IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received August 6, 2020, accepted August 24, 2020, date of publication September 4, 2020, date of current version September 18, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3021721

Enhance the Uncertainty Modeling Ability of Fuzzy Grey Cognitive Maps by General Grey Number

JUN CHEN[1], (Member, IEEE), XUDONG GAO[1], AND JIA RONG[2]
[1]School of Electronics and Information, Northwestern Polytechnical University, Xi'an 710072, China
[2]Department of Data Science and AI, Faculty of IT, Monash University, Clayton, VIC 3800, Australia

Corresponding author: Jun Chen (junchen@nwpu.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 61305133, and in part by the Aeronautical Science Foundation of China under Grant 2016ZC53020.

ABSTRACT In real-life systems, people cannot get precise data. The data are represented in the forms of interval or multiple intervals in many cases. Most intelligent algorithms are designed for precise data in algorithm research. People always use a real number contained in the interval or multiple intervals as the candidate for the precise data. However, such a measure will lose lots of information contained in the interval or multiple intervals. Fuzzy Cognitive Map (FCM) is one of the famous intelligent algorithms. The Fuzzy Grey Cognitive Map (FGCM) was proposed to enable FCM to do interval computation. But FGCM can only deal with a single interval, as for multiple intervals, the FGCM is powerless. Thus, this paper aims to enhance the FGCM to make it can cope with multiple intervals. The paper introduces the general grey number and deduces the new activation functions according to Grey System Theory (GST) and Taylor series. Finally, an industrial process control problem is applied to verify the new algorithm. The results show that the new algorithm is not only compatible with the original FGCM and FCM, but can process more uncertain knowledge and data. In general, the new algorithm inherits most of FGCM's characteristics and can cope with the data expressed by multiple intervals, which means it can be used in environments with more uncertain knowledge and data.

INDEX TERMS Fuzzy grey cognitive map, fuzzy cognitive map, uncertainty modeling, general grey number, grey system theory.

I. INTRODUCTION

Recently, coping with the uncertainty data and information is one of the main tasks of soft computing. Soft computing is different from the ''hard computing'', which need deterministic analytic techniques. Soft computing can deal with imprecision, uncertainty, partial truth, and approximations. The principal constituents of soft computing techniques are probabilistic reasoning, fuzzy logic, neuro-computing, genetic algorithms, belief networks, chaotic systems, as well as learning theory [1]. Fuzzy Cognitive Map (FCM) is one of the soft computing techniques [2]. Due to its simplicity, support of inconsistent knowledge, and circle causalities for knowledge modeling and inference [3], FCM has made many significant achievements in many research areas, including decision-support [4], [5], system simulation [6], system prediction [7], smart city [8], intrusion detection [9], and industrial process control [10]–[13] etc. FCM assigns fuzzy weights to topological graphs, so it is also treated as a fuzzy-neural system [14]. Unlike neural networks, FCM's nodes are explanatory, which gives engineers and data scientists convenience to conduct the what-if analysis of the system [15]. And an FCM can be easily built by exploiting the historical data or the experts' knowledge [16].

The data from real-life system always have uncertainty. People always use intervals to express the data with uncertainty, which leads to the emergence of grey system theory (GST) [17]. To make the use of such uncertainty data, many algorithms have been extended according to the GST, which include the FCM. The model is called Fuzzy Grey Cognitive Map [15] (FGCM), the FGCM can support the interval input,

The associate editor coordinating the review of this manuscript and approving it for publication was Nagarajan Raghavan.

improving the uncertainty modeling ability of the FCM [18]. In this paper, the uncertainty modeling ability refers to the algorithm's ability to represent and process uncertain data. Although FGCM can process the uncertain data represented by intervals, the multiple intervals or discrete data are often encountered in real life, FGCM cannot deal with such data. Consider an occasion: a decision support model of an aircraft needs the data of the ground speed, but the ground speed obtained from the Inertial Navigation System (INS) is always different from the one calculated by the Global Positioning System (GPS). For example, the GPS gives a value falls in the data range of $[215, 220]$ because of the positioning error, but the INS gives a value falls in the data range of $[200, 210]$ caused by the cumulative error, if the decision support model is built as an FCM, a fuzzy number which falls into the union of $[200, 210] \cup [215, 220]$ may be chosen as the input value because the FCM only supports a fuzzy number as its node value. Which number should be chosen as the input? Different numbers may cause totally different results, the pilot has to make a decision according to a number selected randomly in $[200, 210] \cup [215, 220]$, which will involve many fateful consequences. If the decision support model is built as an FGCM, which interval should be chosen as the input of the FGCM? Some methods, like Dempster-Shafer (D-S) evidence theory [19], [20] would like to make a choice between them, but making a choice between $[200, 210]$ and $[215, 220]$ will lose another interval's information, and it should be avoided to make a decision at the cost of losing partly information. Choosing $[200, 220]$ will add the uncertainty of the input data, which can also damage the precision of the decision-making. The best way is to make $[200, 210] \cup [215, 220]$ as the input of FGCM, in this way, no information is lost, and the pilot can make a decision by full information. However, FGCM cannot calculate with such data. The same predicament will be confronted more often when two or more experts were invited to determine the FGCM's weight. For example, expert A believes the weight w should be $[0.1, 0.15]$ whereas expert B thinks it is $[0.35, 0.4]$. How can the $[0.1, 0.15] \cup [0.35, 0.4]$ be used as the values of w? One solution is to run the FGCM multiple times, but the running time is too long to tolerate. Consider an FGCM has 10 nodes and 10 weights, each node has 2 candidate intervals, each weight has 3 candidate weights, to get the candidate results, we have to run FGCM $2^{10} \times 3^{10}$ times! Another idea is to "average" such intervals directly, but how to "average" intervals? Averaging their upper bounds and lower bound is unwise because such measurement will average the length of intervals at the same time. For example, average $[0.1, 0.15]$ and $[0.35, 0.4]$ directly, and we get $[0.225, 0.275]$, simultaneously, the length of $[0.225, 0.275]$ becomes 0.05, but the length of $[0.1, 0.15] \cup [0.35, 0.4]$ is more than 0.05 for sure. Actually, the average is not equal to the union, some information will be lost by averaging the intervals. A reasonable solution to cope with the multiple interval values is to introduce the general grey number into FGCM, change the inference process of the FGCM according to the arithmetic rules of the general grey number, which is the aim of this paper.

A general grey number is the union of interval grey numbers or real numbers [21]. For example, $[200, 210] \cup [215, 220]$ is a general grey number, and $[200, 210] \cup [215, 220] \cup 230$ is also a general grey number. In 2007, the extend grey number [22], [23] was proposed as the precursor of the general grey number. Then in 2012, the general grey number was clarified as the "generalized grey number" [24]. Almost simultaneously, the general grey number and their operations were summarized in [21]. After that, the general grey number can be used in many fields [25]–[27] because of its adequate representation and arithmetic operation system.

The FGCM uses the interval grey number as its basic element. Interval grey number is a special form of the general grey number, for example, $[200, 220]$ can be seen as an interval grey number. However, the general grey number runs different arithmetic operation systems with the interval grey number, which is the main obstacle to introduce the general grey number to the FGCM. To tackle the problem, this paper changed the data structure as well as the activation functions for FGCM. As the innovation is mathematical, one classical application of FGCM is enough to verify it. The new algorithm should be compatible with classical FGCM, and it can operate with the data in the form of the general grey number.

Thus, the paper's main contributions are:

- Introducing the general grey number in the FGCM;
- Formulating the FGCM's activation function in the form of the general grey number;
- Validating the new models compatibility and availability by the industrial process control problem.

The FGCM using general grey number is a novel extension of the FCM, which combines GST and FCM more deeply than FGCM. When the data have more uncertainty, especially contains multiple intervals or discrete values, the FGCM using the general grey number should be applied.

The rest of the paper is organized as follows: Section II reviews the fundamentals of GST followed by the FGCM framework, introduces the general grey number; Section III shows how to apply the general grey number to FGCM and how the activation functions works; Section IV takes an industrial process control problem as an example to verify FGCM using general grey number; Section V analyzes the experiment result and computational complexity; finally, section VI concludes the characteristics of FGCM using general grey number and shows the future work.

II. PRELIMINARY KNOWLEDGE

Before turning to the proposed algorithm, the preliminary knowledge is described here.

A. GREY SYSTEM THEORY

Grey System Theory (GST) was proposed in the 1980s to deal with the uncertainty caused by incomplete information and inaccurate data. The GST has been successfully applied in the areas of environment [28], military [29], business [30], agriculture [31], geology [32], and meteorology [33].

In order to tackle the uncertainty of the system, representation is required firstly. The grey number is the basic component of the GST, which represents a crisp number falling into one or more ranges. The interval grey number is the most commonly used grey number, indicated as $\otimes G \in [\underline{G}, \overline{G}]$, $\underline{G} \leqslant \overline{G}$. If the lower bound is known but the upper bound is unknown, the interval grey number should be written as $\otimes G \in [\underline{G}, +\infty)$; the same as the case if the upper bound is known but the lower bound is unknown, denoted as $\otimes G \in (-\infty, \overline{G}]$. If neither the upper bound nor lower bound is known, the grey number becomes a black number, indicated as $\otimes G \in (-\infty, +\infty)$. No information can be operated from a black number. If a grey number's upper bound and lower bound are equal, then it is a white number, it means the complete certain information is known, indicated as $\otimes G \in [\underline{G}, \overline{G}]$, $\underline{G} = \overline{G}$.

