential to exploiting the special structure in constructing the linear subproblem. The relevant symbolic calculations are carried out using *NCAlgebra*. Here `a**b` stands for noncommutative multiplication, `tp[ ]` is an involution and we think of `tp[x]` as the "transpose" of `x`. The command `NCExpand` expands expressions while `NCCollect[x**h+b**h, h]` collects on `h` to produce `(x+b)**h`, and `DirectionalD[f,x,h]` takes the noncommutative directional derivative of f wrt. x in direction h.

Example 7.3. (1) For the quadratic function $f((a,b,c),x) = a\,x + x\,a^* + x\,b\,x + c$ the left hand side of (7.1) is

$$f((a,b,c),x) + a\,h + h\,a^* + x\,b\,h + h\,b\,x \tag{7.3}$$

and is computed in NCAlgebra as:

```
In[6]:= f[x_] = a ** x + x ** tp[a] + x ** b ** x + c;
In[7]:= Sylvester1 = f[x] + DirectionalD[f[x], x, h];
Out[7]= f[x] +  a ** h + h ** tp[a] + h ** b ** x + x ** b ** h
```

The coefficients of (7.2) are

$$r_1 = a, \qquad s_1 = I \qquad r_2 = x\,b, \qquad s_2 = I,$$
$$r_3 = s_1^*, \qquad s_3 = r_1^*, \qquad r_4 = s_2^*, \qquad s_4 = r_2^*.$$

(2) A different representation of the type (7.2) for (7.3) is obtain by collecting on h. In NCAlgebra:

```
In[10]:= Sylvester2 = NCCollect[Sylvester1, h];
Out[11]= f[x] + h ** (b ** x + tp[a]) + (a + x ** b) ** h
```

Now we have different coefficients

$$r_1 = a + x\,b, \qquad s_1 = I \qquad r_2 = s_1^*, \qquad s_2 = r_1^*.$$

(3) The rational function $f((a,c),x) = a\,x\,a^* - x + c + x(I-x)^{-1}x$ produces a representation (7.2) which has coefficients

$$r_1 = s_1^* = a, \qquad r_2 = -I, \qquad s_2 = I - (I-x)^{-1}x,$$
$$r_3 = x(I-x)^{-1}, \qquad s_3 = I + (I-x)^{-1}x.$$

as produced by *NCAlgebra* `NCExpand/NCCollect`:

```
In[13]:= f[x_] = a ** x ** tp[a] - x + c + x ** inv[1 - x] ** x;
In[14]:= Sylvester3 =
         f[x] + NCCollect[NCExpand[DirectionalD[f[x], x, h]], h];
Out[14]=  f[x] - h ** (1 - inv[1 - x] ** x) + a ** h ** tp[a] +
                x ** inv[1 - x] ** h ** (1 + inv[1 - x] ** x)
```

(4) The same rational function also produces a representation (7.2) with the following coefficients.

$$r_1 = s_1^* = a, \qquad r_2 = -s_2 = -\sqrt{2}I, \qquad r_3 = s_3^* = I + x(I-x)^{-1}.$$

after some manipulation.

□

We define *syl* and *sylT* to be *the Sylvester indices* of equation (7.2). That is the number of terms in each of the summations indicated in equation (7.2). For instance, in example (1) $syl = 4$ and $sylT = 0$. Note that the representation (7.2) is not unique (e.g. examples (1-2) and (3-4) above) so each representation may have its own pair of Sylvester indices (e.g. $syl = 4$ in example (1) and $syl = 2$ in example (2)). It turns out that finding a representation with small, or smallest, Sylvester indices is important for numerics. Note also that the coefficients of the Sylvester equation in example (4) are symmetric while the ones in example (3) are not, which may be of importance.

7.2. **Optimization with Matrix Inequality Constraints.** What often occurs in engineering is an optimization problem subject to matrix inequality constraints. One proceeds by writing down first order Karush-Kuhn-Tucker (KKT) type optimality conditions which are a set of equations of the form $F(a,x) = 0$ but with matrix positivity side conditions, which are then solved numerically often using a barrier type method. We shall illustrate our NC symbolic approach with the following broad and important class of problems:

$$\min_X \{\mathrm{Trace}(C^* X) : \quad f(A,X) \preceq 0\} \tag{7.4}$$

in which $f(a,x)$ is a symmetric noncommutative function. A linear cost function is assumed without loss of generality[2]. Some examples are as follows:

(1) Riccati inequality: maximize Trace(X) such that $A^*X + XA - X\,B\,X + C \succeq 0$ and $X \succeq 0$.

[2]If the cost is not linear one may add a variable so that $\min_X g(X) = \min_{X,\mu}\{\mu : \mu \geq g(X)\}$. The former problem has a linear cost function.

(2) Problem (1) is not in the form (7.4) but can be easily reformulated as

$$\min_X \{-\operatorname{Trace}(X) : \quad \{X\,B\,X - A^*X - XA - C, -X\} \preceq 0\}$$

which is in the form (7.4) for $C = -I$ and

$$f(a, b, c, x) = \{x\,b\,x - a^*x - xa - c,\ -x\}.$$

(3) Static output feedback stabilization: minimize $\operatorname{Trace}(X_1)$ such that

$$(A + BX_2D)^*X_1 + X_1\,(A + BX_2D) + C \preceq 0 \quad \textit{and} \quad X_1 \succeq 0.$$

This problem is in the form (7.4) for $C = \{I, 0\}$ and

$$f(a, b, c, x) = \{(a + bx_2d)^*x_1 + x_1\,(a + bx_2d) + c,\ -x_1\}.$$

Many other examples of problems of the class (7.4) can be found in [BEGFB94, SIG98]. Symbolic NC algorithms are discussed in [dOH06a] that can manipulate and produce instances of problems in the form (7.4) in systems and control engineering.

