Trans. Japan Soc. Aero. Space Sci.
Vol. 62, No. 3, pp. 125–136, 2019
DOI: 10.2322/tjsass.62.125

Enhancement of GPS/INS Navigation System Observability Using a Triaxial Magnetometer*

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Enhancing the observability of an integrated navigation system (i.e., triaxial-magnetometer-aided global positioning system (GPS) and inertial navigation system (INS)) is analyzed utilizing the Earth-centered-Earth-fixed (ECEF) coordinate system. The error states of the extended Kalman filter (EKF)-based GPS/INS integration algorithm are not fully observable given the position and velocity measurements of a single-antenna GPS receiver. Although the manner of maneuvering a vehicle can improve observability, full observability is not guaranteed. Measurements of a triaxial magnetometer provide attitude information and can enhance observability of the GPS/INS system. In this study, enhancing observability through aid of a triaxial magnetometer is investigated applying an analytic approach. The results of the analysis show that using a triaxial magnetometer allows the error states to be fully observable when the vehicle performs maneuvers. In addition, only one unobservable mode exists, even if the vehicle is in a static or non-accelerating condition.

Key Words: Global Positioning System (GPS), Inertial Navigation System (INS), Magnetometer, Observability

Nomenclature

b_a: accelerometer bias vector
b_g: rate gyroscope bias vector
C_a^d: coordinate transformation matrix from a-frame to d-frame, i.e., $x^d = C_a^d x^a$
f: specific force vector
g: gravitational acceleration vector
∇g: gravity gradient with respect to the Earth-centered-Earth-fixed (ECEF) position, $\partial g/\partial p\big|_e$
m: magnetic vector of the Earth
p: position vector
t: time variable
v: velocity vector or measurement noise vector
w: process noise vector
$\dot{x}$: time derivative of vector x with respect to the body frame, $dx/dt\big|_b$
$\hat{x}$: estimate of vector x
x^a: representation of vector x in a-frame
δx: estimation error of vector x
$[x]^\times$: skew symmetric representation of vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$

$$[x]^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$\delta\rho$: attitude estimation error vector
ω_{ad}: angular velocity vector of d-frame with respect to a-frame
Ω_E: rotational rate of the Earth
Ω: skew symmetric representation of rotation vector of the Earth in the ECEF frame

$$\Omega = [\omega_{ie}^e]^\times \text{ where } \omega_{ie}^e = \begin{bmatrix} 0 & 0 & \Omega_E \end{bmatrix}^T$$

Superscripts
b: b-frame, body frame
e: e-frame, ECEF frame
i: i-frame, Earth-fixed inertial (ECI) frame

1. Introduction

The integrated global positioning system (GPS)/inertial navigation system (INS) is widely used in modern navigation equipment. The GPS provides position and velocity measurements with bounded errors that are used for estimating and bounding the error of the INS. The error of the INS is evaluated using various error states according to the complexity of the error model. To establish whether or not the estimation of all the error states is possible, the observability of the system, which corresponds to the order of the error model and measurements, should be analyzed.

The observability of the static initial alignment phase of the INS has been studied by Bar-Itzhack and Berman,[1] and Jiang and Lin.[2] In those studies, it was shown that the error states are not fully observable in the static case. Attitude error, accelerometer bias error and gyroscope bias error are indistinguishable, and their estimation accuracies are limited. Multi-position alignment was proposed by Lee et al.[3] and Chung et al.[4] to improve the static initial alignment. This method manipulates the system matrix of the error dynamics by changing the positions of the system several times.

Alternatively, the error states can be estimated using the velocity measurement of the GPS receiver while the vehicle is moving. This non-static alignment process is called in-flight alignment (IFA). The error dynamics of a slowly moving vehicle with constant velocity is close to that of a static

*Received 30 November 2017; final revision received 11 October 2018; accepted for publication 13 November 2018.
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case. Certain maneuvers that induce acceleration change the system matrix of error dynamics, and this change in the system matrix can improve observability. Several elementary maneuvers were studied by Bar-Itzhack and Porat,[5] and Porat and Bar-Itzhack.[6] An observability analysis of the piece-wise constant system was introduced by Goshen-Meskin and Bar-Itzhack.[7] In their study, the IFA process was divided into several segments of a piece-wise constant acceleration. Since the observability analysis of a time-varying system is intricate, these analyses were based on the numerical covariance simulation or a piece-wise constant system. An analytic observability analysis of the time-varying system for linear acceleration maneuvering and steady-turn maneuvering was conducted by Rhee et al.[8] However, a sign error exists in the system equation for the time-varying case that must be corrected. In addition, analysis based on the corrected system equation is required. The observability of the time-varying system through Kalman filtering of simulation data was studied by Fandino Ospina[9] and Hong et al.[10] In those studies, sinusoidal variations of specific force and attitude were considered.

The magnetic field of the Earth is generally used to determine the heading. As the tilt angles of the north and east directions are determined using the gravity vector, the magnetic vector contains information regarding the two angles around its direction. Additional attitude information measured from the magnetic vector can enhance the observability of the GPS/INS navigation system. Integrating the magnetometer with the GPS/INS navigation system has been previously performed. A magnetometer was integrated into the GPS/INS navigation system of a helicopter UAV system by Wendel et al.[11] and Barczyk and Lynch.[12] Two methods exist for magnetometer integration. One method involves integrating the heading angle from magnetic compassing. The other uses raw three-dimensional magnetic vectors as measurements. Both of the methods were tested on a helicopter UAV by Barczyk and Lynch,[12] and the results revealed that the raw magnetic vector method exhibits higher levels of performance in terms of attitude. This result implies that magnetic vector measurement provides better system observability than the magnetic heading method. However, enhancing the observability by integrating a magnetometer was not discussed in that study.

In this study, enhancing observability of the integrated GPS/INS system using a magnetometer is analyzed utilizing the Earth-centered-Earth-fixed (ECEF) coordinate system. The observability of the integrated GPS/INS system is addressed first. The sign error appearing in the study by Rhee et al.[8] is corrected, and analysis is performed for the corrected system equation. The observability of the INS error states with magnetic vector measurements is also analyzed. The observability for both the time-invariant case and the time-varying case are analyzed because observability is dependent on the vehicle maneuvering. For the time-varying case, a non-constant axial acceleration maneuver and a constant-rate horizontal turn maneuver, which are typical for an aircraft, are considered. Finally, the observability analysis results of the magnetometer are combined with those of the integrated GPS/INS system and discussed.

2. Background of Observability Analysis

This section describes the target system and the method to conduct observability analysis.

