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Enhancement by noise in parallel arrays of sensors with power-law characteristics

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An optimally tuned power-law sensor is shown capable of amplifying the signal-to-noise ratio of a sine wave in Gaussian white noise. When associated in parallel arrays, further improvement can be obtained with independent noises injected on these sensors. This form of stochastic resonance in arrays, obtained here with smooth threshold-free nonlinearities, yields signal-to-noise ratio gains above unity in a true regime of added noise for a sine wave in Gaussian white noise, along with a class of nonlinear devices with useful potentialities for noise-aided information processing.

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In recent years it has been realized that nonlinear devices authorize improvement by noise of signal transmission or processing through stochastic resonance (SR) [1]. Most forms of SR so far reported involve nonlinear devices with a threshold or a potential barrier in their response, driven by a signal which is alone too small or subthreshold, and which needs assistance by the noise to overcome the threshold or barrier [1]. Some other forms of SR have been reported in barrier-free monostabale dynamic systems as an amplification by noise of a sinusoidal signal [2,3], and in nondynamical threshold-free systems as an enhancement of the signal-to-noise ratio (SNR) at the output of a pulse-firing device whose emission rate depends exponentially on a noisy sinusoidal input [4]. In this paper, we will show the possibility of SR in another class of nonlinear nondynamical (static) threshold-free devices: power-law sensors. Such devices can be viewed as models for existing sensors, but also as a basis for a potentially useful generation of "intelligent" sensors, owing to their response to noise.

For the transmission of a periodic signal as considered in most SR studies, in addition to signal amplification and SNR enhancement, SR devices can sometimes produce an output SNR larger than the input SNR [5–7]. Yet, this last property has never been observed with static nonlinearities in the practically very important case of a sine wave buried in Gaussian white noise. We will show that this becomes possible with the power-law sensors. In addition, we will show that further improvement by noise is possible when the power-law sensors are associated in parallel array according to the configuration introduced in [8] for suprathreshold SR. So far, suprathreshold SR has been demonstrated for nonlinear devices with a threshold, essentially comparators or one-bit quantizers; we shall extend this form of SR to threshold-free smooth nonlinearities.

Consider the signal-plus-noise mixture $x(t)=s(t)+\xi(t)$, where $s(t)$ is deterministic with period T_s, and $\xi(t)$ is a stationary white noise, independent of $s(t)$, and with probability density $f_\xi(u)$. Input $x(t)$ is applied to a nonlinear sensor with a static or memoryless characteristic $g(\cdot)$ to produce the output

$$y(t) = g\big(s(t) + \xi(t)\big). \tag{1}$$

The transmission of $s(t)$ is assessed by the output SNR which is standard in SR studies [1,9]. It measures in $y(t)$ the power contained in the coherent spectral line at $1/T_s$ divided by the power contained in the noise background in a small frequency band ΔB around $1/T_s$, and reads [9]

$$\mathcal{R}_{\text{out}} = \frac{|\langle E[y(t)]\exp(-i2\pi t/T_s)\rangle|^2}{\langle \text{var}[y(t)]\rangle \Delta t \Delta B}. \tag{2}$$

In Eq. (2), a time average is defined as

$$\langle \cdots \rangle = \frac{1}{T_s}\int_0^{T_s} \cdots dt, \tag{3}$$

$\mathrm{E}[y(t)]$ and $\text{var}[y(t)]=\mathrm{E}[y^2(t)]-\mathrm{E}^2[y(t)]$ represent the expectation and variance of $y(t)$ at a fixed time t, and Δt is the time resolution of the measurement (i.e., the signal sampling period in a discrete time implementation); throughout this study we take $\Delta t\Delta B=10^{-3}$. The white noise assumption, throughout, models a broadband physical noise with a correlation duration much smaller than the other relevant time scales, i.e., T_s and Δt [9]. Since $x(t)=s(t)+\xi(t)$, the probability density for $x(t)$ is $f_\xi\big(x-s(t)\big)$, and from Eq. (1) one has

$$E[y(t)] = \int_{-\infty}^{+\infty} g(x) f_\xi(x - s(t))dx, \tag{4}$$

and

$$E[y^2(t)] = \int_{-\infty}^{+\infty} g^2(x) f_\xi(x - s(t))dx. \tag{5}$$

Owing to its practical importance, we consider the case of a sinusoidal input

$$s(t) = A\sin(2\pi t/T_s) \tag{6}$$

buried in zero-mean Gaussian noise $\xi(t)$ with variance σ_ξ^2. An input SNR $\mathcal{R}_{\text{in}}$, defined in a similar way as $\mathcal{R}_{\text{out}}$ of Eq. (2), results as

$$\mathcal{R}_{\text{in}} = \frac{A^2/4}{\sigma_\xi^2 \Delta t \Delta B}. \tag{7}$$