In the above examples braces are used to represent a *vector* of NC functions. Inequalities should be taken as applied to each entry of the vector, e.g.

$$\{x_1, x_2\} \preceq 0 \quad \Leftrightarrow \quad \{x_1 \preceq 0, x_2 \preceq 0\}$$

We do this also with certain specific NC functions, namely inverses and ln det, where we define

$$\{x_1, \cdots, x_g\}^{-1} := \{x_1^{-1}, \cdots, x_2^{-1}\} \quad \text{and} \quad \ln\det\{x_1, \cdots, x_2\} := \ln\det x_1 + \cdots + \ln\det x_2.$$

Note transposes * convert columns of symbols to rows of the transposed symbols.

Now we sketch a popular class of methods for computing a solution to optimization problems of the form (7.4) called interior-point methods. (They follow the outline at the beginning of this subsection.) In order to relate one of many variants of such methods to Algorithm 7.1, let us first write problem (7.4) in the equivalent form

$$\min_{X,Y} \{\operatorname{Trace}(C^*X) : \quad f(A, X) + Y = 0, \quad Y \succeq 0\}. \tag{7.5}$$

after introduction of the "slack variable" Y. Consider now the incorporation of a "barrier function" associated with Y into the objective function

$$\phi(X, Y) := \operatorname{Trace}(C^*X) - \mu \log\det Y$$

used to produce the auxiliary problem

$$\min_{X,Y} \{\phi(X, Y) : \quad f(A, X) + Y = 0\}. \tag{7.6}$$

In a nutshell, the idea behind the $-\mu \log\det Y$ barrier function is that this blows up as $Y \succ 0$ gets closer and closer to having a zero eigenvalue. Thus when $\mu > 0$ any iterative algorithm initialized with a positive definite value for the slack variable, i.e. $Y \succ 0$, will keep Y positive definite as long as the objective function is minimized (in a numerical implementation this may require taking short steps).

Now introduce a self-adjoint "Lagrange multiplier" z (i.e. $z = z^*$) and define the NC function

$$g(a,x,z) = \nabla_x \operatorname{trace}(zf(a,x)) = \nabla_{h_x} \operatorname{trace}\left(z\frac{\partial}{\partial x}f(a,x)[h_x]\right). \tag{7.7}$$

Note that in order to manipulate the above expression symbolical we need implement a symbolic operator 'trace', which obeys the familiar linear and cyclic properties present in the Trace functional on square matrices. The KKT optimality conditions for problem (7.6) are

$$c + g(a,x,z) = 0, \qquad z - \mu y^{-1} = 0, \qquad f(a,x) + y = 0. \tag{7.8}$$

For fixed A, C, μ these equations are of the form $F(X,Y,Z) = 0$ in unknowns X, Y, Z and can be solved for X, Y, Z by Algorithm 7.1, the trouble being keeping Y negative semidefinite. However, doing Algorithm 7.1 with a on α to minimize ϕ handles this [3]. Note $Y \succeq 0$ automatically implies $Z \succeq 0$.

Now we turn to the parameter μ which was inserted into the problem. A solution to the original problem (7.4) is found by approximately solving problems of the form (7.8) for a sequence of decreasing positive values of μ. Note that as $\mu \to 0$ the equations (7.8) with $Y \succeq 0$ coincide with the KKT conditions of the modified problem (7.5).

7.3. **The Linear Subproblem and Symbolic Computation.** At this stage in applying Algorithm 7.1 the burning issue is to write down a formula for the linearization of the optimality equations (7.8). It is straightforward to see that the linearization of (7.8), for a given $\mu > 0$, $y_k \succ 0$ and $z_k \succ 0$, are the Generalized Sylvester Equations

$$\begin{aligned} c + g(a,x_k,z_k) + \frac{\partial}{\partial x}g(a,x_k,z_k)[h_x] + g(a,x_k,h_z) &= 0, \\ z_k - \mu y_k^{-1} + h_z + \mu y_k^{-1} h_y y_k^{-1} &= 0, \\ f(a,x_k) + y_k + \frac{\partial}{\partial x}f(a,x_k)[h_x] + h_y &= 0, \end{aligned} \tag{7.9}$$

where we have used the fact that g is linear in z.

[3]For fairness in advertising we reveal that what is often done in practice is a line search on a more complicated function. For instance, in [VS99] the following function is used

$$\phi(X,Y) = \operatorname{Trace}(C^*X) - \mu \log\det Y + \beta \,\|f(A,X) + Y\|^2$$

where there are two adjustable parameters μ and β. The additional penalty term present in ϕ is used with a sufficiently large $\beta > 0$ so as to reenforce the equality constraint in (7.5).

In particular problems this must be computed concretely. Our goal in this subsection is to convince the reader that one can carry this out and should symbolically while keeping the matrix variables intact; in other words there is a big advantage to keeping the computations dimension free. Of course one can always take the dimension dependent approach, disaggregating the variables once the size of the matrices are specified. In this case each entry of each matrix is viewed as a variable. If the matrices are, say 300×300, then the computations will involve about 10^5 variables on which symbolic computation is a joke.

While we have algorithms [CHS06, dOH06b] and have implementations on classes of problems including broad classes of functions f, we shall confine our illustrations to the Riccati inequality optimization

$$f(a,b,c,x) = \{x\,b\,x - a^*x - xa - c, -x\}$$

defined previously as an example.