2.1. Error equation of strapdown INS

The perturbed error equation of the strapdown INS depends on the coordinate system. The navigation frame and ECEF frame are widely used coordinate systems for analyzing and implementing strap-down INS. In this study, the perturbed error equation based on the ECEF frame is used to analyze observability. The error dynamics in the ECEF frame have a simpler form than that of the navigation frame. The error states can easily be transformed from one frame to another.[8]

2.1.1. Perturbed error equations

The perturbed linear error states and equation for the ECEF frame are given as[13]

$$x = \begin{bmatrix} \delta p^e & \delta v^e & \delta \rho^e & \delta b_a^b & \delta b_g^b \end{bmatrix}^T \tag{1}$$

and

$$\dot{x} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ \nabla g^e - \Omega^2 & -2\Omega & -[C_b^e f^b]^\times & C_b^e & 0 \\ 0 & 0 & -\Omega & 0 & C_b^e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + w \tag{2}$$

Since the position and velocity errors are directly measured from the GPS receiver and position error is affected only by the velocity error, as shown in Eq. (2), observability can be analyzed for the reduced system as[8]

$$\dot{x} = Ax + w \tag{3}$$

where,

$$x = \begin{bmatrix} \delta v^e & \delta \rho^e & \delta b_a^b & \delta b_g^b \end{bmatrix}^T \tag{4}$$

$$A = \begin{bmatrix} -2\Omega & -[C_b^e f^b]^\times & C_b^e & 0 \\ 0 & -\Omega & 0 & C_b^e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

2.1.2. Transformed perturbed error equation

Consider a linear transformation,[8]

$$x = T\bar{x} \tag{6}$$

where,

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & T_f \\ 0 & 0 & I & T_\omega \end{bmatrix} \tag{7}$$

$$T^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -T_f & I & 0 \\ 0 & -T_\omega & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix} \tag{8}$$

$$T_f = [f^b]^\times C_e^b \tag{9}$$

$$T_\omega = C_e^b \Omega \tag{10}$$

This transformation yields a new state vector:

$$\bar{x} = \begin{bmatrix} \delta v^e \\ \delta b_a^b - [f^b]^\times C_e^b \delta\rho^e \\ \delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \\ \delta\rho^e \end{bmatrix} \tag{11}$$

By substituting Eq. (6) into Eq. (3), the system equation based on the new state vector is given by

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{w} \tag{12}$$

where,

$$\bar{A} = T^{-1}\big(AT - \dot{T}\big) = \begin{bmatrix} -2\Omega & C_b^e & 0 & 0 \\ 0 & 0 & -[f^b]^\times & -\dot{T}_f \\ 0 & 0 & -[\omega_{ie}^b]^\times & -\dot{T}_\omega \\ 0 & 0 & C_b^e & 0 \end{bmatrix} \tag{13}$$

$$\dot{T}_f = \big([\dot{f}^b]^\times - [f^b]^\times[\omega_{eb}^b]^\times\big)C_e^b \tag{14}$$

$$\dot{T}_\omega = -[\omega_{eb}^b]^\times[\omega_{ie}^b]^\times C_e^b \tag{15}$$

$$\bar{w} = T^{-1}w \tag{16}$$

Note that the term $\dot{T}$ and the fourth column elements of Eq. (13) have minus signs. In Rhee et al.,[8] these minus signs are missing: there are no minus signs in term $\dot{T}$ of Eq. (13a) and the fourth column of definition of $\bar{A}$. In this study, observability is analyzed based on the correct system matrix in Eq. (13).

If the observation equation is given by

$$z = Hx + v \tag{17}$$

then the observation equation based on the new state vector can be obtained as

$$z = \bar{H}\bar{x} + v \tag{18}$$

where,

$$\bar{H} = HT \tag{19}$$

This transformation replaces the last column of the system matrix of Eq. (13) with zero for the time-invariant case. For the time-varying case, the time-varying component of the system matrix appears only in the last column. Therefore, analyzing the observability of the transformed system can be more easily performed than that for the original system.

2.2. Observability analysis method

Consider the following linear time-varying system:

$$\dot{x}(t) = A(t)x(t) \tag{20}$$

$$z(t) = H(t)x(t) \tag{21}$$

It is assumed that the state vector is n-dimensional. For the system in Eq. (20), the observability matrix is defined as

$$O(t) = \begin{bmatrix} N_0(t) \\ N_1(t) \\ \vdots \\ N_{n-1}(t) \end{bmatrix} \tag{22}$$

where,

$$N_0(t) = H(t) \tag{23}$$

$$N_{k+1}(t) = N_k(t)A(t) + \frac{d}{dt}N_k(t) \qquad k = 0, 1, 2, \cdots, n-1 \tag{24}$$

The system in Eq. (20) is said to be instantaneously observable on the interval (t_0, t_1) if the observability matrix has rank n for all $t \in (t_0, t_1)$.[11] If the system is instantaneously observable, then the state of the system at any time can be determined instantaneously from the measurements and their derivatives.[8] Therefore, instantaneous observability is important for minimizing the time required and amount of IFA maneuvering, and improving the accuracies of the estimated error states of the integrated GPS/INS system. If the observability matrix has a rank less than n, then the system is not fully observable. In this case, the bases of the null space of the observability matrix represent unobservable modes.

3. Observability of an Integrated GPS/INS Navigation System

The GPS receiver provides position and velocity measurements. As previously mentioned, the position error can be taken directly from GPS position measurements. Therefore, only GPS velocity measurement requires consideration. The measurement equation for velocity error is given by

$$z_{\mathrm{GPS}} = H_{\mathrm{GPS}}x + v_{\mathrm{GPS}} \tag{25}$$

where,

$$H_{\mathrm{GPS}} = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

and for the transformed system,

$$z_{\mathrm{GPS}} = \bar{H}_{\mathrm{GPS}}\bar{x} + v_{\mathrm{GPS}} \tag{27}$$

where,

$$\bar{H}_{\mathrm{GPS}} = H_{\mathrm{GPS}}T = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

The observability of the integrated GPS/INS navigation system can be analyzed by constructing the observability matrix of Eq. (22) using the system matrix of Eq. (13) and the observation matrix of Eq. (28). The observability matrix for a general case is given by

$$O = \begin{bmatrix} I & 0 & 0 & 0 \\ -2\Omega & C_b^e & 0 & 0 \\ O_{2,1} & O_{2,2} & O_{2,3} & O_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ O_{11,1} & O_{11,2} & O_{11,3} & O_{11,4} \end{bmatrix} \tag{29}$$

where,

$$O_{1,1} = -2\Omega, \quad O_{1,2} = C_b^e, \quad O_{1,3} = 0, \quad O_{1,4} = 0 \tag{30}$$

$$O_{k,1} = (-2\Omega)^k \tag{31}$$

$$O_{k,2} = O_{k-1,1}C_b^e + \dot{O}_{k-1,2} \tag{32}$$

$$O_{k,3} = -O_{k-1,2}[f^b]^\times - O_{k-1,3}[\omega_{ie}^b]^\times + O_{k-1,4}C_b^e + \dot{O}_{k-1,3} \tag{33}$$

$$O_{k,4} = O_{k-1,2}\dot{T}_f + O_{k-1,3}\dot{T}_\omega + \dot{O}_{k-1,4} \quad k = 2, \cdots, 11 \tag{34}$$

3.1. Time-invariant case

The observability of an integrated GPS/INS navigation system for time-invariant or static cases has already been studied in Bar-Itzhack and Berman,[1] Rhee et al.,[8] and Jiang and Lin.[2] In this section, the results for the time-invariant case are briefly summarized. The proof of the summary can be found in Rhee et al.[8]

For the time-invariant case, the vehicle maintains a constant specific force ($\dot{f}^b = 0$) and does not rotate with respect to the Earth ($\omega_{eb}^b = 0$). The time derivative components of Eq. (13) and Eq. (24) become zero. The observability matrix is given as

$$O = \begin{bmatrix} I & 0 & 0 & 0 \\ -2\Omega & C_b^e & 0 & 0 \\ (-2\Omega)^2 & (-2\Omega)C_b^e & O_{2,3} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-2\Omega)^{11} & (-2\Omega)^{10}C_b^e & O_{11,3} & 0 \end{bmatrix} \tag{35}$$

where,

$$\begin{aligned} O_{2,3} &= -C_b^e[f^b]^\times, \\ O_{3,3} &= 2\Omega C_b^e[f^b]^\times + C_b^e[f^b]^\times[\omega_{ie}^b]^\times \end{aligned} \tag{36}$$

$$O_{k,2} = (-2\Omega)^{k-1}C_b^e \tag{37}$$

$$O_{k,3} = -O_{k-1,3}[\omega_{ie}^b]^\times - (-2)^{k-2}C_b^e\{[\omega_{ie}^b]^\times\}^{k-2}[f^b]^\times \quad k = 3, \cdots, 11 \tag{38}$$

Through a rank test and a null space analysis of the observability matrix, as shown in Eq. (35), the observability of the integrated GPS/INS system for the time-invariant case is summarized in Table 1.

