Enhancement of near cloaking for the full Maxwell equations*

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Abstract

In this paper, we consider near cloaking for the full Maxwell equations. We extend the method of [5, 6], where the quasi-static limit case and the Helmholtz equation are considered, to electromagnetic scattering problems. We construct very effective near cloaking structures for the electromagnetic scattering problem at a fixed frequency. These new structures are, before using the transformation optics, layered structures and are designed so that their first scattering coefficients vanish. Inside the cloaking region, any target has near-zero scattering cross section for a band of frequencies. We analytically show that our new construction significantly enhances the cloaking effect for the full Maxwell equations.

AMS subject classifications. 35R30, 35B30

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1 Introduction

The cloaking problem is to make a target invisible from far-field electromagnetic measurements [29, 20, 23, 13, 12, 21]. Many schemes for cloaking are under active current investigation. These include exterior cloaking in which the cloaking region is outside the cloaking device [25, 1, 24, 11, 10, 2], active cloaking [15], and interior cloaking, which is the focus of our study.

In interior cloaking, the difficulty is to construct electromagnetic material parameter distributions of a cloaking structure such that any target placed inside the structure is undetectable to waves. One approach is to use transformation optics [29, 13, 30, 33, 16]. It takes advantage of the fact that the equations governing electromagnetism have transformation laws under change of variables. This allows one to design structures that steel waves around a hidden region, returning them to their original path on the far side. The change of variables based cloaking method uses a singular transformation to boost the material properties so that it makes a cloaking region look like a point to outside measurements. However, this transformation induces the singularity of material constants in the transversal direction (also in the tangential direction in two dimensions), which causes difficulty both in the theory and applications. To overcome this weakness, so called 'near cloaking' is naturally considered, which is a regularization or an approximation of singular cloaking. In [19], instead of the singular transformation, the authors use a regular one to push forward the material constant in the conductivity equation describing the quasi-static limit of electromagnetism, in which a small ball is blown up to the cloaking region. In [18], this regularization

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point of view is adopted for the Helmholtz equation. See also [22, 28]. More recently, Bao and Liu [8] considered near cloaking for the full Maxwell equations. They derived sharp estimates for the boundary effect due to a small inclusion with an arbitrary material parameters enclosed by a thin high-conducting layer. Their results show that the near cloaking scheme can be applied to cloak targets from electromagnetic boundary measurements.

In [5, 6] it is shown that the near cloaking, from measurements of the Dirichlet-to-Neumann map for the conductivity equation and of the scattering cross section for the Helmholtz equation, can be drastically enhanced by using multi-layered structures. The structures are designed so that their generalized polarization tensors (GPTs) or scattering coefficients vanish (up to a certain order). GPTs are building blocks of the far-field behavior of solutions in the quasi-static limits (conductivity equations) and the scattering coefficients are 'Fourier coefficients' of the scattering amplitude. The multi-layered structures combined with the usual change of variables (transformation optics) greatly reduce the visibility of an object. This fact is also confirmed by numerical experiments [3].

The purpose of this paper is to extend the results of [5, 6] to Maxwell's equation and show that the near cloaking from cross section scattering measurements at a fixed frequency can be enhanced by using layered structures together with the change of variables. Again the layered structures are designed so that their first scattering coefficients vanish. It is also shown that inside the cloaking region, any target has near-zero scattering cross section for a band of (low) frequencies. We analytically show that our new construction significantly enhances the near cloaking effect for the full Maxwell equations. It is worth mentioning that even if the basic scheme of this work is parallel to that of [6], the analysis is much more complicated due to the vectorial nature of the Maxwell equations.

The paper is organized as follows. In Section 2, we recall some fundamental results on the scattering problem for the full Maxwell equations. In Section 3, we introduce the scattering coefficients of an electromagnetic inclusion and prove that the scattering coefficients are basically the spherical harmonic expansion coefficients of the far-field pattern. Section 4 is devoted to the construction of layered structures with vanishing scattering coefficients. We also present some numerical examples of the scattering coefficient vanishing structures. In Section 5, we show that the near cloaking is enhanced if a scattering coefficient vanishing structure is used.

2 Multipole solutions to the Maxwell equations

In this section, we recall a few fundamental results related to electromagnetic scattering, which will be essential in the sequel.

Consider the time-dependent Maxwell equations

$$\begin{cases} \nabla \times \mathcal{E} = -\mu \frac{\partial}{\partial t} \mathcal{H}, \\ \nabla \times \mathcal{H} = \epsilon \frac{\partial}{\partial t} \mathcal{E}, \end{cases}$$

where μ is the magnetic permeability and ϵ is the electric permittivity.

In the time-harmonic regime, we look for the electromagnetic fields of the form

$$\begin{cases} \mathcal{H}(\mathbf{x}, t) = \mathbf{H}(\mathbf{x}) e^{-i\omega t}, \\ \mathcal{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}) e^{-i\omega t}, \end{cases}$$

where ω is the frequency. The couple $(\mathbf{E}, \mathbf{H})$ is a solution to the harmonic Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E} = i\omega\mu \mathbf{H}, \\ \nabla \times \mathbf{H} = -i\omega\epsilon \mathbf{E}. \end{cases} \tag{2.1}$$

We say that $(\mathbf{E}, \mathbf{H})$ is radiating if it satisfies the Silver-Müller radiation condition:

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| (\sqrt{\mu} \mathbf{H} \times \hat{\mathbf{x}} - \sqrt{\epsilon} \mathbf{E}) = 0,$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. In the sequel, we set $k = \omega\sqrt{\epsilon\mu}$, which is called the wave number.

For $m = -n, \dots, n$ and $n = 1, 2, \dots$, set Y_n^m to be the spherical harmonics defined on the unit sphere S. For a wave number $k > 0$, the following function

$$v_{n,m}(k; \mathbf{x}) = h_n^{(1)}(k|\mathbf{x}|)Y_n^m(\hat{\mathbf{x}}) \tag{2.2}$$

satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^3 \setminus \{0\}$ and the Sommerfeld radiation condition:

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|\Big(\frac{\partial v_{n,m}}{\partial|\mathbf{x}|}(k; \mathbf{x}) - ikv_{n,m}(k; \mathbf{x})\Big) = 0.$$

Here, $h_n^{(1)}$ is the spherical Hankel function of the first kind and order n which satisfies the Sommerfeld radiation condition. Similarly, $\tilde{v}_{n,m}(\mathbf{x})$ is defined as

$$\tilde{v}_{n,m}(k; \mathbf{x}) = j_n(k|\mathbf{x}|)Y_n^m(\hat{\mathbf{x}}), \tag{2.3}$$

where j_n is the spherical Bessel function of the first kind. The function $\tilde{v}_{n,m}$ satisfies the Helmholtz equation in all $\mathbb{R}^3$.

In the same manner, we can make solutions to the Maxwell system with the vector version of spherical harmonics. Define the vector spherical harmonics as

$$\mathbf{U}_{n,m} = \frac{1}{\sqrt{n(n+1)}}\nabla_S Y_n^m(\hat{\mathbf{x}}) \quad \text{and} \quad \mathbf{V}_{n,m} = \hat{\mathbf{x}} \times \mathbf{U}_{n,m}, \tag{2.4}$$

for $m = -n, \dots, n$ and $n = 1, 2, \dots$. Here, $\hat{\mathbf{x}} \in S$ and ∇_S denotes the surface gradient on the unit sphere S. The vector spherical harmonics defined in (2.4) form a complete orthogonal basis for $L^2_T(S)$, where $L^2_T(S) = \{\mathbf{u} \in (L^2(S))^3 \mid \boldsymbol{\nu} \cdot \mathbf{u} = 0\}$ and $\boldsymbol{\nu}$ is the outward unit normal to S.

Multiplying the vector spherical harmonics to the Hankel function, we can make the so-called multipole solutions to the Maxwell system. To make the analysis simple, we separate the solutions into transverse electric, $(\mathbf{E} \cdot \mathbf{x}) = 0$, and transverse magnetic, $(\mathbf{H} \cdot \mathbf{x}) = 0$. Define the exterior transverse electric multipoles to (2.1) as

$$\begin{cases} \mathbf{E}^{TE}_{n,m}(k; \mathbf{x}) = -\sqrt{n(n+1)}h_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}}), \\ \mathbf{H}^{TE}_{n,m}(k; \mathbf{x}) = -\dfrac{i}{\omega\mu}\nabla\times\Big(-\sqrt{n(n+1)}h_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}})\Big), \end{cases} \tag{2.5}$$

and the exterior transverse magnetic multipoles as

$$\begin{cases} \mathbf{E}^{TM}_{n,m}(k; \mathbf{x}) = \dfrac{i}{\omega\epsilon}\nabla\times\Big(-\sqrt{n(n+1)}h_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}})\Big), \\ \mathbf{H}^{TM}_{n,m}(k; \mathbf{x}) = -\sqrt{n(n+1)}h_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}}). \end{cases} \tag{2.6}$$

The exterior electric and magnetic multipole satisfies the radiation condition. By the same way, we define the interior multipoles $(\widetilde{\mathbf{E}}^{TE}_{n,m}, \widetilde{\mathbf{H}}^{TE}_{n,m})$ and $(\widetilde{\mathbf{E}}^{TM}_{n,m}, \widetilde{\mathbf{H}}^{TM}_{n,m})$ with $h_n^{(1)}$ replaced by j_n, *i.e.*,

$$\begin{cases} \widetilde{\mathbf{E}}^{TE}_{n,m}(k; \mathbf{x}) = -\sqrt{n(n+1)}j_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}}), \\ \widetilde{\mathbf{H}}^{TE}_{n,m}(k; \mathbf{x}) = -\dfrac{i}{\omega\mu}\nabla \times \widetilde{\mathbf{E}}^{TE}_{n,m}(k; \mathbf{x}), \end{cases} \tag{2.7}$$

and

$$\begin{cases} \widetilde{\mathbf{H}}^{TM}_{n,m}(k; \mathbf{x}) = -\sqrt{n(n+1)}j_n^{(1)}(k|\mathbf{x}|)\mathbf{V}_{n,m}(\hat{\mathbf{x}}), \\ \widetilde{\mathbf{E}}^{TM}_{n,m}(k; \mathbf{x}) = \dfrac{i}{\omega\epsilon}\nabla \times \widetilde{\mathbf{H}}^{TM}_{n,m}(k; \mathbf{x}). \end{cases} \tag{2.8}$$

We will sometimes omit the wave number k in the notation of the multipoles.

Note that we have

$$\nabla \times \mathbf{E}_{n,m}^{TE}(k;\mathbf{x}) = \frac{\sqrt{n(n+1)}}{|\mathbf{x}|}\mathcal{H}_n(k|\mathbf{x}|)\mathbf{U}_{n,m}(\hat{\mathbf{x}}) + \frac{n(n+1)}{|\mathbf{x}|}h_n^{(1)}(k_0|\mathbf{x}|)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}}, \tag{2.9}$$

$$\nabla \times \widetilde{\mathbf{E}}_{n,m}^{TE}(k;\mathbf{x}) = \frac{\sqrt{n(n+1)}}{|\mathbf{x}|}\mathcal{J}_n(k|\mathbf{x}|)\mathbf{U}_{n,m}(\hat{\mathbf{x}}) + \frac{n(n+1)}{|\mathbf{x}|}j_n^{(1)}(k_0|\mathbf{x}|)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}}, \tag{2.10}$$

where $\mathcal{H}_n(t) = h_n^{(1)}(t) + t\left(h_n^{(1)}\right)'(t)$ and $\mathcal{J}_n(t) = j_n(t) + tj_n'(t)$.