The input-output characteristic of the sensor is chosen as the power-law function, sometimes referred to as γ-law correction in technologies,

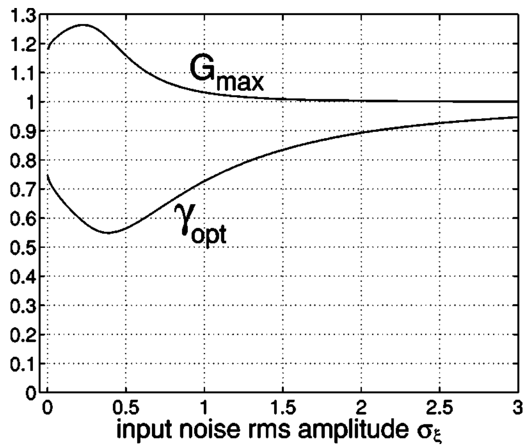

FIG. 1. Optimal exponent γ_{opt} in Eq. (8) and maximum input-output SNR gain G_{max} at γ_{opt}, as a function of the rms amplitude σ_ξ of the zero-mean Gaussian noise $\xi(t)$, for the transmission of the sinusoidal $s(t)$ of Eq. (6) with $A=1$.

$$g(u) = \text{sgn}(u)|u|^\gamma, \tag{8}$$

parametrized by $\gamma \geqslant 0$. The choice $\gamma=1$ is the purely linear sensor. $\gamma=0$ gives the signum function, which is standard in SR studies. At $\gamma=0$, with an extra threshold θ making $g(u)=\text{sgn}(u-\theta)$, previous SR studies have shown that for a subthreshold input $s(t)<\theta, \forall t$, the output SNR $\mathcal{R}_{out}$ of Eq. (2) can be improved by increasing the noise level σ_ξ. At $\gamma=0$ with no extra threshold (i.e., $\theta=0$), the input $s(t)$ of Eq. (6) is suprathreshold, and no SNR improvement is observed but a monotonic decay of $\mathcal{R}_{out}$ as σ_ξ grows. This is also what we have observed here at any $\gamma>0$ in Eq. (8): a monotonic decay of $\mathcal{R}_{out}$ as σ_ξ grows.

Another useful measure in SR studies is the input-output SNR gain $G=\mathcal{R}_{out}/\mathcal{R}_{in}$. With hard-threshold nonlinearities, comparable to Eq. (8) at $\gamma=0$, SNR gains G raised above unity by increasing σ_ξ have been shown possible, separately with a sinusoidal input $s(t)$ and non-Gaussian noise $\eta(t)$, or with Gaussian noise $\eta(t)$ and a periodic nonsinusoidal input $s(t)$ [5,6]. No SNR gain G above unity has been found with simultaneously sinusoidal $s(t)$ and Gaussian $\eta(t)$. We observe that this becomes possible by varying γ. At $\gamma>0$ in Eq. (8), with $s(t)$ of Eq. (6) and as σ_ξ grows, although $\mathcal{R}_{out}$ experiences a monotonic decay, $G=\mathcal{R}_{out}/\mathcal{R}_{in}$ can experience a nonmonotonic evolution, culminating at a maximum which can be above unity. Such evolutions can readily be obtained from Eqs. (1)–(8) for a given $\gamma>0$, in a similar way as done in [5] at $\gamma=0$. A complementary characterization of the influence of γ is provided by Fig. 1 which shows, for each noise level σ_ξ, the optimal value γ_{opt} of γ that maximizes the SNR gain G, along with the value G_{max} of this maximum.

Figure 1 clearly shows that for each noise level σ_ξ (evaluated in units of the signal amplitude $A=1$), there exists a value γ_{opt} of γ that realizes a maximum G_{max} above unity for the gain G. $\gamma=0$ is the hard threshold and $\gamma=1$ is the linear sensor, and both cases are unable to produce a gain $G>1$. The optimal γ's associated to with $G_{max}>1$ are found between 0 and 1, depending on the noise level σ_ξ. Values of γ above 1 are not found as optimal conditions to maximize the gain G.