In general, if the aircraft is not located at the poles of the Earth, then all states are observable except the attitude error state, $\delta\rho^e$. However, the bias errors of the accelerometer and the gyroscope are not necessarily observable. States $\delta b_a^b - [f^b]^\times C_e^b\delta\rho^e$ and $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b\delta\rho^e$ are observable; however, since both states contain $\delta\rho^e$, the biases of the accelerometer and gyroscope are not directly observable. There is an uncertainty of as much as $\delta\rho^e$. In the case of the gyroscope bias, the uncertainty is not great because the cross-product is executed between $\delta\rho^e$ and the rotational rate of the Earth. However, the estimation error of the accelerometer bias due to the attitude error cannot be ignored because the cross-product is executed between $\delta\rho^e$ and the specific force, f^b. f^b corresponds to the gravitational acceleration and is relatively large, which explains why accelerometer bias cannot be easily estimated accurately. Thus, the uncertainty of the biases of the accelerometer and gyroscope is reduced when the observability of the attitude error is improved, and it is possible to make an accurate estimation of the biases.

Table 1. Observability analysis results of GPS/INS for the time-invariant case.

Case	Rank(O)	Unobservable modes
Not on the poles ($\omega_{ie} \times f \neq 0$)	9	$\delta\rho^e$
On the poles ($\omega_{ie} \times f = 0$ and $f \neq 0$)	8	$\delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b\delta\rho^e \parallel \omega_{ie}^b$
Freefall ($f = 0$)	6	$\delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b\delta\rho^e$

3.2. Time-varying case

Since the system matrix of Eq. (13) changes with maneuvering, the observability of the GPS/INS navigation system also changes according to maneuvering. It is known that to change the system matrix through maneuvering improves observability. Analysis of observability is thus performed for linear acceleration and constant-rate horizontal turn maneuvers, which are typical for aircraft.

3.2.1. Linear acceleration maneuver

It is assumed that the linear acceleration satisfies the following conditions:

$$\omega_{eb}^b = 0 \tag{39}$$

and

$$\dot{f}^b \neq 0, \quad f^{(k)b} = \left.\frac{d^k}{dt^k}\right|_b f^b = a_k \cdot \dot{f} \text{ for } a_k \in \mathbb{R} \quad k = 2, \cdots, 10 \tag{40}$$

The aircraft does not rotate with respect to the Earth, and the differential of the specific force is not zero. In addition, the 2nd to the 10th differential of specific forces are all parallel to the differential of the specific force. When these assumptions are satisfied, the observability matrix is given as follows.

$$O = \begin{bmatrix} I & 0 & 0 & 0 \\ -2\Omega & C_b^e & 0 & 0 \\ O_{2,1} & O_{2,2} & O_{2,3} & O_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ O_{11,1} & O_{11,2} & O_{11,3} & O_{11,4} \end{bmatrix} \tag{41}$$

where,

$$O_{2,1} = (-2\Omega)^2, \quad O_{2,2} = (-2\Omega)C_b^e, \quad O_{2,3} = -C_b^e[f^b]^\times, \quad O_{2,4} = -C_b^e[\dot{f}^b]^\times C_e^b \tag{42}$$

$$O_{k,1} = (-2\Omega)^k \tag{43}$$

$$O_{k,2} = (-2\Omega)^{k-1}C_b^e \tag{44}$$

$$O_{k,3} = -O_{k-1,2}[f^b]^\times - O_{k-1,3}[\omega_{ie}^b]^\times + O_{k-1,4}C_b^e + \dot{O}_{k-1,3} \tag{45}$$

$$O_{k,4} = -O_{k-1,2}[\dot{f}^b]^\times C_e^b + \dot{O}_{k-1,4} \tag{46}$$

In the observability matrix in Eq. (41), I of the first row and C_b^e of the second row ensure that the first and second columns of the observability matrix have the rank of 6 and that the corresponding first and second error states are observable. O_{sub} is constructed using $O_{2,3}$, $O_{2,4}$, $O_{3,3}$ and $O_{3,4}$ in the third and fourth columns to perform the rank test and null space analysis as follows:

$$O_{\mathrm{sub}} = \begin{bmatrix} O_{2,3} & O_{2,4} \\ O_{3,3} & O_{3,4} \end{bmatrix} \tag{47}$$

where,

$$O_{2,3} = -C_b^e[f^b]^\times \tag{48}$$

$$O_{2,4} = -C_b^e[\dot{f}^b]^\times C_e^b \tag{49}$$

$$O_{3,3} = C_b^e\big([f^b]^\times[\omega_{ie}^b]^\times + 2[\omega_{ie}^b]^\times[f^b]^\times - 2[\dot{f}^b]^\times\big) \tag{50}$$

$$O_{3,4} = -C_b^e\big(a_2[\dot{f}^b]^\times - 2[\omega_{ie}^b]^\times[\dot{f}^b]^\times\big)C_e^b \tag{51}$$

By performing elementary row operations, the rank of O_{sub} can be represented as

$$\mathrm{rank}(O_{\mathrm{sub}}) = \mathrm{rank}\left(\begin{bmatrix} -[f^b]^\times & -[\dot{f}^b]^\times \\ \big([f^b]^\times[\omega_{ie}^b]^\times - 2[\dot{f}^b]^\times\big) & -a_2[\dot{f}^b]^\times \end{bmatrix}\right) \tag{52}$$

The results based on Eq. (52) depend on the states of f^b and $\dot{f}^b$. The analysis is performed for two cases of f^b. The first case is on the Equator ($\omega_{ie}^b \cdot f^b = 0$) and the second case is not on the Equator ($\omega_{ie}^b \cdot f^b \neq 0$). The results are summarized for each case in Table 2 and Table 3, respectively.

The above results are the analytical analysis results from the first row to the fourth row of the observability matrix, and the unobservable modes can be seen in the remaining fifth to twelfth rows. For the remaining parts of the observability matrix, a numerical analysis is performed due to the complexity of the analytical analysis. The numerical analysis method proposed is as follows.

1. Create an unobservable basis y_{null} for each mode, $|y_{\mathrm{null}}| = 1$, corresponding to the unit vector
2. Calculate the numerical observability matrix

$$O = \begin{bmatrix} I & 0 & 0 & 0 \\ -2\Omega & C_b^e & 0 & 0 \\ O_{2,1} & O_{2,2} & O_{2,3} & O_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ O_{11,1} & O_{11,2} & O_{11,3} & O_{11,4} \end{bmatrix}$$

3. Check the numerical observability based on the definition of the null space

Table 2. Analytic observability analysis results for a linear acceleration, Case 1: on the Equator ($\omega_{ie}^b \cdot f^b = 0$).