The solutions to the Maxwell system can be represented as separated variable sums of the multipole solutions, see [27, Section 5.3]. With multipole solutions and the Helmholtz solutions in (2.2) and (2.3), it is also possible to expand the fundamental solution to the Helmholtz operator. For $k > 0$, the fundamental solution Γ_k to the Helmholtz operator $(\Delta + k^2)$ in $\mathbb{R}^3$ is

$$\Gamma_k(\mathbf{x}) = -\frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}. \tag{2.11}$$

Let $\mathbf{p}$ be a fixed vector in $\mathbb{R}^3$. For $|\mathbf{x}| > |\mathbf{y}|$, the following addition formula holds (see [26, Section 9.3.3]):

$$\begin{aligned}\Gamma_k(\mathbf{x}-\mathbf{y})\mathbf{p} = &-\sum_{n=1}^{\infty}\frac{ik}{n(n+1)}\frac{\epsilon}{\mu}\sum_{m=-n}^{n}\mathbf{E}_{n,m}^{TM}(k;\mathbf{x})\overline{\widetilde{\mathbf{E}}_{n,m}^{TM}(k;\mathbf{y})}\cdot\mathbf{p}\\ &+\sum_{n=1}^{\infty}\frac{ik}{n(n+1)}\sum_{m=-n}^{n}\mathbf{E}_{n,m}^{TE}(k;\mathbf{x})\overline{\widetilde{\mathbf{E}}_{n,m}^{TE}(k;\mathbf{y})}\cdot\mathbf{p}\\ &-\frac{i}{k}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\nabla v_{n,m}(k;\mathbf{x})\overline{\nabla\tilde{v}_{n,m}(k;\mathbf{y})}\cdot\mathbf{p},\end{aligned} \tag{2.12}$$

with $v_{n,m}$ and $\tilde{v}_{n,m}$ being defined by (2.2) and (2.3).

Plane wave solutions to the Maxwell equations have the expansion using the multipole solutions as well (see [17]). The incoming wave $\mathbf{E}^i(\mathbf{x}) = ik(\mathbf{q}\times\mathbf{p})\times\mathbf{q}e^{ik\mathbf{q}\cdot\mathbf{x}}$, where $\mathbf{q}\in S$ is the direction of propagation and the vector $\mathbf{p}\in\mathbb{R}^3$ is the direction of polarization, is expressed as

$$\mathbf{E}^i(\mathbf{x}) = ik\sum_{p=1}^{\infty}\frac{4\pi i^p}{\sqrt{p(p+1)}}\sum_{q=-p}^{p}\left[\big(\mathbf{V}_{p,q}(\mathbf{q})\cdot\mathbf{c}\big)\widetilde{\mathbf{E}}_{p,q}^{TE}(\mathbf{x}) - \frac{1}{i\omega\mu}\big(\mathbf{U}_{p,q}(\mathbf{q})\cdot\mathbf{c}\big)\widetilde{\mathbf{E}}_{p,q}^{TM}(\mathbf{x})\right], \tag{2.13}$$

where $\mathbf{c} = (\mathbf{q}\times\mathbf{p})\times\mathbf{q}$.

3 Scattering coefficients of an inclusion

Let D be a bounded domain in $\mathbb{R}^3$ with $\mathcal{C}^{1,\alpha}$ boundary for some $\alpha > 0$, and let (ϵ_0, μ_0) be the pair of electromagnetic parameters (permittivity and permeability) of $\mathbb{R}^3\setminus\overline{D}$ and (ϵ_1,μ_1) be that of D. We assume that $\epsilon_0, \epsilon_1, \mu_0$, and μ_1 are positive constants. Then the permittivity and permeability distributions are given by

$$\epsilon = \epsilon_0\chi(\mathbb{R}^3\setminus\overline{D}) + \epsilon_1\chi(D) \quad\text{and}\quad \mu = \mu_0\chi(\mathbb{R}^3\setminus\overline{D}) + \mu_1\chi(D),$$

where χ denotes the characteristic function. In the sequel, we set $k = \omega\sqrt{\epsilon_1\mu_1}$ and $k_0 = \omega\sqrt{\epsilon_0\mu_0}$.

For a given solution $(\mathbf{E}^i, \mathbf{H}^i)$ to the Maxwell equations

$$\begin{cases}\nabla\times\mathbf{E}^i = i\omega\mu_0\mathbf{H}^i & \text{in } \mathbb{R}^3,\\ \nabla\times\mathbf{H}^i = -i\omega\epsilon_0\mathbf{E}^i & \text{in } \mathbb{R}^3,\end{cases}$$

let $(\mathbf{E}, \mathbf{H})$ be the solution to the following Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E} = i\omega\mu\mathbf{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} & \text{in } \mathbb{R}^3, \\ (\mathbf{E} - \mathbf{E}^i, \mathbf{H} - \mathbf{H}^i) & \text{satisfies the Silver-Müller radiation condition.} \end{cases} \tag{3.1}$$

We emphasize that along the interface ∂D, the following transmission condition holds:

$$[\boldsymbol{\nu} \times \mathbf{E}] = [\boldsymbol{\nu} \times \mathbf{H}] = 0. \tag{3.2}$$

Here, $[\nu \times \mathbf{E}]$ denotes the jump of $\nu \times \mathbf{E}$ along ∂D, namely,

$$[\nu \times \mathbf{E}] = (\nu \times \mathbf{E})\big|_{\partial D}^{+} - (\nu \times \mathbf{E})\big|_{\partial D}^{-}.$$

Let $\nabla_{\partial D} \cdot$ denote the surface divergence. We introduce the function space

$$TH(\mathrm{div}, \partial D) := \Big\{ \boldsymbol{u} \in L_T^2(\partial D) : \nabla_{\partial D} \cdot \boldsymbol{u} \in L^2(\partial D) \Big\},$$

equipped with the norm

$$\|\boldsymbol{u}\|_{TH(\mathrm{div}, \partial D)} = \|\boldsymbol{u}\|_{L^2(\partial D)} + \|\nabla_{\partial D} \cdot \boldsymbol{u}\|_{L^2(\partial D)}.$$

For a density $\boldsymbol{\varphi} \in TH(\mathrm{div}, \partial D)$, we define the single layer potential associated with the fundamental solutions Γ_k given in (2.11) by

$$\mathcal{S}_D^k[\boldsymbol{\varphi}](\mathbf{x}) := \int_{\partial D} \Gamma_k(\mathbf{x} - \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3.$$

For a scalar density contained in $L^2(\partial D)$, the single layer potential is defined by the same way. We also define boundary integral operators:

$$\begin{aligned} \mathcal{L}_D^k[\boldsymbol{\varphi}](\mathbf{x}) &:= \Big(\boldsymbol{\nu} \times \big(k^2 \mathcal{S}_D^k[\boldsymbol{\varphi}] + \nabla \mathcal{S}_D^k[\nabla_{\partial D} \cdot \boldsymbol{\varphi}]\big)\Big)(\mathbf{x}), \\ \mathcal{M}_D^k[\boldsymbol{\varphi}](\mathbf{x}) &:= \mathrm{p.v.} \int_{\partial D} \boldsymbol{\nu}(\mathbf{x}) \times \Big(\nabla_{\mathbf{x}} \times \big(\Gamma_k(\mathbf{x} - \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y})\big)\Big) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial D. \end{aligned}$$

In the same way, we define $\mathcal{S}_D^{k_0}$, $\mathcal{L}_D^{k_0}$, and $\mathcal{M}_D^{k_0}$ associated with Γ_{k_0} instead of Γ_k. Then the solution to (3.1) can be represented as the following:

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \mathbf{E}^i(\mathbf{x}) + \mu_0 \nabla \times \mathcal{S}_D^{k_0}[\boldsymbol{\varphi}](\mathbf{x}) + \nabla \times \nabla \times \mathcal{S}_D^{k_0}[\boldsymbol{\psi}](\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D}, \\ \mu_1 \nabla \times \mathcal{S}_D^{k}[\boldsymbol{\varphi}](\mathbf{x}) + \nabla \times \nabla \times \mathcal{S}_D^{k}[\boldsymbol{\psi}](\mathbf{x}), & \mathbf{x} \in D, \end{cases} \tag{3.3}$$

and

$$\mathbf{H}(\mathbf{x}) = -\frac{i}{\omega\mu}\big(\nabla \times \mathbf{E}\big)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial D,$$

where the pair $(\boldsymbol{\varphi}, \boldsymbol{\psi}) \in TH(\mathrm{div}, \partial D) \times TH(\mathrm{div}, \partial D)$ is the unique solution to

$$\begin{bmatrix} \dfrac{\mu_1 + \mu_0}{2} I + \mu_1 \mathcal{M}_D^k - \mu_0 \mathcal{M}_D^{k_0} & \mathcal{L}_D^k - \mathcal{L}_D^{k_0} \\ \mathcal{L}_D^k - \mathcal{L}_D^{k_0} & \left(\dfrac{k^2}{2\mu_1} + \dfrac{k_0^2}{2\mu_0}\right) I + \dfrac{k^2}{\mu_1} \mathcal{M}_D^k - \dfrac{k_0^2}{\mu_0} \mathcal{M}_D^{k_0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} \mathbf{E}^i \times \boldsymbol{\nu} \\ i\omega \mathbf{H}^i \times \boldsymbol{\nu} \end{bmatrix}\Bigg|_{\partial D}. \tag{3.4}$$

The invertibility of the system of equations (3.4) on $TH(\mathrm{div}, \partial D) \times TH(\mathrm{div}, \partial D)$ was proved in [32]. Moreover, there exists a constant $C = C(\epsilon, \mu, \omega)$ such that

$$\|\boldsymbol{\varphi}\|_{TH(\mathrm{div}, \partial D)} + \|\boldsymbol{\psi}\|_{TH(\mathrm{div}, \partial D)} \le C\big(\|\mathbf{E}^i \times \boldsymbol{\nu}\|_{TH(\mathrm{div}, \partial D)} + \|\mathbf{H}^i \times \boldsymbol{\nu}\|_{TH(\mathrm{div}, \partial D)}\big). \tag{3.5}$$

From (2.12) (with k_0 in the place of k) and (3.3) it follows that, for sufficiently large $|\mathbf{x}|$,

$$(\mathbf{E} - \mathbf{E}^i)(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{ik_0}{n(n+1)} \sum_{m=-n}^{n} \Big(\alpha_{n,m} \mathbf{E}_{n,m}^{TE}(k_0; \mathbf{x}) \ + \ \beta_{n,m} \mathbf{E}_{n,m}^{TM}(k_0; \mathbf{x}) \Big), \tag{3.6}$$

where

$$\alpha_{n,m} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}(\mathbf{y}) + k_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}(\mathbf{y}),$$
$$\beta_{n,m} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}(\mathbf{y}) - \omega^2\epsilon_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}(\mathbf{y}).$$

