ENHANCED TRANSPORT IN TOKAMAKS DUE TO TOROIDAL RIPPLE

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ABSTRACT

A method for evaluating transport in non-symmetric systems is developed and applied to a previously unstudied ripple collisionality regime of tokamaks. This collisionality regime, the ripple plateau, is the regime of primary importance both for present day and reactor scale tokamaks. The results can be directly applied to related systems like the toroidal z pinch.

I. INTRODUCTION

The lack of toroidal symmetry in tokamaks due to the finite number of toroidal field coils results in enhanced transport coefficients. The enhancement has been evaluated by a number of authors.[1-4] Unfortunately, the ripple collisionality regime of most importance to tokamaks, the ripple plateau, was not studied. In this paper, a general method of calculating transport coefficients in non-symmetric geometries is developed and the ripple plateau transport coefficients are evaluated. The methods developed here imply corrections are required to the earlier work on the banana drift[3,4] transport. However, these results will be given in a later paper. The methods of evaluating the transport can be applied to other non-symmetric devices like stellarators and the results are given in a form which is applicable to other systems closely related to the tokamak, like the toroidal Z pinch.

The fundamental problem in the study of ripple transport is the separation of the ordinary neo-classical and the ripple effects. In practice, this means separating the particle drift into a part which returns to the same magnetic surface after a bounce cycle of the trapped particles and a part which does not. The drift of passing particles will be shown not to be significantly affected by ripple provided good magnetic surfaces exist. This separation of neo-classical and ripple effects is the subject of Section II.

In Section III the ripple plateau transport coefficients are evaluated. Although the transport coefficients maintain constant values, as a function of collision frequency throughout the ripple plateau, the mathematical method changes. In Section III A, the more collisional ripple plateau is evaluated. In that section we assume the collision frequency ν lies in the range $Nv/R >> \nu >> \nu_*/(Nq)^{\frac{1}{2}}$ with v the particle velocity, N the toroidal mode number of the ripple, $\nu_* = \varepsilon^{3/2} v/qR$ the critical collisionality for the ordinary banana regime, ε the inverse aspect ratio, and R the major radius. In Section III B the less collisional part of the ripple plateau regime is evaluated assuming $\nu_*/(Nq)^{\frac{1}{2}} >> \nu >> \nu_*/(Nq)^2$. The dominant ripple plateau transport coefficient is the ion thermal conductivity which will be found to be Eq. 55

$$K = 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}} nN <\delta^2> \rho_\theta^2 \frac{v_{th}}{R} \frac{B}{B_\phi}$$

with n the plasma density $<\delta^2>$ twice the average of the ripple over the magnetic surface, $\rho_\theta = v_{th}/(eB_\theta/mc)$ the poloidal gyroradius, $v_{th} = (T/m)^{\frac{1}{2}}$ the thermal velocity and B and B_ϕ the total toroidal component of the magnetic field.

II. BASIC EQUATIONS

The study of ripple diffusion is made easier by the separability of the ripple and neo-classical transport. In this section, the drift kinetic equation for a tokamak with ripple is broken into two equations

$$\vec{v}_{NC}\cdot\vec{\nabla}F = C(F) \quad , \quad \vec{v}_R\cdot\vec{\nabla}f = C(f) \ .$$

The first equation gives the usual neo-classical distribution function. The second equation gives the response of the distribution function to ripple effects. Remarkably, the separability of the kinetic equation is not based on the smallness of the ripple. The formalism developed in this section can be applied to other non-symmetric geometries such as the stellarator.

To demonstrate separability, the drift velocity is written as[5]

$$\vec{v} = \frac{v_{\parallel}}{B}\,(\vec{B} + \vec{\nabla}\times\rho_{\parallel}\vec{B}) \tag{1}$$

with $\rho_{\parallel} = v_{\parallel}/(eB/mc)$. The derivatives of $v_{\parallel}$ are defined by the energy equation

$$E = \tfrac{1}{2} m v_{\parallel}^2 + \mu B + e\Phi \tag{2}$$

with the energy and magnetic moment considered as constants.

To divide $\vec{v}$ into a neoclassical and a ripple part, coordinates ψ, θ, ϕ_o, are defined so that

$$\vec{B} = \vec{\nabla}\phi_o \times \vec{\nabla}\psi \tag{3}$$

and the field strength in the related symmetric system depends only on ψ and θ. The quantity ψ is used to label the pressure surfaces and can be thought of as the poloidal magnetic flux. In a tokamak one would normally define θ so that the toroidal angle ϕ is given by

$$\phi = \phi_o + q(\psi)\theta$$

with $q(\psi)$ the safety factor. This choice is not required, however, by the formalism. Using vector identities one can write

$$\rho_{\parallel}\vec{B} = \rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\psi \times \vec{\nabla}\theta)}{\vec{B}\cdot\vec{\nabla}\theta}\,\vec{\nabla}\phi_o + \rho_{\parallel}\frac{\vec{B}\cdot(\vec{\nabla}\theta \times \vec{\nabla}\phi_o)}{\vec{B}\cdot\vec{\nabla}\theta}\,\nabla\psi + \rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\phi_o \times \vec{\nabla}\psi)}{\vec{B}\cdot\vec{\nabla}\theta}\,\vec{\nabla}\theta \,. \tag{4}$$

Using $\vec{B}\cdot(\vec{\nabla}\phi_o \times \vec{\nabla}\psi) = B^2$, neoclassical and ripple drift velocities are defined by

$$\vec{v}_{NC} \equiv \frac{v_{\parallel}}{B}\left[\vec{B} + \vec{\nabla}\times\left\{\rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\psi \times \vec{\nabla}\theta)}{\vec{B}\cdot\vec{\nabla}\theta}\vec{\nabla}\phi_o + \rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\theta \times \vec{\nabla}\phi_o)}{\vec{B}\cdot\vec{\nabla}\theta}\vec{\nabla}\psi\right\}\right] \tag{5}$$

$$\vec{v}_R \equiv \frac{v_{\parallel}}{B}\left[\vec{B} + \vec{\nabla}\times\left\{\rho_{\parallel}\,\frac{B^2}{\vec{B}\cdot\vec{\nabla}\theta}\,\vec{\nabla}\theta\right\}\right] \,. \tag{6}$$

The justification of the division of the drift velocity into two parts requires an examination of the components of $\vec{v}_{NC}$ and $\vec{v}_R$ as well a study of the drift kinetic equation. Retaining only the lowest nonvanishing terms in gyroradius to system-size

$$\vec{v}\cdot\vec{\nabla}\theta = \vec{v}_{NC}\cdot\vec{\nabla}\theta = \vec{v}_R\cdot\vec{\nabla}\theta = v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B} \tag{7}$$

$$\vec{v}\cdot\vec{\nabla}\psi = \vec{v}_{NC}\cdot\vec{\nabla}\psi + \vec{v}_R\cdot\vec{\nabla}\psi \quad , \quad \vec{v}\cdot\vec{\nabla}\phi_o = \vec{v}_{NC}\cdot\vec{\nabla}\phi_o + \vec{v}_R\cdot\vec{\nabla}\phi_o \ . \tag{8}$$