Condition on $\dot{f}^b$	Rank(O_{sub})	Unobservable modes
$\dot{f}^b = 1/2 \cdot a_2 \cdot f^b$	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} -1/2 \cdot a_2 \cdot \omega_{ie}^b \times f^b \\ \omega_{ie}^b \times f^b \end{bmatrix}$
$\dot{f}^b = 1/2 \cdot (a_2 \cdot f^b - \omega_{ie}^b \times f^b)$	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} -1/2 \cdot \omega_{ie}^b \times f^b \\ f^b \end{bmatrix}$
$\dot{f}^b = \frac{1}{2}a_2 \cdot f^b + n \cdot \omega_{ie}^b \times f^b$ ($n \neq 0$ and $-1/2$)	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} -1/2 \cdot a_2 \cdot \omega_{ie}^b \times f^b \\ \omega_{ie}^b \times f^b \end{bmatrix}$
$\dot{f}^b = m \cdot f^b + n \cdot \omega_{ie}^b \times f^b$ ($m \neq 0$ and $1/2 \cdot a_2$)	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} m(a_2 - 2m) \cdot f^b \\ -m(1+2n) \cdot \omega_{ie}^b \times f^b \\ (1+2n) \cdot \omega_{ie}^b \times f^b \end{bmatrix}$
$\dot{f}^b = l \cdot \omega_{ie}^b + 1/2 \cdot a_2 \cdot f^b + n \cdot \omega_{ie}^b \times f^b$ ($l \neq 0$)	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} l \cdot \omega_{ie}^b + n \cdot \omega_{ie}^b \times f^b \\ f^b \end{bmatrix}$
Otherwise	5	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$

Table 3. Analytic observability analysis results for linear acceleration, Case 2: not on the Equator ($\omega_{ie}^b \cdot f^b \neq 0$).

Condition on $\dot{f}^b$	Rank(O_{sub})	Unobservable modes
$\dot{f}^b = m \cdot f^b$ ($m \neq 0$)	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} m(\omega_{ie}^b \cdot f^b) \cdot \omega_{ie}^b \\ +m(a_2 - 2m)^2 \cdot f^b \\ -m(a_2 - 2m) \cdot \omega_{ie}^b \times f^b \\ (a_2 - 2m) \cdot \omega_{ie}^b \times f^b \\ -(\omega_{ie}^b \cdot f^b) \cdot \omega_{ie}^b \end{bmatrix}$
$\dot{f}^b = m \cdot f^b - 1/2 \cdot \omega_{ie}^b \times f^b$	4	$\begin{bmatrix} 0 \\ \dot{f}^b \end{bmatrix}$, $\begin{bmatrix} f^b \\ 0 \end{bmatrix}$
$\dot{f}^b = l \cdot \omega_{ie}^b + 1/2 \cdot a_2 \cdot f^b$ ($l \neq 0$)	4	$\begin{bmatrix} 0 \\ \dot{f}^b \end{bmatrix}$, $\begin{bmatrix} l \cdot \omega_{ie}^b \\ f^b \end{bmatrix}$
$\dot{f}^b = l \cdot \omega_{ie}^b + 1/2 \cdot a_2 \cdot f^b - 1/2 \cdot \omega_{ie}^b \times f^b$	4	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$, $\begin{bmatrix} l^2 \cdot \omega_{ie}^b \\ +1/4 \cdot (\omega_{ie}^b \cdot f^b) \cdot f^b \\ -1/2 \cdot l \cdot \omega_{ie}^b \times f^b \\ l \cdot f^b \end{bmatrix}$
Otherwise	5	$\begin{bmatrix} 0 & \dot{f}^b \end{bmatrix}^T$

$\kappa = \max\{O \cdot y_{\text{null}}\}$
$\kappa > \varepsilon \Rightarrow$ the mode is observable
$\kappa \le \varepsilon \Rightarrow$ the mode is unobservable

where
ε: threshold for numerical observability

Case 1 is the analysis for a latitude of 0 deg, and Case 2 is the analysis for a latitude of 45 deg. The longitude and attitude, which do not affect observability, are both zero. The constants, $a_2, \ldots, a_{10}$, l, m and n determining the mode shape, are all assumed to be 1. The general mode shapes of Cases 1 and 2 are analyzed. The value of ε is important for determining the observability. In this analysis, ε is not fixed to a specific value, and the observability is determined by comparing the value κ of every other mode because the magnitudes of κ increase with the specific force and its differentials. Therefore, the relative observability between modes can be examined by comparing the value of κ. The results are shown in Table 4.

It is clear that the common mode, the attitude error vector of the differential specific force direction, is not observable because κ of this mode is small relative to the other modes. The modes whose mode shapes are characterized by the dominance of the specific force vector in the third row have a κ value of approximately 10^{-2}. These modes correspond to the gyroscope bias. The other modes are the error modes dominated by attitude error and have a κ value of approximately 10^{-4}. The differences in the value of κ indicates that gyroscope bias can be observed approximately 100 times easier than the attitude error.

The numerical analysis results show that all of the modes are observable except for the attitude error of the differential specific force direction when the aircraft performs a linear acceleration maneuver.

Table 4. Numerical observability analysis results for linear acceleration.

Mode	Mode shape	Observability
Common mode	For arbitrary f^b and $\dot{f}^b$, $y_{\text{null}} = \begin{bmatrix} 0 & 0 & 0 & \dot{f}^b \end{bmatrix}^T$	$\kappa \sim 10^{-17}$ $\Rightarrow$ unobservable
General mode 1 on the Equator	For $\dot{f}^b = m \cdot f^b + n \cdot \omega_{ie}^b \times f^b$, $y_{\text{null}} = \begin{bmatrix} 0 \\ 0 \\ m(a_2 - 2m) \cdot f^b \\ -m(1+2n) \cdot \omega_{ie}^b \times f^b \\ (1+2n) \cdot \omega_{ie}^b \times f^b \end{bmatrix}$	$\kappa \sim 10^{-2}$ $\Rightarrow$ observable
General mode 2 on the Equator	For $\dot{f}^b = l \cdot \omega_{ie}^b + \frac{1}{2} a_2 \cdot f^b + n \cdot \omega_{ie}^b \times f^b$, $y_{\text{null}} = \begin{bmatrix} 0 \\ 0 \\ l \cdot \omega_{ie}^b + n \cdot \omega_{ie}^b \times f^b \\ f^b \end{bmatrix}$	$\kappa \sim 10^{-4}$ $\Rightarrow$ observable
General mode 1 not on the Equator	For $\dot{f}^b = m \cdot f^b$, $y_{\text{null}} = \begin{bmatrix} 0 \\ 0 \\ l \cdot (\omega_{ie}^b \cdot f^b) \cdot \omega_{ie}^b \\ +m(a_2 - 2m)^2 \cdot f^b \\ -m(a_2 - 2m) \cdot \omega_{ie}^b \times f^b \\ (a_2 - 2m) \cdot \omega_{ie}^b \times f^b \\ -\left(\omega_{ie}^b \cdot f^b\right) \cdot \omega_{ie}^b \end{bmatrix}$	$\kappa \sim 10^{-2}$ $\Rightarrow$ observable
General mode 2 not on the Equator	For $\dot{f}^b = m \cdot f^b - 1/2 \cdot \omega_{ie}^b \times f^b$, $y_{\text{null}} = \begin{bmatrix} 0 & 0 & f^b & 0 \end{bmatrix}^T$	$\kappa \sim 10^{-2}$ $\Rightarrow$ observable
General mode 3 not on the Equator	For $\dot{f}^b = l \cdot \omega_{ie}^b + 1/2 \cdot a_2 \cdot f^b$, $y_{\text{null}} = \begin{bmatrix} 0 & 0 & l \cdot \omega_{ie}^b & f^b \end{bmatrix}^T$	$\kappa \sim 10^{-4}$ $\Rightarrow$ observable
General mode 4 not on the Equator	For $\dot{f}^b = l \cdot \omega_{ie}^b + 1/2 \cdot a_2 \cdot f^b - 1/2 \cdot \omega_{ie}^b \times f^b$, $y_{\text{null}} = \begin{bmatrix} 0 \\ 0 \\ l \cdot \omega_{ie}^b \\ +1/4 \cdot (\omega_{ie}^b \cdot f^b) \cdot f^b \\ -1/2 \cdot \omega_{ie}^b \times f^b \\ f^b \end{bmatrix}$	$\kappa \sim 10^{-4}$ $\Rightarrow$ observable