The possibility of $G>1$ with the power-law sensor of Eq. (8) with appropriate γ, means that these sensors are able to amplify the periodic input $s(t)$ more than they amplify the noise. Moreover, this preferential amplification of $s(t)$ is at its maximum of efficacy for a nonzero noise level σ_ξ located at $\sigma_{\xi,opt}\approx 0.22$ in Fig. 1, where G_{max} culminate at 1.27 with $\gamma_{opt}\approx 0.59$. In practice, one will usually work at a given noise level $\sigma_\xi>0$ imposed by the input signal-noise mixture and fixing $\mathcal{R}_{in}$. It will then be beneficial to tune the sensor of Eq. (8) at the optimal γ given by Fig. 1, and this will provide a maximally amplified output SNR $\mathcal{R}_{out}>\mathcal{R}_{in}$. Furthermore, if σ_ξ happens to be below $\sigma_{\xi,opt}$, purposeful addition of noise can be envisaged at the input to raise the noise level at $\sigma_{\xi,opt}$. This will improve the maximal SNR gain G_{max} reachable in Fig. 1 and simultaneously degrade $\mathcal{R}_{in}$. Yet, it can be verified from Fig. 1 that $G_{max}\mathcal{R}_{in}$ is a decreasing function of σ_ξ. So the purposeful addition of noise, although it can improve G_{max}, will always degrade $\mathcal{R}_{out}$.

A nonlinearity of Eq. (8) can therefore always be used as an SNR amplifier, through an appropriate choice of γ. This property afforded by simple static nonlinearities as Eq. (8) is quite remarkable since it is proved that linear systems, whatever their complexity, are incapable of $G>1$. As an SR device, noise addition on Eq. (8) can only improve the gain G but not the output SNR $\mathcal{R}_{out}$. We shall now show that it is possible to recover a systematic improvement of the output SNR $\mathcal{R}_{out}$ through noise addition, when the sensors of Eq. (8) are associated in a parallel array.

The input signal $x(t)=s(t)+\xi(t)$ is applied onto a parallel array of N identical sensors with the same input-output characteristic $g(\cdot)$ of Eq. (8). A noise $\eta_i(t)$, independent of $x(t)$, can be added to $x(t)$ at each sensor i so as to produce the output

$$y_i(t) = g[x(t) + \eta_i(t)], \quad i = 1,2, \ldots ,N. \tag{9}$$

The N noises $\eta_i(t)$ are white, mutually independent, and identically distributed (i.i.d.) with probability density $f_\eta(u)$. The response $y(t)$ of the array is obtained by averaging the outputs of all the sensors, as

$$y(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{10}$$

The transmission by the array is assessed in the same way by the output SNR $\mathcal{R}_{out}$ of Eq. (2).

At time t, for a fixed given value x of the input $x(t)$, one has, according to Eq. (10), the conditional expectations

$$E[y(t)|x] = E[y_i(t)|x] \tag{11}$$

and

$$E[y^2(t)|x] = \frac{1}{N}E[y_i^2(t)|x] + \frac{N-1}{N}E^2[y_i(t)|x] \tag{12}$$

which are both independent of i since the $\eta_i(t)$ are i.i.d. The large array limit $N=\infty$ will be simply accessible by letting $E[y^2(t)|x]=E^2[y_i(t)|x]$ in Eq. (12). Next, one obtains

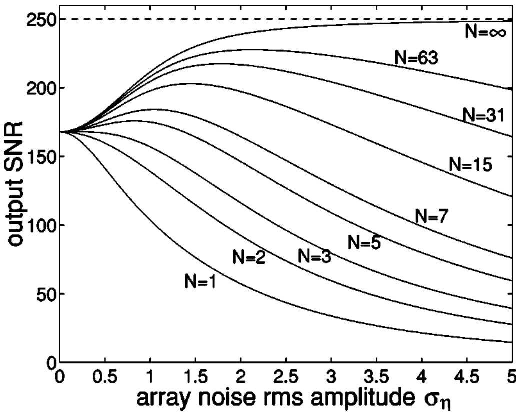

FIG. 2. Output SNR $\mathcal{R}_{\rm out}$ of Eq. (2), as a function of the rms amplitude $\sigma_\eta = a/\sqrt{3}$ of the uniform array noises $\eta_i(t)$, with $\gamma=2$. $\sigma_\xi=1$ whence the input SNR in Eq. (7) $\mathcal{R}_{\rm in}=250$ (dotted line).