The interval grey numbers' operation rules are defined as below:

$$\otimes G_1 + \otimes G_2 \in \left[\underline{G_1} + \underline{G_2}, \overline{G_1} + \overline{G_2}\right] \tag{1}$$

$$-\otimes G \in \left[-\overline{G}, -\underline{G}\right] \tag{2}$$

$$\begin{aligned} \otimes G_1 - \otimes G_2 &= \otimes G_1 + (-\otimes G_2) \\ &\in \left[\underline{G_1} - \overline{G_2}, \overline{G_1} - \underline{G_2}\right] \end{aligned} \tag{3}$$

$$\begin{aligned} &\otimes G_1 \times \otimes G_2 \\ &\quad \in \big[\min(\underline{G_1} \cdot \underline{G_2}, \overline{G_1} \cdot \overline{G_2}, \underline{G_1} \cdot \overline{G_2}, \overline{G_1} \cdot \underline{G_2}), \\ &\quad \max(\underline{G_1} \cdot \underline{G_2}, \overline{G_1} \cdot \overline{G_2}, \underline{G_1} \cdot \overline{G_2}, \overline{G_1} \cdot \underline{G_2})\big] \end{aligned} \tag{4}$$

$$\frac{1}{\otimes G} \in \left[\frac{1}{\overline{G}}, \frac{1}{\underline{G}}\right] \tag{5}$$

$$\begin{aligned} &\otimes G_1 \div \otimes G_2 = \otimes G_1 \times \frac{1}{\otimes G_2} \\ &\quad \in \Big[\min(\underline{G_1}\frac{1}{\underline{G_2}}, \overline{G_1}\frac{1}{\overline{G_2}}, \underline{G_1}\frac{1}{\overline{G_2}}, \overline{G_1}\frac{1}{\underline{G_2}}), \\ &\quad \max(\underline{G_1}\frac{1}{\underline{G_2}}, \overline{G_1}\frac{1}{\overline{G_2}}, \underline{G_1}\frac{1}{\overline{G_2}}, \overline{G_1}\frac{1}{\underline{G_2}})\Big] \end{aligned} \tag{6}$$

$$\lambda \cdot \otimes G \in \left[\lambda \cdot \underline{G}, \lambda \cdot \overline{G}\right] \tag{7}$$

The degree of greyness represents the level of uncertainty a grey number contains. For convenience, the degree of greyness will be called greyness in this paper. Given a grey number $\otimes$ generated from the domain Ω, its greyness can be defined as:

$$g^{\circ}(\otimes) = \frac{\mu(\otimes)}{\mu(\Omega)}. \tag{8}$$

The $\mu()$ calculates the measure of the grey number or the domain. The greyness is valid in [0, 1], i.e., $0 \leqslant g^{\circ}(\otimes) \leqslant 1$. The higher the greyness is, the higher the uncertainty of the grey number. A white number's greyness is 0 and a black number's greyness is 1 obvious.

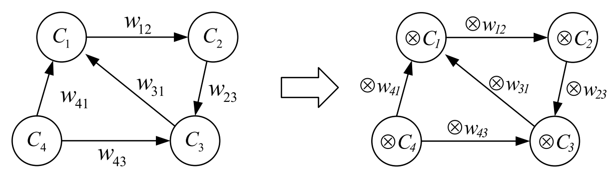


FIGURE 1. **Mapping of the FCM into the FGCM.**

B. THE FUZZY GREY COGNITIVE MAP

The main idea of the FGCM is to use greyness to replace the fuzziness in the FCM, because a grey set can be specified to a fuzzy set [24]. The FGCM model keeps the same topological structure and the inference process as those in the FCM (Fig. 1). The nodes in FGCM represent the variables or concepts that have uncertainty. And grey weight (weight shown by grey numbers) can represent the uncertainty of the experts' suggestions or the incomplete data sets.

To carry out the iteration in the FGCM model, the node values and weight are often written as grey vectors and grey matrix. The grey vectors' and matrix's operation rules are the same with the real vectors and matrix. For example, Fig. 1 (right) shows a simple FGCM, its initial state vector is

$$\begin{aligned} \otimes\vec{C}_0 &= \left(\otimes C_0^{[1]}, \otimes C_0^{[2]}, \otimes C_0^{[3]}, \otimes C_0^{[4]}\right) \\ &= \left(\left[\underline{C_0^{[1]}}, \overline{C_0^{[1]}}\right], \left[\underline{C_0^{[2]}}, \overline{C_0^{[2]}}\right], \right. \\ &\quad \left. \left[\underline{C_0^{[3]}}, \overline{C_0^{[3]}}\right], \left[\underline{C_0^{[4]}}, \overline{C_0^{[4]}}\right]\right). \end{aligned} \tag{9}$$

And the grey weight matrix is

$$\otimes w = \begin{pmatrix} 0 & \otimes w_{12} & 0 & 0 \\ 0 & 0 & \otimes w_{23} & 0 \\ \otimes w_{31} & 0 & 0 & 0 \\ \otimes w_{41} & 0 & \otimes w_{43} & 0 \end{pmatrix}. \tag{10}$$

Like the FCM, the FGCM gets its new node values by

$$\begin{aligned} \otimes\vec{C}_{t+1} &= S\left(\otimes\vec{C}_t \cdot \otimes w\right) \\ &= S\left(\otimes\vec{C}'_{t+1}\right) \\ &= \left(S\left(\otimes C_t^{[1]'}\right), S\left(\otimes C_t^{[2]'}\right), \right. \\ &\quad \left. \cdots, S\left(\otimes C_t^{[n]'}\right)\right), \end{aligned} \tag{11}$$

where $S(\otimes)$ is the activation function to compress the node value into [0, 1] or [−1, 1], when the node value is in [0, 1], the activation function is

$$S(\otimes G) = \left[\frac{1}{1 + e^{-\lambda \underline{G}}}, \frac{1}{1 + e^{-\lambda \overline{G}}}\right], \tag{12}$$

when the node value falls in [−1, 1], the $S(\otimes)$ is the hyperbolic tangent:

$$\begin{aligned} S(\otimes G) &= \tanh(\otimes G) \\ &= \left[\frac{e^{\lambda \underline{G}} - e^{-\lambda \underline{G}}}{e^{\lambda \underline{G}} + e^{-\lambda \underline{G}}}, \frac{e^{\lambda \overline{G}} - e^{-\lambda \overline{G}}}{e^{\lambda \overline{G}} + e^{-\lambda \overline{G}}}\right]. \end{aligned} \tag{13}$$

The positive parameter λ is used to control the steepness of the curve. The lager the λ is, the steeper the curve is.

The inference results reach a steady state, a limit cycle or chaotic results, which depends on the FGCM's structure [15].

C. THE GENERAL GREY NUMBER

This section introduces the general grey number's concept and arithmetic operations. Following Liu *et al.*'s work [21], we present some definitions and theorems are given here to provide a better understanding of general grey number [34]. They are fundamentals to enhance the FGCM's uncertainty modeling ability.

Definition 1: Assume $\otimes \in [\underline{a}, \overline{a}]$ is an interval grey number, with the distribution of the grey number $\otimes$ is unknown:

1) *if $\otimes$ is continuous, the kernel of the $\otimes$ is $\hat{\otimes} = \frac{1}{2}(\underline{a} + \overline{a})$;*
2) *if $\otimes$ is discrete, the kernel of the $\otimes$ is $\hat{\otimes} = \frac{1}{n}(\sum_{i=1}^{n} a_i)$.*

Definition 2: Assume $\otimes \in [\underline{a}, \overline{a}]$ is an interval grey number, with the distribution of the grey number $\otimes$ is already known, the kernel of the $\otimes$ is $\hat{\otimes} = E(\otimes)$. The $E(\otimes)$ is the expectation, calculated as $E(\otimes) = \int p \otimes \mathrm{d}\otimes$ if the $\otimes$ is a continuous grey number, or $E(\otimes) = \sum p_i \otimes_i$ if the $\otimes$ is a discrete grey number.

Definition 3: Assume $\hat{\otimes}$ is the kernel of the grey number $\otimes$, g° is the greyness of the grey number, then the $\hat{\otimes}_{(g^\circ)}$ is called the simplified form of the grey number.

For an interval grey number, the simplified form of it contains all information about itself. The simplified form of interval grey number can replace the interval grey number itself.

Definition 4: The interval grey number and white number are the basic elements of the general grey number.

Definition 5: A general grey number is defined as

$$g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a}_i, \overline{a}_i] \tag{14}$$

for all interval grey number $\otimes_i \in [\underline{a}_i, \overline{a}_i] \subset \bigcup_{i=1}^{n} [\underline{a}_i, \overline{a}_i]$, $\underline{a}_i, \overline{a}_i \in \mathbb{R}$ and $\overline{a}_{i-1} \leqslant \underline{a}_i \leqslant \overline{a}_i \leqslant \underline{a}_{i+1}$,

$$g^{-} = \inf_{\underline{a}_i \in g^{\pm}} \underline{a}_i, \quad g^{+} = \sup_{\overline{a}_i \in g^{\pm}} \overline{a}_i$$

are called the lower bound and upper bound of $g^{\pm}$.

The general grey number is defined as the union of interval grey number or white number, it can express data have more uncertainty. The next step is to determine the operation of the general grey number. Before achieving this objective, define the kernel and greyness of the general grey number.