3.2.2. Constant-rate horizontal turn maneuver

It is assumed that a constant-rate horizontal turn satisfies the following conditions:

$$\omega_{eb}^b \neq 0, \quad \dot{\omega}_{eb}^b = 0 \tag{53}$$

and

$$f^b \neq 0, \quad \dot{f}^b = 0 \tag{54}$$

The angular velocity of the aircraft with respect to the Earth is not zero and is constant. The specific force is not zero and is constant. The observability matrix for the constant-rate horizontal turn maneuver satisfying these conditions is given by

$$O = \begin{bmatrix} I & 0 & 0 & 0 \\ -2\Omega & C_b^e & 0 & 0 \\ O_{2,1} & O_{2,2} & O_{2,3} & O_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ O_{11,1} & O_{11,2} & O_{11,3} & O_{11,4} \end{bmatrix} \tag{55}$$

where,

$$O_{2,1} = (-2\Omega)^2, \quad O_{2,2} = (-2\Omega)C_b^e + C_b^e[\omega_{eb}^b]^\times, \\ O_{2,3} = -C_b^e[f^b]^\times, \quad O_{2,4} = C_b^e[f^b]^\times[\omega_{eb}^b]^\times C_e^b \tag{56}$$

$$O_{k,1} = (-2\Omega)^k \tag{57}$$

$$O_{k,2} = (-2\Omega)^{k-1} C_b^e + \dot{O}_{k-1,2} \tag{58}$$

$$O_{k,3} = -O_{k-1,2}[f^b]^\times - O_{k-1,3}[\omega_{ie}^b]^\times + O_{k-1,4}C_b^e + \dot{O}_{k-1,3} \tag{59}$$

$$O_{k,4} = O_{k-1,2}[f^b]^\times[\omega_{eb}^b]^\times C_e^b + O_{k-1,3}[\omega_{eb}^b]^\times[\omega_{ie}^b]^\times C_e^b + \dot{O}_{k-1,4} \tag{60}$$

As in the case of linear acceleration analyzed in the previous section, in Eq. (55), I of the first row and C_b^e of the second row ensure that the first and second columns of the observability matrix have the rank of 6, and that the corresponding first and second error states are observable. O_{sub} is constructed using $O_{2,3}$, $O_{2,4}$, $O_{3,3}$ and $O_{3,4}$ in the third and fourth columns to perform the rank test and null space analysis as follows:

$$O_{sub} = \begin{bmatrix} O_{2,3} & O_{2,4} \\ O_{3,3} & O_{3,4} \end{bmatrix} \tag{61}$$

where,

$$O_{2,3} = -C_b^e[f^b]^\times \tag{62}$$

$$O_{2,4} = C_b^e[f^b]^\times[\omega_{eb}^b]^\times C_e^b \tag{63}$$

$$\begin{aligned} O_{3,3} = C_b^e\{&2([\omega_{ie}^b]^\times - [\omega_{eb}^b]^\times)[f^b]^\times \\ &+ [f^b]^\times([\omega_{ie}^b]^\times + [\omega_{eb}^b]^\times)\} \end{aligned} \tag{64}$$

$$\begin{aligned} O_{3,4} = C_b^e\{&-2([\omega_{ie}^b]^\times - [\omega_{eb}^b]^\times)[f^b]^\times[\omega_{eb}^b]^\times \\ &- [f^b]^\times[\omega_{eb}^b]^\times([\omega_{ie}^b]^\times + [\omega_{eb}^b]^\times)\}C_e^b \end{aligned} \tag{65}$$

By performing elementary row operations, the rank of O_{sub} can be represented as

$$\begin{aligned} &\text{rank}(O_{sub}) \\ &= \text{rank}\left(\begin{bmatrix} [f^b]^\times & 0 \\ 0 & [f^b]^\times \end{bmatrix} \times \begin{bmatrix} -I & [\omega_{eb}^b]^\times \\ ([\omega_{ie}^b]^\times + [\omega_{eb}^b]^\times) & -[\omega_{eb}^b]^\times([\omega_{ie}^b]^\times + [\omega_{eb}^b]^\times) \end{bmatrix}\right) \end{aligned} \tag{66}$$

The analysis results based on Eq. (66) are shown in Table 5.

$$y_{null,1} = \begin{bmatrix} \left(\omega_{ie}^b \cdot \omega_{eb}^b + \left|\omega_{eb}^b\right|^2\right)\left(\omega_{ie}^b \times \omega_{eb}^b\right) \\ -\left(\omega_{ie}^b + \omega_{eb}^b\right) \times \left(\omega_{ie}^b \times \omega_{eb}^b\right) \end{bmatrix} \tag{67}$$

$$y_{null,2} = \begin{bmatrix} \left\{l \cdot \left(\omega_{ie}^b \cdot \omega_{eb}^b\right) + m \cdot \left|\omega_{eb}^b\right|^2\right\}\left(\omega_{ie}^b \times \omega_{eb}^b\right) \\ -\left(l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b\right) \times \left(\omega_{ie}^b \times \omega_{eb}^b\right) \end{bmatrix} \tag{68}$$

$$y_{null,3} = \begin{bmatrix} \left(l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b\right) \\ -\dfrac{m-l}{n\left|\omega_{ie}^b \times \omega_{eb}^b\right|^2} \\ \cdot\left\{l \cdot \left(\omega_{ie}^b \cdot \omega_{eb}^b\right) + m \cdot \left|\omega_{eb}^b\right|^2\right\}\left(\omega_{ie}^b \times \omega_{eb}^b\right) \\ \dfrac{m-l}{n\left|\omega_{ie}^b \times \omega_{eb}^b\right|^2} \\ \cdot\begin{bmatrix} \left\{l \cdot \left(\omega_{ie}^b \cdot \omega_{eb}^b\right) + m \cdot \left|\omega_{eb}^b\right|^2\right\} \cdot \omega_{ie}^b \\ -\left\{l \cdot \left|\omega_{ie}^b\right|^2 + m \cdot \left(\omega_{ie}^b \cdot \omega_{eb}^b\right)\right\} \cdot \omega_{eb}^b \end{bmatrix} \\ +n \cdot \left(\omega_{ie}^b + \omega_{eb}^b\right) \end{bmatrix} \tag{69}$$

For the remaining fifth to twelfth rows of the observability matrix, a numerical analysis is performed in the same manner as the linear acceleration case. The angular rate of the aircraft's turn is assumed to be 10 deg/s. The constants l, m and n, determining the mode shape, are all assumed to be 1. The results are shown in Table 6.