The first aspect is the NCAlgebra computation of the function $g(a,x,y)$ of equation (7.7), and one easily gets using `DirectionalD` and a command which invokes the cyclic property of *trace*.:

$$\begin{aligned} g(a,b,c,x,z) &= \nabla_{h_x} \operatorname{trace}\left(z_1\left(h_x\,b\,x + x\,b\,h_x - a^*h_x - h_x a\right) - z_2 h_x\right) \\ &= (b\,x - a)z_1 + z_1(x\,b - a^*) - z_2. \end{aligned}$$

Once g has been computed symbolically, it is straightforward to plug f and g into (7.9) and get concrete formulas. Here, in the Riccati inequality example this gives:

$$\begin{aligned} -I + (b\,x_k - a)z_{1_k} + z_{1_k}(x_k\,b - a^*) - z_{2k} \\ +b\,h_x z_{1_k} + z_{1_k} h_x\,b + (b\,x_k - a)h_{z_1} + h_{z_1}(x_k\,b - a^*) - h_{z_2} &= 0, \\ z_{1_k} - \mu y_{1_k}^{-1} + h_{z_1} + \mu y_{1_k}^{-1} h_{y_1} y_{1_k}^{-1} &= 0, \\ z_{2_k} - \mu y_{2_k}^{-1} + h_{z_2} + \mu y_{2_k}^{-1} h_{y_2} y_{2_k}^{-1} &= 0, \\ x_k\,b\,x_k - a^*x_k - x_k a - c + y_{1_k} + (x\,b - a^*)h_x + h_x(b\,x - a) + h_{y_1} &= 0, \\ -x_k + y_{2_k} - h_x + h_{y_2} &= 0. \end{aligned}$$

which can all be computed in NCAlgebra using little more than `DirectionalD`. For example the third equation is gotten from

```
In[36]:= f2[y_,z_] := z - mu inv[y];
In[37]:= Sylvester22 =  f2[y2,z2] + DirectionalD[f2[y2,z2], y2, hy2]
                                  + DirectionalD[f2[y2,z2], z2, hz2];
Out[37] = f2[y2,z2] + hz2 + mu inv[y2] ** hy2 ** inv[y2]
```

These Sylvester equations are often "reduced" by solving for some of the unknowns *apriori*. For instance, solving for h_y and h_z

$$
\begin{aligned}
h_y &= -f(a, x_k) - y_k - \frac{\partial}{\partial x} f(a, x_k)[h_x], \\
h_z &= \mu y_k^{-1} - z_k - \mu y_k^{-1} h_y y_k^{-1} = \mu y_k^{-1} \left[2y_k + f(a, x_k) \right] y_k^{-1} - z_k + \mu y_k^{-1} \frac{\partial}{\partial x} f(a, x_k)[h_x] y_k^{-1}
\end{aligned}
$$

as a function of h_y we obtain, for some properly defined $q(a, x_k, y_k, z_k)$, the reduced Generalized Sylvester Equation

$$
q(a, x_k, y_k, z_k) + \frac{\partial}{\partial x} g(a, x_k, z_k)[h_x] + \mu\, g\left(a, y_k^{-1} \frac{\partial}{\partial x} f(a, x_k)[h_x] y_k^{-1}\right) = 0, \tag{7.10}
$$

whose only unknown is h_x. Again note that (7.10) can be computed automatically with NCAlgebra mainly because solving symbolically for the $h's$ is straightforward. Jumping a bit ahead we mention that, for this example, a version of Algorithm 7.1 can be constructed in which step (3), the line search, is not needed.

7.4. **Numerical Linear Solvers.** The symbolics of the preceding subsection are run at the beginning of an optimization computation and the formulas one gets are stored effectively symbolically (they are extremely short compared to prevailing methods where the matrices have been disaggregated). Next comes the numerical iteration and at each step the coefficients in our Sylvester linear problem

$$
S(H) = Q
$$

e.g. (7.1), (7.9) and (7.10), are matrices, say $n \times n$. So far we are exploiting the matrix structure of our original problem.

Now comes the challenge which to a large extent is open. *Find efficient numerical Sylvester linear solvers.* One can ignore the special Sylvester structure of these linear equations, that is, one can "vectorizes" the computation, then one has "unstructured" linear equations in n^2 variables, so for $n = 300$ even storing and accessing the matrix scales like 300^4 which is outlandish. (Our problems are certainly not sparse.) On the other hand, if we keep the Sylvester structure, then storing iterates $H_{k+1} := S(H_k)$ scales like n^2 and computing these iterates scales like $2(syl + sylT)n^3$, which is cheap. This disposes one to linear solvers based on such iteration, for example, Conjugate-Gradient type algorithms. What matters is getting good accuracy with not too many iterations.

We have implemented versions of the Conjugate-Gradient algorithm for the classes of problems described here. They are still being tested and tried with various "preconditioners". Some examples run (on one a gigahertz PC with 1 gigabyte of RAM) on problems of remarkable size, eg. 300×300, but some do not. A likely divide is how well conditioned the original problem is, in which case preconditioners can play a big role.

When f is not convex, then various difficulties of classical type emerge that must be combated. For example, one may have to modify the linear subproblem so that it become positive definite, which is traditionally done by adding a multiple of the identity to the coefficient matrix (see for instance [VS99, BHN99]). The Generalized Sylvester Equation (7.10) structure survives most such modifications.

7.4.1. *Matrix convexity of f and the linear subproblem.* Seriously effecting the numerics is whether or not the purely linear part of (7.10) in h_x is given by a positive semidefinite operator. We check both terms of this linear part. For the first term use the definition of the function g in (7.7) and set $z = y_k^{-1} \frac{\partial}{\partial x} f(a, x_k)[h_x] y_k^{-1}$ to get

$$\text{trace}\left(h_x^* \, g\,(a, x, z)\right) = \text{trace}\left(y_k^{-1} \frac{\partial}{\partial x} f(a, x_k)[h_x] y_k^{-1} \frac{\partial}{\partial x} f(a, x_k)[h_x]\right). \tag{7.11}$$

If all variables are substituted with matrices with $y_k \to Y_k$ a positive definite matrix, then the right side is clearly positive for all A, X_k, H_k. For the second term note that

$$\text{trace}\left(h_x^* \frac{\partial}{\partial x} g(a, x_k, z_k)[h_x]\right) = \frac{\partial}{\partial x} \text{trace}\left(h_x^* \, g(a, x_k, z_k)\right)[h_x] = \text{trace}\left(z_k \frac{\partial^2}{\partial^2 x} f(a, x_k)[h_x]\right).$$

Therefore, from the discussion in Section 3, if f is NC convex, then

$$\frac{\partial^2}{\partial^2 x} f(A, X_k)[H_x] \succeq 0$$

whenever there are matrix substitutions $\{a, x_k\} \to \{A, X_k\}$. Thus

$$\text{Trace}\left(Z_k \frac{\partial^2}{\partial^2 x} f(A, X_k)[H_x]\right) \geq 0 \quad \text{for all } H_x, Z_k \succeq 0.$$

The conclusion is that when the NC function $f(a, x)$ is convex the purely linear part of the Generalized Sylvester Equation (7.10) is positive semidefinite.