Definition 1. *Let* $(\boldsymbol{\varphi}_{p,q}^{TE}, \boldsymbol{\psi}_{p,q}^{TE})$ *be the solution to (3.4) when* $\mathbf{E}^i = \widetilde{\mathbf{E}}_{p,q}^{TE}(k_0; \mathbf{y})$ *and* $\mathbf{H}^i = \widetilde{\mathbf{H}}_{p,q}^{TE}(k_0; \mathbf{y})$, *and* $(\boldsymbol{\varphi}_{p,q}^{TM}, \boldsymbol{\psi}_{p,q}^{TM})$ *when* $\mathbf{E}^i = \widetilde{\mathbf{E}}_{p,q}^{TM}(k_0; \mathbf{y})$ *and* $\mathbf{H}^i = \widetilde{\mathbf{H}}_{p,q}^{TM}(k_0; \mathbf{y})$. *The scattering coefficients* $\Big(W_{(n,m)(p,q)}^{TE,TE}, W_{(n,m)(p,q)}^{TE,TM}, W_{(n,m)(p,q)}^{TM,TE}, W_{(n,m)(p,q)}^{TM,TM} \Big)$ *associated with the permittivity and the permeability distributions* ϵ, μ *and the frequency* ω *(or* k, k_0, D*) is defined to be*

$$W_{(n,m)(p,q)}^{TE,TE} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}_{p,q}^{TE}(\mathbf{y}) \ d\sigma(\mathbf{y}) + k_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}_{p,q}^{TE}(\mathbf{y}) \ d\sigma(\mathbf{y}),$$
$$W_{(n,m)(p,q)}^{TE,TM} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}_{p,q}^{TM}(\mathbf{y}) \ d\sigma(\mathbf{y}) + k_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}_{p,q}^{TM}(\mathbf{y}) \ d\sigma(\mathbf{y}),$$
$$W_{(n,m)(p,q)}^{TM,TE} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}_{p,q}^{TE}(\mathbf{y}) \ d\sigma(\mathbf{y}) - \omega^2\epsilon_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}_{p,q}^{TE}(\mathbf{y}) \ d\sigma(\mathbf{y}),$$
$$W_{(n,m)(p,q)}^{TM,TM} = -i\omega\epsilon_0\mu_0 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TE}}(k_0; \mathbf{y}) \cdot \boldsymbol{\varphi}_{p,q}^{TM}(\mathbf{y}) \ d\sigma(\mathbf{y}) - \omega^2\epsilon_0^2 \int_{\partial D} \overline{\widetilde{\mathbf{E}}_{n,m}^{TM}}(k_0; \mathbf{y}) \cdot \boldsymbol{\psi}_{p,q}^{TM}(\mathbf{y}) \ d\sigma(\mathbf{y}).$$

As we see it now, the scattering coefficients appear naturally in the expansion of the scattering amplitude. We first obtain the following estimates of the scattering coefficients.

Lemma 3.1. *There exists a constant* C *depending on* (ϵ, μ, ω) *such that*

$$\Big| W_{(n,m)(p,q)}^{TE,TE}[\epsilon, \mu, \omega] \Big| \leq \frac{C^{n+p}}{n^n p^p}, \tag{3.7}$$

for all $n, m, p, q \in \mathbb{N}$. *The same estimates hold for* $W_{(n,m)(p,q)}^{TE,TM}$, $W_{(n,m)(p,q)}^{TM,TE}$, *and* $W_{(n,m)(p,q)}^{TM,TM}$.

Proof. Let $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ be the solution to (3.4) with $\mathbf{E}^i(\mathbf{y}) = \widetilde{\mathbf{E}}_{p,q}^{TE}(k_0; \mathbf{y})$ and $\mathbf{H}^i = -\frac{i}{\omega\mu_0}\nabla \times \mathbf{E}^i$. Recall that the spherical Bessel function j_p behaves as

$$j_p(t) = \frac{t^p}{1 \cdot 3 \cdots (2p+1)} \Big(1 + O\left(\frac{1}{p}\right) \Big) \quad \text{as } p \to \infty,$$

uniformly on compact subsets of $\mathbb{R}$. Using Stirling's formula $p! = \sqrt{2\pi p}(p/e)^p(1 + o(1))$, we have

$$j_p(t) = O\left(\frac{C^p t^p}{p^p}\right) \quad \text{as } p \to \infty, \tag{3.8}$$

uniformly on compact subset of $\mathbb{R}$ with a constant C independent of p. Thus we have

$$\|\mathbf{E}^i\|_{TH(\mathrm{div}, \partial D)} + \|\mathbf{H}^i\|_{TH(\mathrm{div}, \partial D)} \leq \frac{C'^p}{p^p}$$

for some constant C'. It then follows from (3.5) that

$$\|\boldsymbol{\varphi}\|_{L^2(\partial D)} + \|\boldsymbol{\psi}\|_{L^2(\partial D)} \leq \frac{C^p}{p^p}$$

for another constant C. So we get (3.7) from the definition of the scattering coefficients. $\square$

Suppose that the incoming wave is of the form

$$\mathbf{E}^i(\mathbf{x}) = \sum_{p=1}^{\infty}\sum_{q=-p}^{p}\Big(a_{p,q}\widetilde{\mathbf{E}}_{p,q}^{TE}(k_0;\mathbf{x}) + b_{p,q}\widetilde{\mathbf{E}}_{p,q}^{TM}(k_0;\mathbf{x})\Big) \tag{3.9}$$

for some constants $a_{p,q}$ and $b_{p,q}$. Then the solution $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ to (3.4) is given by

$$\boldsymbol{\varphi} = \sum_{p=1}^{\infty}\sum_{q=-p}^{p}\Big(a_{p,q}\boldsymbol{\varphi}_{p,q}^{TE} + b_{p,q}\boldsymbol{\varphi}_{p,q}^{TM}\Big),$$
$$\boldsymbol{\psi} = \sum_{p=1}^{\infty}\sum_{q=-p}^{p}\Big(a_{p,q}\boldsymbol{\psi}_{p,q}^{TE} + b_{p,q}\boldsymbol{\psi}_{p,q}^{TM}\Big).$$

By (3.6) and Definition 1, the solution $\mathbf{E}$ to (3.1) can be represented as

$$(\mathbf{E}-\mathbf{E}^i)(\mathbf{x}) = \sum_{n=1}^{\infty}\frac{ik_0}{n(n+1)}\sum_{m=-n}^{n}\Big(\alpha_{n,m}\mathbf{E}_{n,m}^{TE}(k_0;\mathbf{x}) \ + \ \beta_{n,m}\mathbf{E}_{n,m}^{TM}(k_0;\mathbf{x})\Big), \quad |\mathbf{x}|\to\infty, \tag{3.10}$$

where

$$\begin{cases} \alpha_{n,m} = \displaystyle\sum_{p=1}^{\infty}\sum_{q=-p}^{p}\Big(a_{p,q}W_{(n,m)(p,q)}^{TE,TE} + b_{p,q}W_{(n,m)(p,q)}^{TE,TM}\Big), \\ \beta_{n,m} = \displaystyle\sum_{p=1}^{\infty}\sum_{q=-p}^{p}\Big(a_{p,q}W_{(n,m)(p,q)}^{TM,TE} + b_{p,q}W_{(n,m)(p,q)}^{TM,TM}\Big). \end{cases} \tag{3.11}$$

Using (3.10), (3.11) and the behavior of the spherical Bessel functions, we can estimate the far-field pattern of the scattered wave $(\mathbf{E}-\mathbf{E}^i)$. The far-field pattern (also called the scattering amplitude) $\mathbf{A}_\infty[\epsilon,\mu,\omega]$ is defined by

$$\mathbf{E}(\mathbf{x}) - \mathbf{E}^i(\mathbf{x}) = \frac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}\mathbf{A}_\infty[\epsilon,\mu,\omega](\hat{\mathbf{x}}) + o(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}|\to\infty. \tag{3.12}$$

Since the spherical Bessel function $h_n^{(1)}$ behaves like

$$\begin{cases} h_n^{(1)}(t) \sim \dfrac{1}{t}e^{it}e^{-i\frac{n+1}{2}\pi} & \text{as } t\to\infty, \\ (h_n^{(1)})'(t) \sim \dfrac{1}{t}e^{it}e^{-i\frac{n}{2}\pi} & \text{as } t\to\infty, \end{cases}$$

one can easily see by using (2.9) that

$$\begin{cases} \mathbf{E}_{n,m}^{TE}(k_0;\mathbf{x}) \sim \dfrac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}e^{-i\frac{n+1}{2}\pi}\ \big(-\sqrt{n(n+1)}\big)\mathbf{V}_{n,m}(\hat{\mathbf{x}}) & \text{as } |\mathbf{x}|\to\infty, \\ \mathbf{E}_{n,m}^{TM}(k_0;\mathbf{x}) \sim \dfrac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}\sqrt{\dfrac{\mu_0}{\epsilon_0}}e^{-i\frac{n+1}{2}\pi}\big(-\sqrt{n(n+1)}\big)\mathbf{U}_{n,m}(\hat{\mathbf{x}}) & \text{as } |\mathbf{x}|\to\infty. \end{cases}$$

The following result holds.

Proposition 3.2. *If $\mathbf{E}^i$ is given by (3.9), then the corresponding scattering amplitude can be expanded as*

$$\mathbf{A}_\infty[\epsilon,\mu,\omega](\hat{\mathbf{x}}) = \sum_{n=1}^{\infty}\frac{-i^{-n}k_0}{\sqrt{n(n+1)}}\sum_{m=-n}^{n}\Big(\alpha_{n,m}\mathbf{V}_{n,m}(\hat{\mathbf{x}}) + \beta_{n,m}\sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{U}_{n,m}(\hat{\mathbf{x}})\Big), \tag{3.13}$$

where $\alpha_{n,m}$ and $\beta_{n,m}$ are defined by (3.11).

We emphasize that since $\{\mathbf{V}_{n,m}, \mathbf{U}_{n,m}\}$ forms an orthogonal basis of $L^2_T(S)$, the conversion of the far-field to the near field is achieved via formula (3.10).

We now consider the case where the incident wave $\mathbf{E}^i$ is given by a plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{c}$ with $|\mathbf{k}| = k_0$ and $\mathbf{k}\cdot\mathbf{c} = 0$. It follows from (2.13) that

$$e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{c} = \sum_{p=1}^{\infty} \frac{4\pi i^p}{\sqrt{p(p+1)}} \sum_{q=-p}^{p} \left[\big(\mathbf{V}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}\big)\widetilde{\mathbf{E}}^{TE}_{p,q}(k_0;\mathbf{x}) - \frac{1}{i\omega\mu_0}\big(\mathbf{U}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}\big)\widetilde{\mathbf{E}}^{TM}_{p,q}(k_0;\mathbf{x}) \right],$$

where $\hat{\mathbf{k}} = \mathbf{k}/k_0 \in S$, and therefore,

$$a_{p,q} = \frac{4\pi i^p}{\sqrt{p(p+1)}}(\mathbf{V}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}) \quad \text{and} \quad b_{p,q} = -\frac{4\pi i^p}{\sqrt{p(p+1)}}\frac{1}{i\omega\mu_0}(\mathbf{U}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}).$$

Hence, the scattering amplitude, which we denote by $\mathbf{A}_\infty[\epsilon,\mu,\omega](\boldsymbol{c},\hat{\boldsymbol{k}};\hat{\boldsymbol{x}})$, is given by (3.13) with the coefficients $\alpha_{n,m}$ and $\beta_{n,m}$

$$\begin{cases} \alpha_{n,m} = \displaystyle\sum_{p=1}^{\infty}\sum_{q=-p}^{p} \frac{4\pi i^p}{\sqrt{p(p+1)}} \left[(\mathbf{V}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}) W^{TE,TE}_{(n,m)(p,q)} - \frac{1}{i\omega\mu_0}(\mathbf{U}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}) W^{TE,TM}_{(n,m)(p,q)} \right], \\ \beta_{n,m} = \displaystyle\sum_{p=1}^{\infty}\sum_{q=-p}^{p} \frac{4\pi i^p}{\sqrt{p(p+1)}} \left[(\mathbf{V}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}) W^{TM,TE}_{(n,m)(p,q)} - \frac{1}{i\omega\mu_0}(\mathbf{U}_{p,q}(\hat{\mathbf{k}})\cdot\mathbf{c}) W^{TM,TM}_{(n,m)(p,q)} \right]. \end{cases} \tag{3.14}$$

These formulas tell us that the scattering coefficients appear in the expansion of the scattering amplitude.