$$E[y(t)] = \int_{-\infty}^{+\infty} E[y(t)|x] f_\xi(x - s(t))dx, \qquad (13)$$

and

$$E[y^2(t)] = \int_{-\infty}^{+\infty} E[y^2(t)|x] f_\xi(x - s(t))dx. \qquad (14)$$

Because of Eq. (9), one has for any i,

$$E[y_i(t)|x] = \int_{-\infty}^{+\infty} g(x+u) f_\eta(u)du \qquad (15)$$

and

$$E[y_i^2(t)|x] = \int_{-\infty}^{+\infty} g^2(x+u) f_\eta(u)du. \qquad (16)$$

Since the η_i's can be considered as purposely added noises for the operation of the array, rather than noises imposed by the physical world, we choose for analytical tractability their probability density $f_\eta(u)$ uniform over $[-a,a]$. This allows, with the characteristic of Eq. (8), an explicit evaluation of the integrals (15) and (16) as

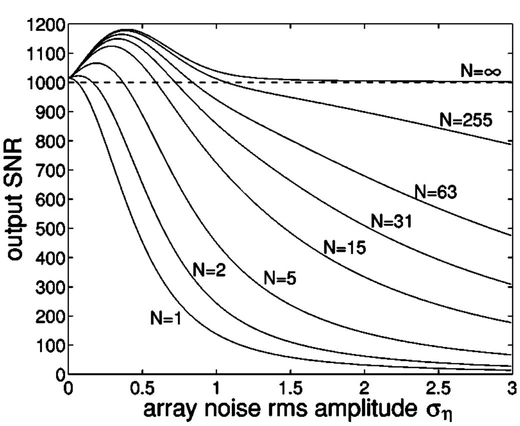

FIG. 3. Same as in Fig. 2, except $\gamma=0.25$ and $\sigma_\xi=0.5$ whence $\mathcal{R}_{\rm in}=1000$ (dotted line).

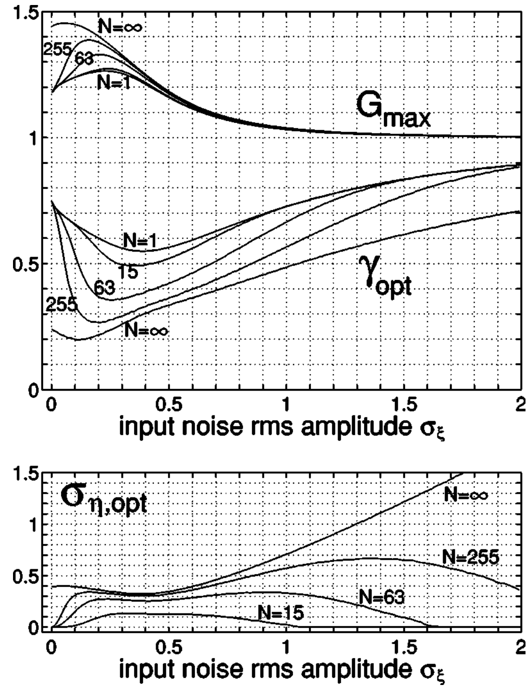

FIG. 4. Behavior of the array of size N, as a function of the rms amplitude σ_ξ of the zero-mean Gaussian input noise $\xi(t)$, for the transmission of the sinusoidal $s(t)$ of Eq. (6) with $A=1$. Upper panel: The optimal exponent $\gamma_{\rm opt}$ in Eq. (8) and maximum input-output SNR gain $G_{\rm max}$ at $(\gamma_{\rm opt}, \sigma_{\eta,{\rm opt}})$. Lower panel: The corresponding optimal value $\sigma_{\eta,{\rm opt}}$ of the rms amplitude σ_η of the added array noises $\eta_i(t)$.

$$E[y_i(t)|x] = \frac{1}{2a}\frac{1}{\gamma+1}(|x+a|^{\gamma+1} - |x-a|^{\gamma+1}) \qquad (17)$$

and

$$E[y_i^2(t)|x] = \frac{1}{2a}\frac{1}{2\gamma+1}[(x+a)|x+a|^{2\gamma} - (x-a)|x-a|^{2\gamma}], \qquad (18)$$

completing the theoretical derivation of the output SNR $\mathcal{R}_{\rm out}$ of Eq. (2).

Figure 2 displays evolutions of the output SNR $\mathcal{R}_{\rm out}$ of Eq. (2), as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$, for sensors with $\gamma=2$.