As an aside note that the notion of positivity, expressed in the above inequalities and earlier in this section and in Section 3, can be formalized at the level of symbolics. We do not describe this here.

8. Answers to the free sample

Here is an answer to the standard problem of H^∞ control which was stated in §4.1. Recall the key question: *Is there is a set of noncommutative convex inequalities with an equivalent set of solutions?*

This is a question in algebra and the answer after a lot of work is yes. The path to success is:

(1) *Firstly, one must eliminate unknowns and change variables to get a new set of inequalities $\mathcal{K}$.*
(2) *Secondly, one must check that $\mathcal{K}$ is "convex" in the unknowns.*

This outline transcends our example and applies to very many situations. Issue (2) is becoming reasonably understood, for as we saw earlier, a convex polynomial with real coefficients has degree two or less, so these are trivial to identify. Issue (2), changing variables, is still a collection of isolated tricks with which mathematical theory has not caught up. For the particular problem in our example we shall not derive the solution since it is long. However, we do state the classical answer in the next subsection.

8.0.2. *Solution to the Problem.* The textbook solution is as follows, due to Doyle-Glover-Kargonekar-Francis. It appeared in [DGKF89] which won the 1991 annual prize for the best paper to appear in an IEEE journal. Roughly speaking it was deemed the best paper in electrical engineering in that year.

We denote

$$DGKF_X := (A - B_2C_1)'X + X(A - B_2C_1) + X(\gamma^{-2}B_1B_1' - B_2^{-1}B_2')X,$$
$$DGKF_Y := A^{\times}Y + YA^{\times\prime} + Y(\gamma^{-2}C_1'C_1 - C_2'C_2)Y,$$

where $A^{\times} := A - B_1C_2$.

Theorem 8.1. [DGKF89] *There is a system solving the control problem if there exist solutions*

$$X \succeq 0 \qquad \text{and} \qquad Y \succ 0$$

to inequalities the

$$DGKF_Y \preceq 0 \;\; \text{and} \;\; DGKF_X \preceq 0$$

which satisfy the coupling condition

$$X - Y^{-1} \prec 0.$$

This is if and only if provided $Y \succ 0$ is replaced by $Y \succeq 0$ and Y^{-1} is interpreted correctly.

This set of inequalities while not usually convex in X, Y are convex in the new variables $W = X^{-1}$ and $Z = Y^{-1}$, since $DGKF_X$ and $DGKF_Y$ are linear in W and Z and $X - Y^{-1} = W^{-1} - Z$ has second derivative $2W^{-1}HW^{-1}HW^{-1}$ which is non negative in H for each $W^{-1} = X \succ 0$. These inequalities are also equivalent to LMIs which we do not write down.

9. Classical RAG extended to free-* algebras

At this point one might think of the emerging area of free $*$- semi-algebraic geometry as having two main paths. One is an analog of classical commutative semi-algebraic geometry and focuses on general polynomial inequalities generally and Positivstellensätze - algebraic identities involving sums of squares - in particular. This noncommutative semi-algebraic geometry is the focus of this section where we shall sketch some basic ideas behind the emerging theory of inequalities involving polynomials on a free $*$-algebra. As we shall see, for strict inequalities the theory gives satisfying theorems, while when vanishing occurs (as in the real Negativstellensatz) the results are less definitive.

The second area of free $*$- semi-algebraic geometry has little analog classically and focuses on noncommutative functions with positive second derivatives or more generally varieties (not in this paper) whose curvature meets inequality constraints. Such convexity issues have been the main topic of this paper so far. The proofs can be done with the "middle matrix" representation in §6, which is a type of special Positivstellensatz for quadratics, thereby avoiding the much more generally applicable Positivstellensätze discussed in this section. However, the theory of noncommutative semi-algebraic geometry exposited in this section is expanding rapidly and may someday find applications.

9.1. **Sums of squares in a free $*$-algebra.** Let $\mathbb{R}\langle x, x^*\rangle$ denote the algebra of polynomials with real coefficients, in the free variables $x_1, ..., x_g, x_1^*, ..., x_g^*$. These variables do not commute, but they are associated with an involution:

$$(fq)^* = q^* f^*, \qquad (x_j)^* = x_j^*.$$

Thus, we are now breaking with the convention of the rest of this survey in that the variables x_j are now **not** symmetric; i.e., $x^* \neq x$. We will call $\mathbb{R}\langle x, x^*\rangle$ the *real free* $*-$ *algebra* on generators x, x^*. Let Σ^2 denote the cone of sums of squares:

$$\Sigma^2 = \mathrm{co}\{f^* f;\ f \in \mathbb{R}\langle x, x^*\rangle\},$$

where "co" denotes convex hull.

9.2. **A basic technique.** Call a linear functional $L \in \mathbb{R}\langle x, x^*\rangle'$ *symmetric* provided that $L(f) = L(f^*)$ for all $f \in \mathbb{R}\langle x, x^*\rangle$. A symmetric linear functional $L \in \mathbb{R}\langle x, x^*\rangle'$ satisfying $L|_{\Sigma^2} \geq 0$ produces a positive semi-definite bilinear form

$$\langle f, q\rangle = L(q^* f)$$

on $\mathbb{R}\langle x, x^*\rangle$. A standard use of Cauchy-Schwarz inequality shows that the set of null-vectors

$$N = \{f \in \mathbb{R}\langle x, x^*\rangle;\ \langle f, f\rangle = 0\}$$

is a vector subspace of $\mathbb{R}\langle x, x^*\rangle$. Whence one can endow the quotient $\mathcal{D} = \mathbb{R}\langle x, x^*\rangle/N$ with a positive definite Hermitian form, and pass to the Hilbert space completion H, with $\mathcal{D}$ the dense subspace of H generated by $\mathbb{R}\langle x, x^*\rangle$. The separable Hilbert space H carries the multiplication operators $M_{x_j} : \mathcal{D} \longrightarrow \mathcal{D}$:

$$M_{x_j} f = x_j f, \quad f \in \mathcal{D},\ 1 \le j \le n.$$

One verifies from the definition that each M_{x_j} is well defined and

$$\langle M_{x_j} f, q\rangle = \langle x_j f, q\rangle = \langle f, x_j^* q\rangle, \quad f, q \in \mathcal{D}.$$