The mode on the second row of Table 6 appears to be unobservable because it has a value of κ 10,000 times smaller than that of the other modes. The other modes also have smaller values of κ than the linear acceleration cases.

The numerical analysis results show that there is one unobservable mode when the specific force during a constant-rate horizontal turn lies on the plane formed by the Earth's rotational rate vector and the angular velocity vector of the aircraft with respect to the Earth. This mode occurs at the time point when the heading of the aircraft is aligned to the east or the west during a turn maneuver. Therefore, observations are possible except at those time points. Numerical results show that observability is generally lower than during

Table 5. Analytic observability analysis results for a constant-rate horizontal turn.

Condition on f^b	Rank(O_{sub})	Unobservable modes
$f^b = l \cdot (\omega_{ie}^b + \omega_{eb}^b)$	3	$\begin{bmatrix} \omega_{eb}^b \times (\omega_{ie}^b \times \omega_{eb}^b) \\ \omega_{ie}^b \times \omega_{eb}^b \end{bmatrix}$, $\begin{bmatrix} \omega_{ie}^b + \omega_{eb}^b \\ 0 \end{bmatrix}$, $y_{null,1}$
$f^b = l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b$	4	$\begin{bmatrix} \omega_{eb}^b \times (\omega_{ie}^b \times \omega_{eb}^b) \\ \omega_{ie}^b \times \omega_{eb}^b \end{bmatrix}$, $y_{null,2}$
$f^b = l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b + n \cdot (\omega_{ie}^b \times \omega_{eb}^b)$ ($n \neq 0$)	4	$\begin{bmatrix} \omega_{eb}^b \times (\omega_{ie}^b \times \omega_{eb}^b) \\ \omega_{ie}^b \times \omega_{eb}^b \end{bmatrix}$, $y_{null,3}$

Table 6. Numerical observability analysis results for a constant-rate horizontal turn.

Mode	Mode shape	Observability
Common mode	For arbitrary f^b, $y_{null} = \begin{bmatrix} 0 \\ 0 \\ \omega_{eb}^b \times (\omega_{ie}^b \times \omega_{eb}^b) \\ \omega_{ie}^b \times \omega_{eb}^b \end{bmatrix}$	$\kappa \sim 10^{-6}$ $\Rightarrow$ observable
f^b lies on ω_{ie}^b–ω_{eb}^b plane	For $f^b = l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b$, $y_{null} = \begin{bmatrix} 0 \\ 0 \\ y_{null,2} \end{bmatrix}$	$\kappa \sim 10^{-10}$ $\Rightarrow$ unobservable
f^b is parallel with $\omega_{ie}^b + \omega_{eb}^b$	For $f^b = l \cdot (\omega_{ie}^b + \omega_{eb}^b)$, $y_{null} = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie}^b + \omega_{eb}^b \\ 0 \end{bmatrix}$	$\kappa \sim 10^{-6}$ $\Rightarrow$ observable
f^b does not lie on ω_{ie}^b–ω_{eb}^b plane	For $f^b = l \cdot \omega_{ie}^b + m \cdot \omega_{eb}^b + n \cdot (\omega_{ie}^b \times \omega_{eb}^b)$ ($n \neq 0$), $y_{null} = \begin{bmatrix} 0 \\ 0 \\ y_{null,3} \end{bmatrix}$	$\kappa \sim 10^{-6}$ $\Rightarrow$ observable

a linear acceleration maneuver.

4. Observability by Magnetic Vector Measurement

Integrating the magnetic heading is based on the fact that the heading information can be obtained from the principle of the compass through the horizontal direction of the magnetic field. The magnetic heading is widely used as an additional measurement for AHRS or INS.[11,12,14–17] The magnetic heading is determined using only the horizontal component of the magnetic vector. The vertical component of the magnetic vector, which contains directional information, is not integrated generally as like the magnetic heading. Alternatively, integrating the magnetic vector has also been proposed by Barczyk and Lynch.[12] Magnetic vector integration combines the three-dimensional vector measurement from the magnetometer. This method uses the horizontal and vertical components of the magnetic vector. As the gravity vector measurement of the accelerometer provides two tilt angles about the local vertical, magnetic vector measurement also contains information regarding the two tilt angles around its direction. Therefore, the magnetic vector provides more information about the attitude than the magnetic heading and can ultimately enhance the observability of the integrated GPS/INS system. The measurement equation for magnetic vector measurement is given as

$$z_{\mathrm{mag}} = H_{\mathrm{mag}} x + v_{\mathrm{mag}} \tag{70}$$

where,

$$z_{\mathrm{mag}} = \hat{C}_b^e m_{\mathrm{mag}}^b - m^e \tag{71}$$

$$H_{\mathrm{mag}} = \begin{bmatrix} 0 & [m^e]^\times & 0 & 0 \end{bmatrix} \tag{72}$$

The difference between the measured magnetic vector transformed to the e-frame and the magnetic vector of the Earth from the geomagnetic model[18] provides information about the attitude error. For the system transformed using Eq. (6), the measurement equation is given as

$$z_{\mathrm{mag}} = \bar{H}_{\mathrm{mag}} \bar{x} + v_{\mathrm{mag}} \tag{73}$$

where,

$$\bar{H}_{\mathrm{mag}} = H_{\mathrm{mag}} T = \begin{bmatrix} 0 & 0 & 0 & [m^e]^\times \end{bmatrix} \tag{74}$$

This method was also implemented for various navigation systems reported in Barczyk and Lynch,[12] Gebre-Egziabher and Elkaim,[19] Yun et al.,[20] and Psiaki et al.[21] However, magnetic vector aiding is not as common as magnetic heading aiding. The magnetic field of the Earth can be distorted by environmental objects, such as metals or electronic devices. Because the accuracy of the roll and pitch angles is important, magnetic vector attitude information should be used with caution. Despite these concerns, the magnetic vector aiding method has been successfully implemented in many cases. In the following sections, the usability of the magnetic vector as an aiding source is described, and enhancing its observability for the INS is analyzed. For simplification, in the analysis, the direction of the magnetic vector is assumed to be vertical at the poles and aligned to the rotational axis of the Earth at the Equator. Detailed information about the magnetic field is described in the world magnetic model (WMM) provided by the National Geographic Data Center (NGDC).[22]

4.1. Usability of magnetic vector measurement

Information about the magnetic vector is required for implementing magnetic vector measurement because the magnetic vector varies according to the position on the Earth. The magnetic models in various resolutions are available from the NGDC, and the user can select a model suitable to the purpose of use. In the case of navigation systems, a magnetic model should be selected according to the level of performance of the sensors equipped, operational range, and storage capacity.

The error of the magnetic vector measurement arises not only from the internal sensor error, but also from external metal objects or electromagnetic fields. The internal sensor error and influence of metal objects in a vehicle, such as metal structures or equipment, can be eliminated using various calibration methods for the magnetometer.[23–25] The influence of ground objects, such as cars or buildings, cannot be calibrated beforehand because the influence changes with the relative position between the system and the various objects. The influence of the ground objects decreases with increasing distance between the system and the objects. For an aircraft, the influence of the ground objects is negligible if the altitude is maintained at a sufficiently high level. External disturbances can be detected and isolated by monitoring the magnitude of the magnetic field.