Figure 2 reveals that, thanks to the added array noises $\eta_i(t)$, moderately large arrays ($N \gtrsim 5$) perform better than a single sensor with no array noise. An optimal nonzero amount of the array noises $\eta_i(t)$ maximizes the output SNR $\mathcal{R}_{\rm out}$, and the improvement of $\mathcal{R}_{\rm out}$ by noise gets more pronounced as the array size N increases.

Improvement by the noises $\eta_i(t)$ in the array persists for supralinear sensors ($\gamma > 1$), as well as for sublinear sensors ($\gamma < 1$) as shown by Fig. 3. The effect would disappear for linear sensors ($\gamma=1$), and it starts to appear as γ departs away from 1, from above or below.

A comparable behavior of $\mathcal{R}_{\rm out}$ as in Figs. 2 and 3 had already been observed in [10] at $\gamma=0$ for hard-threshold nonlinearities and interpreted as a form of suprathreshold SR as introduced in [8]. By contrast, Figs. 2 and 3, at $\gamma > 0$, characterize smooth threshold-free nonlinearities. The effect of suprathreshold SR as observed in [8,10] can therefore be extended to smooth threshold-free nonlinearities, and is consequently not essentially linked to the presence of a threshold. It is more a collective effect of the nonlinear array. The essential ingredient is the nonlinear transformation, after injection of the noises $\eta_i(t)$ and prior to averaging, that leads to a richer capability of representation by the array. The effect does not take place with a linear transformation $(\gamma=1)$, but it does not require a threshold nonlinearity $(\gamma=0)$. Any smooth nonlinearity, supralinear $(\gamma>1)$ or sublinear $(\gamma<1)$, can also produce this effect of array SR.

When the array is operated as in Figs. 2 and 3, the input noise level σ_ξ is fixed, as is the input SNR $\mathcal{R}_{\text{in}}$ of Eq. (7). For a given input noise level σ_ξ and array size N, one can look for the best choice of γ that will produce the highest output SNR $\mathcal{R}_{\text{out}}$ at the optimal level of the array noises σ_η. We have undertaken this double optimization, according to (γ, σ_η), for maximizing the SNR gain $G=\mathcal{R}_{\text{out}}/\mathcal{R}_{\text{in}}$ of the array, at a given σ_ξ fixing $\mathcal{R}_{\text{in}}$. The results are presented in Fig. 4 and reveal several interesting properties specific to these arrays of power-law sensors.

Figure 4 shows that for any given input noise level $\sigma_\xi > 0$ and array size N, an optimal tuning γ_{opt} exists for the exponent γ with always $\gamma_{\text{opt}} \in (0,1)$; this means in particular that neither the hard threshold $(\gamma=0)$, nor the linear sensor $(\gamma=1)$, nor supralinear sensors $(\gamma>1)$ are capable of the best performance. In addition in Fig. 4, at the optimal tuning γ_{opt}, the maximum output SNR gain $G_{\max}$ achieved by the array is always larger than unity and also larger than the SNR gain $G_{\max}$ achieved by a single sensor as shown in Fig. 1; assembling several sensors into arrays always leads to an improved $G_{\max}>1$. Finally in Fig. 4, the best SNR gain $G_{\max}$ achieved by the array is usually obtained at a nonzero optimal level of the array noises $\eta_i(t)$. This is true for any input noise level $\sigma_\xi > 0$, even at large σ_ξ, provided the array size N is also appropriately large. Thanks to the mechanism of array SR, here SNR gains above unity are achieved in an SR regime, i.e., where the best output SNR $\mathcal{R}_{\text{out}}$, maximally amplifying above $\mathcal{R}_{\text{in}}$, is obtained at a nonzero amount of added noise.

Owing to their response to noise exhibited by the above results, power-law sensors, possibly associated in arrays, offer a class of useful devices with specific potentialities for noise-aided processing of noisy signals. In particular, they allow SNR gains above unity through a true regime of added noise for a sine wave in Gaussian white noise. Several possibilities are open for further improving the efficacy of these devices, especially for array SR. For instance, preferable forms for $f_\eta(u)$, other than uniform, can be sought. Other classes of smooth nonlinearities can be tested for similar properties as inaugurated here with the power-law sensors. Further perspectives are formed by the investigation of such parallel arrays of nonlinear sensors for optimal dectection and estimation of signals in noise, or as candidates for devising new generations of smart arrays for information processing.

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