Thus $M_{x_j}^* = M_{x_j^*}$. The vector 1 is still cyclic, in the sense that the linear span $\vee_{p\in\mathbb{R}\langle x,x^*\rangle} p(M, M^*)1$ is dense in H. The above is known in the operator theory community as the *Gelfand-Naimark-Segal construction.*

Lemma 9.1. *There exists a bijective correspondence between symmetric positive linear functionals, namely*

$$L \in \mathbb{R}\langle x, x^*\rangle' \quad \text{and} \quad L|_{\Sigma^2} \ge 0,$$

and g-tuples of unbounded linear operators T with a cyclic vector ξ, established by the formula

$$L(f) = \langle f(T, T^*)\xi, \xi\rangle, \quad f \in \mathbb{R}\langle x, x^*\rangle.$$

We stress that the above operators do not commute, and might be unbounded. The calculus $f(T, T^*)$ is the noncommutative functional calculus: $x_j(T) = T_j,\ x_j^*(T) = T_j^*$.

An important feature of the above correspondence is that it can be restricted by the degree filtration. Specifically, let $\mathbb{R}\langle x, x^*\rangle_k = \{f;\ \deg f \le k\}$, and similarly, for a quadratic form L as in the lemma, let $\mathcal{D}_k$ denote the finite dimensional subspace of H generated by the polynomials of $\mathbb{R}\langle x, x^*\rangle_k$. Define also

$$\Sigma_k^2 = \Sigma^2 \cap \mathbb{R}\langle x, x^*\rangle_k.$$

Start with a symmetric functional $L \in \mathbb{R}\langle x, x^*\rangle_{2k}'$ satisfying $L|_{\Sigma_{2k}^2} \ge 0$. One can still construct a finite dimensional Hilbert space H, as the completion of $\mathbb{R}\langle x, x^*\rangle_k$ with respect to the inner product $\langle f, q\rangle = L(q^* f),\ f, q \in \mathbb{R}\langle x, x^*\rangle_k$. The multipliers

$$M_{x_j} : \mathcal{D}_{k-1} \longrightarrow H, \qquad M_{x_j} f = x_j f,$$

are well defined and can be extended by zero (on the orthogonal complement of $\mathcal{D}_{k-1}$) to the whole H. Let

$$N(k) = \dim \mathbb{R}\langle x, x^*\rangle_k = 1 + (2g) + (2g)^2 + ... + (2g)^k = \frac{(2g)^{k+1} - 1}{2g - 1}.$$

In short, we have proved the following specialization of the main Lemma.

Lemma 9.2. *Let symmetric functional $L \in \mathbb{R}\langle x, x^*\rangle_{2k}'$ satisfy $L|_{\Sigma_{2k}^2} \geq 0$. There exists a Hilbert space of dimension less than or equal to $N(k)$ and an g-tuple of linear operators M on H, with a distinguished vector $\xi \in H$, such that*

$$L(p) = \langle p(M, M^*)\xi, \xi\rangle, \quad p \in \mathbb{R}\langle x, x^*\rangle_{2k-2}. \tag{9.1}$$

Note that we do not exclude in the above lemma $L = 0$, in which case one can take $\xi = 0$.

9.3. **Positivstellensätze.** This subsection gives an indication of various free *-algebra analogs to the classical theorems characterizing polynomial inequalities in a purely algebraic way. We will start with an easily stated and fundamental Nichtnegativstellensatz.

Theorem 9.3 ([Hel02]). *Let $p \in \mathbb{R}\langle x, x^*\rangle_d$ be a noncommutative polynomial. If $p(M, M^*) \succeq 0$ for all g-tuples of linear operators M acting on a Hilbert space of dimension at most $N(k)$, $2k \geq d + 2$, then $p \in \Sigma^2$.*

Proof. Note that a polynomial p satisfying the hypothesis automatically satisfies $p = p^*$. The only necessary technical result we need is the closedness of the cone Σ_k^2 in the Euclidean topology of the finite dimensional space $\mathbb{R}\langle x, x^*\rangle_k$. This is done as in the commutative case, using Carathédodory's convex hull theorem, more exactly, every polynomial of Σ_k^2 is a convex combination of at most $\dim \mathbb{R}\langle x, x^*\rangle_k + 1$ polynomials. On the other hand the positive functionals on Σ_k^2 separate the points of $\mathbb{R}\langle x, x^*\rangle_k$. See for details [HMP04].

Assume that $p \notin \Sigma^2$ and let $k \geq (d + 2)/2$, so that $p \in \mathbb{R}\langle x, x^*\rangle_{2k-2}$. Once we know that Σ_{2k}^2 is a closed cone, we can invoke Minkowski separation theorem and find a symmetric functional $L \in \mathbb{R}\langle x, x^*\rangle_{2k}'$ providing the strict separation:

$$L(p) < 0 \leq L(f), \quad f \in \Sigma_{2k}^2.$$

According to Lemma 9.2 there exists a tuple M of operators acting on a Hilbert space H of dimension $N(k)$ and a vector $\xi \in H$, such that

$$0 \leq \langle p(M, M^*)\xi, \xi\rangle = L(p) < 0,$$

a contradiction. ∎

When compared to the commutative framework, this theorem is stronger in the sense that it does not assume a strict positivity of p on a well chosen "spectrum". Variants with supports (for instance for spherical tuples $M: \ M_1^* M_1 + ... + M_g^* M_g \succeq I$) of the above result are discussed in [HMP04].