Definition 6: Assume $g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a}_i, \overline{a}_i]$ is a general grey number, its kernel is calculated by

$$\hat{g} = \frac{1}{n}(\sum_{i=1}^{n} \hat{a}_i). \tag{15}$$

If the distribution of the general grey number is already known, and the probability of $g^{\pm} \in [\underline{a}_i, \overline{a}_i]$, $(i = 1, 2, \cdots, n)$ is p_i, and

$$\sum_{i=1}^{n} p_i = 1, p_i > 0, \quad i = 1, 2, \cdots, n$$

then the kernel is calculated by

$$\hat{g} = \sum_{i=1}^{n} p_i \hat{a}_i. \tag{16}$$

Definition 7: Assume Ω is the domain of $g^{\pm} \in \bigcup_{i=1}^{n} [\underline{a}_i, \overline{a}_i]$, $\mu()$ is the measure function in Ω, then the greyness is defined as

$$g^\circ(g^{\pm}) = \frac{1}{|\hat{g}|} \sum_{i=1}^{n} |\hat{a}_i| \mu(\otimes_i)/\mu(\Omega). \tag{17}$$

There are two axioms prepared for the operation rules of general grey number [21].

Axiom 1 (The Synthesis Axiom for Greyness): When adding or subtracting a set of n general grey numbers $g_1, g_2, \cdots, g_n$, the greyness g° of the operation result $g^{\pm}$ can be calculated as follows:

$$g^\circ(g^{\pm}) = \frac{1}{\sum_{i=1}^{n} |\hat{g}_i|} \sum_{i=1}^{n} g_i^\circ |\hat{g}_i| = \sum_{i=1}^{n} w_i g_i^\circ, \tag{18}$$

the w_i is the weight of g_i°, $w_i = \frac{|\hat{g}_i|}{\sum_{i=1}^{n} |\hat{g}_i|}$, $i = 1, 2, \cdots, n$.

Axiom 2 (Axiom of Greyness Undiminished): When dividing or multiplying a set of n general grey numbers, the greyness of the operation result is no less than the maximum greyness of the operated general grey number in the set.

Based on these two axioms, the basic operating rules of the general grey number are summarized and proven in [35], and shown as follows:

$$\hat{g}_{1(g_1^\circ)} = \hat{g}_{2(g_2^\circ)} \Longleftrightarrow \hat{g}_1 = \hat{g}_2 \text{ and } g_1^\circ = g_2^\circ \tag{19}$$

$$\hat{g}_{1(g_1^\circ)} + \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 + \hat{g}_2)_{(w_1 g_1^\circ + w_2 g_2^\circ)} \tag{20}$$

where

$$w_1 = \frac{|\hat{g}_1|}{|\hat{g}_1| + |\hat{g}_2|}, \quad w_2 = \frac{|\hat{g}_2|}{|\hat{g}_1| + |\hat{g}_2|}$$

$$-\hat{g}_{(g^\circ)} = (-\hat{g})_{(g^\circ)} \tag{21}$$

$$\hat{g}_{1(g_1^\circ)} - \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 - \hat{g}_2)_{(w_1 g_1^\circ + w_2 g_2^\circ)} \tag{22}$$

where

$$w_1 = = \frac{|\hat{g}_1|}{|\hat{g}_1| + |\hat{g}_2|}, \quad w_2 = \frac{|\hat{g}_2|}{|\hat{g}_1| + |\hat{g}_2|}$$

$$\hat{g}_{1(g_1^\circ)} \times \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 \times \hat{g}_2)_{(\max(g_1^\circ, g_2^\circ))} \tag{23}$$

$$\frac{1}{\hat{g}_{1(g_1^\circ)}} = \left(\frac{1}{\hat{g}_1}\right)_{(g_1^\circ)}, \quad (\hat{g}_1 \neq 0) \tag{24}$$

$$\frac{\hat{g}_{1(g_1^\circ)}}{\hat{g}_{2(g_2^\circ)}} = \left(\frac{\hat{g}_1}{\hat{g}_2}\right)_{(\max(g_1^\circ, g_2^\circ))}, \quad (\hat{g}_2 \neq 0) \tag{25}$$

$$k \cdot \hat{g}_{(g^\circ)} = (k \cdot \hat{g})_{(g^\circ)}, \quad k \in \mathbb{R} \tag{26}$$

$$(\hat{g}_{(g^\circ)})^k = (\hat{g})^k_{(g^\circ)}, \quad k \in \mathbb{R} \tag{27}$$

III. THE PROPOSED ALGORITHM

Comparing to the interval grey number, the general grey number can handle more uncertainties carried from the incomplete information. For the node value, two or more interval grey numbers can often be chosen as the input. If the FGCM is applied to such occasions, only one interval grey number can be used, or one uses the minimum (the greyness is minimum) interval grey number that contains all node values gotten before. For example, $g^\pm \in [0.5, 0.6] \bigcup [0.8, 0.9]$ is a general grey number. Its underlying white number may get its value from [0.5, 0.6] or [0.8, 0.9]. However, we cannot get its value within (0.6, 0.8). This is different from an interval [0.5, 0.9] where (0.6, 0.8) is a valid part for candidate values [24]. What's more, a general grey number allows candidate values in a discrete set, such as $g^\pm \in \{0.3, 0.4, 0.5\}$, which cannot be represented as an interval. Because of the conflict opinions raised by experts or the data from the environment, the weight and node values of the FGCM can also encounter multiple interval or discrete values cases. To make use of the general grey number in FGCM, the activation function of the general grey number's form is needed.

A. THE SIGMOID ACTIVATION FUNCTION IN THE FORM OF GENERAL GREY NUMBER

If the domain is [0, 1], then the sigmoid activation function is

$$S(g^\pm) = \frac{1}{1 + e^{-\lambda g^\pm}}. \tag{28}$$

The result of $S(g^\pm)$ is a new general grey number, the kernel of the $\hat{S}(g^\pm)$ is calculated as

$$\hat{S}(g^\pm) = \frac{1}{1 + e^{-\lambda \hat{g}}}, \tag{29}$$

the greyness is calculated as

$$S^\circ(g^\pm) = \frac{1}{1 + e^{-\lambda \hat{g}}} g^\circ. \tag{30}$$

The proof of (29) and (30) is shown below:

Proof: According to the Taylor's formula,

$$e^{-\lambda g^\pm} = \frac{(-\lambda)^0 (g^\pm)^0}{0!} + \frac{(-\lambda)^1 (g^\pm)^1}{1!} + \frac{(-\lambda)^2 (g^\pm)^2}{2!} + \cdots \tag{31}$$

according to the equation (27), the kernel of $(g^\pm)^n$ is $\hat{g}^n$, the greyness is g°; thus, the kernel of $\frac{(-\lambda)^n (g^\pm)^n}{n!}$ is $\frac{(-\lambda)^n \hat{g}^n}{n!}$, $n = 1, 2, 3 \cdots$, the greyness is g°; therefore, the kernel of $e^{-\lambda g^\pm}$ is

$$\begin{aligned} \hat{e}^{-\lambda g^\pm} &= 1 + \frac{(-\lambda)\hat{g}}{1!} + \frac{(-\lambda)^2 (\hat{g})^2}{2!} + \frac{(-\lambda)^3 (\hat{g})^3}{3!} + \cdots \\ &= e^{-\lambda \hat{g}}, \end{aligned} \tag{32}$$

according to axiom 1, the greyness of $e^{-\lambda g^\pm}$ is

$$\begin{aligned} &(e^{-\lambda g^\pm})^\circ \\ &= \frac{1 \times g^\circ}{1 + \left|\frac{(-\lambda)\hat{g}}{1!}\right| + \left|\frac{(-\lambda)^2(\hat{g})^2}{2!}\right| + \left|\frac{(-\lambda)^3(\hat{g})^3}{3!}\right| + \cdots} \\ &\quad + \frac{\left|\frac{(-\lambda)\hat{g}}{1!}\right| \times g^\circ}{1 + \left|\frac{(-\lambda)\hat{g}}{1!}\right| + \left|\frac{(-\lambda)^2(\hat{g})^2}{2!}\right| + \left|\frac{(-\lambda)^3(\hat{g})^3}{3!}\right| + \cdots} \\ &\quad + \frac{\left|\frac{(-\lambda)^2(\hat{g})^2}{2!}\right| \times g^\circ}{1 + \left|\frac{(-\lambda)\hat{g}}{1!}\right| + \left|\frac{(-\lambda)^2(\hat{g})^2}{2!}\right| + \left|\frac{(-\lambda)^3(\hat{g})^3}{3!}\right| + \cdots} \\ &\quad + \cdots \\ &= \frac{1 \times g^\circ + \left|\frac{(-\lambda)\hat{g}}{1!}\right| \times g^\circ + \left|\frac{(-\lambda)^2(\hat{g})^2}{2!}\right| \times g^\circ + \cdots}{e^{|\lambda \hat{g}|}} \\ &= g^\circ. \end{aligned} \tag{33}$$

According to the (20) and (24), the greyness of the $S(g^\pm)$ is

$$\begin{aligned} S^\circ(g^\pm) &= (1 + e^{-\lambda g^\pm})^\circ \\ &= \frac{1 \times 0}{1 + e^{-\lambda \hat{g}}} + \frac{g^\circ}{1 + e^{-\lambda \hat{g}}} \\ &= \frac{1}{1 + e^{-\lambda \hat{g}}} g^\circ. \end{aligned} \tag{34}$$

And using (32), the kernel of $S(g^\pm)$ is

$$\begin{aligned} \hat{S}(g^\pm) &= \frac{1}{1 + \hat{e}^{-\lambda g^\pm}} \\ &= \frac{1}{1 + e^{-\lambda \hat{g}}} \end{aligned} \tag{35}$$

The proof is end. ■

B. THE HYPERBOLIC TANGENT ACTIVATION FUNCTION IN THE FORM OF GENERAL GREY NUMBER

If the domain is [−1, 1], then the hyperbolic tangent is often used as activation function:

$$\tanh(g^\pm) = \frac{e^{\lambda g^\pm} - e^{-\lambda g^\pm}}{e^{\lambda g^\pm} + e^{-\lambda g^\pm}}. \tag{36}$$

The kernel of $\tanh(g^\pm)$ is

$$\tanh(\hat{g}) = \frac{e^{\lambda \hat{g}} - e^{-\lambda \hat{g}}}{e^{\lambda \hat{g}} + e^{-\lambda \hat{g}}}, \tag{37}$$

the greyness is calculated as

$$(\tanh(g^\pm))^\circ = g^\circ \tag{38}$$

The proof of it is very similar to the sigmoid activation function's, so it will not be shown here.