We now investigate the low frequency behavior of the scattering coefficients. Let $\Gamma(\mathbf{x}) := -1/(4\pi|\mathbf{x}|)$ denote the fundamental solution corresponding to the case $k = 0$, and $\mathcal{M}_D$ the associated boundary integral operator:

$$\mathcal{M}_D[\boldsymbol{\varphi}](\mathbf{x}) := \text{p.v.} \int_{\partial D} \boldsymbol{\nu}(\mathbf{x}) \times \Big(\nabla_{\mathbf{x}} \times \big(\Gamma(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\big)\Big) d\sigma(\mathbf{y}), \quad \boldsymbol{\varphi} \in TH(\text{div}, \partial D).$$

Analogously to (3.4), one can prove that there is a unique solution $(\boldsymbol{\varphi}^{(0)}, \boldsymbol{\psi}^{(0)}) \in TH(\text{div}, \partial D) \times TH(\text{div}, \partial D)$ to the following equations:

$$\begin{bmatrix} (\mu_1-\mu_0)\Big(\dfrac{\mu_1+\mu_0}{2(\mu_1-\mu_0)}I + \mathcal{M}_D\Big) & 0 \\ 0 & (\epsilon_1-\epsilon_0)\Big(\dfrac{\epsilon_1+\epsilon_0}{2(\epsilon_1-\epsilon_0)}I + \mathcal{M}_D\Big) \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}^{(0)} \\ \omega\boldsymbol{\psi}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{E}^i \times \boldsymbol{\nu} \\ i\mathbf{H}^i \times \boldsymbol{\nu} \end{bmatrix}\Bigg|_{\partial D}. \tag{3.15}$$

In fact, since ∂D is $\mathcal{C}^{1,\alpha}$, $\mathcal{M}_D$ is compact and we may apply the Fredholm alternative to prove unique solvability of above equation. Moreover, we have

$$\|\boldsymbol{\varphi}^{(0)}\|_{TH(\text{div},\partial D)} + \omega\|\boldsymbol{\psi}^{(0)}\|_{TH(\text{div},\partial D)} \leq C(\|\mathbf{E}^i \times \boldsymbol{\nu}\|_{TH(\text{div},\partial D)} + \|\mathbf{H}^i \times \boldsymbol{\nu}\|_{TH(\text{div},\partial D)}), \tag{3.16}$$

with a constant $C = C(\epsilon, \mu)$.

Let ρ be a small positive number and consider the boundary integral equation (3.4) with k, k_0, and ω replaced by ρk, ρk_0, and $\rho\omega$, respectively. Then, we have (see [14])

$$\mathcal{M}_D^{\rho k} - \mathcal{M}_D = O(\rho^2), \quad \mathcal{M}_D^{\rho k_0} - \mathcal{M}_D = O(\rho^2),$$

and

$$\mathcal{L}_D^{\rho k} - \mathcal{L}_D^{\rho k_0} = O(\rho^2).$$

Since

$$\left(\frac{k^2}{2\mu_1}+\frac{k_0^2}{2\mu_0}\right)I+\frac{k^2}{\mu_1}\mathcal{M}_D^k-\frac{k_0^2}{\mu_0}\mathcal{M}_D^{k_0}=\rho^2\omega^2\left[\frac{\epsilon_1+\epsilon_0}{2}I+(\epsilon_1-\epsilon_0)\mathcal{M}_D+O(\rho^2)\right],$$

if we express the solution $(\boldsymbol{\varphi},\boldsymbol{\psi})$ to (3.4) as $(\boldsymbol{\varphi},\boldsymbol{\psi}):=(\boldsymbol{\varphi}^\rho,\rho\omega\boldsymbol{\psi}^\rho)$, then it satisfies

$$\Big(A+O(\rho)\Big)\begin{bmatrix}\boldsymbol{\varphi}^\rho\\ \rho\omega\boldsymbol{\psi}^\rho\end{bmatrix}=\begin{bmatrix}\mathbf{E}^i\times\boldsymbol{\nu}\\ i\mathbf{H}^i\times\boldsymbol{\nu}\end{bmatrix}\Bigg|_{\partial D},$$

where A is the 2-by-2 matrix appeared in the left-hand side of (3.15). From the invertibility of A, it follows that there are constants ρ_0 and $C=C(\epsilon,\mu,\omega)$ independent of ρ as long as $\rho\le\rho_0$ such that

$$\|\boldsymbol{\varphi}^\rho\|_{TH(\mathrm{div},\partial D)}+\rho\omega\|\boldsymbol{\psi}^\rho\|_{TH(\mathrm{div},\partial D)}\le C\big(\|\mathbf{E}^i\times\boldsymbol{\nu}\|_{TH(\mathrm{div},\partial D)}+\|\mathbf{H}^i\times\boldsymbol{\nu}\|_{TH(\mathrm{div},\partial D)}\big). \tag{3.17}$$

Lemma 3.3. *There exists ρ_0 such that, for all $\rho\le\rho_0$,*

$$\left|W_{(n,m)(p,q)}^{TE,TE}[\epsilon,\mu,\rho\omega]\right|\le\frac{C^{n+p}}{n^np^p}\rho^{n+p+1}, \tag{3.18}$$

for all $n,m,p,q\in\mathbb{N}$, where the constant C depends on (ϵ,μ,ω) but is independent of ρ. The same estimate holds for $W_{(n,m)(p,q)}^{TE,TM}$, $W_{(n,m)(p,q)}^{TM,TE}$, and $W_{(n,m)(p,q)}^{TM,TM}$.

Proof. Let $(\boldsymbol{\varphi},\boldsymbol{\psi})$ be the solution to (3.4) with $\mathbf{E}^i(\mathbf{y})=\widetilde{\mathbf{E}}_{p,q}^{TE}(\rho k_0;\mathbf{y})$ and $\mathbf{H}^i=-\frac{i}{\rho\omega\mu_0}\nabla\times\mathbf{E}^i$. Then, from (3.8), it follows that

$$\left\|\mathbf{E}^{i,\rho}\right\|_{TH(\mathrm{div},\partial D)}+\left\|\mathbf{H}^{i,\rho}\right\|_{TH(\mathrm{div},\partial D)}\le\frac{C^p}{p^p}\rho^p,$$

where C is independent of ρ, and hence

$$\left\|\boldsymbol{\varphi}^\rho\right\|_{L^2(\partial D)}+\rho\left\|\boldsymbol{\psi}^\rho\right\|_{L^2(\partial D)}\le\frac{C^p}{p^p}\rho^p,$$

for $\rho\le\rho_0$ for some ρ_0. So we get (3.18) from the definition of the scattering coefficients in Definition 1. $\square$

4 S-vanishing structures

The purpose of this section is to construct multilayered structures whose scattering coefficients vanish, which we call *S-vanishing structures.* The multi-layered structure is defined as follows: For positive numbers $r_1,\ldots,r_{L+1}$ with $2=r_1>r_2>\cdots r_{L+1}=1$, let

$$A_j:=\{\mathbf{x}:r_{j+1}\le|\mathbf{x}|<r_j\},\quad j=1,\ldots,L,\quad A_0:=\mathbb{R}^2\setminus\overline{A_1},\quad A_{L+1}(=D):=\{\mathbf{x}:|\mathbf{x}|<1\},$$

and

$$\Gamma_j=\{|\mathbf{x}|=r_j\},\quad j=1,\ldots,L+1.$$

Let (μ_j,ϵ_j) be the pair of permeability and permittivity parameters of A_j for $j=1,\ldots,L+1$. Set $\mu_0=1$ and $\epsilon_0=1$. We then define

$$\mu=\sum_{j=0}^{L+1}\mu_j\chi(A_j)\quad\text{and}\quad\epsilon=\sum_{j=0}^{L+1}\epsilon_j\chi(A_j), \tag{4.1}$$

which are permeability and permittivity distributions of the layered structure.

The scattering coefficients $\left(W^{TE,TE}_{(n,m)(p,q)}, W^{TE,TM}_{(n,m)(p,q)}, W^{TM,TE}_{(n,m)(p,q)}, W^{TM,TM}_{(n,m)(p,q)}\right)$ are defined as before, namely, if $\mathbf{E}^i$ given as in (3.9), the scattered field $\mathbf{E} - \mathbf{E}^i$ can be expanded as (3.10) and (3.11). The transmission condition on each interface Γ_j is given by

$$[\hat{\mathbf{x}} \times \mathbf{E}] = [\hat{\mathbf{x}} \times \mathbf{H}] = 0. \tag{4.2}$$

We assume that the core A_{L+1} is perfectly conducting (PEC), namely,

$$\mathbf{E} \times \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{L+1} = \partial A_{L+1}. \tag{4.3}$$

Thanks to the symmetry of the layered (radial) structure, the scattering coefficients are much simpler than the general case. In fact, if the incident field is given by $\mathbf{E}^i = \widetilde{\mathbf{E}}^{TE}_{n,m}$, then the solution $\mathbf{E}$ to (3.1) takes the form

$$\mathbf{E}(\mathbf{x}) = \tilde{a}_j \widetilde{\mathbf{E}}^{TE}_{n,m}(\mathbf{x}) + a_j \mathbf{E}^{TE}_{n,m}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, \dots, L, \tag{4.4}$$

with $\tilde{a}_0 = 1$. From (2.9) and (2.10), the interface condition (4.2) amounts to

$$\begin{aligned} &\begin{bmatrix} j_n(k_j r_j) & h^{(1)}_n(k_j r_j) \\ \dfrac{1}{\mu_j} \mathcal{J}_n(k_j r_j) & \dfrac{1}{\mu_j} \mathcal{H}_n(k_j r_j) \end{bmatrix} \begin{bmatrix} \tilde{a}_j \\ a_j \end{bmatrix} \\ &= \begin{bmatrix} j_n(k_{j-1} r_j) & h^{(1)}_n(k_{j-1} r_j) \\ \dfrac{1}{\mu_{j-1}} \mathcal{J}_n(k_{j-1} r_j) & \dfrac{1}{\mu_{j-1}} \mathcal{H}_n(k_{j-1} r_j) \end{bmatrix} \begin{bmatrix} \tilde{a}_{j-1} \\ a_{j-1} \end{bmatrix}, \quad j = 1, \dots, L, \end{aligned} \tag{4.5}$$

where $\mathcal{H}_n(t) = h^{(1)}_n(t) + t\left(h^{(1)}_n\right)'(t)$ and $\mathcal{J}_n(t) = j_n(t) + t j'_n(t)$, and the PEC boundary condition on Γ_{L+1} is

$$\begin{bmatrix} j_n(k_L) & h^{(1)}_n(k_L) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{a}_L \\ a_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.6}$$

Since the matrices appeared in (4.5) are invertible, one can see that there are a_j and $\tilde{a}_j$, $j = 0, 1, \dots L$ satisfying (4.5) and (4.6). Similarly, one can see that if the incident field is given by $\mathbf{E}^i = \widetilde{\mathbf{E}}^{TM}_{n,m}(\mathbf{x})$, then the solution $\mathbf{E}$ takes the form

$$\mathbf{E}(\mathbf{x}) = \tilde{b}_j \widetilde{\mathbf{E}}^{TM}_{n,m}(\mathbf{x}) + b_j \mathbf{E}^{TM}_{n,m}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, 1, ..., L \tag{4.7}$$

for some constants b_j and $\tilde{b}_j$ ($\tilde{b}_0 = 1$). One can see now from (4.4) and (4.7) that the scattering coefficients satisfy

$$\begin{aligned} W^{TE,TM}_{(n,m)(p,q)} &= W^{TM,TE}_{(n,m)(p,q)} = 0 \quad \text{for all } (m,n) \text{ and } (p,q), \\ W^{TE,TE}_{(n,m)(p,q)} &= W^{TM,TM}_{(n,m)(p,q)} = 0 \quad \text{if } (m,n) \neq (p,q), \end{aligned}$$

and, since (4.4) and (4.7) hold independently of m, we have

$$\begin{aligned} W^{TE,TE}_{(n,0)(n,0)} &= W^{TE,TE}_{(n,m)(n,m)}, \\ W^{TM,TM}_{(n,0)(n,0)} &= W^{TM,TM}_{(n,m)(n,m)} \qquad \text{for } -n \leq m \leq n. \end{aligned}$$