To draw a very general conclusion from the above computations: when dealing with positivity in a free-* algebra, the standard point evaluations (or more precisely prime spectrum evaluations) of the commutative case are replaced by matrix evaluations of the free variables. The positivity can be tailored to "evaluations in a supporting set". The results pertaining to the resulting algebraic decompositions are called Positivstellensätze, see again [PD01] for details in the commutative setting. We state below an illustrative and generic result, from [HM04], for sums of squares decompositions in a free $*$-algebra.

Theorem 9.4. *Let $p = p^* \in \mathbb{R}\langle x, x^* \rangle$ and let $q = \{q_1, ..., q_k\} \subset \mathbb{R}\langle x, x^* \rangle$ be a set of symmetric polynomials, so that*

$$QM(q) = \mathrm{co}\{f^* q_k f; \ f \in \mathbb{R}\langle x, x^* \rangle, \ 0 \leq i \leq k\}, \ q_0 = 1,$$

contains $1 - x_1^* x_1 - ... - x_g^* x_g$. If for all tuples of linear bounded Hilbert space operators $X = (X_1, ..., X_g)$ subject to the conditions

$$q_i(X, X^*) \succeq 0, \ 1 \leq i \leq k, \tag{9.2}$$

we have

$$p(X, X^*) \succ 0,$$

then $p \in QM(q)$.

Henceforth, call $QM(q)$ the **quadratic module** generated by the set of polynomials q.

Some interpretation is needed in degenerate cases, such as those where no bounded operators satisfy the relations $q_i(X, X^*) \succeq 0$. Suppose for example, if ϕ_i denotes the defining relations for the Weyl algebra and the q_i include $-\phi^* \phi$. In this case, we would say $p(X, X^*) \succ 0$, since there are no X satisfying $q(X, X^*)$, and voila $p \in QM(q)$ as the theorem says.

Proof Assume that p does not belong to the convex cone $QM(q)$. Since the latter contains the constants in its algebraic interior, by Minkovski's separation principle there exists a symmetric linear functional $L \in \mathbb{R}\langle x, x^* \rangle'$, such that

$$L(p) \leq 0 \leq L(f), \quad f \in QM(q).$$

Define the Hilbert space H associated to L, and remark that the left multipliers M_{x_i} on $\mathbb{R}\langle x, x^*\rangle$ give rise to linear bounded operators (denoted by the same symbols) on H. Then

$$q_i(M, M^*) \succeq 0, \quad 1 \leq i \leq k,$$

by construction, and

$$0 < \langle p(M, M^*)1, 1\rangle = L(p) \leq 0,$$

a contradiction. See for full definitions and more details [HM04] or the survey [HP07]. ■

A paradigm practical question with matrix inequalities is:

Given a NC symmetric polynomial $p(a, x)$ and a $n \times n$ matrix tuple A, find $X \geq 0$ if possible which makes $p(A, X) \succeq 0$.

This fails in a region defined by a noncommutative symmetric polynomial $q(a, x)$ for a given matrix tuple A, means that $p(A, X) \not\succeq 0$ for any X satisfying $q(A, X) \succeq 0$. There is a great thrust of research aimed at numerical solution of such problems (see §7), but it is not clear how Posivstellensätze can aid with solving this particular problem. The next theorem informs us that the main problem here is the matrix coefficients A, in that the theorem gives a "certificate of infeasibility" for the problem when the coefficients are real numbers rather than polynomials.

Theorem 9.5 (The Klep-Schweighofer Nirgendsnegativsemidefinitheitsstellensatz [KS07]). *Let $p = p^* \in \mathbb{R}\langle x, x^*\rangle$ and let $q = \{q_1, ..., q_k\} \subset \mathbb{R}\langle x, x^*\rangle$ be a set of symmetric polynomials, so that $QM(q)$ contains $1 - x_1^*x_1 - ... - x_g^*x_g$. If for all tuples of linear bounded Hilbert space operators $X = (X_1, ..., X_g)$ subject to the conditions*

$$q_i(X, X^*) \succeq 0, \; 1 \leq i \leq k,$$

we have

$$p(X, X^*) \not\succeq 0,$$

then there exists an integer r and $h_1, \ldots, h_r \in \mathbb{R}\langle x, x^*\rangle$ with $\sum_{i=1}^r h_i^* f h_i \in 1 + QM(q)$.

Proof The thread of the argument again uses a separating linear functional and the GNS construction, but adorned with clever constructions. ■

9.4. **Quotient Algebras.** The results from Section 9.3 allow a variety of specializations to quotient algebras. In this subsection we consider a two sided ideal $\mathcal{I}$ of $\mathbb{R}\langle x, x^*\rangle$ which need not be invariant under $*$. Then one can replace the quadratic module QM in the statement of the Positivstellensätze with $QM(q) + \mathcal{I}$, and apply similar arguments as above. For instance, the next simple observation can be deduced.

Corollary 9.6. *Assume, in the hypotheses of Theorem 9.4, that the relations* (9.2) *include some relations of the form $r(X, X^*) = 0$, even with r not symmetric, then*

$$p \in QM(q) + \mathcal{I}_r \tag{9.3}$$

where $\mathcal{I}_r$ denotes the two sided ideal generated by r.

Proof This follows immediately from $p \in QM(q,\ -r^*r)$ which is a consequence of Theorem 9.4 and the fact

$$QM(q,\ -r^*r) \subset QM(q) + \mathcal{I}_r.$$

■

For instance, we can look at the situation where $r(x) :=\ [x_i, x_j]$ as insisting on positivity of $q(X)$ only on commuting tuples of operators, in which case the ideal $\mathcal{I}$ generated by $[x_j^*, x_i^*],\ [x_i, x_j]$ is added to $QM(q)$. The classical commuting case is captured by the corollary applied to the "commutator ideal": $\mathcal{I}_{[x_j^*,x_i^*],\ [x_i,x_j],\ [x_i,x_j^*]}$ for $i, j =\ 1, \cdots, g$ which requires testing only on commuting tuples of operators drawn from a commuting C^* algebra. The classical Spectral Theorem, then converts this to testing only on $\mathbb{R}^g$, cf [HP07].