4.2. Time-invariant case

The observability matrix for magnetic vector aiding is constructed using the system matrix in Eq. (13) and the observation matrix of the magnetic vector in Eq. (74). The assumptions for the time-invariant case are the same as those of the GPS/INS. Much like the time-invariant case of the GPS/INS, the time derivative terms in Eq. (13) and Eq. (22) are ignored. The observability matrix for the time-invariant case is given as:

$$O = \begin{bmatrix} 0 & 0 & 0 & [m^e]^\times \\ 0 & 0 & [m^e]^\times C_b^e & 0 \\ 0 & 0 & O_{2,3} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & O_{11,3} & 0 \end{bmatrix} \tag{75}$$

where,

$$O_{k,3} = [m^e]^\times C_b^e (-[\omega_{ie}^b]^\times)^{k-1} \quad k = 2, \cdots, 11 \tag{76}$$

The last column of Eq. (75) has a rank of 2. The third column of Eq. (75) should be evaluated to calculate the rank of the complete observability matrix. The rank of the observability matrix can be expressed as follows:

$$\mathrm{rank}(O) = 2 + \mathrm{rank}(O_{\mathrm{sub}}) \tag{77}$$

where,

Table 7. Observability analysis results of magnetic vector measurement for the time-invariant case.

Case	Rank(O)	Unobservable modes
Not on the poles or the Equator ($\omega_{ie}^e \times m^e \neq 0$)	5	$\delta v^e, \delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta\rho^e \parallel m^e$
On the poles or the Equator ($\omega_{ie}^e \times m^e = 0$)	4	$\delta v^e, \delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$

$$O_{\text{sub}} = \begin{bmatrix} [m^e]^\times C_b^e \\ O_{2,3} \\ \vdots \\ O_{11,3} \end{bmatrix} \tag{78}$$

Performing several elementary row operations on O_{sub} yields the rank of the observability matrix as follows:

$$\text{rank}(O) = 2 + \text{rank}(\bar{O}_{\text{sub}}) \tag{79}$$

where,

$$\bar{O}_{\text{sub}} = \begin{bmatrix} [m^b]^\times \\ [m^b]^\times(-[\omega_{ie}^b]^\times) \\ \vdots \\ [m^b]^\times(-[\omega_{ie}^b]^\times)^{10} \end{bmatrix} \tag{80}$$

The results of the rank test and the null space analysis for Eq. (79) are summarized in Table 7. Note that the Earth's magnetic field has been simplified to a dipole model with the poles as both poles of the magnetic field to facilitate analysis.

Magnetic vector measurement cannot provide observability for the velocity error or the bias of the accelerometer. However, the attitude error and gyroscope bias can be observed using the magnetic vector. If the position of the aircraft is not at one of the poles or the Equator, then the use of the magnetic vector guarantees observability of the attitude error, which is not parallel to the magnetic vector. The state vector $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$, which corresponds to the gyroscope bias, is also observable. At the poles or the Equator, the magnetic field vector and angular velocity vector of the Earth are parallel, and $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$ of the Earth's rotation vector direction is additionally unobservable, in contrast to the case of not being at the poles.

4.3. Time-varying case

For the time-varying case, observability is analyzed for two types of maneuvers, as in the case of the GPS/INS. The assumptions for each maneuver are also identical to those of the GPS/INS.

4.3.1. Linear acceleration maneuver

For the linear acceleration maneuver, the observability matrix is identical to that of the time-invariant case. The rank of the observability matrix and unobservable modes are exactly equal to those of the time-invariant case. Note that the unobservable attitude error mode for magnetic vector aid is not parallel with that of the GPS/INS. Therefore, using the magnetic vector can significantly improve observability of the GPS/INS system in the case of linear acceleration maneuvers.

4.3.2. Constant-rate horizontal turn maneuver

For the constant-rate horizontal turn maneuver, the observability matrix is given as:

$$O = \begin{bmatrix} 0 & 0 & 0 & [m^e]^\times \\ 0 & 0 & [m^e]^\times C_b^e & 0 \\ 0 & 0 & O_{2,3} & O_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & O_{11,3} & O_{11,4} \end{bmatrix} \tag{81}$$

where,

$$\begin{aligned} O_{2,3} &= [m^e]^\times C_b^e\{[\omega_{eb}^b]^\times - [\omega_{ie}^b]^\times\}, \\ O_{2,4} &= -[m^e]^\times C_b^e[\omega_{eb}^b]^\times[\omega_{ie}^b]^\times C_b^e \end{aligned} \tag{82}$$

$$O_{k,3} = O_{k-1,3}(-[\omega_{ie}^b]^\times) + O_{k-1,4}C_b^e + \dot{O}_{k-1,3} \tag{83}$$

$$O_{k,4} = O_{k-1,3}(-[\omega_{eb}^b]^\times[\omega_{ie}^b]^\times C_e^b) + \dot{O}_{k-1,4} \quad \text{for } k = 3, \cdots, 11 \tag{84}$$

The rank of the observability matrix can be expressed as:

$$\text{rank}(O) = \text{rank}(O_{\text{sub}}) \tag{85}$$

where,

$$O_{\text{sub}} = \begin{bmatrix} 0 & [m^e]^\times \\ [m^e]^\times C_b^e & 0 \\ O_{2,3} & O_{2,4} \\ \vdots & \vdots \\ O_{11,3} & O_{11,4} \end{bmatrix} \tag{86}$$

Consider the rank of O_{sub} from the first to the third rows. Several elementary row operations yield:

$$\text{rank}\left(\begin{bmatrix} 0 & [m^e]^\times \\ [m^e]^\times C_b^e & 0 \\ O_{2,3} & O_{2,4} \end{bmatrix}\right) = \text{rank}(\bar{M}\bar{O}) \tag{87}$$

where,

$$\bar{M} = \begin{bmatrix} [m^b]^\times & 0 & 0 \\ 0 & [m^b]^\times & 0 \\ 0 & 0 & [m^b]^\times \end{bmatrix} \tag{88}$$

$$\bar{O} = \begin{bmatrix} 0 & I \\ I & 0 \\ [\omega_{eb}^b]^\times - [\omega_{ie}^b]^\times & [\omega_{eb}^b]^\times[\omega_{ie}^b]^\times \end{bmatrix} \tag{89}$$

For the first and second rows of $\bar{M}\bar{O}$, there are two bases of the null space given by:

$$y_{\text{null},1} = \begin{bmatrix} m^b & 0 \end{bmatrix}^T \tag{90}$$

$$y_{\text{null},2} = \begin{bmatrix} 0 & m^b \end{bmatrix}^T \tag{91}$$

For $y_{\text{null},1}$ and $y_{\text{null},2}$ to be the bases of the third row of $\bar{M}\bar{O}$, the following equations must be satisfied:

$$[m^b]^\times \Phi_1^b m^b = 0 \tag{92}$$

$$[m^b]^\times \Phi_2^b m^b = 0 \tag{93}$$

where,

$$\Phi_1^b = [\omega_{eb}^b]^\times - [\omega_{ie}^b]^\times, \quad \Phi_2^b = [\omega_{eb}^b]^\times [\omega_{ie}^b]^\times \tag{94}$$