Moreover, if we write

$$W^{TE}_n := W^{TE}_{(n,0)(n,0)} \quad \text{and} \quad W^{TM}_n := W^{TM}_{(n,0)(n,0)},$$

then we have

$$W_n^{TE} = -\frac{in(n+1)}{k_0} a_0 \quad \text{and} \quad W_n^{TE} = -\frac{in(n+1)}{k_0} b_0. \tag{4.8}$$

Suppose now that $\widetilde{\mathbf{E}}_{n,0}^{TE}$ is the incident field and the solution $\mathbf{E}$ is given by

$$\mathbf{E}(\mathbf{x}) = \tilde{a}_j \widetilde{\mathbf{E}}_{n,0}^{TE}(\mathbf{x}) + a_j \mathbf{E}_{n,0}^{TE}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, \ldots, L,$$

with $\tilde{a}_0 = 1$, where the coefficients $\tilde{a}_j$'s and a_j's are determined by (4.5) and (4.6). We have from (4.5) that

$$\begin{bmatrix} \tilde{a}_j \\ a_j \end{bmatrix} = \begin{bmatrix} j_n(k_j r_j) & h_n^{(1)}(k_j r_j) \\ \frac{1}{\mu_j}\mathcal{J}_n(k_j r_j) & \frac{1}{\mu_j}\mathcal{H}_n(k_j r_j) \end{bmatrix}^{-1} \begin{bmatrix} j_n(k_{j-1} r_j) & h_n^{(1)}(k_{j-1} r_j) \\ \frac{1}{\mu_{j-1}}\mathcal{J}_n(k_{j-1} r_j) & \frac{1}{\mu_{j-1}}\mathcal{H}_n(k_{j-1} r_j) \end{bmatrix} \begin{bmatrix} \tilde{a}_{j-1} \\ a_{j-1} \end{bmatrix},$$

for $j = 1, \ldots, L$. Substituting these relations into (4.6) yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P_n^{TE}[\varepsilon, \mu, \omega] \begin{bmatrix} \tilde{a}_0 \\ a_0 \end{bmatrix}, \tag{4.9}$$

where

$$\begin{aligned} P_n^{TE}[\varepsilon, \mu, \omega] &:= \begin{bmatrix} p_{n,1}^{TE} & p_{n,2}^{TE} \\ 0 & 0 \end{bmatrix} = (-i\omega)^L \left(\prod_{j=1}^{L} \mu_j^{\frac{3}{2}} \varepsilon_j^{\frac{1}{2}} r_j \right) \begin{bmatrix} j_n(k_L) & h_n^{(1)}(k_L) \\ 0 & 0 \end{bmatrix} \\ &\times \prod_{j=1}^{L} \begin{bmatrix} \frac{1}{\mu_j}\mathcal{H}_n(k_j r_j) & -h_n^{(1)}(k_j r_j) \\ -\frac{1}{\mu_j}\mathcal{J}_n(k_j r_j) & j_n(k_j r_j) \end{bmatrix} \begin{bmatrix} j_n(k_{j-1} r_j) & h_n^{(1)}(k_{j-1} r_j) \\ \frac{1}{\mu_{j-1}}\mathcal{J}_n(k_{j-1} r_j) & \frac{1}{\mu_{j-1}}\mathcal{H}_n(k_{j-1} r_j) \end{bmatrix}. \end{aligned} \tag{4.10}$$

We then have from (4.9)

$$W_n^{TE} = -\frac{in(n+1)}{k_0} a_0 = -\frac{in(n+1)}{k_0} \frac{p_{n,1}^{TE}}{p_{n,2}^{TE}}. \tag{4.11}$$

Similarly, for W_n^{TM}, we look for another solution $\mathbf{E}$ of the form

$$\mathbf{E}(\mathbf{x}) = \tilde{b}_j \widetilde{\mathbf{E}}_{n,0}^{TM}(\mathbf{x}) + b_j \mathbf{E}_{n,0}^{TM}(\mathbf{x}), \quad \mathbf{x} \in A_j, \quad j = 0, ..., L,$$

with $\tilde{b}_0 = 1$. The transmission conditions become

$$\begin{aligned} &\begin{bmatrix} \frac{1}{\varepsilon_j}\mathcal{J}_n(k_j r_j) & \frac{1}{\varepsilon_j}\mathcal{H}_n(k_j r_j) \\ j_n(k_j r_j) & h_n^{(1)}(k_j r_j) \end{bmatrix} \begin{bmatrix} \tilde{b}_j \\ b_j \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\varepsilon_{j-1}}\mathcal{J}_n(k_{j-1} r_j) & \frac{1}{\varepsilon_{j-1}}\mathcal{H}_n(k_{j-1} r_j) \\ j_n(k_{j-1} r_j) & h_n^{(1)}(k_{j-1} r_j) \end{bmatrix} \begin{bmatrix} \tilde{b}_{j-1} \\ b_{j-1} \end{bmatrix}, \quad j = 1, \ldots, N+1, \end{aligned} \tag{4.12}$$

and the PEC boundary condition on the inner most layer is

$$\begin{bmatrix} \mathcal{J}_n(k_L) & \mathcal{H}_n(k_L) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{b}_L \\ b_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.13}$$

From (4.12) and (4.13), we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P_n^{TM}[\varepsilon, \mu, \omega] \begin{bmatrix} \tilde{b}_0 \\ b_0 \end{bmatrix}, \tag{4.14}$$

where

$$P_n^{TM}[\varepsilon,\mu,\omega] := \begin{bmatrix} p_{n,1}^{TM} & p_{n,2}^{TM} \\ 0 & 0 \end{bmatrix} = (i\omega)^L \left(\prod_{j=1}^{L} \mu_j^{\frac{1}{2}} \varepsilon_j^{\frac{3}{2}} r_j \right) \begin{bmatrix} \mathcal{J}_n(k_L) & \mathcal{H}_n(k_L) \\ 0 & 0 \end{bmatrix}$$

$$\times \prod_{j=1}^{L} \begin{bmatrix} h_n^{(1)}(k_j r_j) & -\frac{1}{\epsilon_j}\mathcal{H}_n(k_j r_j) \\ -j_n(k_j r_j) & \frac{1}{\epsilon_j}\mathcal{J}_n(k_j r_j) \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon_{j-1}}\mathcal{J}_n(k_{j-1} r_j) & \frac{1}{\epsilon_{j-1}}\mathcal{H}_n(k_{j-1} r_j) \\ j_n(k_{j-1} r_j) & h_n^{(1)}(k_{j-1} r_j) \end{bmatrix}. \tag{4.15}$$

From the definition of W_n^{TM} and (4.14), we have

$$W_n^{TE} = -\frac{in(n+1)}{k_0}\frac{b_0}{\tilde{b}_0} = -\frac{in(n+1)}{k_0}\frac{p_{n,1}^{TM}}{p_{n,2}^{TM}}. \tag{4.16}$$

It should be emphasized that $p_{n,2}^{TE} \neq 0$ and $p_{n,2}^{TM} \neq 0$. In fact, if $p_{n,2}^{TE} = 0$, then (4.9) can be fulfilled with $\tilde{a}_0 = 0$ and $a_0 = 1$. This means that there exists (μ, ϵ) on $\mathbb{R}^3 \setminus \overline{D}$ such that the following problem has a solution:

$$\begin{cases} \nabla \times \mathbf{E} = i\omega\mu\mathbf{H} & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\mathbf{x} \times \mathbf{E})\big|_+ = 0 & \text{on } \partial D, \\ \mathbf{E}(\mathbf{x}) = \mathbf{E}_{n,0}^{TE}(\mathbf{x}) & \text{for } |\mathbf{x}| > 2. \end{cases}$$

Applying the following Green's theorem on $\Omega = \{\mathbf{x} \mid 1 < |\mathbf{x}| < R\}$,

$$\begin{aligned} &\int_\Omega \big(\mathbf{E} \cdot \Delta\mathbf{F} + \mathrm{curl}\mathbf{E} \cdot \mathrm{curl}\mathbf{F} + \mathrm{div}\mathbf{E}\ \mathrm{div}\mathbf{F}\big) d\mathbf{x} \\ &= \int_{\partial\Omega} \big(\nu \times \mathbf{E} \cdot \mathrm{curl}\mathbf{F} + \nu \cdot \mathbf{E}\ \mathrm{div}\mathbf{F}\big) d\sigma(\mathbf{x}) \end{aligned}$$

with $\mathbf{F} = \overline{\mathbf{E}_{n,0}^{TE}}(\mathbf{x})$ and the PEC boundary condition on $\{\mathbf{x}| = 1\}$, we have

$$\int_{|\mathbf{x}|=R} (\nu \times \mathbf{E}) \cdot \overline{\mathbf{H}} d\sigma(\mathbf{x}) = ik_0 \int_\Omega (|\mathbf{H}|^2 - |\mathbf{E}|^2) d\mathbf{x}.$$

In particular, the left-hand side is real-valued. Hence,

$$\begin{aligned} \int_{|\mathbf{x}|=R} |\mathbf{H} \times \nu - \mathbf{E}|^2 d\sigma(\mathbf{x}) &= \int_{|\mathbf{x}|=R} \big(|\mathbf{H} \times \nu|^2 + |\mathbf{E}|^2 - 2\Re((\nu \times \mathbf{E}) \cdot \overline{\mathbf{H}}\big) d\sigma(\mathbf{x}) \\ &= \int_{|\mathbf{x}|=R} \big(|\mathbf{H} \times \nu|^2 + |\mathbf{E}|^2\big) d\sigma(\mathbf{x}). \end{aligned}$$

From the radiation condition, the left-hand side goes to zero as $R \to \infty$, and it contradicts the behavior of the hankel functions. One can show that $p_{n,2}^{TM} \neq 0$ in a similar way. $\square$

To construct the S-vanishing structure at a fixed frequency ω we look for (μ, ϵ) such that

$$W_n^{TE}[\varepsilon,\mu,\omega] = 0,\ W_n^{TM}[\varepsilon,\mu,\omega] = 0, \quad n = 1, \ldots, N$$

for some N. More ambitiously we may look for a structure (μ, ϵ) for a fixed ω such that

$$W_n^{TE}[\mu,\epsilon,\rho\omega] = 0, \quad W_n^{TM}[\mu,\epsilon,\rho\omega] = 0$$

for all $1 \leq n \leq N$ and $\rho \leq \rho_0$ for some ρ_0. Such a structure may not exist. So instead we look for a structure such that

$$W_n^{TE}[\mu, \epsilon, \rho\omega] = o(\rho^{2N+1}), \quad W_n^{TM}[\mu, \epsilon, \rho\omega] = o(\rho^{2N+1}), \tag{4.17}$$

for all $1 \leq n \leq N$ and $\rho \leq \rho_0$ for some ρ_0. We call such a structure a *S-vanishing structure of order N at low frequencies*. In the following section, we expand the scattering coefficients for low frequencies and derive conditions for the magnetic permeability and the electric permittivity to be a S-vanishing structure.