In a very different vein of proof is the theorem due to Igor Klep (private communication) below. A special case, $z = [x_1 + x_1^*, x_2 + x_2^*]$, was stated without proof in [HP07].

Theorem 9.7 (I. Klep). *Let $\mathcal{I}$ be the two sided ideal of $\mathbb{R}\langle x, x^*\rangle$ generated by $z - 1$ with $z^* = -z$. Then $\mathcal{I} + \Sigma^2 = \mathbb{R}\langle x, x^*\rangle$.*

Proof First of all, $-z^2 = z^*z \in \Sigma^2$, so $-1 = z(z-1) - z^2 \in \mathcal{I} + \Sigma^2$. For a symmetric polynomial $s \in \mathbb{R}\langle x, x^*\rangle$,

$$s = \left(\frac{s+1}{2}\right) - \left(\frac{s-1}{2}\right), \tag{9.4}$$

showing that $\mathcal{I} + \Sigma^2$ contains all symmetric polynomials $\operatorname{Sym} \mathbb{R}\langle x, x^*\rangle$. Also, for $j \in \mathcal{I}$,

$$j = (j + j^*) - j \in \operatorname{Sym} \mathbb{R}\langle x, x^*\rangle + \mathcal{I} \subset J + \Sigma^2.$$

Let $t \in \mathbb{R}\langle x, x^*\rangle$ be an arbitrary skew symmetric polynomial. Then $(-t)(z-1) = -tz + t \in \mathcal{I}$. Likewise, $(z-1)t = zt - t \in \mathcal{I}$ and thus by the above, $tz + t = (zt - t)^* \in \mathcal{I}^* \subset \mathcal{I} + \Sigma^2$. Adding the first and the last relation yields $t \in \mathcal{I} + \Sigma^2$.

As every polynomial f is a sum of a symmetric and a skew symmetric polynomial ($f = \frac{f+f^*}{2} + \frac{f-f^*}{2}$), this concludes the proof. ■

9.5. **A Nullstellensatz.** With similar techniques (well chosen, separating, $*$-representations of the free algebra) and a rather different "dilation type" of argument, one can prove a series of Nullstellensätze.

We state for information one of them. For an early version see [HMP05].

Theorem 9.8. *Let $q_1(x), ..., q_m(x) \in \mathbb{R}\langle x\rangle$ be polynomials not depending on the x_j^* variables and let $p(x,x^*) \in \mathbb{R}\langle x,x^*\rangle$. Assume that for every g tuple X of linear operators acting on a finite dimensional Hilbert space H, and every vector $v \in H$, we have:*

$$(q_j(X)v = 0,\ 1 \le j \le m) \ \Rightarrow\ (p(X,X^*)v = 0).$$

Then p belongs to the left ideal $\mathbb{R}\langle x,x^*\rangle q_1 + ... + \mathbb{R}\langle x,x^*\rangle q_m$.

Again, this proposition is stronger than its commutative counterpart. For instance there is no need of taking higher powers of p, or of adding a sum of squares to p. Note that here $\mathbb{R}\langle x\rangle$ has a different meaning than earlier, since, unlike previously, the variables are nonsymmetric.

We refer the reader to [HMP07] for the proof of Theorem 9.8. However, we say a few words about the intuition behind it. We are assuming

$$q_j(X)v = 0, \forall j \qquad \Longrightarrow \qquad p(X,X^*)v = 0.$$

On a very large vector space, if X is determined on a small number of vectors, then X^* is not heavily constrained; it is almost like being able to take X^* to be a completely independent tuple Y. If it were independent, we would have

$$q_j(X)v = 0, \forall j \qquad \Longrightarrow \qquad p(X,Y)v = 0.$$

Now, in the free algebra $\mathbb{R}\langle x,y\rangle$, it is much simpler to prove that this implies $p \in \sum_j^m \mathbb{R}\langle x,y\rangle\, q_j$, as required. We isolate this fact in a separate lemma.

Lemma 9.9. *Fix a finite collection $q_1, ..., q_m$ of polynomials in noncommuting variables $\{x_1, \ldots, x_g$ and let p be a given polynomial in $\{x_1, \ldots, x_g\}$. Let d denote the maximum of the* $\deg(p)$ *and* $\{\deg(q_j) : 1 \le j \le m\}$.

There exists a real Hilbert space $\mathcal{H}$ of dimension $\sum_{j=0}^d g^j$, such that, if

$$p(X)v = 0$$

whenever $X = (X_1, \dots, X_g)$ *is a tuple of operators on* $\mathcal{H}$, $v \in \mathcal{H}$, *and*

$$q_j(X)v = 0 \text{ for all } j,$$

then p *is in the left ideal generated by* $q_1, ..., q_m$.

Proof (of Lemma). We sketch a proof based on an idea of G. Bergman, see [HM04].

Let $\mathcal{I}$ be the left ideal generated by $q_1, ..., q_m$ in $F = \mathbb{R}\langle x_1, ..., x_g\rangle$. Define $\mathcal{V}$ to be the vector space $F/\mathcal{I}$ and denote by $[f]$ the equivalence class of $f \in F$ in the quotient $F/\mathcal{I}$. Define X_j on the vector space $F/\mathcal{I}$ by $X_j[f] = [x_j f]$ for $f \in F$, so that $x_j \mapsto X_j$ implements a quotient of the left regular representation of the free algebra F.

If $\mathcal{V} := F/\mathcal{I}$ is finite dimensional, then the linear operators $X = (X_1, \dots, X_g)$ acting on it can be viewed as a tuple of matrices and we have, for $f \in F$,

$$f(X)[1] = [f].$$

In particular, $q_j(X)[1] = 0$ for all j. If we do not worry about the dimension counts, by assumption, $0 = p(X)[1]$, so $0 = [p]$ and therefore $p \in \mathcal{I}$. Minus the precise statement about the dimension of $\mathcal{H}$ this establishes the result when $F/\mathcal{I}$ is finite dimensional.