One also finds

$$\vec{v}_{NC}\cdot\vec{\nabla}\psi = v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial}{\partial\theta}\left[\rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\psi\times\vec{\nabla}\theta)}{\vec{B}\cdot\vec{\nabla}\theta}\right] \tag{9}$$

$$\vec{v}_{NC}\cdot\vec{\nabla}\phi_o = -v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial}{\partial\theta}\left[\rho_{\parallel}\,\frac{\vec{B}\cdot(\vec{\nabla}\theta\times\vec{\nabla}\phi_o)}{\vec{B}\cdot\vec{\nabla}\theta}\right] \tag{10}$$

$$\vec{v}_R\cdot\vec{\nabla}\psi = -v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial}{\partial\phi_o}\left(\rho_{\parallel}\,\frac{B^2}{\vec{B}\cdot\vec{\nabla}\theta}\right) \tag{11}$$

$$\vec{v}_R\cdot\vec{\nabla}\phi_o = v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial}{\partial\psi}\left(\rho_{\parallel}\,\frac{B^2}{\vec{B}\cdot\vec{\nabla}\theta}\right) \ . \tag{12}$$

Assuming the distribution function is close to Maxwellian, the drift kinetic equation can be written

$$v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial F}{\partial\theta} + \vec{v}\cdot\vec{\nabla}\phi_o\,\frac{\partial F}{\partial\phi_o} - C(F) = -\vec{v}\cdot\vec{\nabla}\psi\,\frac{\partial f_m}{\partial\psi} \ . \tag{13}$$

The distribution function F is written in two parts F and f . The first function F is driven by the inhomogeneous term $\vec{v}_{NC}\cdot\vec{\nabla}\psi\,\partial f_M/\partial\psi$. The second function f is driven by $\vec{v}_R\cdot\vec{\nabla}\psi\,\partial f_M/\partial\psi$. That is

$$v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial F}{\partial\theta}+\vec{v}\cdot\vec{\nabla}\phi_o\,\frac{\partial F}{\partial\phi_o}-C(F)=-\vec{v}_{NC}\cdot\vec{\nabla}\psi\,\frac{\partial f_M}{\partial\psi}\tag{14}$$

$$v_{\parallel}\,\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\,\frac{\partial f}{\partial\theta}+\vec{v}\cdot\vec{\nabla}\phi_o\,\frac{\partial f}{\partial\phi_o}-C(f)=-\vec{v}_R\cdot\vec{\nabla}\psi\,\frac{\partial f_M}{\partial\psi}\quad .\tag{15}$$

The parallel motion $v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta/B$ is much larger than the drift within the magnetic surface provided the poloidal gyroradius is small compared to system dimensions. This implies $\vec{v}\cdot\vec{\nabla}\phi_o$ can be neglected in the neoclassical equation. However, this term cannot be neglected in the ripple equation. The difference is the average of the driving term, $v\cdot\nabla\psi\,\partial f_M/\partial\psi$, over a particle's transit or bounce motion. It is zero for the neoclassical case but not for the ripple case. That is

$$\oint \vec{v}_{NC}\cdot\vec{\nabla}\psi\,\frac{1}{v_{\parallel}}\,\frac{B}{\vec{B}\cdot\vec{\nabla}\theta}\,d\theta=0\quad .$$

Of course only the transit or bounce average of $\vec{v}\cdot\vec{\nabla}\phi_o$ affects the ripple kinetic equation and this average of $\vec{v}_{NC}\cdot\vec{\nabla}\phi_o$ is zero. If negligible terms are eliminated from the kinetic equations , it is clear they can be written in the desired forms

$$\vec{v}_{NC}\cdot\vec{\nabla}F=C(F)\quad ,\qquad \vec{v}_R\cdot\vec{\nabla}f=C(f)\quad .\tag{16}$$

The dependence of the particle and heat flux on F , f , $\vec{v}_{NC}$, and $\vec{v}_R$ still requires clarification. Let us consider the particle flux. The total flux of particles crossing a ψ surface is

$$\Gamma_T = \oint_{\psi} d\vec{S} \cdot \oint d^3 v \; \vec{v} F_E \tag{17}$$

with

$$d\vec{S} = \frac{\vec{\nabla}\psi}{\vec{B}\cdot\vec{\nabla}\theta} \, d\theta \, d\phi_o \quad , \qquad d^3 v = \frac{4\pi}{m^2} \frac{B}{|v_{||}|} \, dE \, d\mu \quad . \tag{18}$$

and F_E the even part of the distribution function in $v_{||}$. This total flux can be divided into a neoclassical and a ripple flux by using $\vec{v}\cdot\vec{\nabla}\psi = \vec{v}_{NC}\cdot\vec{\nabla}\psi + \vec{v}_R\cdot\vec{\nabla}\psi$. The neoclassical flux turns out to depend only on the parallel current while the ripple flux is closely related to the pressure tensor or parallel viscosity. First consider the neoclassical flux,

$$\Gamma_{NC} = \oint_{\psi} d\vec{S} \cdot \int d^3 v \; \vec{v}_{NC} F_E \quad . \tag{19}$$

By integration by parts this can be written

$$\Gamma_{NC} = \oint_{\psi} d\vec{S} \cdot \int dE \, d\mu \; \frac{4\pi c}{me} (\vec{B} \times \vec{\nabla}\theta) \; \frac{|v_{||}|}{B} \frac{\partial F_E}{\partial \theta} \quad .$$

The drift kinetic equation implies

$$|v_{||}| \frac{\vec{B}\cdot\vec{\nabla}\theta}{B} \frac{\partial F_E}{\partial \theta} = C(F_o)$$

with F_o the odd part of the distribution function in $v_{\parallel}$.
Therefore,

$$\Gamma_{NC} = \oint_{\psi} d\vec{S} \cdot \left[\frac{\vec{B} \times \vec{\nabla}\theta}{\vec{B} \cdot \vec{\nabla}\theta} \frac{mc}{eB} \int v_{\parallel}\, C(F)\, d^3v \right]$$

but the parallel resistivity and current are given by

$$\eta_{\parallel} J_{\parallel} = - \frac{m}{ne} \int v_{\parallel}\, C(F)\, d^3v \qquad ,$$

therefore

$$\Gamma_{NC} = \oint_{\psi} d\vec{S} \cdot \left[\frac{\vec{\nabla}\theta}{\vec{B} \cdot \vec{\nabla}\theta} \times (nc\, \eta_{\parallel} J_{\parallel} \frac{\vec{B}}{B}) \right] \quad . \tag{20}$$

Since $\eta_{\parallel} J_{\parallel}$ is the same for electrons and ions the neoclassical flux is intrinsically ambipolar. A similar expression can be found for the total radial energy flux. The energy flux expression depends on the parallel heat flux $q_{\parallel}$ and the parallel thermal conductivity $K_{\parallel}$.