Suppose that (μ, ϵ) is an S-vanishing structure of order N at low frequencies. Let the incident wave $\mathbf{E}^i$ be given by a plane wave $e^{i\rho\mathbf{k}\cdot\mathbf{x}}\mathbf{c}$ with $|\mathbf{k}| = k_0$ and $\mathbf{k}\cdot\mathbf{c} = 0$. From (3.14), the corresponding scattering amplitude, $\mathbf{A}_\infty[\mu, \epsilon, \rho\omega](\boldsymbol{c}, \hat{\boldsymbol{k}}; \hat{\boldsymbol{x}})$, is given by (3.13) with the following $\alpha_{n,m}$ and $\beta_{n,m}$.

$$\begin{cases} \alpha_{n,m} = \dfrac{4\pi i^n}{\sqrt{n(n+1)}}(\mathbf{V}_{n,m}(\hat{\mathbf{k}}) \cdot \mathbf{c})W_n^{TE}[\mu, \epsilon, \rho\omega], \\ \beta_{n,m} = -\dfrac{4\pi i^n}{\sqrt{n(n+1)}}\dfrac{1}{i\omega\mu_0}(\mathbf{U}_{n,m}(\hat{\mathbf{k}}) \cdot \mathbf{c})W_n^{TM}[\mu, \epsilon, \rho\omega]. \end{cases}$$

Applying (3.18) and (4.17), we have

$$\boldsymbol{A}_\infty[\mu, \epsilon, \rho\omega](\mathbf{c}, \hat{\mathbf{k}}; \hat{\mathbf{x}}) = o(\rho^{2N+1}) \tag{4.18}$$

uniformly in $(\hat{\mathbf{k}}, \hat{\mathbf{x}})$ if $\rho \leq \rho_0$. Thus using such a structure the visibility of scattering amplitude is greatly reduced.

4.1 Asymptotic expansion of the scattering coefficients

The spherical Bessel functions of the first and second kinds have the series expansions:

$$j_n(t) = \sum_{l=0}^{\infty} \frac{(-1)^l t^{n+2l}}{2^l l! 1 \cdot 3 \cdots (2n+2l+1)},$$

and

$$y_n(t) = -\frac{(2n)!}{2^n n!}\sum_{l=0}^{\infty} \frac{(-1)^l t^{2l-n-1}}{2^l l!(-2n+1)(-2n+3)\cdots(-2n+2l-1)}.$$

So, using the notation of double factorials, which is defined by

$$n!! := \begin{cases} n \cdot (n-2) \ldots 3 \cdot 1 & \text{if } n > 0 \text{ is odd}, \\ n \cdot (n-2) \ldots 4 \cdot 2 & \text{if } n > 0 \text{ is even}, \\ 1 & \text{if } n = -1, 0, \end{cases}$$

we have

$$j_n(t) = \frac{t^n}{(2n+1)!!}\big(1 + o(t)\big) \quad \text{for } t \ll 1, \tag{4.19}$$

and

$$y_n(t) = -\big((2n-1)!!\big)t^{-n+1}\big(1 + o(t)\big) \quad \text{for } t \ll 1. \tag{4.20}$$

We now compute $P_n^{TE}[\epsilon,\mu,t]$ for small t. For $n \geq 1$,

$$P_n^{TE}[\varepsilon,\mu,t] = (-it)^L \left(\prod_{j=1}^{L} \mu_j^{\frac{3}{2}} \varepsilon_j^{\frac{1}{2}} r_j\right) \begin{bmatrix} \frac{z_L^n}{(2n+1)!!} t^n + o(t^n) & \frac{-iQ(n)}{z_L^{n+1}} t^{-n-1} \\ 0 & 0 \end{bmatrix}$$
$$\times \prod_{j=1}^{L} \left(\begin{bmatrix} \frac{iQ(n)n}{\mu_j (z_j r_j)^{n+1}} t^{-n-1} + o(t^{-n-1}) & \frac{iQ(n)}{(z_j r_j)^{n+1}} t^{-n-1} + o(t^{-n-1}) \\ \frac{-(n+1)(z_j r_j)^n}{\mu_j (2n+1)!!} t^n + o(t^n) & \frac{(z_j r_j)^n}{(2n+1)!!} t^n + o(t^n) \end{bmatrix} \begin{bmatrix} \frac{(z_{j-1} r_j)^n}{(2n+1)!!} t^n + o(t^n) & \frac{-iQ(n)}{(z_{j-1} r_j)^{n+1}} t^{-n-1} + o(t^{-n-1}) \\ \frac{(n+1)(z_{j-1} r_j)^n}{\mu_{j-1}(2n+1)!!} t^n + o(t^n) & \frac{iQ(n)n}{\mu_{j-1}(z_{j-1} r_j)^{n+1}} t^{-n-1} + o(t^{-n-1}) \end{bmatrix} \right),$$

where $z_j = \sqrt{\varepsilon_j \mu_j}$ and $Q(n) = (2n-1)!!$. We then have

$$P_n^{TE}[\varepsilon,\mu,t] = \begin{bmatrix} \frac{z_L^n}{(2n+1)!!} t^n + o(t^n) & \frac{-iQ(n)}{z_L^{n+1}} t^{-n-1} + o(t^{-n-1}) \\ 0 & 0 \end{bmatrix} \times$$
$$\prod_{j=1}^{L} \begin{bmatrix} \frac{Q(n) z_{j-1}^n}{(2n+1)!! z_j^n} \big(n + \frac{(n+1)\mu_j}{\mu_{j-1}}\big)\big(1+o(1)\big) & (-i)\frac{(Q(n))^2 n}{z_j^n z_{j-1}^{n+1} r_j^{2n+1}} \big(1 - \frac{\mu_j}{\mu_{j-1}}\big) t^{-2n-1}\big(1+o(1)\big) \\ i\frac{z_{j-1}^n z_j^{n+1} r_j^{2n+1}(n+1)}{((2n+1)!!)^2} \big(1 - \frac{\mu_j}{\mu_{j-1}}\big) t^{2n+1}\big(1+o(1)\big) & \frac{Q(n) z_j^{n+1}}{(2n+1)!! z_{j-1}^{n+1}} \big(n+1+\frac{n\mu_j}{\mu_{j-1}}\big)\big(1+o(1)\big) \end{bmatrix}.$$

Similarly, for the transverse magnetic case, we have

$$P_n^{TM}[\varepsilon,\mu,t] = \begin{bmatrix} \frac{(n+1) z_L^n}{(2n+1)!!} t^n + o(t^n) & \frac{-inQ(n)}{z_L^{n+1}} t^{-n-1} + o(t^{-n-1}) \\ 0 & 0 \end{bmatrix} \times$$
$$\prod_{j=1}^{L} \begin{bmatrix} \frac{Q(n) z_{j-1}^n}{(2n+1)!! z_j^n} \left((n + \frac{\varepsilon_j}{\varepsilon_{j-1}}(n+1)\right)\big(1+o(1)\big) & (-i)\frac{(Q(n))^2 n}{z_j^n z_{j-1}^{n+1} r_j^{2n+1}} \left(1 - \frac{\varepsilon_j}{\varepsilon_{j-1}}\right) t^{-2n-1}\big(1+o(1)\big) \\ i\frac{z_{j-1}^n z_j^{n+1} r_j^{2n+1}(n+1)}{((2n+1)!!)^2} \left(1 - \frac{\varepsilon_j}{\varepsilon_{j-1}}\right) t^{2n+1}\big(1+o(1)\big) & \frac{Q(n) z_j^{n+1}}{(2n+1)!! z_{j-1}^{n+1}} \left(n+1+\frac{\varepsilon_j}{\varepsilon_{j-1}} n\right)\big(1+o(1)\big) \end{bmatrix}.$$

Using the behavior of spherical bessel functions for small arguments, we see that $p_{n,1}^{TE}$ and $p_{n,2}^{TE}$ admit the following expansions:

$$p_{n,1}^{TE}[\mu,\varepsilon,t] = t^n \left(\sum_{l=0}^{N-n} f_{n,l}^{TE}(\mu,\varepsilon) t^{2l} + o(t^{2N-2n})\right) \tag{4.21}$$

and

$$p_{n,2}^{TE}[\mu,\varepsilon,t] = t^{-n-1} \left(\sum_{l=0}^{N-n} g_{n,l}^{TE}(\mu,\varepsilon) t^{2l} + o(t^{2N-2n})\right). \tag{4.22}$$

Similarly, $p_{n,1}^{TM}$ and $p_{n,2}^{TM}$ have the following expansion:

$$p_{n,1}^{TM}[\mu,\varepsilon,t] = t^n \left(\sum_{l=0}^{N-n} f_{n,l}^{TM}(\mu,\varepsilon) t^{2l} + o(t^{2N-2n})\right) \tag{4.23}$$

and

$$p_{n,2}^{TM}[\mu,\varepsilon,t] = t^{-n-1} \left(\sum_{l=0}^{N-n} g_{n,l}^{TM}(\mu,\varepsilon) t^{2l} + o(t^{2N-2n})\right). \tag{4.24}$$

for $t = \rho\omega$ and some functions $f_{n,l}^{TE}, g_{n,l}^{TE}, f_{n,l}^{TM}$, and $g_{n,l}^{TM}$ independent of t.

Lemma 4.1. *For any pair of* (μ, ϵ)*, we have*

$$g_{n,0}^{TE}(\mu, \varepsilon) \neq 0 \tag{4.25}$$

and

$$g_{n,0}^{TM}(\mu, \varepsilon) \neq 0. \tag{4.26}$$

Proof. Assume that there exists a pair of (μ, ϵ) such that $g_{n,0}^{TE}(\mu, \varepsilon) = 0$. Since $p_{n,2}^{TE}[\mu, \varepsilon, \rho\omega] = o(\rho^{-n-1})$, the solution given by (4.4) with $a_0 = 1$ and $\tilde{a}_0 = 0$ satisfies

$$\begin{cases} \nabla \times \left(\dfrac{1}{\mu} \nabla \times \mathbf{E} \right) - \rho^2 \omega^2 \epsilon \mathbf{E} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \nabla \cdot \mathbf{E} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\boldsymbol{\nu} \times \mathbf{E})\big|_+ = o(\rho^{-(n+1)}) & \text{on } \partial D, \\ \mathbf{E}(\mathbf{x}) = h_n^{(1)}(\rho k_0 |\mathbf{x}|) \boldsymbol{V}_{n,0}(\hat{\mathbf{x}}) & \text{for } |\mathbf{x}| > 2. \end{cases}$$

Let $\boldsymbol{V}(\mathbf{x}) = \lim_{\rho \to 0} \rho^{n+1} \mathbf{E}(\mathbf{x})$. Using (4.20) we know that the limit $\boldsymbol{V}$ satisfies

$$\begin{cases} \nabla \times \left(\dfrac{1}{\mu} \nabla \times \boldsymbol{V} \right) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \nabla \cdot \boldsymbol{V} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\boldsymbol{\nu} \times \boldsymbol{V})\big|_+ = 0 & \text{on } \partial D, \\ \boldsymbol{V}(\mathbf{x}) = -\big((2n-1)!!\big) \boldsymbol{V}_{n,0}(\hat{\mathbf{x}}) & \text{for } |\mathbf{x}| > 2. \end{cases}$$

Since $\boldsymbol{V}_{n,0}(\hat{\mathbf{x}}) = O(|x|^{-1})$, we get $\boldsymbol{V}(\mathbf{x}) = 0$ by Green's formula, which is a contradiction. Thus $g_{n,0}^{TE}(\mu, \varepsilon) \neq 0$. In a similar way, (4.26) can be proved. □

From Lemma 4.1, we have the following theorem.