The figures of these two functions are shown in the Fig. 2 and 3. Like the FGCM's activation functions, the lager the λ is, the steeper the surface is.

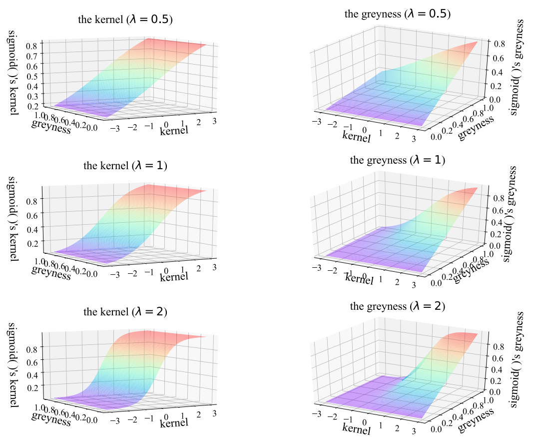


FIGURE 2. The figures of $S(g^{\pm})$ with different λs.

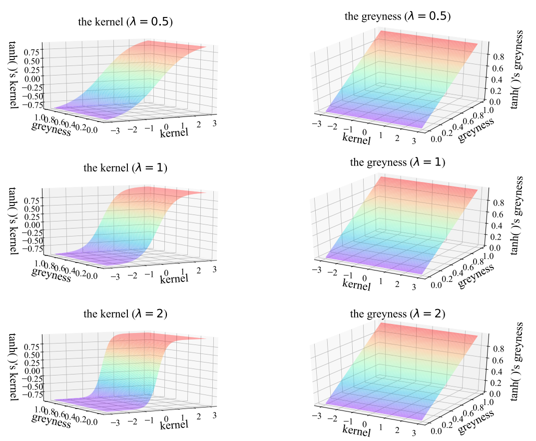


FIGURE 3. The figures of $tanh(g^{\pm})$ with different λs.

Algorithm 1 FGCM Using General Grey Number

Data: the initial vector in the form of general grey number $\otimes\vec{C}$

Result: the output vector in the form of general grey number

while t < *iteration_times* **do**

$$\otimes\vec{C}_{t+1} = S\left(\otimes\vec{C}_t \cdot \otimes w\right) = S\left(\otimes\vec{C}'_{t+1}\right) = \left(S\left(\otimes C_t^{[1]'}\right), S\left(\otimes C_t^{[2]'}\right), \cdots, S\left(\otimes C_t^{[n]'}\right)\right),$$

$t = t + 1$

the $S\left(\otimes C_t^{[i]'}\right)$ is calculated by (28),(29),(30) or (36),(37),(38);

end

C. FUZZY GREY COGNITIVE MAP USING GENERAL GREY NUMBER

After getting the activation functions, the general grey number can be applied in FGCM. The FGCM using general grey number algorithm is shown in Algorithm 1.

Like FCMs, the new algorithm iterates until the node values reach a steady point, a cycle contains several general grey numbers or chaotic results.

Compared with the FGCM, the new algorithm's data structure is general grey number whereas the FGCM's is interval grey number. And the activation functions' computing methods are different, the new algorithm's are (28),(29),(30) or (36),(37),(38), the kernel and greyness are calculated in order, but FGCMs use (12) or (13). The new model is more complex than original FGCM.

In next section, the validation experiments are conducted to show how the FGCM using the general grey number works.

IV. EXPERIMENTS AND RESULTS

The proposed algorithm can be used universally. And it is enough to validate it using a classical application of FCM or FGCM. We choose the industrial process control problem as an example to show how the general grey number works in FGCM.

To evaluate the performance of the proposed algorithm, three experiments are designed to test the compatibility and availability. In Section IV-A, we show the background of the industrial process control problem and how the FCM solves this problem, this FCM is the benchmark of the comparison experiments shown in Section IV-B. And then we build an original FGCM and FGCM using general grey number to prove the compatibility of the proposed algorithm in Section IV-B. In Section IV-C, two cases are specially designed to test the unique characteristics of the proposed algorithm, in which a single FCM or FGCM is not applicable. These two cases also proved that the proposed algorithm has a wider application domain than the FCM and FGCM. Finally, a more complexed situation where both nodes and weights values are two or more intervals is designed in Section IV-D to show the proposed algorithm's availability, at the same time, we exploit the original FGCM multiple times to solve this situation, the results range of the original FGCM is consistent with the proposed model, but so many original FGCM's runtime is too long to be tolerant.

A. THE INDUSTRIAL PROCESS CONTROL SYSTEM

The industrial process control problem was first discussed using FCM by C.D. Stylios in 1998 [36], [37]. Then, it was used in many validations of new FCM-relevant algorithms. In 2006, this problem was used as a case study to compare two unsupervised learning methods for fine-tuning fuzzy cognitive map causal links [38]. In 2011, Extended Great Deluge Algorithm (EGDA) has been considered as a training algorithm for FCM, the industrial process control problem was one of the case studies to validate the EGDA [39]. And the FGCM was applied to analyze the industrial process control problem in 2014 [13]. Therefore, the FGCM using general grey number is validated by the industrial process control problem here.

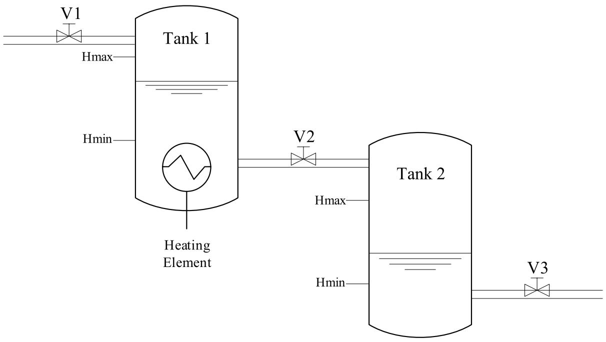


FIGURE 4. The industrial process control system.

This industrial process control system aims to control the height and temperature of the liquid in 2 tanks in a suitable range. It contains two tanks, three valves, and a heating element, as shown in Fig. 4. V1 is the inlet valve, and its outlet valve (V2) is the inlet valve of tank 2. The heating element lies in tank 1, thus, the hot liquid heated in tank 1 can flow into the tank 2. V3 is the outlet valve of the tank 2. If the height of the liquid in tank 2 is too high, the valve 3 is needed to open.

The aims of such a control system are:

$$\begin{cases} H^1_{min} \leqslant H^1 \leqslant H^1_{max} \\ H^2_{min} \leqslant H^2 \leqslant H^2_{max} \\ T^1_{min} \leqslant T^1 \leqslant T^1_{max} \\ T^2_{min} \leqslant T^2 \leqslant T^2_{max} \end{cases}$$

where, the $H^i, (i = 1, 2)$ is the height of liquid in tank $i, (i = 1, 2)$; the $T^i, (i = 1, 2)$ is the temperature of liquid in tank $i, (i = 1, 2)$.

Stylios and Salmeron have constructed the FCM and the FGCM models correspondingly. However, Salmeron applied a different set of weights with the Stylios' [13]. To make a convincing comparison, we use the original FCM built by Stylios as the benchmark. The FCM contains 8 nodes. Each node's definition [36] is shown below:

C1 The height of the liquid in tank 1.
C2 The height of the liquid in tank 2.
C3 The state of valve 1.
C4 The state of valve 2.
C5 The state of valve 3.
C6 The temperature of the liquid in tank 1.
C7 The temperature of the liquid in tank 2.
C8 The state of the heating element.