The ripple particle flux can be shown to depend only on the parallel viscosity of the plasma. To do this, the ripple velocity is written

$$\vec{v}_R \cdot \vec{\nabla}\psi = \frac{c}{e} \left[\mu \frac{\partial B}{\partial \phi_o} - m v_{\parallel}^2 \frac{\vec{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial}{\partial \phi_o} \frac{B}{\vec{B} \cdot \vec{\nabla}\theta} + e \frac{\partial \Phi}{\partial \phi_o} \right] \quad . \tag{21}$$

Using

$$P_{\|} = \int mv_{\|}^2 F_E\, d^3v \quad , \qquad P_{\perp} = \oint \mu B\, F_E\, d^3v \tag{22}$$

and integrating the term depending $\partial(B/\vec{B}\cdot\vec{\nabla}\theta)/\partial\phi_o$ by parts, one finds

$$\Gamma_R = \frac{c}{e}\oint_{\psi} dS\left[\frac{\partial P_{\|}}{\partial\phi_o} - (P_{\|} - P_{\perp})\frac{1}{B}\frac{\partial B}{\partial\phi_o} + en\frac{\partial\Phi}{\partial\phi_o}\right]$$

with $dS = d\theta\, d\phi_o/\vec{B}\cdot\vec{\nabla}\theta$. The pressure tensor $\overleftrightarrow{P}$ is defined by

$$\overleftrightarrow{P} = P_{\|}\hat{b}\hat{b} + P_{\perp}(\overleftrightarrow{\delta} - \hat{b}\hat{b})$$

with $\overleftrightarrow{\delta}$ the unit tensor. It can then be shown that

$$\vec{\nabla}\cdot\overleftrightarrow{P} = \vec{\nabla}P_{\|} - (P_{\|} - P_{\perp})\frac{\vec{\nabla}B}{B} - \vec{B}\times\left[\vec{\nabla}\times\left(\frac{P_{\|} - P_{\perp}}{B^2}\right)\right] .$$

If both sides of this equation are dotted with $(\vec{\nabla}\psi\times\vec{\nabla}\theta)/\vec{B}\cdot\vec{\nabla}\theta$, one finds

$$\frac{\partial P_{\|}}{\partial\phi_o} - (P_{\|} - P_{\perp})\frac{1}{B}\frac{\partial B}{\partial\phi_o} = \vec{\nabla}\psi\cdot\left[\frac{\vec{\nabla}\theta}{B\cdot\nabla\theta}\times\vec{\nabla}\cdot\overleftrightarrow{P} - \vec{\nabla}\times\left(\frac{P_{\|} - P_{\perp}}{B^2}\vec{B}\right)\right] .$$

Since the integral of a curl over a closed surface vanishes,

$$\Gamma_R = \frac{c}{e}\oint_{\psi} d\vec{S}\cdot\left[\frac{\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\times(\vec{\nabla}\cdot\overleftrightarrow{P} + en\vec{\nabla}\Phi)\right] . \tag{23}$$

Since like particle collisions can cause an anisotropic pressure, the ripple transport is not intrinsically ambipolar.

The expression for the total particle flux crossing a magnetic surface, $\Gamma_T = \Gamma_{NC} + \Gamma_R$ has a simple physical interpretation. Consider the usual fluid equation

$$-\vec{\nabla}\phi + \frac{1}{c}\vec{v}\times\vec{B} = \eta\vec{J} + \frac{1}{en}\vec{\nabla}\cdot\overleftrightarrow{P} \quad . \tag{24}$$

If this expression is crossed with $\vec{\nabla}\theta/\vec{B}\cdot\vec{\nabla}\theta$, the result is

$$\vec{v} - \frac{\vec{v}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\vec{B} = c\,\frac{\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\times\left[\eta\vec{J} + \frac{1}{en}\vec{\nabla}\cdot\overleftrightarrow{P} + \vec{\nabla}\phi\right]$$

and

$$\Gamma_T = \oint_\psi n\vec{v}\cdot d\vec{S}$$

$$= \oint_\psi d\vec{S}\cdot\left[\frac{\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\times\left(nc\,\eta\vec{J} + \frac{c}{e}\vec{\nabla}\cdot\overleftrightarrow{P} + cn\,\vec{\nabla}\phi\right)\right] \quad . \tag{25}$$

The drift kinetic and fluid results for the total transport agree except for the well known fact that drift kinetic approach leaves out the contribution of the perpendicular current to the transport. The various terms in the integral for Γ_T represent toroidal torques placed on the species being calculated. To see this, note that in a torus a force $\vec{f}$ and the toroidal torque T are related by $T = Rf_\phi$ with R the local major radius. Consequently, $\vec{f}$ can be written using $\vec{\nabla}\phi = \hat{\phi}/R$ as

$$\vec{f} = T\,\vec{\nabla}\phi + f_\theta \vec{\nabla}\theta + f_\psi \vec{\nabla}\psi$$

and

$$T = \vec{\nabla}\psi \cdot \left(\frac{\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta} \times \vec{f} \right) \quad . \tag{26}$$

The evaluation of the particle and heat fluxes would be considerably simplified if only the neoclassical distribution function were needed to calculate the neoclassical transport and the ripple function for the ripple transport. Unfortunately, this is not in general true. However, one finds this separation is often a very good approximation.

If one is using the approximation that the ripple transport depends only on the ripple distribution function, one can obtain the ripple diffusion coefficients by imagining a plasma in which the drift velocity is $\vec{v}_R$. This drift velocity has an interesting feature of being derivable from a Hamiltonian in ψ, θ, ϕ_o coordinates for an arbitrary magnetic field. Let $\rho_* \equiv \rho_\parallel B^2/B\cdot\vec{\nabla}\theta$. Physically ρ_* is c/e times the poloidal angular momentum in the parallel motion. The Hamiltonian is then c/e times $mv_\parallel/2 + \mu B + e\Phi$ or

$$H(\psi\, , \theta\, , \phi_o\, , \rho_*) = \frac{1}{2}\,\frac{e}{mc}\left(\frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\right)^2 \rho_*^2 + \frac{c}{e}\,\mu B + c\Phi \tag{27}$$

with

$$\dot{\theta} = \frac{\partial H}{\partial \rho_*} \quad , \quad \dot{\rho}_* = -\frac{\partial H}{\partial \theta}$$

$$\dot{\psi} = \frac{\partial H}{\partial \phi_o} \quad , \quad \dot{\phi}_o = -\frac{\partial H}{\partial \psi} \quad . \tag{28}$$

The longitudinal invariant J is

$$J = \frac{e}{c} \oint \rho_* \, d\theta = \oint m v_{\parallel} d\ell \tag{29}$$

noting that the differential distance along the field lines is $d\ell = (B/\vec{B}\cdot\vec{\nabla}\theta)d\theta$.