Proposition 4.2. *We have*

$$W_n^{TE}[\mu, \varepsilon, t] = t^{2n+1} \sum_{l=0}^{N-n} W_{n,l}^{TE}[\mu, \varepsilon] t^{2l} + o(t^{2N+1}),$$

and

$$W_n^{TM}[\mu, \varepsilon, t] = t^{2n+1} \sum_{l=0}^{N-n} W_{n,l}^{TM}[\mu, \varepsilon] t^{2l} + o(t^{2N+1}),$$

where $t = \rho\omega$ *and the coefficients* $W_{n,l}^{TE}[\mu, \varepsilon]$ *and* $W_{n,l}^{TM}[\mu, \varepsilon]$ *are independent of* t*.*

Hence, if we have (μ, ϵ) such that

$$W_{n,l}^{TE}[\mu, \epsilon] = W_{n,l}^{TM}[\mu, \varepsilon] = 0, \quad \text{for all } 1 \leq n \leq N, \ 0 \leq l \leq (N-n), \tag{4.27}$$

(μ, ϵ) satisfies (4.17), in other words, it is a *S-vanishing structure of order* N *at low frequencies.* It is quite challenging to construct (μ, ϵ) analytically satisfying (4.27). In the next section we get some numerical examples of such structures.

4.2 Numerical examples

In this section we provide numerical examples of S-vanishing structures of order N at low frequencies based on (4.27). To do this, the gradient descent method for the suitable energy functional is used, as used in [5] and [6] to compute the enhanced near-cloaking structures for the conductivity problem and the Helmholtz problem. As in [6], we symbolically compute the scattering coefficients.

In the place of spherical Bessel functions and spherical Hankel functions, we put its low frequency asymptotic expansions in (4.10) and (4.15), and symbolically compute W_n^{TE} and W_n^{TM} to have $W_{n,l}^{TE}[\mu,\epsilon]$ and $W_{n,l}^{TM}[\mu,\varepsilon]$.

The following example is a S-vanishing structure of order $N = 2$ made of 6 multilayers. The radii of the concentric disks are $r_j = 2 - \frac{j-1}{6}$ for $j = 1, \dots, 7$. From Proposition 4.2, the nonzero leading terms of $W_n^{TE}[\mu,\epsilon,t]$ and $W_n^{TM}[\mu,\epsilon,t]$ up to t^5 are

- $[t^3, t^5]$ terms in $W_1^{TE}[\mu,\epsilon,t]$, *i.e.*, $W_{1,0}^{TE}, W_{1,1}^{TE}$,
- $[t^3, t^5]$ terms in $W_1^{TM}[\mu,\epsilon,t]$, *i.e.*, $W_{1,0}^{TM}, W_{1,1}^{TM}$,
- $[t^5]$ term in $W_2^{TE}[\mu,\epsilon,t]$, *i.e.*, $W_{2,0}^{TE}$,
- $[t^5]$ term in $W_2^{TM}[\mu,\epsilon,t]$, *i.e.*, $W_{2,0}^{TM}$.

Consider the mapping

$$(\boldsymbol{\mu}, \boldsymbol{\varepsilon}) \longrightarrow (W_{1,0}^{TE}, W_{1,1}^{TE}, W_{1,0}^{TM}, W_{1,1}^{TM}, W_{2,0}^{TE}, W_{2,0}^{TM}), \tag{4.28}$$

where, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_6)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_6)$. We look for $(\boldsymbol{\mu}, \boldsymbol{\varepsilon})$ which has the right-hand side of (4.28) as small as possible. Since (4.28) is a nonlinear equation, we solve it iteratively. Initially, we wet $\boldsymbol{\mu} = \boldsymbol{\mu}^{(0)}$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)}$. We iteratively modify $(\boldsymbol{\mu}^{(i)}, \boldsymbol{\varepsilon}^{(i)})$

$$[\boldsymbol{\mu}^{(i+1)}\ \boldsymbol{\varepsilon}^{(i+1)}]^T = [\boldsymbol{\mu}^{(i)}\ \boldsymbol{\varepsilon}^{(i)}]^T - A_i^{\dagger}\mathbf{b}^{(i)}, \tag{4.29}$$

where $A_i^{\dagger}$ is the pseudoinverse of

$$A_i := \frac{\partial(W_{1,0}^{TE}, W_{1,1}^{TE}, \dots, W_{2,0}^{TM})}{\partial(\boldsymbol{\mu}, \boldsymbol{\varepsilon})}\Big|_{(\boldsymbol{\mu},\boldsymbol{\varepsilon})=(\boldsymbol{\mu}^{(i)},\boldsymbol{\varepsilon}^{(i)})},$$

and

$$\mathbf{b}^{(i)} = \begin{bmatrix} W_{1,0}^{TE} \\ W_{1,1}^{TE} \\ \vdots \\ W_{2,0}^{TM} \end{bmatrix}\Bigg|_{(\boldsymbol{\mu},\boldsymbol{\varepsilon})=(\boldsymbol{\mu}^{(i)},\boldsymbol{\varepsilon}^{(i)})}.$$

Example 1. Figure 4.1 and Figure 4.2 show computational results of 6-layers S-vanishing structure of order $N = 2$. We set $\boldsymbol{r} = (2, \frac{11}{6}, \dots, \frac{7}{6})$, $\boldsymbol{\mu}^{(0)} = (3,6,3,6,3,6)$ and $\boldsymbol{\varepsilon}^{(0)} = (3,6,3,6,3,6)$ and modify them following (4.29) with the constraints that μ and ε belongs to the interval between 0.1 and 10. The obtained material parameters are $\boldsymbol{\mu} = (0.1000, 1.1113, 0.2977, 2.0436, 0.1000, 1.8260)$ and $\boldsymbol{\varepsilon} = (0.4356, 1.1461, 0.2899, 1.8199, 0.1000, 3.1233)$, respectively. Differently from the no-layer structure with PEC condition at $|\mathbf{x}| = 1$, the obtained multilayer structure has the nearly zero coefficients of $W_n^{TE}[\mu,\epsilon,t]$ and $W_n^{TM}[\mu,\epsilon,t]$ up to t^5.

5 Enhancement of near cloaking

We make a cloaking structure based on the following lemma.

Lemma 5.1. *Let F be a diffeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^3$ such that $F(\mathbf{x})$ is identity for $|\mathbf{x}|$ large enough. If $(\mathbf{E},\mathbf{H})$ is a solution to*

$$\begin{cases} \nabla \times \mathbf{E} = i\omega\mu\mathbf{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} & \text{in } \mathbb{R}^3, \\ (\mathbf{E} - \mathbf{E}^i, \mathbf{H} - \mathbf{H}^i) \text{ is radiating,} & \end{cases} \tag{5.1}$$

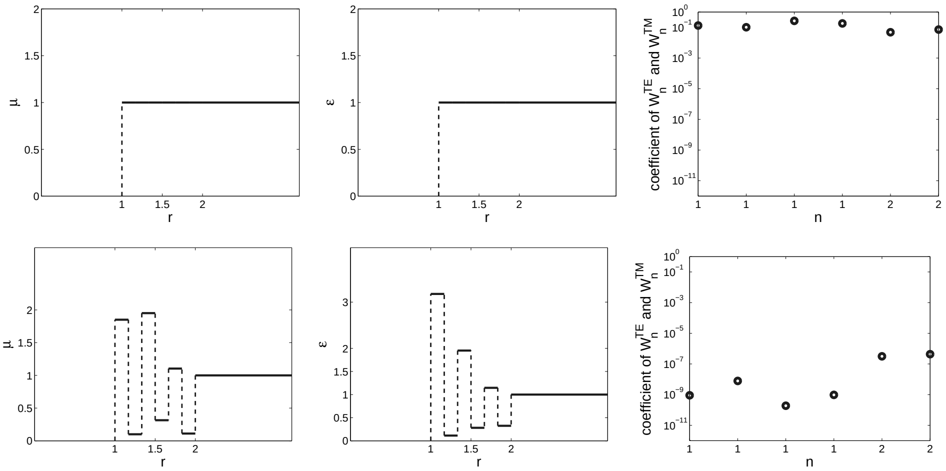

Figure 4.1: This figure shows the graph of the material parameters and the corresponding coefficients in $W_n^{TE}[\mu,\epsilon,t]$ and $W_n^{TM}[\mu,\epsilon,t]$ up to t^5. The first row is of the no-layer case, and the second row is of 6-layers S-vanishing structure of order $N = 2$ which is explained in Example 1. In the third column, the y-axis shows $(W_{1,0}^{TE}, W_{1,1}^{TE}, W_{1,0}^{TM}, W_{1,1}^{TM}, W_{2,0}^{TE}, W_{2,0}^{TM})$ from the left to the right.

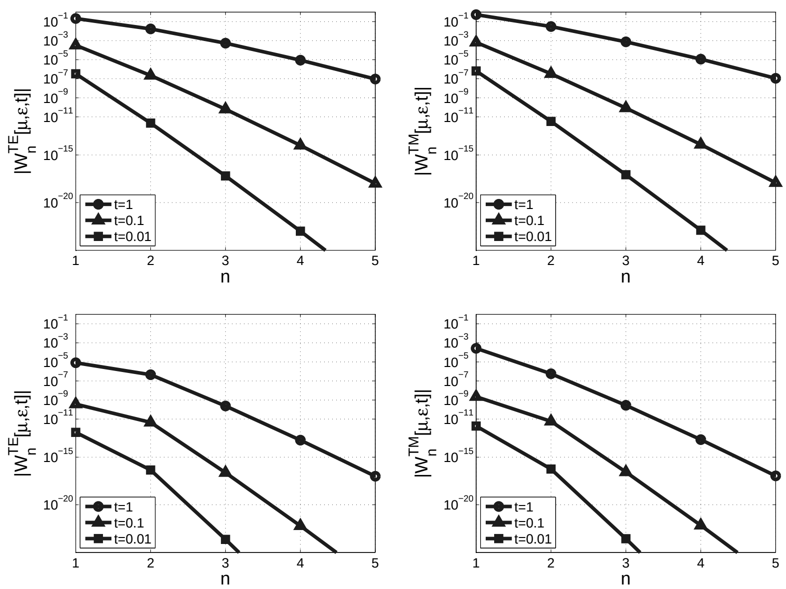

Figure 4.2: This figure shows the graph of $W_n^{TE}[\mu,\epsilon,t]$ and $W_n^{TM}[\mu,\epsilon,t]$ for various values of t. The first row is of the no-layer case, and the second row if of 6-layers S-vanishing structure of order $N = 2$ which is explained in Example 1. The values of W_n^{TE} and W_n^{TM} are much smaller in the S-vanishing structure than in the no-layer structure.

then $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ *defined by* $(\widetilde{\mathbf{E}}(\mathbf{y}), \widetilde{\mathbf{H}}(\mathbf{y})) = \big(\mathbf{E}(F^{-1}(\mathbf{y})), \mathbf{H}(F^{-1}(\mathbf{y}))\big)$ *satisfies*

$$\begin{cases} \nabla \times \widetilde{\mathbf{E}} = i\omega(F_*\mu)\widetilde{\mathbf{H}} & \textit{in } \mathbb{R}^3, \\ \nabla \times \widetilde{\mathbf{H}} = -i\omega(F_*\epsilon)\widetilde{\mathbf{E}} & \textit{in } \mathbb{R}^3, \\ (\widetilde{\mathbf{E}}\widetilde{\mathbf{E}}^i, \widetilde{\mathbf{H}} - \widetilde{\mathbf{H}}^i) \textit{ is radiating}, & \end{cases}$$

where $(\widetilde{\mathbf{E}}^i(\mathbf{y}), \widetilde{\mathbf{H}}^i(\mathbf{y})) = \big(\mathbf{E}^i(F^{-1}(\mathbf{y})), \mathbf{H}^i(F^{-1}(\mathbf{y}))\big)$,

$$(F_*\mu)(\mathbf{y}) = \frac{DF(\mathbf{x})\mu(\mathbf{x})DF^T(\mathbf{x})}{\det(DF(\mathbf{x}))}, \quad \textit{and} \quad (F_*\epsilon)(\mathbf{y}) = \frac{DF(\mathbf{x})\epsilon(\mathbf{x})DF^T(\mathbf{x})}{\det(DF(\mathbf{x}))},$$

with $\mathbf{x} = F^{-1}(\mathbf{y})$ *and* DF *is the Jacobian matrix of* F.