Equation (92) and Eq. (93) imply that m^b is an eigenvector that corresponds to the real eigenvalues of Φ_1^b and Φ_2^b. Here, a coordinate system is introduced and is referred to as the w-frame, whose z-axis is aligned with ω_{eb} and whose x-axis is aligned with a vector that is obtained by projecting ω_{ie} on the plane perpendicular to the z-axis. Note that ω_{eb} is parallel with the local vertical when the aircraft performs a horizontal turn. In the w-frame, ω_{eb}, ω_{ie}, Φ_1^w and Φ_2^w are expressed as:

$$\omega_{eb}^w = \begin{bmatrix} 0 & 0 & \omega \end{bmatrix}^T \tag{95}$$

$$\omega_{ie}^w = \begin{bmatrix} \Omega_1 & 0 & \Omega_3 \end{bmatrix}^T \tag{96}$$

$$\Phi_1^w = \begin{bmatrix} 0 & -(\omega - \Omega_3) & 0 \\ (\omega - \Omega_3) & 0 & \Omega_1 \\ 0 & -\Omega_1 & 0 \end{bmatrix} \tag{97}$$

$$\Phi_2^w = \begin{bmatrix} -\omega\Omega_3 & 0 & \omega\Omega_1 \\ 0 & -\omega\Omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{98}$$

For Φ_1^w, the eigenvector is $(\omega_{eb}^w - \omega_{ie}^w)$, and the corresponding eigenvalue is zero. Generally, the angular rate of the aircraft is much higher than the rotational rate of the Earth. Therefore, ignoring the rotational rate of the Earth, $y_{\text{null},1}$ is not a basis of the null space of Φ_1^w if the magnetic vector is not parallel with ω_{eb}. For Φ_2^w, the eigenvectors are ω_{ie} and any vector perpendicular to ω_{eb}, and the corresponding eigenvalues are zero and $\omega\Omega_3$. Therefore, the eigenvectors of Φ_2^w can span m^w, and $y_{\text{null},2}$ is the basis of the null space of Φ_2^w. Moreover, the rest of the observability matrix does not provide additional observability for the error states because all components of the remaining part are also pre-multiplied by $[m^b]^\times$ and manipulated by the skew symmetric matrices of $[\omega_{ie}^b]^\times$ and $[\omega_{eb}^b]^\times$ that have a rank lower than 2. The results of the rank test and null space analysis for Eq. (81) are summarized in Table 8.

The magnetic vector cannot provide attitude information in its direction even if an aircraft performs a time-varying maneuver. The constant-rate horizontal turn maneuver provides additional observability for gyroscope bias, in contrast to the static case or linear acceleration case.

4.4. Observability of GPS/INS/magnetic vector integrated navigation system

In this section, the analysis results of previous sections are combined, and enhanced observability using the magnetic vector to assist a GPS/INS navigation system is summarized. The results of analysis for the time-invariant case are shown in Table 9.

For the time-invariant case, the integrated GPS/INS system cannot observe all the error states, even when using the magnetic vector. However, the magnetic vector improves observability of the system greatly for all cases. For non-freefall cases of the integrated GPS/INS system, the attitude error $\delta\rho^e$ is commonly unobservable; and the gyroscope bias error vector, which is parallel to the rotation axis of the Earth, is additionally unobservable when located on the poles. The observability of attitude error can be enhanced by the magnetic vector because it provides observability for attitude error in all cases, except for the attitude error vector along the direction of the magnetic field. Therefore, the rank of the GPS/INS system aided by the magnetic vector is improved by two for non-freefall cases. These improvements correspond to the observability of attitude provided by the magnetic vector. For freefall cases of the integrated GPS/INS system, $\delta\rho^e$ and $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$ are not observable. The unobservable mode $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$ mainly corresponds to the gyroscope bias error since the attitude error $\delta\rho^e$ and ro-

Table 8. Observability analysis results of magnetic vector measurement for a constant-rate horizontal turn.

Case	Rank(O)	Unobservable modes
Not on the poles ($\omega_{eb}^e \times m^e \neq 0$)	5	$\delta v^e, \delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta\rho^e \parallel m^e$
On the poles ($\omega_{eb}^e \times m^e = 0$)	4	$\delta v^e, \delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$

Table 9. Comparison of observability for the time-invariant case.

Case	GPS/INS		INS with magnetic vector		GPS/INS with magnetic vector	
	Rank(O)	Unobservable modes	Rank(O)	Unobservable modes	Rank(O)	Unobservable modes
Not on the poles or the Equator	9	$\delta\rho^e$	5	δv^e, $\delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta\rho^e \parallel m^e$	11	$\delta\rho^e \parallel m^e$
On the poles	8	$\delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel \omega_{ie}^b$	4	δv^e, $\delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$	10	$\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$
On the Equator	9	$\delta\rho^e$	4	δv^e, $\delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$	11	$\delta\rho^e \parallel m^e$
Freefall, not on the poles or the Equator	6	$\delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$	5	δv^e, $\delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta\rho^e \parallel m^e$	11	$\delta\rho^e \parallel m^e$
Freefall on the poles or the Equator	6	$\delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e$	4	δv^e, $\delta b_a^b - [f^b]^\times C_e^b \delta\rho^e$, $\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$	10	$\delta b_g^b - [\omega_{ie}^b]^\times C_e^b \delta\rho^e \parallel m^e$, $\delta\rho^e \parallel m^e$

Table 10. Comparison of observability for the time-varying case.

Case	GPS/INS	GPS/INS/ with magnetic vector	
	Rank(O)	Rank(O)	Unobservable modes
Linear acceleration ($\dot{f}^b \nparallel m^b$)	11	12	None
Constant-rate horizontal turn (f^b lies on $\omega_{ie}^b - \omega_{eb}^b$ plane)	11	12	None

tational rate of the Earth ω_{ie}^b are small quantities. The magnetic vector provides observability for all gyroscope bias error when not located on the poles or the Equator. Even if the system is located on the poles or the Equator, the observability of gyroscope bias error, which is not parallel to the magnetic vector, is provided. Therefore, the rank of the GPS/INS system aided by the magnetic vector is improved by five when not being located on the poles or the Equator, and by four when located on the poles or the Equator. The results of analysis for the time-varying case are shown in Table 10.

In the case of linear acceleration, the error states are fully observable when the differential specific force vector is not aligned to the magnetic field vector. The error states are always fully observable in the case of a constant-rate horizontal turn. The results of the analysis indicate that combining magnetic vector measurements with the GPS/INS integrated navigation system can ensure observability for all states in almost all situations under dynamic conditions.

5. Conclusion

In this study, observability of the GPS/INS navigation system and enhancing said observability via aid enabled by the magnetic vector was investigated. For the time-invariant case, two additional attitude error states become observable. Moreover, the observability of the system is not influenced by freefall. For the time-varying case, non-constant axial acceleration and constant-rate horizontal turn maneuvers were considered. The results of the analysis showed that the observability of all error states is guaranteed for the two maneuvers considered.

Enhanced observability reduces the initial alignment time required for integrated GPS/INS systems and improves the estimation performance of the navigation solution and sensor error. This enhanced sensor error estimation performance decreases the divergence speed of the navigation solution in the case of a GPS outage. Even if GPS is not available, magnetic vector measurement is still valid and useful for maintaining attitude accuracy.

The analyses of this study are not limited to magnetic vector measurement; they are applicable to any vector measurement for which the general characteristics are compatible. Therefore, the results of this study enable easier integration of additional vector measurements to GPS/INS navigation systems.

Acknowledgments

This research was supported by Lockheed Martin Corporation through the Republic of Korea Science and Technology (RoKST&R) Program, contracted through the Institute of Advanced Aerospace Technology at Seoul National University.

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