In this paper we assume exact magnetic surfaces exist. That is, we assume there exists a well-behaved and non-trivial function of position P such that $\vec{B}\cdot\vec{\nabla}P = 0$. Under this circumstance, we will show that highly passing particles have no systematic radial drift due to ripple, which generally implies they contribute negligibly to the ripple transport. What is actually shown is that highly passing particles have an additional constant of the motion besides E and μ , which we call K . The drift velocity is parallel to the vector[5] $\vec{H} = \vec{B} + \vec{\nabla} \times (\rho_{\parallel} \vec{B})$. Since $\vec{H}$ is divergence free it can be thought of as a magnetic field. In particular, its "magentic" differential equation $\vec{H}\cdot\vec{\nabla}K = 0$ can be solved for its "magnetic" surfaces. In mechanics terminology K is just a constant of the motion. In toroidal symmetry, $\vec{B} = g\vec{\nabla}\phi + \vec{\nabla}\phi \times \vec{\nabla}\psi$ and one can easily show K , which is a function of $\psi_* = \psi - g\rho_{\parallel}$, satisfies the "magnetic" differential equation. The constant of the motion ψ_* is just $(c/e)\,P_\phi$ with P_ϕ the conserved toroidal component of the cannonical momentum. Now the vector $\vec{H}$ is very close to the vector $\vec{B}$ for highly passing particles provided the gyroradius ρ_θ in the field component $\vec{B}\cdot\vec{\nabla}\theta$ is small compared to the system length scale a . Indeed, the difference between $\vec{H}$ and $\vec{B}$ is of order $(\rho_\theta/a)(v/v_\parallel)$. Consequently, if $|v_\parallel/v| >> \rho_\theta/a$ one expects $\vec{H}$ to have good surfaces when $\vec{B}$ does. This condition is, of course, never satisfied for trapped particles near turning points.

III. PLATEAU TRANSPORT

The ripple transport coefficients in the collisionality regimes of primary importance for a tokamak reactor are of the plateau form. That is, they are independent of the collision frequency. The collisionality range of the ripple plateau regime is broad. For this regime, one requires the mean free path v/ν be much longer than R/N with R the major radius and N the toroidal mode number of the ripple, but $\nu_*/\nu < (Nq)^2$ with $\nu_* = \epsilon^{1/2}\, v/qR$, ϵ the inverse aspect ratio, v the particle velocity, q the safety factor, and ν the collision frequency. The factor ν_*/ν is essentially the number of banana orbits a trapped particle executes before scattering into the passing orbit part of phase space. For calculation purposes, the ripple plateau consists of two collisionality regimes. In the higher collisionality regime, particles with small pitch, $v_\parallel/v$, suffer significant collisions while crossing a single phase of the ripple. In the lower collisionality regime, even particles with zero pitch can cross a number of phases of the ripple without suffering significant collisions, however, significant collisional effects remain during the time trapped particles execute their banana orbits. Passing particles do not contribute in any signficant way to ripple plateau transport. The division between the two ripple plateau regimes is $\nu_*/\nu \simeq (Nq)^{1/2}$.

IIIA RIPPLE PLATEAU (MORE COLLISIONAL CASE)

First, let us consider a heuristic derivation of the more collisional type of ripple plateau transport. The more collisional ripple plateau is caused by particles of such low pitch $\lambda = v_\parallel/v$ that they suffer significant collisions while crossing a single phase of the ripple. Let λ_c be the critical value of the pitch. The effective collision time τ_{eff} is the time a particle with $|\lambda| < \lambda_c$ remains at such low pitch. This time is roughly $\tau_{eff} = \lambda_c^2/\nu$ with ν the 90^0 collision frequency. The transit time through a phase of the ripple is $\tau_T = R/N\lambda_c v$ with R the major radius, N the mode number of the ripple, and v the particle velocity. Equating τ_{eff} and τ_T and including a factor of two for future consistency, one finds $\lambda_c = (\nu R/2vN)^{1/3}$.

In a tokamak reactor $\lambda_c \approx 0.05$. The radial drift due to the ripple will be shown to be

$$v_r \simeq N\delta \frac{\rho_\theta}{R} v \cos(N\phi)$$

with δ the ripple magnitude and ρ_θ the gyroradius in the poloidal component of the magnetic field. The diffusion coefficient is

$$D \simeq \lambda_c \; (v_r \tau_T)^2 / \tau_{eff}$$

$$\simeq N \;\; \delta^2 \;\; \rho_\theta^2 \, \frac{v}{R} \tag{30}$$

The factor λ_c in D is the fraction of the particles participating in the transport and $v_r \tau_T$ their radial excursion. Of course, $\tau_T = \tau_{eff} = \lambda_c^2/\nu$.

Ripple transport is evaluated with the ripple kinetic equation

$$\vec{v}_R \cdot \vec{\nabla} f = C(f) \;\; .$$

The spacial coordinates we will use are ψ , θ , ϕ_o so $\vec{B} = \vec{\nabla}\phi_c \times \vec{\nabla}\psi$. The ordinary toroidal angle is given by $\phi = \phi_o + q(\psi)\theta$. The coordinate ϕ_o is physically the toroidal position of a field line when it crosses the mid-plane of the torus.

Assuming $\rho_\theta/a \ll 1$, $\vec{v}_R\cdot\vec{\nabla}\theta$ is given by Eq. 7 and $\vec{v}_R\cdot\vec{\nabla}\phi_o$ is negligible throughout the plateau regime. An expression for $\vec{v}_R\cdot\vec{\nabla}\psi$ is given by Eq. 21 . However, this expression can be considerably simplified. The term proportional to $mv_\parallel^2$ can be neglected, for trapped particles dominate the transport. The term $\partial\Phi/\partial\phi_o$ is also negligible, but for a somewhat more subtle reason. Eq. 23 implies that the magnitude of $\vec{\nabla}\cdot\overleftrightarrow{P}$ within a magnetic surface, $|\vec{\nabla}\cdot\overleftrightarrow{P}|_S$, is of order

$$|\vec{\nabla}\cdot\overleftrightarrow{P}|_S \sim \frac{eB_\theta}{c\delta} D \frac{dn}{dr} .$$