The node values need to be normalized before processing by the FCM. For example, if the height of liquid in tank 1 is 20%, then the C1's value is 0.2. In practice, the normalized approach was selected by the engineers. If the C3 is 0, it means the valve 1 is totally closed. If the C3 is 1, it means the valve 1 is totally open. And the weight matrix is shown below.

$$w = \begin{pmatrix} 0 & 0 & 0.21 & 0.38 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.70 & 0.60 & 0 & 0 & 0 \\ 0.76 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.80 & 0.80 & 0 & 0 & 0 & 0 & 0 & 0.09 \\ 0 & -0.42 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0.53 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

TABLE 1. The weights in interval grey number and general grey number forms.

weights	interval grey number	general grey number
$\otimes w_{13}$	$[0.01, 0.41]$	$0.21_{(0.20)}$
$\otimes w_{14}$	$[0.28, 0.48]$	$0.38_{(0.10)}$
$\otimes w_{24}$	$[0.60, 0.80]$	$0.70_{(0.10)}$
$\otimes w_{25}$	$[0.50, 0.70]$	$0.60_{(0.10)}$
$\otimes w_{31}$	$[0.51, 1]$	$0.755_{(0.245)}$
$\otimes w_{41}$	$[-0.90, -0.70]$	$-0.80_{(0.10)}$
$\otimes w_{42}$	$[0.70, 0.90]$	$0.8_{(0.10)}$
$\otimes w_{48}$	$[-0.16, 0.34]$	$0.09_{(0.25)}$
$\otimes w_{52}$	$[-0.52, -0.32]$	$-0.42_{(0.1)}$
$\otimes w_{67}$	$[0.5, 0.7]$	$0.6_{(0.1)}$
$\otimes w_{73}$	$[0.3, 0.5]$	$0.4_{(0.1)}$
$\otimes w_{76}$	$[0.43, 0.63]$	$0.53_{(0.1)}$
$\otimes w_{83}$	$[0.05, 0.55]$	$0.3_{(0.25)}$

The initial vector is

$$C_{initial} = \begin{pmatrix} 0.2 & 0.01 & 0.55 & 0.58 & 0 & 0.05 & 0.2 & 0.1 \end{pmatrix}. \quad (39)$$

After 5 iterations, the output contracted to a steady state, which is

$$C_{output} = \begin{pmatrix} 0.48 & 0.58 & 0.58 & 0.68 & 0.59 & 0.58 & 0.59 & 0.52 \end{pmatrix}. \quad (40)$$

Next compatibility and availability experiments are conducted. The FGCM of this problem has been built in [40] and it was compatible with the original FCM we referred. To validate the compatibility, keep the weights be same with the FGCM, but transfer the weights into general grey number and build a new FGCM using general grey number. Given the same input vector, if the results are approximately equal, then the new model is compatible with FCM and FGCM.

B. COMPATIBILITY CHECK

As Salmeron has built the FGCM for this problem, we refer to the weight greyness in [40] and get a matrix of interval grey number. The weights are listed in Table 1 and transferred to the general grey numbers by Definition 1, 2 and 3.

Apply the weight above, the FGCM using general grey number is obtained; the topological graph is shown in Fig. 5, and the inputs are the same as FCM's. The only difference is the data structure. After 10 iterations, all three models: FCM, FGCM and FGCM using general grey number, reach a steady state, as shown in Table 2 and Fig. 6.

To make a comparison, we transfer the FGCM's outputs into the form of the general grey number, they are listed

TABLE 2. The results of the FGCM and FGCM using general grey number.

concepts	FCM	FGCM	FGCM (in the simplified form)	FGCM using general grey number
C_1	0.48	[0.41, 0.56]	$0.4857_{(0.1571)}$	$0.4817_{(0.0806)}$
C_2	0.57	[0.53, 0.62]	$0.5739_{(0.0853)}$	$0.5734_{(0.0573)}$
C_3	0.58	[0.55, 0.70]	$0.6241_{(0.1493)}$	$0.6201_{(0.1042)}$
C_4	0.68	[0.63, 0.73]	$0.6788_{(0.1001)}$	$0.6768_{(0.0718)}$
C_5	0.59	[0.57, 0.61]	$0.5861_{(0.0402)}$	$0.5852_{(0.0585)}$
C_6	0.58	[0.56, 0.59]	$0.5773_{(0.0328)}$	$0.5770_{(0.0577)}$
C_7	0.59	[0.57, 0.60]	$0.5860_{(0.0328)}$	$0.5857_{(0.0586)}$
C_8	0.52	[0.48, 0.56]	$0.5163_{(0.0908)}$	$0.5152_{(0.1288)}$

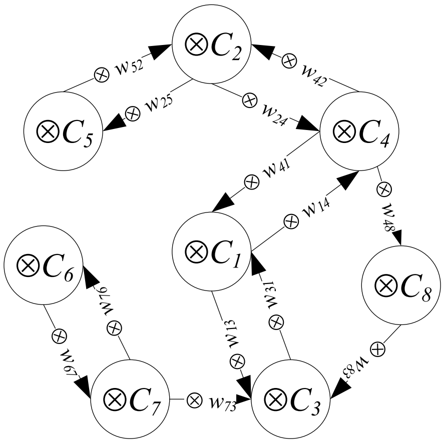


FIGURE 5. The topological graph of FGCM using general grey number.

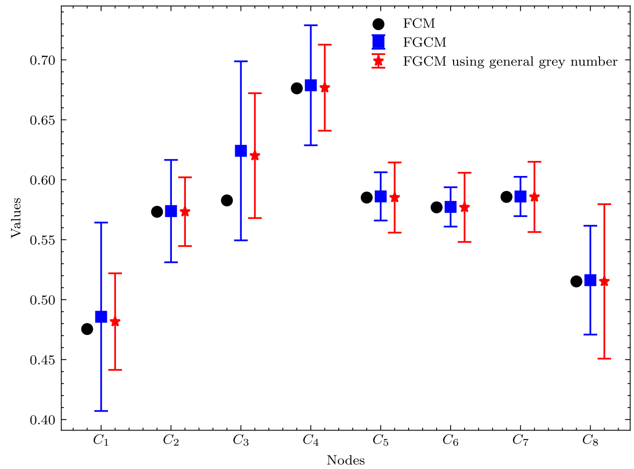


FIGURE 6. The proposed algortihm is compatible with the original FCM and FGCM.

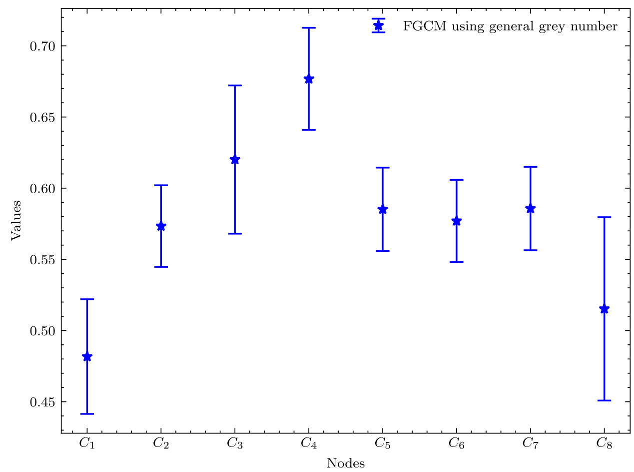


FIGURE 7. The results of FGCM using general grey number with the general grey initial vector (41).

correspondingly in the fourth column (Table 2). Findings from Table 2 are summarized as following:

- Both FGCM and FGCM using general grey number are compatible with the original FCM as their results always cover the FCM's results;
- The kernels of the two FGCM's results are appropriately equal to the FCM's;
- The greyness calculated by the FGCM using general grey number is appropriately equal to the ones calculated by FGCM.

C. THE UNIQUE CHARACTERISTICS OF THE PROPOSED ALGORITHM

Characteristics of FGCM using general grey number are explored here by setting the following 2 cases: the input contains the general grey number and the weights contain the general grey number. The FCM and the original FGCM cannot cope with these 2 cases, but the algorithm this paper proposed can tackle them.

1) THE INPUT CONTAINS GENERAL GREY NUMBERS

Consider a case following: there are two liquid height sensors in tank 1, both sensors are not broken down, but one gives the height of the liquid in [0.19, 0.21], another gives [0.29, 0.31], the engineers cannot determine which scale is more appropriate as they cannot measure it directly, but they ensure the true liquid height falls either in [0.19, 0.21] or [0.29, 0.31]. So the $[0.19, 0.21] \cup [0.29, 0.31]$ is the input value, it is transferred into the form of general grey number $0.25_{(0.04)}$ using (15) and (17). Thus the initial vector is

$$C_{initial1} = \begin{pmatrix} 0.25_{(0.04)} & 0.01 & 0.55 & 0.58 & 0 & 0.05 & 0.2 & 0.1 \end{pmatrix}. \quad (41)$$

Using (41) as the input, the same output with the (39) should be obtained according to the research about the convergence

TABLE 3. The results of the FGCM using general grey number, whose $\otimes w_{13} = 0.352_{(0.1864)}$, compared with the FGCM using general grey number with $\otimes w_{13} = 0.21_{(0.20)}$.

nodes	$\otimes w_{13} = 0.352_{(0.1864)}$	$\otimes w_{13} = 0.21_{(0.20)}$
C_1	$0.4847_{(0.0815)}$	$0.4817_{(0.0806)}$
C_2	$0.5734_{(0.0573)}$	$0.5734_{(0.0573)}$
C_3	$0.6363_{(0.1068)}$	$0.6201_{(0.1042)}$
C_4	$0.6770_{(0.0718)}$	$0.6768_{(0.0718)}$
C_5	$0.5852_{(0.0585)}$	$0.5852_{(0.0585)}$
C_6	$0.5770_{(0.0577)}$	$0.5770_{(0.0577)}$
C_7	$0.5857_{(0.0586)}$	$0.5857_{(0.0586)}$
C_8	$0.5152_{(0.1288)}$	$0.5152_{(0.1288)}$

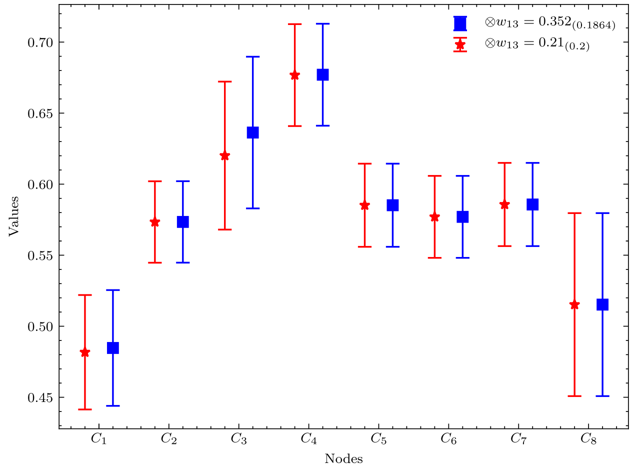


FIGURE 8. The results of FGCM using general grey number with $\otimes w_{13} = 0.352_{(0.1864)}$.

of the FCM [41]. The simulation results are shown in Fig. 7, the model gets the same results with the FGCM using general grey number in Section IV-B.