Now we treat the general case where we do not assume finite dimensionality of the quotient. Let $\mathcal{V}$ and $\mathcal{W}$ denote the vector spaces

$$\mathcal{V} := \{[f] : \ f \in F, \ \deg(f) \le d\},$$

$$\mathcal{W} := \{[f] : \ f \in F, \ \deg(f) \le d - 1\}.$$

Note that the dimension of $\mathcal{V}$ is at most $\sum_{j=0}^{d} g^j$. We define X_j on $\mathcal{W}$ to be multiplication by x_j. It maps $\mathcal{W}$ into $\mathcal{V}$. Any linear extension of X_j to the whole $\mathcal{V}$ will satisfy: if f has degree at most d, then $f(X)[1] = [f]$. The proof now proceeds just as in the part 1 of the proof above. ■

With this observation we can return and finish the proof of Theorem 9.8. Since X^* is dependent on X, an operator extension with properties stated in the lemma below gives just enough structure to make the above free algebra Nullstellensatz apply.

Lemma 9.10. *Let* $x = \{x_1, \dots, x_m\}$, $y = \{y_1, \dots, y_m\}$ *be free, noncommuting variables. Let* H *be a finite dimensional Hilbert space, and let* X, Y *be two* m*-tuples of linear operators acting on* H. *Fix a degree* $d \ge 1$.

Then there exists a larger Hilbert space $K \supset H$, *an* m*-tuple of linear transformations* $\tilde{X}$ *acting on* K, *such that*

$$\tilde{X}_j|_H = X_j, \quad 1 \le j \le g,$$

and for every polynomial $p \in \mathbb{R}\langle x, x^*\rangle$ *of degree at most* d *and vector* $v \in H$,

$$p(\tilde{X}, \tilde{X}^*)v = 0 \;\Rightarrow p(X, Y)v = 0.$$

For the construction of the larger Hilbert space $K \supset H$ and $\tilde{X}$ on it, see the proof in [HMP07].

Here is a theorem which could be regarded as a very different type of noncommutative Nullstellensatz.

Theorem 9.11 (Theorem 2.1 [KS08]). *Let* $p = p^* \in \mathbb{R}\langle x, x^*\rangle_d$ *be a non-commutative polynomial satisfying* $\operatorname{tr} p(M, M^*) = 0$ *for all* g*-tuples of linear operators* M *acting on a Hilbert space of dimension at most* d*. Then* p *is a sum of commutators of noncommutative polynomials.*

9.6. **A typical noncommutative phenomenon.** We end this subsection with an example which goes against any intuition we would carry from the commutative case, see [HM04].

Example 9.12. Let $q = (x^*x + xx^*)^2$ and $p = x + x^*$ where x is a single variable. Then, for every matrix X and vector v (belonging to the space where X acts), $q(X)v = 0$ implies $p(X)v = 0$; however, there does not exist a positive integer m and $r, r_j \in \mathbb{R}\langle x, x^*\rangle$, so that

$$p^{2m} + \sum r_j^* r_j = qr + r^*q. \tag{9.5}$$

Moreover, we can modify the example to add the condition $q(X)$ is positive semi-definite implies $p(X)$ is positive semi-definite and still not obtain this representation. □

Proof Since $A := XX^* + X^*X$ is self-adjoint, $A^2v = 0$ if and only if $Av = 0$. It now follows that if $q(X)v = 0$, then $Xv = 0 = X^*v$ and therefore $p(X)v = 0$.

For $\lambda \in \mathbb{R}$, let

$$X = X(\lambda) = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

viewed as an operator on $\mathbb{R}^3$ and let $v = e_1$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$.

We begin by calculating the first component of even powers of the matrix $p(X)$. Let $Q = p(X)^2$ and verify,

$$Q = \begin{pmatrix} \lambda^2 & 0 & \lambda \\ 0 & 1+\lambda^2 & 0 \\ \lambda & 0 & 1 \end{pmatrix}. \tag{9.6}$$

For each positive integer m there exist a polynomial p_m so that

$$Q^m e_1 = \begin{pmatrix} \lambda^2(1+\lambda p_m(\lambda)) \\ 0 \\ \lambda(1+\lambda p_m(\lambda)) \end{pmatrix} \tag{9.7}$$

which we now establish by an induction argument. In the case $m = 1$, from equation (9.6), it is evident that $p_1 = 0$. Now suppose equation (9.7) holds for m. Then, a computation of $QQ^m e_1$ shows that equation (9.7) holds for $m+1$ with $p_{m+1} = \lambda(p_m + \lambda + \lambda p_m)$. Thus, for any m,

$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} < Q^m e_1, e_1 >= \lim_{\lambda\to 0}(1+\lambda p_m(\lambda)) = 1. \tag{9.8}$$

Now we look at q and get

$$q(X) = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & (1+\lambda^2)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} (< r(X)^* q(X) e_1, e_1 > + < q(X) r(X) e_1, e_1 >) = 0.$$

If the representation of equation (9.5) holds, then apply $< \cdot\, e_1, e_1 >$ to both sides and take λ to 0. We just saw that the right side is 0, so the left side is 0, which because

$$< \sum r_j(X)^* r_j(X) e_1, e_1 > \geq 0$$

forces

$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} < Q^m e_1, e_1 > \leq 0$$

a contradiction to equation (9.8). Hence the representation of equation (9.5) does not hold.

The last sentence claimed in the example is true when we use the same polynomial q and replace p with p^2. ■

9.7. **Non-free Semi-Algebraic Geometry.** A series of natural structures:

- Weyl algebra, enveloping algebras of Lie algebras (see Schmüdgen's article in this volume, and separately the work of Cimprič [Ci]),
- the classical area of PI rings, (e.g. $N \times N$ matrices for fixed N, as studied by Procesi-Schacher [PS76], or the Nullstellensatz for PI rings as discussed in [Ami57]),

are calling for a general framework incorporating all known Positiv- and Nullstellensätze in the literature. Such a construct is missing at the time of writing the present survey.

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