The factor of δ , the ripple magnitude, occurs in this formula due to the assumption that $\vec{\nabla}\cdot\overleftrightarrow{P}$ averaged over ϕ_o vanishes; so the transport arises from the ripple variation of the field interacting with $\vec{\nabla}\cdot\overleftrightarrow{P}$. The diffusion coefficient is given by Eq. 30 . Within a surface $e\vec{\nabla}\Phi$ is of the same order of magnitude as $\vec{\nabla}\cdot\overleftrightarrow{P}$ due to Eq. 24; so $e|\vec{\nabla}\Phi|_S \approx |\vec{\nabla}\cdot\overleftrightarrow{P}|_S/n \approx \delta(\rho_\theta/a)\, TN/R$. In this formula a is the minor radius and T the temperature. Therefore,
$(e\,\partial\Phi/\partial\phi_o)/(\mu\partial B/\partial\phi_o) \approx \rho_\theta/a \ll 1$ and

$$\vec{v}_R\cdot\vec{\nabla}\psi = \frac{c}{e}\,\mu\,\frac{\partial B}{\partial\phi} \tag{31}$$

In this paper, the magnetic field strength is assumed to be

$$B = B_o(\psi)\,[1 + 2\epsilon\sin^2\theta/2 + \delta\sin N\phi] \tag{32}$$

with the inverse aspect ratio ε a function of ψ and the ripple δ a function of ψ and θ . Using $\phi = \phi_o + q(\psi)\theta$, one finds

$$\frac{\partial B}{\partial \phi_o} = N\delta B_o \cos (N\phi_o + Nq\theta) \;.$$

For trapped particles $\mu B_o \simeq mv^2/2$; so

$$\dot{\vec{v}}_R \cdot \vec{\nabla}\psi = N\delta \frac{mc}{2e} v^2 \cos(N\phi_o + Nq\theta) \;. \tag{33}$$

In ordinary r , θ , ϕ toroidal coordinates $\vec{\nabla}\psi \simeq \hat{r} RB_\theta$; so the radial ripple drift

$$(v_R)_r = \frac{N\delta}{2} \frac{\rho_\theta}{R} v \cos(N\phi) \tag{34}$$

with $\rho_\theta = v/(mc/eB_\theta)$.

While evaluating the more collisional part of the ripple plateau regime, two further approximations can be made. The θ dependence of $v_\parallel$ $(E, \mu, \psi, \theta, \phi_o)$ can be ignored and $C(f)$ can be approximated by

$$C(f) = \frac{\nu}{2} \frac{\partial^2 f}{\partial \lambda^2} \tag{35}$$

with $\lambda = v_\parallel / v$ the pitch. The approximation that $v_\parallel$ varies slowly in θ in comparison to its variation due to collisions, will be examined more carefully later. It is just this approximation which defines the transition from the more to the less collisional plateau regime.

With the approximations we have discussed, the ripple kinetic equation can be written

$$v_\parallel \frac{\vec{B}\cdot\vec{\nabla}\theta}{B} \frac{\partial f}{\partial \theta} + N\delta \frac{mc}{2e} v^2 \frac{\partial f_M}{\partial \psi} \cos(N\phi_o + Nq\theta) = \frac{\nu}{2} \frac{\partial^2 f}{\partial \lambda^2} \tag{36}$$

Define $g(s)$, λ_c , and s so

$$\begin{aligned} f &= -\frac{\delta}{\lambda_c} \frac{mc}{2e} \frac{v}{q} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \frac{\partial f_M}{\partial \psi} \left[g\, e^{i(N\phi_o + Nq\theta)} \right]_{Re} \\ \lambda_c &= \left(\frac{\nu}{2Nqv} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \right)^{1/3} , \quad s = \lambda/\lambda_c \quad . \end{aligned} \tag{37}$$

The critical pitch λ_c agrees with that of the heuristic derivation for $B/B\cdot\nabla\theta \approx qR$. The braclets $[\ \ldots\ ]_{Re}$ mean the real part. The ripple kinetic equation can then be simply written

$$\frac{d^2 g}{ds^2} - i s g = -1 \tag{38}$$

which has the solution

$$g(s) = \int_0^\infty \exp\left(-\frac{1}{3}t^3 - i\,s\,t\right) dt \quad . \tag{39}$$

Actually only the even part in $v_\parallel$ of the distribution function contributes to the transport. Let G(s) be the even part of g (s)

$$G(s) = \int_0^\infty \cos(st) e^{-t^3/3} dt \tag{40}$$

then the even part of the distribution function f_E is

$$f_E = -\delta \frac{mc}{2e} \frac{v}{q} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \frac{\partial f_M}{\partial \psi} \frac{G(\lambda/\lambda_c)}{\lambda_c} \cos(N\phi_o + Nq\theta) \tag{41}$$

The total particle flux crossing a magnetic surface is

$$\Gamma = \int_\psi d\vec{S} \cdot \int \vec{v}_R f\, d^3 v$$

with $d\vec{S} = (\vec{\nabla}\psi/\vec{B}\cdot\vec{\nabla}\theta)\, d\theta\, d\phi_o$ .We ignore the cross terms between neoclassical and ripple transport since the cos(Nϕ) averaging makes these cross terms small. Using Eq. 33 for $\vec{v}_R\cdot\vec{\nabla}\psi$ and Eq. 41 for f

$$\Gamma = \int \left(N\delta \frac{mc}{2e} v^2 \cos N\phi \right)\left(-\delta \frac{mc}{2e} \frac{v}{q} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \frac{\partial f_M}{\partial \psi} \frac{G(\lambda/\lambda_c)}{\lambda_c} \cos N\phi \right)$$

$$\times \frac{d\theta\, d\phi}{\vec{B}\cdot\vec{\nabla}\theta} 2\pi v^2 dv d\lambda \quad . \tag{42}$$

The pitch integral is just

$$\int_{-1}^{1} G\left(\frac{\lambda}{\lambda_c}\right)\frac{1}{\lambda_c}\,d\lambda = \int_{-1/\lambda c}^{1/\lambda c} ds \int_0^\infty dt \cos st\; e^{-t^3/3}$$

$$= 2\int_0^\infty \frac{1}{t}\sin(t/\lambda c)\,e^{-t^3/3}\,dt$$

$$\simeq \pi \quad (\lambda_c << 1) \tag{43}$$

while the average of $\cos^2(N\phi)$ is just $\frac{1}{2}$. Therefore,

$$\Gamma = -\int \frac{d\theta\, d\phi}{\vec{B}\cdot\vec{\nabla}\theta}\left[\frac{\pi}{16}\,\frac{N}{q}\,\delta^2\left(\frac{mc}{e}\right)^2\frac{B}{\vec{B}\cdot\vec{\nabla}\theta}\left(\int v^3\,\frac{\partial f_M}{\partial\psi}\,d^3v\right)\right] \quad . \tag{44}$$