2) THE WEIGHTS CONFRONT CONTRADICTORY EXPERTS' SUGGESTIONS

Now consider another case: one engineer believes that $\otimes w_{13} = [0.01, 0.25]$, another thinks $\otimes w_{13} = [0.4, 0.6]$ is more appropriate. According to the experience, the first engineer is more authoritative than the second, so the weight of the first engineer's weight is 0.6, and the weight of the second engineer's is 0.4. Use the (16) and (17), the $\otimes w_{13}$ should be $0.352_{(0.1864)}$. In this way, both two of the engineer's advice is considered. So we update the weight matrix accordingly and use the initial vector (39) in the FGCM using general grey number. The nodes vector gets into a steady state at the sixth iteration. The steady state is shown in Table 3 and Fig. 8. Because the $\otimes w_{13}$ is the only changed weight, it can only influence the node 1 and node 3: the values of these two nodes have changed accordingly.

Next, a more complex situation is designed to show the proposed algorithm can handle that most of the nodes and weights are in the form of multiple intervals.

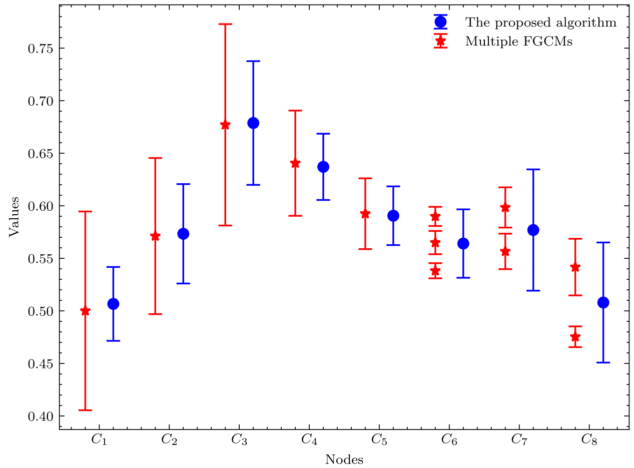


FIGURE 9. The results of FGCM and proposed method in a more complex situation.

D. MORE COMPLEX SITUATION

This experiment is designed to show the availability of the proposed algorithm and why we should use the FGCM using general grey number instead of simply run many different FGCMs when we encounter the multiple intervals data.

If most of nodes confront the the situation like section IV-C1, i.e., most nodes' input is multiple intervals or discrete data, as shown in Table 4. And most of the weights are made up by multiple intervals, as shown in Table 5. Using the proposed model, the results are show in Fig. 9 and Table 6.

If we want to use FGCM to solve the problem, we need to run $\prod_{i=1}^{n} Cn_i \times \prod_{i=1,j=1}^{n\times n} wn_{ij} = 35831808$ times. Cn_i means the node C_i contains Cn_i intervals. If C_i is a real number, $Cn_i = 1$. n is the number of nodes. wn_i is the number of intervals w_{ij} contained, if w_{ij} is a real number, $wn_i = 1$. It is so time consuming that using many FGCMs to solve the cases that the data contains multiple intervals. Actually, under the platform of MacOS 10.15.5, using Python 3.7, the CPU is intel core i7, 2.3 GHz, the total runtime of all FGCMs is 110980.1357s. On the other hand, the FGCM using general grey number only needs to compute one time, and the runtime is 0.0056s.

The results of multiple FGCMs are overlapped with each other, after unioning the 35831808 intervals, the results are also summarized in Fig. 9 and Table 6.

Whether can we average the multiple intervals before running the FGCM? The answer is no. There is no meaning to average a set of interval grey numbers. For example, consider the C_1's value $[0.19, 0.21] \cup [0.29, 0.31]$, if average the $[0.19, 0.21]$ and $[0.29, 0.31]$ directly, the average result is $[0.24, 0.26]$, the greyness becomes 0.02. Obviously, the greyness of $[0.19, 0.21]\cup[0.29, 0.31]$ is more than 0.02. Actually, we can calculate the greyness of $[0.19, 0.21] \cup [0.29, 0.31]$ is 0.04 according to Axiom 1. Thus, there is no meaning to average intervals represented by interval grey numbers.

TABLE 4. A more complex input.

nodes	nodes values	nodes values in the simplified form
C_1	$[0.19, 0.21] \cup [0.29, 0.31]$	$0.2500_{(0.0400)}$
C_2	$[0.01, 0.01]$	$0.0100_{(0)}$
C_3	$[0.49, 0.53] \cup [0.54, 0.57] \cup [0.58, 0.59]$	$0.55_{(0.0780)}$
C_4	$[0.47, 0.52] \cup [0.54, 0.59] \cup [0.61, 0.62] \cup [0.65, 0.69]$	$0.5862_{(0.1466)}$
C_5	$[0, 0]$	$0.0_{(0)}$
C_6	$[0.01, 0.04] \cup [0.05, 0.08] \cup [0.09, 0.10]$	$0.0633_{(0.0711)}$
C_7	$[0.10, 0.15] \cup [0.20, 0.30]$	$0.1875_{(0.1667)}$
C_8	$[0.02, 0.09] \cup [0.1, 0.15]$	$0.0900_{(0.1122)}$

TABLE 5. More complex weights.

nodes	weights values	weights values in the simplified form
w_{13}	$[0.01, 0.41] \cup [0.21, 0.31] \cup [0.5, 0.6]$	$0.3400_{(0.2426)}$
w_{14}	$[0.28, 0.35] \cup [0.36, 0.39] \cup [0.47, 0.49]$	$0.3900_{(0.0550)}$
w_{24}	$[0.5, 0.55] \cup [0.6, 0.67] \cup [0.7, 0.8]$	$0.6367_{(0.1144)}$
w_{25}	$[0.47, 0.52] \cup [0.61, 0.62] \cup [0.65, 0.69] \cup [0.75, 0.8]$	$0.6387_{(0.0755)}$
w_{31}	$[0.45, 0.6] \cup [0.75, 0.9] \cup [0.97, 1]$	$0.7783_{(0.1491)}$
w_{41}	$[-0.95, -0.8] \cup [-0.75, -0.65]$	$-0.7875_{(0.1278)}$
w_{42}	$[0.6, 0.8] \cup [0.85, 0.95]$	$0.8000_{(0.1438)}$
w_{48}	$[-0.2, -0.1] \cup [0.1, 0.4]$	$0.0500_{(0.2250)}$
w_{52}	$[-0.6, -0.4] \cup [-0.35, -0.1]$	$-0.3625_{(0.2155)}$
w_{67}	$[0.3, 0.5] \cup [0.6, 0.8]$	$0.5500_{(0.2000)}$
w_{73}	$[0.6, 0.8] \cup [0.85, 0.95]$	$0.8000_{(0.1438)}$
w_{76}	$[0.23, 0.3] \cup [0.4, 0.5] \cup [0.6, 0.65]$	$0.4467_{(0.1061)}$
w_{83}	$[0, 0.1] \cup [0.3, 0.5]$	$0.2250_{(0.1889)}$

TABLE 6. The results of the FGCM and FGCM using general grey number in a complex situation.

concepts	FGCM's results	FGCM's results in the simplified form	FGCM using general grey number
C_1	$[0.4055, 0.5946]$	$0.5000_{(0.1891)}$	$0.5067_{(0.0703)}$
C_2	$[0.4969, 0.6455]$	$0.5712_{(0.1485)}$	$0.5734_{(0.0946)}$
C_3	$[0.5812, 0.7729]$	$0.6771_{(0.1916)}$	$0.6788_{(0.1177)}$
C_4	$[0.5905, 0.6906]$	$0.6405_{(0.1001)}$	$0.6371_{(0.0630)}$
C_5	$[0.5588, 0.6261]$	$0.5924_{(0.0673)}$	$0.5905_{(0.0559)}$
C_6	$[0.5310, 0.5454] \cup [0.5539, 0.5761] \cup [0.5808, 0.5900]$	$0.5644_{(0.0550)})$	$0.5641_{(0.0651)}$
C_7	$[0.5397, 0.5735] \cup [0.5793, 0.6176]$	$0.5775_{(0.0722)}$	$0.5769_{(0.1154)}$
C_8	$[0.4655, 0.4852] \cup [0.5148, 0.5686]$	$0.5085_{(0.0758)}$	$0.5080_{(0.1143)}$

V. THE RESULTS ANALYSIS

In this section, a more thorough analysis of the experiments above is given.