The derivative of a Maxwellian with respect to ψ *with the energy* held constant is

$$\frac{\partial f_M}{\partial\psi} = \left[\frac{1}{n}\frac{dn}{d\psi} + \frac{e}{T}\frac{d\Phi}{d\psi} + \left(\frac{1}{2}\frac{mv^2}{T} - \frac{3}{2}\right)\frac{1}{T}\frac{dT}{d\psi}\right] f_M$$

and

$$\int v^3\,\frac{\partial f_M}{\partial\psi}\,d^3v = \frac{16}{\sqrt{2\pi}}\left(\frac{T}{m}\right)^{3/2}\left(\frac{dn}{d\psi} + \frac{3}{2}\,\frac{n}{T}\,\frac{dT}{d\psi} + \frac{en}{T}\,\frac{d\Phi}{d\psi}\right) \quad .$$

One then finds

$$\Gamma = -\int \frac{d\theta\, d\phi}{\vec{B}\cdot\vec{\nabla}\theta} \left[\left(\frac{\pi}{2}\right)^{\frac{1}{2}} N\delta^2 \left(\frac{mc}{e}\right)^2 \frac{1}{q} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \left(\frac{T}{m}\right)^{3/2} \right] \left(\frac{dn}{d\psi} + \frac{3}{2}\frac{n}{T}\frac{dT}{d\psi} + \frac{cn}{T}\frac{d\phi}{d\psi}\right) \qquad (45)$$

For the case of almost circular flux surfaces so $\psi = \psi(r)$, $RB_\theta = d\psi/dr$, and $q = rB_\phi/RB_\theta$, one can evaluate a diffusion coefficient

$$D = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} N \langle\delta^2\rangle \rho_\theta^2 \frac{v_{th}}{R} \frac{B}{B_\phi} \quad , \quad \langle\delta^2\rangle \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta^2 d\theta \qquad (46)$$

with $v_{th}^2 = T/m$ and $\rho_\theta = v_{th}/(eB_\theta/mc)$. The average radial particle flux per unit area, γ , is then

$$\gamma = -D\left[\frac{dn}{dr} + \frac{3}{2}\frac{n}{T}\frac{dT}{dr} + \frac{en}{T}\frac{d\phi}{dr}\right] \quad . \qquad (47)$$

The total heat flux Q crossing a surface is

$$Q = \int_\psi d\vec{S} \cdot \int v_R \left(\frac{1}{2} mv^2\right) f\, d^3v \qquad (48)$$

which gives

$$Q = -\int \frac{d\theta\, d\phi}{\vec{B}\cdot\vec{\nabla}\theta} \left[\frac{\pi}{16} \frac{N\delta^2}{q} \left(\frac{mc}{e}\right)^2 \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \int \left(\frac{1}{2} m v^2\right) v^3 \frac{\partial f_M}{\partial\psi} d^3v \right] \qquad (49)$$

and

$$\int \left(\frac{1}{2} m v^2\right) v^3 \frac{\partial f_M}{\partial\psi} d^3v = \frac{16\times 6}{\sqrt{2\pi}} \frac{T}{2} \left(\frac{T}{m}\right)^{3/2} \left[\frac{dn}{d\psi} + \frac{5}{2}\frac{n}{T}\frac{dT}{d\psi} + \frac{en}{T}\frac{d\phi}{d\psi}\right] \quad .$$

This means

$$Q = -\int \frac{d\theta\, d\phi_o}{\vec{B}\cdot\vec{\nabla}\theta} \left[3\left(\frac{\pi}{2}\right)^{1/2} T N\delta^2 \left(\frac{mc}{e}\right)^2 \frac{1}{q} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \left(\frac{T}{m}\right)^{3/2}\right]\left(\frac{dn}{d\psi} + \frac{5}{2}\frac{n}{T}\frac{dT}{d\psi} + \frac{en}{T}\frac{d\phi}{d\psi}\right) . \tag{50}$$

For the almost circular flux surface case, the thermal conductivity is

$$K = \frac{15}{2}\left(\frac{\pi}{2}\right)^{1/2} n N \langle\delta^2\rangle \rho_\theta^2 \frac{v_{th}}{R}\frac{B}{B_\phi} . \tag{51}$$

The average heat flux per unit area q is

$$q = -K\left[\frac{dT}{dr} + \frac{2}{5}\left(T\frac{1}{n}\frac{dn}{dr} + e\frac{d\phi}{dr}\right)\right] . \tag{52}$$

The ion ripple plateau diffusion coefficient exceeds that of the electrons by the square root of the mass ratio. Hence, we might expect ambipolarity would require the potential to adjust to hold in the ions. That is

$$-\frac{e}{T}\frac{d\phi}{dr} = \frac{1}{n}\frac{dn}{dr} + \frac{3}{2}\frac{1}{T}\frac{dT}{dr} . \tag{53}$$

If this occurs the ion heat flux q_i is given by

$$q_i = -K_A \frac{dT_i}{dr} \tag{54}$$

with the ambipolar thermal conductivity $K_A = 2K/5$. That is

$$K_A = 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}} n\, N <\delta^2> \rho_\theta^2 \frac{v_{th}}{R} \frac{B}{B_\phi} \tag{55}$$

With the transport evaluated, let us examine the approximation that $v_\parallel$ varies slowly with θ compared to its variation due to collisions. As is clear from the calculation, the only critical region of velocity space in near $v_\parallel = 0$. That is near the turning point of a particle. The turning point region remains critical even in the more collisionless regimes; so an expression for $v_\parallel$ is required which is valid near turning points. Letting $v_\parallel(0) = v_\parallel(\theta = 0)$, one finds from energy conservation Eq. 2 and the expression for B, Eq. 32

$$\left(\frac{v_\parallel}{v}\right)^2 = \left(\frac{v_\parallel(0)}{v}\right)^2 - 2\varepsilon \sin^2 \theta/2 - \delta[\sin(N\phi_o + Nq\theta) - \sin N\phi_o] \tag{56}$$

where we have assumed $|v_\parallel(0)/v| << 1$. To simplify the expression for $v_\parallel^2$, one would like to ignore the term proportional to δ . To obtain the correct θ dependence of $v_\parallel^2$ near a turning point at $\theta = \theta_T$ one must have the correct derivative of $v_\parallel^2$ there.

$$\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{v}\right)^2 = -\varepsilon \sin \theta_T - N\, q\, \delta \cos(N\phi_o + Nq\theta_T) \quad .$$

Clearly $\alpha_* = (\varepsilon/Nq\delta)\sin\theta_T$ must be greater than one for the neglect of the term proportional to δ to be valid. The condition $\alpha_* > 1$ is also the criterion for the ripple to be too weak to form local magnetic walls. Although α_* generally is not larger than one near $\theta = 0$ we assume it is larger than one over most of the θ range (i.e. $\alpha = \varepsilon/Nq\delta >> 1$) . Let us compare the approximate expression for $v_\parallel^2$

$$\left(\frac{v_\parallel}{v}\right)^2 = \left(\frac{v_\parallel(0)}{v}\right)^2 - 2\varepsilon\sin^2\theta/2 \tag{57}$$

with collisional terms. Define s so $v_\parallel/v = \lambda_c s$ as before, then near a turning point θ_T , $s^2 = \varepsilon\sin\theta_T(\theta_T - \theta)/\lambda_c^2$. The dominant contribututuions to plateau transport come from $s^2 \approx 1$, $\theta_T - \theta \approx 1/Nq$. The critical cross over in the $v_\parallel$ variation due to collisions and collisionless orbital effects is then

$$\frac{\varepsilon\sin\theta_T}{N q} \approx \lambda_c^2 \quad . \tag{58}$$

Using $\nu_* = \varepsilon^{3/2} v/qR$ this condition can be put in a simpler form

$$\nu_*/\nu \lesssim \frac{\sqrt{Nq}}{2(\sin\theta_T)^{3/2}} \tag{59}$$

for the collisional part of the ripple plateau. ν_*/ν is the approximate number of times a trapped particle bounces before scattering out of the trapped particle region of velocity space.

IIIB. RIPPLE PLATEAU (LESS COLLISIONAL CASE)

The physics and the calculational method changes considerably as we move from the more to the less collisional ripple plateau. However, the transport coefficients remain unchanged. The physics is this. As a particle goes along a field line it averages the radial drift due to the ripple to zero except near its turning points. Its radial drift near the turning point Δr is just the radial ripple velocity $(v_R)_r \simeq N\delta(\rho_\theta/R)v$ times the time to cross the last ripple which is $\tau_R = R/Nv_\parallel \simeq qR/v\sqrt{\varepsilon Nq}$. The reason $v_\parallel/v \simeq \sqrt{\varepsilon Nq}$ is that the particle is such a small distance from its turning point. Consequently, $\Delta r \simeq \delta\,\rho_\theta\,\sqrt{Nq/\varepsilon}$. The fraction of the particles which take part in the transport is $\varepsilon^{1/2}$, for all trapped particles participate. Each time the particle bounces it picks up a random Δr for we assume there are enough collisions in a bounce time $\tau_B = qR/\sqrt{\varepsilon}\,v$ so the turning point of the banana θ_T has changed by more than $1/Nq$. This means $\nu_*/\nu < (Nq)^2$ is the criterion for remaining in the ripple plateau with $\nu_* = \varepsilon^{3/2}v/qR$. The diffusion coefficient is then

$$D = \varepsilon^{1/2}(\Delta r)^2/\tau_B$$

$$= N\delta^2\rho_\theta^2\,\frac{v}{R}$$

as before.

The appropriate drift kinetic equation for the less collisional ripple plateau is

$$v_\parallel \frac{\vec{B}\cdot\vec{\nabla}\theta}{B} \frac{\partial f}{\partial\theta} + \vec{v}_R\cdot\vec{\nabla}\psi \frac{\partial f_M}{\partial\psi} = C(f) \tag{60}$$

with the collision operator of the Lorentz form

$$C(f) = \nu \frac{v_\parallel}{B} \frac{\partial}{\partial\mu}\left(m v_\parallel \mu \frac{\partial f}{\partial\mu}\right) \quad . \tag{61}$$

Let

$$\Delta(\theta) = \int_\theta^{\theta_T} \vec{v}_R\cdot\vec{\nabla}\psi \frac{1}{|v_\parallel|} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} d\theta \quad . \tag{62}$$

Physically $\Delta(\theta)$ is the displacement in ψ of a particle from its position at the turning point $\theta = \theta_T$. The important feature of $\Delta(\theta)$ is that it approaches a constant Δ_* for $|\theta_T - \theta| > 1/Nq$. Letting $\sigma = v_\parallel/|v_\parallel|$ be the sign of the parallel velocity the drift kinetic equation becomes

$$\sigma \frac{\partial f}{\partial\theta} - \frac{\partial\Delta}{\partial\theta}\frac{\partial f_M}{\partial\psi} = \frac{1}{|v_\parallel|} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} C(f) \quad . \tag{63}$$

Over the range $|\theta_T - \theta| \lesssim 1/Nq$ collisions have little effect in this less collisional regime; so

$$f_\sigma(\theta) = \sigma\Delta(\theta) \frac{\partial f_M}{\partial\psi} + f(\theta_T) \quad , \quad |\theta_T - \theta| \lesssim \frac{1}{Nq} \quad . \tag{64}$$

For the collisional problem, $|\theta_T - \theta| >> 1/Nq$, we write

$$f_\sigma(\theta) = g_\sigma(\theta)\left[\sigma\Delta_* \frac{\partial f_M}{\partial\psi} + f(\theta_T)\right] , \quad |\theta_T - \theta| >> \frac{1}{Nq} \tag{65}$$

with $g_\sigma(\theta_T) = 1$. The plan is to substitute this expression for f into Eq. 63 to find $g_\sigma(\theta)$. Before doing this, we must evaluate Δ_* and change from using μ as a velocity variable to θ_T . While carrying out the calculations, we assume $\theta_T > 0$. The calculations for $\theta_T < 0$ follow in an obvious manner.

To evaluate Δ_* an expression is needed for $v_\parallel$. This is given by Eq. 57 which we further simplify by assuming $\theta \lesssim 1$ so $\sin\theta/2 \simeq \theta/2$. The turning point θ is then

$$\theta_T^2 = \frac{2}{\varepsilon}\left(\frac{v_\parallel(0)}{v}\right)^2 \tag{66}$$

with $v_\parallel(0) = v_\parallel(\theta = 0)$. Hence

$$\left(\frac{v_\parallel}{v}\right)^2 = \frac{\varepsilon}{2}(\theta_T^2 - \theta^2) . \tag{67}$$

Using Eq. 33 for $\vec{v}_R\cdot\vec{\nabla}\psi$ and assuming $|\theta_T| >> |\theta_T - \theta| >> 1/Nq$ we have

$$\Delta_* = N\delta \frac{mc}{2e} \frac{v}{(\frac{\varepsilon}{2}2\theta_T)^2} \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \int_{-\infty}^{\theta_T} \frac{\cos(N\phi_o + Nq\theta)}{(\theta_T - \theta)^{1/2}} d\theta . \tag{68}$$