A. COMPATIBILITY AND AVAILABILITY

In Section IV-B, we run the FGCM and the proposed algorithm independently. The FGCM is extended from the FCM directly according to [40], and the FGCM using general grey number is obtained from FGCM directly. The comparison fairness is ensured by:

- The FCM, FGCM, and the proposed method share with the same structure.
- The inputs and weights of FGCM are extended from FCM directly according to [40].
- The inputs and weights of FGCM and proposed method are the same.
- The FCM, FGCM, and the proposed method iterates 10 times correspondingly.
- The FCM, FGCM, and the proposed method using the same type of activation function: sigmoid, and the parameter λ is set as 1 for all models.
- The only difference is the data structure.

The comparation results show the proposed method are compatible with the FGCM and FCM as expected: most of the results of the FGCM and the proposed method cover the results of FCM. And the kernel of the results of FGCM and the proposed algorithm is approximately equal to the FCMs.

The results of the proposed algorithm and the original FGCM have almost the same greyness, which ensures compatibility in the perspective of greyness. When FGCM processing the data, it calculates the upper bound and lower bound of the interval grey number, the greyness is calculated in an implicit and passive way. The proposed model calculates the greyness in an explicit and active way: first it calculates the kernel and then calculates the greyness directly. The uncertainty propagation of the FGCM using general grey number is another interesting problem deserving more explorations.

In Section IV-C and IV-D, we test if FGCM using general grey number can handle the multiple intervals in nodes, weights, and both of them. The results show that the proposed algorithm can cope with the occasions that the nodes and weights contain multiple intervals. A more complex case

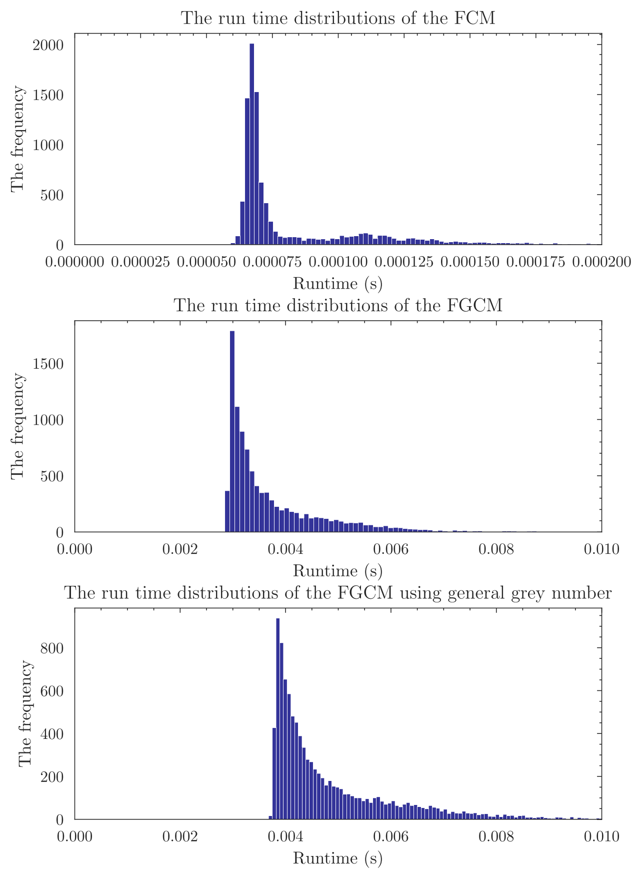


FIGURE 10. The histograms of the runtime of three algorithms.

is conducted also, it shows that the FGCM using general grey number can handle the situation where both nodes and weights contain multiple intervals. But when the original FGCM encounter such occasions, it has to run many times, the time consumed cannot be tolerant.

B. THE COMPUTATIONAL COMPLEXITY

In this part, the computational complexity is analyzed. Consider a single FCM, FGCM, and FGCM using general grey number. Because their structures are same with each other, their computational complexity is the same, it is $O(m \times n \times l)$, where m is the number of weights, n is the number of nodes, l is the number of iteration times.

Although a single FCM, FGCM, and FGCM using general grey number share the same time complexity, the runtime is different from each other. With the same m, n, and l, the proposed algorithm's runtime is longest, and the FGCM's is medium, the shortest is FCM's. The reason is that the arithmetical operations of general grey numbers are more complex than the operations of real numbers and interval grey numbers, especially for the addition. For the real number, it just needs an addition operation of float number. For the interval grey number, an addition operation needs 2 float addition operations. For the general grey number, an addition operation needs 2 *abs*() functions, 2 multiplications, 2 divisions, and three additions. So, with the same n, m, and l, the runtime of the proposed algorithm is longest.

We run the experiment in Section IV-B to show the runtime of each algorithm because the fairness is ensured in that experiment. The platform is MacOS 10.15.5, using Python 3.7, the CPU is intel core i7, 2.3 GHz. To eliminate the randomness of a single simulation, we run 10000 times experiments. The runtime distributions of each algorithm are shown in Fig. 10. It can be found that the runtime of the three algorithms obeys right-skewed distribution. The the medians of FCM's, FGCM's, and proposed algorithm's runtime are 0.000069, 0.003333, and 0.004359 correspondingly. We did a hypothesis-testing to confirm the runtimes order. Because the runtime data obey skewed distribution, Wilcoxon tests were used. The results show, in our simulation environment, the runtime of FGCM is longer than 46 times of FCM's, with the confidence of 99.96%; the runtime of FGCM using general grey number is longer than 1.297 times of the original FGCM, with the confidence of 99.93%.

However, when encountered the multiple interval input or weights, The program's time complexity becomes $O(\prod_{i=1}^{n} Cn_i \times \prod_{i=1 j=1}^{n \times n} wn_{ij} \times m \times n \times l)$, where Cn_i means the node C_i contains Cn_i intervals, if C_i is a real number, $Cn_i = 1$, n is the number of nodes, wn_i is the number of intervals w_{ij} contained, if w_{ij} is a real number, $wn_{ij} = 1$. But apply the proposed algorithm, the program's time complexity is still $O(m \times n \times l)$. Thus, in the perspective of runtime, when $\prod_{i=1}^{n} Cn_i \times \prod_{i=1 j=1}^{n \times n} wn_{ij} \geq 1.297$, i.e., once the multiple intervals encountered, the proposed model is more appropriate to be used.

VI. CONCLUSIONS AND FUTURE WORK

As a soft computing algorithm, FCM's uncertainty modeling ability has been improved by FGCM. But FGCM's uncertain modeling ability is limited due to the fact that it can only deal with an interval grey number. Thus, the aim of this paper is to enhance the FGCM's uncertainty modeling ability.

This paper introduces the general grey number into FGCM to make it have the ability to deal with multiple interval data. And two activation functions are deduced according to GST and Taylor series. In this way, the FGCM using general grey number is proposed. It transfers FGCM's interval grey number into the general grey number. The new algorithm's arithmetic operation and activation function are different from the FGCM. Finally, three experiments are conducted to validate the proposed algorithm of which the industrial process control problem is the background. The first experiment is employed to show the new model is compatible with FCM and FGCM. The second experiment shows the new model's characteristics which the original FCM and FGCM do not have. And the third experiment shows that the proposed algorithm can handle the more complex situation: both nodes and weights values are comprised of multiple intervals.

In general, the main finding of this paper are as follows:

- The new model has a stronger uncertainty modeling ability since the general grey number is applied in the FGCM.
- Two activation functions in the form of the general grey number: sigmoid and tanh, make the new model can do the reasoning under the uncertainty represented by the general grey number.
- The reasoning ability and availability are proven by the industrial process control problem.

The new algorithm combines the new theoretical achievements of GST and FCM. The FCM provides the basic algorithm framework, and GST provides the representation and arithmetic operation method. The general grey number's characteristics make the FGCM can process multiple intervals, and the greyness will be kept during the inference. It can be used in the situation where exists more uncertainty, such as two or more sensors' data, experts' suggestions.

The proposed algorithm with general grey number increases the complexity of the single FGCM, but when encountered with the multiple interval input or weights, the time complexity keeps the same. The other limitation is that the results of the proposed method will be influenced by the extreme values, but this can be easily overcome by data cleaning or imputing in the data pre-processing.

Finally, there is still a lot of work to be undertaken in the future, such as the investigation of convergence property, the greyness propagation, the greyness's convergence, the applications of the FGCM using general grey number *et al*. And more theoretical and application researches should be conducted in order to construct an integrated theoretical system of FGCM using general grey number, i.e., the Fuzzy General Grey Cognitive Map (FGGCM).

REFERENCES

[1] D. Ibrahim, "An overview of soft computing," *Procedia Comput. Sci.*, vol. 102, pp. 34–38, Jan. 2016.

[2] B. Kosko, "Fuzzy cognitive maps," *Int. J. Man-Mach. Stud.*, vol. 24, no. 1, pp. 65–75, 1986.

[3] E. I. Papageorgiou and J. L. Salmeron, "A review of fuzzy cognitive maps research during the last decade," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 1, pp. 66–79, Feb. 2013.

[4] H. Doukas and A. Nikas, "Decision support models in climate policy," *Eur. J. Oper. Res.*, vol. 280, no. 1, pp. 1–24, Jan. 2020.

[5] L. V. R. Arruda, M. Mendonca, F. Neves, I. R. Chrun, and E. I. Papageorgiou, "Artificial life environment modeled by dynamic fuzzy cognitive maps," *IEEE Trans. Cognit. Develop. Syst.*, vol. 10, no. 1, pp. 88–101, Mar. 2018.