Letting $x = Nq(\theta_T - \theta)$ the integral becomes

$$\int_{\infty}^{\theta_T} \frac{\cos(N\phi_o + Nq\theta)}{(\theta_T - \theta)^{1/2}}\, d\theta = \frac{1}{\sqrt{Nq}} \int_0^{\infty} \cos(N\phi_o + Nq\theta_T - x)\, \frac{dx}{\sqrt{x}}$$

$$= \left(\frac{\pi}{2Nq}\right)^{1/2} \left[\cos(N\phi_o + Nq\theta_T) + \sin(N\phi_o + Nq\theta_T)\right]$$

and

$$\Delta_* = \frac{1}{2}\left(\frac{\pi N}{2q\varepsilon\theta_T}\right)^{1/2} \delta\, \frac{mc}{e}\, v\, \frac{B}{\vec{B}\cdot\vec{\nabla}\theta} \left[\cos(N\phi_o + Nq\theta_T) + \sin(N\phi_o + Nq\theta_T)\right] . \tag{70}$$

The switch from μ to θ_T as a coordinate is carried out using Eq. 66 and energy conservation. That is $E = \varepsilon\theta_T^2 mv^2/4 + \mu B_o + e\Phi$ with B_o the magnetic field at $\theta = 0$. This means

$$B_o\, d\mu = -\varepsilon\, \frac{1}{2}\, m v^2\, \theta_T d\theta_T \quad . \tag{71}$$

The velocity element d^3v for example can be written with θ_T positive

$$d^3v = \frac{4\pi B}{m^2 |v_{\parallel}|}\, dE\, d\mu$$

$$= -\frac{\varepsilon}{2}\, \frac{v}{|v_{\parallel}|}\, \theta_T d\theta_T\, 4\pi v^2\, dv \tag{72}$$

and the collision operator (Eq. 61) becomes

$$C(f) = \frac{\nu}{2}\left|\frac{v_\parallel}{v}\right|\left(\frac{2}{\varepsilon}\right)^{3/2}\frac{1}{\theta_T}\frac{\partial}{\partial\theta_T}\left(\theta_T^2-\theta^2\right)^{\frac{1}{2}}\frac{\partial f}{\partial\theta_T} \tag{73}$$

where we have assumed $B_o \simeq B$ and $\mu B \simeq mv^2/2$.

We can now insert Eq. 65 for $f_\sigma(\theta)$ into Eq. 63 , the kinetic equation. In doing this, we assume $\partial\Delta/\partial\theta$ is zero for it has zero average over the range of θ for which collisions are important. We also assume $f(\theta_T) \sim \Delta_*$ as will be shown later and $Nq \gg 1$. Then

$$C(f_\sigma) \simeq \frac{\nu}{2}\left|\frac{v_\parallel}{v}\right|\left(\frac{2}{\varepsilon}\right)^{3/2}\frac{\left[\theta_T^2-\theta^2\right]^{\frac{1}{2}}}{\theta_T}\frac{\partial^2 f_\sigma}{\partial\theta_T^2} \tag{74}$$

but $\partial^2 f_\sigma/\partial\theta_T^2 = -(Nq)^2 f_\sigma$. This means

$$g_\sigma(\theta) = \exp\left[\sigma(Nq)^2\frac{\nu}{2v}\frac{B}{\vec{B}\cdot\vec{\nabla}\theta}\left(\frac{2}{\varepsilon}\right)^{3/2}\int_\theta^{\theta_T}\left(\theta_T^2-\theta^2\right)^{\frac{1}{2}}\frac{1}{\theta_T}\,d\theta\right] \cdot \tag{75}$$

For $\sigma = +1$, g_σ becomes exponentially large; so $f(\theta_T)$ must be chosen to make the factor multiplying it zero or $f(\theta_T) = -\Delta_*\partial f_M/\partial\psi$. The expression for g_σ for $\sigma = -1$ becomes exponentially small before the other turning point is reached provided $\nu_*/\nu \lesssim (Nq)^2$ which is just the criterion for being in the ripple plateau regime. The interpretation is obvious. Particles with $v_\parallel > 0$ (and hence $\sigma = 1$) have not yet hit the ripple near the turning point $\theta_T > 0$.

Particles will $v_{\parallel} < 0$ have been deflected from the ψ surface by ripple an amount $2\Delta_*$ which depends sensitively on their turning point. Collisions intermix particles with different turning points causing the shift to go to zero away from the turning points.

For purposes of calculating the transport, we will actually only need the even part of the distribution function (in $v_{\parallel}$), f_E, and only for $|\theta_T - \theta| \lesssim 1/Nq$. This expression is

$$f_E = -\Delta_* \frac{\partial f_M}{\partial \psi} \quad . \tag{76}$$

The particle transport is using Eq. 72 for d^3v

$$\Gamma = \int \frac{d\theta\, d\phi_o}{\vec{B}\cdot\vec{\nabla}\theta} \; \frac{\varepsilon}{2} \; \frac{v}{|v_{\parallel}|}\, \theta_T\, d\theta_T\, 4\pi v^2 dv \left(\vec{v}_R\cdot\vec{\nabla}\psi\, \Delta_* \frac{\partial f_M}{\partial \psi}\right) \quad . \tag{77}$$

Now $(\vec{v}_R\cdot\vec{\nabla}\psi)(B/B\cdot\vec{\nabla}\theta)/|v_{\parallel}| = -\partial\Delta/\partial\theta$, so integrating over θ one finds with a factor of two coming from the two turning points

$$\Gamma = -\int \frac{d\phi_o}{B}\, \varepsilon\, v\, \theta_T\, d\theta_T\, 4\pi\, v^2 dv\, \Delta_*^2\, \frac{\partial f_M}{\partial \psi} \quad , \tag{78}$$

Using Eq. 70 for Δ_* and averaging over ϕ_o,

$$\Gamma = -\int \frac{d\phi_o}{\vec{B}\cdot\vec{\nabla}\theta}\, \frac{\pi}{8}\, \frac{N}{q}\, \delta^2 \left(\frac{mc}{e}\right)^2 \frac{B}{\vec{B}\cdot\vec{\nabla}\theta}\, v^3\, \frac{\partial f_M}{\partial \psi}\, d\theta_T\, 4\pi\, v^2 dv \quad . \tag{79}$$

Now the integrand is evaluated at $\theta = \theta_T$ so we can drop the "T" from θ . However, when this is done the equation must be divided by two for $\theta_T \geq 0$ while $-\pi \leq \theta \leq \pi$.

$$\Gamma = -\int \frac{d\theta\, d\phi_o}{\vec{B}\cdot\vec{\nabla}\theta} \left[\frac{\pi}{16}\frac{N}{q}\,\delta^2\left(\frac{mc}{e}\right)^2 \frac{B}{\vec{B}\cdot\vec{\nabla}\theta}\left(\int v^3\,\frac{\partial f_M}{\partial\psi}\,d^3v\right)\right] \tag{80}$$

which is exactly Eq. 44.

ACKNOWLEDGMENTS

The author would like to express his thanks to P.H. Rutherford, and R.J. Goldston for stimulating discussions.

This work was supported by the United States Department of Energy Contract No. EY-76-C-02-3073.

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