[6] A. Awasthi and H. Omrani, "A scenario simulation approach for sustainable mobility project evaluation based on fuzzy cognitive maps," *Int. J. Model. Simul.*, vol. 38, no. 4, pp. 262–272, 2018.

[7] N. Manoharan and A. Thangavelu, "An experimental evaluation of integrated dematal and fuzzy cognitive maps for cotton yield prediction," *Cogn. Sci. Artif. Intell.*, pp. 31–43, 2018.

[8] H. S. Firmansyah, S. H. Supangkat, A. A. Arman, and P. J. Giabbanelli, "Identifying the components and interrelationships of smart cities in Indonesia: Supporting policymaking via fuzzy cognitive systems," *IEEE Access*, vol. 7, pp. 46136–46151, 2019.

[9] F. J. Aparicio-Navarro, K. G. Kyriakopoulos, Y. Gong, D. J. Parish, and J. A. Chambers, "Using Pattern-of-Life as contextual information for anomaly-based intrusion detection systems," *IEEE Access*, vol. 5, pp. 22177–22193, 2017.

[10] C. D. Stylios and P. P. Groumpos, "Fuzzy cognitive maps in modeling supervisory control systems," *J. Intell. Fuzzy Syst.*, vol. 8, pp. 83–98, Mar. 2000.

[11] P. P. Groumpos and C. D. Stylios, "Modelling supervisory control systems using fuzzy cognitive maps," *Chaos, Solitons Fractals*, vol. 11, nos. 1–3, pp. 329–336, Jan. 2000.

[12] C. D. Stylios and P. P. Groumpos, "Fuzzy cognitive maps: A model for intelligent supervisory control systems," *Comput. Ind.*, vol. 39, no. 3, pp. 229–238, Jul. 1999.

[13] J. L. Salmeron and E. I. Papageorgiou, "Using fuzzy grey cognitive maps for industrial processes control," *Fuzzy Cogn. Maps Appl. Sci. Eng.*, pp. 237–252, 2014.

[14] Y. Boutalis, T. L. Kottas, and M. Christodoulou, "Adaptive estimation of fuzzy cognitive maps with proven stability and parameter convergence," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 874–889, Aug. 2009.

[15] J. L. Salmeron, "Modelling grey uncertainty with fuzzy grey cognitive maps," *Expert Syst. Appl.*, vol. 37, no. 12, pp. 7581–7588, Dec. 2010.

[16] J. L. Salmeron and E. I. Papageorgiou, "A fuzzy grey cognitive maps-based decision support system for radiotherapy treatment planning," *Knowl.-Based Syst.*, vol. 30, pp. 151–160, Jun. 2012.

[17] J. L. Deng, "Introduction to grey system theory," *J. Grey Syst.*, vol. 1, no. 1, pp. 1–24, 1989.

[18] J. L. Salmeron and P. R. Palos-Sanchez, "Uncertainty propagation in fuzzy grey cognitive maps with Hebbian-like learning algorithms," *IEEE Trans. Cybern.*, vol. 49, no. 1, pp. 211–220, Jan. 2019.

[19] F. Xiao, "CED: A distance for complex mass functions," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Apr. 17, 2020, doi: 10.1109/TNNLS.2020.2984918.

[20] F. Xiao, "A new divergence measure for belief functions in D–S evidence theory for multisensor data fusion," *Inf. Sci.*, vol. 514, pp. 462–483, Apr. 2020.

[21] S. Liu, Z. Fang, Y. Yang, and J. Forrest, "General grey numbers and their operations," *Grey Syst., Theory Appl.*, vol. 2, no. 3, pp. 341–349, Oct. 2012, doi: 10.1108/20439371211273230.

[22] Y. Yang, "Extended grey numbers and their operations," in *Proc. IEEE Int. Conf. Syst., Man Cybern.*, Oct. 2007, pp. 2181–2186.

[23] Y. Yang, "Extended grey numbers," *Adv. Grey Syst. Res.*, 2010, pp. 73–85.

[24] Y. Yang and R. John, "Grey sets and greyness," *Inf. Sci.*, vol. 185, no. 1, pp. 249–264, Feb. 2012.

[25] S.-Q. Jiang, S.-F. Liu, Z.-G. Fang, and Z.-X. Liu, "Study on distance measuring and sorting method of general grey number," *Grey Syst., Theory Appl.*, vol. 7, no. 3, pp. 320–328, Nov. 2017.

[26] H. Kuang, K. W. Hipel, D. M. Kilgourt, and M. A. Bashar, "A case study of grey-based preference in a graph model for conflict resolution with two decision makers," in *Proc. IEEE Int. Conf. Syst., Man, Cybern. (SMC)*, Oct. 2013, pp. 2037–2041.

[27] L. Qian, S. Liu, and Z. Fang, "Grey risky multi-attribute decision-making method based on regret theory and EDAS," *Grey Syst., Theory Appl.*, vol. 9, no. 1, pp. 101–113, Feb. 2019.

[28] Z.-X. Wang and Q. Li, "Modelling the nonlinear relationship between CO_2 emissions and economic growth using a PSO algorithm-based grey Verhulst model," *J. Cleaner Prod.*, vol. 207, pp. 214–224, Jan. 2019.

[29] J. Chen, X. Gao, X. Chen, and Q. He, "A shifting method for intelligent operational mode of UAVs," in *Proc. 15th Int. Conf. Control, Autom., Robot. Vis. (ICARCV)*, Nov. 2018, pp. 526–531.

[30] H.-C. Liu, L.-E. Wang, X.-Y. You, and S.-M. Wu, "Failure mode and effect analysis with extended grey relational analysis method in cloud setting," *Total Qual. Manage. Bus. Excellence*, vol. 30, nos. 7–8, pp. 745–767, May 2019.

[31] S.-L. Ou, "Forecasting agricultural output with an improved grey forecasting model based on the genetic algorithm," *Comput. Electron. Agricult.*, vol. 85, pp. 33–39, Jul. 2012.

[32] Z. Deng, Y. Ke, H. Gong, X. Li, and Z. Li, "Land subsidence prediction in Beijing based on PS-InSAR technique and improved grey-Markov model," *GISci. Remote Sens.*, vol. 54, no. 6, pp. 797–818, Nov. 2017.

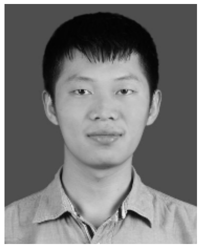
[33] H.-X.-T. Ho, "Forecasting of CO_2 emissions, renewable energy consumption and economic growth in Vietnam using grey models," in *Proc. 4th Int. Conf. Green Technol. Sustain. Develop. (GTSD)*, Nov. 2018, pp. 452–455.

[34] S. Liu, Y. Yang, and J. Forrest, "Grey numbers and their operations," *Grey Data Anal.*, pp. 29–43, 2017.

[35] Y. Yang and S. Liu, "Reliability of operations of grey numbers using kernels," *Grey Syst., Theory Appl.*, vol. 1, no. 1, pp. 57–71, Jan. 2011.

[36] C. D. Stylios and P. P. Groumpos, "The challenge of modelling supervisory systems using fuzzy cognitive maps," *J. Intell. Manuf.*, vol. 9, pp. 339–345, Aug. 1998.
[37] C. D. Stylios, P. P. Groumpos, and V. C. Georgopoulos, "Fuzzy cognitive maps modelling supervisory large scale control systems," *IFAC Proc. Volumes*, vol. 31, no. 20, pp. 93–98, Jul. 1998.
[38] E. I. Papageorgiou, C. Stylios, and P. P. Groumpos, "Unsupervised learning techniques for fine-tuning fuzzy cognitive map causal links," *Int. J. Hum.-Comput. Stud.*, vol. 64, no. 8, pp. 727–743, Aug. 2006.
[39] A. Baykasoglu, Z. D. U. Durmusoglu, and V. Kaplanoglu, "Training fuzzy cognitive maps via extended great deluge algorithm with applications," *Comput. Ind.*, vol. 62, no. 2, pp. 187–195, Feb. 2011.
[40] J. L. Salmeron and E. I. Papageorgiou, "Fuzzy grey cognitive maps and nonlinear Hebbian learning in process control," *Int. J. Speech Technol.*, vol. 41, no. 1, pp. 223–234, Jul. 2014.
[41] G. Nápoles, E. Papageorgiou, R. Bello, and K. Vanhoof, "On the convergence of sigmoid fuzzy cognitive maps," *Inf. Sci.*, vols. 349–350, pp. 154–171, Jul. 2016.

JUN CHEN (Member, IEEE) was born in Jiangxi, China, in 1979. He received the B.S., M.S., and Ph.D. degrees in system engineering from Northwestern Polytechnical University, China, in 2001, 2005, and 2009, respectively. Since 2012, he has been an Associate Professor with the School of Electronics and Information, Northwestern Polytechnical University. His current research interests include modeling and application based on fuzzy cognitive map and intelligent decision-making for complex autonomous system.



XUDONG GAO was born in Shanxi, China, in 1996. He received the B.S. degree in detection guidance and control technology from Northwestern Polytechnical University, China, in 2018, where he is currently pursuing the M.S. degree. His research interests include complex system modeling, data mining, and machine learning.

JIA RONG was born in 1981. She received the Ph.D. degree from Deakin University, in 2012. Her current research interests are data mining, pattern recognition, and complex data analysis.