Hence,

$$\boldsymbol{A}[\mu, \epsilon, \omega] = \boldsymbol{A}[F_*\mu, F_*\epsilon, \omega].$$

To compute the scattering amplitude which corresponds to the material parameters before the transformation, we consider the following scaling function, for small parameter ρ,

$$\Psi_{\frac{1}{\rho}}(\mathbf{x}) = \frac{1}{\rho}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Then we have the following relation between the scattering amplitudes which correspond to two sets of differently scaled material parameters and frequency:

$$\boldsymbol{A}_\infty \left[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\right] = \boldsymbol{A}_\infty[\mu, \epsilon, \rho\omega]. \tag{5.2}$$

To see this, consider $(\mathbf{E}, \mathbf{H})$ which satisfies

$$\begin{cases} (\nabla \times \mathbf{E})(\mathbf{x}) = i\omega\big(\mu \circ \Psi_{\frac{1}{\rho}}\big)(\mathbf{x})\mathbf{H}(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_\rho}, \\ (\nabla \times \mathbf{H})(\mathbf{x}) = -i\omega\big(\epsilon \circ \Psi_{\frac{1}{\rho}}\big)(\mathbf{x})\mathbf{E}(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{B_\rho}, \\ \hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0 & \text{on } \partial B_\rho, \\ (\mathbf{E} - \mathbf{E}^i, \mathbf{H} - \mathbf{H}^i) \text{ is radiating,} \end{cases}$$

with the incident wave $\mathbf{E}^i(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{c}}$ and $\mathbf{H}^i = \frac{1}{i\omega\mu_0}\nabla \times \mathbf{E}^i$ with $\mathbf{k}\cdot\hat{\mathbf{c}} = 0$ and $|\mathbf{k}| = k_0$. Here B_ρ is the ball of radius ρ centered at the origin. Set $\mathbf{y} = \frac{1}{\rho}\mathbf{x}$ and define

$$\big(\widetilde{\mathbf{E}}(\mathbf{y}), \widetilde{\mathbf{H}}(\mathbf{y})\big) := \Big(\big(\mathbf{E} \circ \Psi_{\frac{1}{\rho}}^{-1}\big)(\mathbf{y}), \big(\mathbf{H} \circ \Psi_{\frac{1}{\rho}}^{-1}\big)(\mathbf{y})\Big) = \Big(\big(\mathbf{E} \circ \Psi_\rho\big)(\mathbf{y}), \big(\mathbf{H} \circ \Psi_\rho\big)(\mathbf{y})\Big)$$

and

$$\big(\widetilde{\mathbf{E}}^i(\mathbf{y}), \widetilde{\mathbf{H}}^i(\mathbf{y})\big) := \Big(\big(\mathbf{E}^i \circ \Psi_\rho\big)(\mathbf{y}), \big(\mathbf{H}^i \circ \Psi_\rho\big)(\mathbf{y})\Big).$$

Then, we have

$$\begin{cases} \left(\nabla_{\mathbf{y}} \times \widetilde{\mathbf{E}}\right)(\mathbf{y}) = i\rho\omega\mu(\mathbf{y})\widetilde{\mathbf{H}}(\mathbf{y}) & \text{for } \mathbf{y} \in \mathbb{R}^3 \setminus \overline{B_1} \\ \left(\nabla_{\mathbf{y}} \times \widetilde{\mathbf{H}}\right)(\mathbf{y}) = -i\rho\omega\epsilon(\mathbf{y})\widetilde{\mathbf{E}}(\mathbf{y}) & \text{for } \mathbf{y} \in \mathbb{R}^3 \setminus \overline{B_1}, \\ \hat{\mathbf{y}} \times \widetilde{\mathbf{E}}(\mathbf{y}) = 0 & \text{on } \partial B_1, \\ (\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}^i, \widetilde{\mathbf{H}} - \widetilde{\mathbf{H}}^i) \text{ is radiating} \end{cases}$$

Remind that the scattered wave can be represented using the scattering amplitude as follows:

$$(\mathbf{E} - \mathbf{E}^i)(\mathbf{x}) \sim \frac{e^{ik_0|\mathbf{x}|}}{k_0|\mathbf{x}|}\boldsymbol{A}_\infty\left[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\right](\mathbf{c}, \hat{\mathbf{k}}; \hat{\mathbf{x}}) \quad \text{as } |\mathbf{x}| \to \infty,$$

and

$$(\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}^i)(\mathbf{y}) \sim \frac{e^{ik_0\rho|\mathbf{y}|}}{k_0\rho|\mathbf{y}|}\boldsymbol{A}_\infty\left[\mu, \epsilon, \omega\right](\mathbf{c}, \hat{\mathbf{k}}; \hat{\mathbf{x}}) \quad \text{as } |\mathbf{y}| \to \infty.$$

Since the left-hand sides of the previous equations are coincide, we have (5.2).

Suppose that (μ, ϵ) is a S-vanishing structure of order N at low frequencies as in Section 4. From (4.18) and (5.2), we have

$$\boldsymbol{A}_\infty\left[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\right](\mathbf{c}, \hat{\mathbf{k}}; \hat{\mathbf{x}}) = o(\rho^{2N+1}) \tag{5.3}$$

Then, we define the diffeomorphism F_ρ as

$$F_\rho(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{for } |\mathbf{x}| \geq 2, \\ \Big(\dfrac{3-4\rho}{2(1-\rho)} + \dfrac{1}{4(1-\rho)}|\mathbf{x}|\Big)\dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{for } 2\rho \leq |\mathbf{x}| \leq 2, \\ \Big(\dfrac{1}{2} + \dfrac{1}{2\rho}|\mathbf{x}|\Big)\dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{for } \rho \leq |\mathbf{x}| \leq 2\rho, \\ \dfrac{\mathbf{x}}{\rho} & \text{for } |\mathbf{x}| \leq \rho. \end{cases}$$

We then get from (5.3) and Lemma 5.1 the main result of this paper.

Theorem 5.2. *If (μ, ϵ) is a S-vanishing structure of order N at low frequencies, then there exists ρ_0 such that*

$$\boldsymbol{A}_\infty \left[(F_\rho)_*(\mu \circ \Psi_{\frac{1}{\rho}}), (F_\rho)_*(\epsilon \circ \Psi_{\frac{1}{\rho}}), \omega \right] (\mathbf{c}, \hat{\mathbf{k}}; \hat{\mathbf{x}}) = o(\rho^{2N+1}),$$

for all $\rho \leq \rho_0$, uniformly in $(\hat{\mathbf{k}}, \hat{\mathbf{x}})$.

Remark that the cloaking structure $\big((F_\rho)_*(\mu \circ \Psi_{\frac{1}{\rho}}), (F_\rho)_*(\epsilon \circ \Psi_{\frac{1}{\rho}})\big)$ in Theorem 5.2 satisfies the PEC boundary condition on $|\mathbf{x}| = 1$.

6 Conclusion

We have shown near-cloaking examples for the Maxwell equation. We have designed a cloaking device that achieves enhanced cloaking effect based on the method of [5, 6] to electromagnetic scattering problems. Any target placed inside the cloaking device has an approximately zero scattering amplitude. Such cloaking device is obtained by the blow up using the transformation optics of a multi-coated inclusion with PEC boundary condition. The cloaking device has anisotropic permittivity and permeability parameters.

References

[1] A. Alú and N. Engheta, Achieving transparency with plasmonic and metamaterial coatings, Phys. Rev. E., 72 (2005), 016623, 9 pp.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and G. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, Arch. rat. Mech. Anal., to appear.

[3] H. Ammari, J. Garnier, V. Jugnon, H. Kang, H. Lee, and M. Lim, Enhancement of near cloaking. Part III: numerical simulations, statistical stability, and related questions, Contemp. Math., 577 (2012), 1–24.

[4] H. Ammari and H. Kang, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences, Vol. 162, Springer-Verlag, New York, 2007.

[5] H. Ammari, H. Kang, H. Lee, and M. Lim, Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: the conductivity problem, Comm. Math. Phys., to appear.

[6] H. Ammari, H. Kang, H. Lee, and M. Lim, Enhancement of near cloaking. Part II: the Helmholtz equation, Comm. Math. Phys., to appear.

[7] H. Ammari, M. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations, J. Math. Pures Appl., 80 (2001), 769–814.

[8] G. Bao and H. Liu, Nearly cloaking the full Maxwell equations, ArXiv:1210.2447v1.

[9] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, Cambridge University Press, 6 edition, Cambridge, 1997.

[10] G. Bouchitté and B. Schweizer, Cloaking of small objects by anomalous localized resonance, Quart. J. Mech. Appl. Math. 63 (2010), 437–463.

[11] O. P. Bruno and S. Lintner, Superlens-cloaking of small dielectric bodies in the quasi-static regime, J. Appl. Phys. 102 (2007), 124502.

[12] K. Bryan and T. Leise, Impedance imaging, inverse problems, and Harry Potter's cloak, SIAM Rev., 52 (2010), 359–377.

[13] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, Cloaking devices, electromagnetic wormholes, and transformation optics, SIAM Rev., 51 (2009), 3–33.

[14] R. Griesmaier, An asymptotic factorization method for inverse electromagnetic scattering in layered media, SIAM J. Appl. Math., 68 (2008), 1378–1403.

[15] F. Guevara Vasquez, G. W. Milton, and D. Onofrei, Broadband exterior cloaking, Opt. Express 17 (2009), 14800–14805.

[16] H. Hashemi, A. Oskooi, J. D. Joannopoulos, and S. G. Johnson, General scaling limitations of ground-plane and isolated-object cloaks, Phys. Rev. A, 84 (2011), 023815.

[17] L. Klinkenbusch, Scattering of an arbitrary plane electromagnetic wave by a finite elliptic cone, Archiv für Elektrotechnik, 76 (1993), 181–193.

[18] R. V. Kohn, D. Onofrei, M. S. Vogelius, and M. I. Weinstein, Cloaking via change of variables for the Helmholtz equation, Comm. Pure Appl. Math., 63 (2010), 973–1016.

[19] R. V. Kohn, H. Shen, M. S. Vogelius, and M. I. Weinstein, Cloaking via change of variables in electric impedance tomography, Inverse Problems, 24 (2008), article 015016.

[20] U. Leonhardt, Optical conforming mapping, Science, 312 (2006), 5781, 1777–1780.

[21] U. Leonhardt and T. Tyc, Broadband invisibility by non-euclidean cloaking, Science, 323 (2009), 110–111.

[22] H. Liu, *Virtual reshaping and invisibility in obstacle scattering*, Inverse Problems, 25 (2009), 044006, 16 pp.

[23] D. A. B. Miller, On perfect cloaking, Optics Express, 14 (2006), 12457–12466.

[24] G. Milton and N. A. Nicorovici, On the cloaking effects associated with anomalous localized resonance, Proc. R. Soc. A 462 (2006), 3027-3059.

[25] G. Milton, N. A. Nicorovici, R. C. McPhedran, and V. A. Podolskiy, A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance, Proc. R. Soc. A 461 (2005), 3999-4034.

[26] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, 2003.

[27] J.-C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Springer, 2001.

[28] H. M. Nguyen, Cloaking via change of variables for the Helmholtz equation in the whole space, Comm. Pure Appl. Math., 63 (2010), 1505–1524.

[29] J. B. Pendry, D. Schurig, and D. R. Smith, Controlling electromagnetic fields, Science, 312 (2006), 1780–1782.

[30] C. W. Qiu, L. Hu, B. Zhang, B.-I. Wu, S. G. Johnson, and J. D. Joannopoulos, Spherical cloaking using nonlinear transformations for improved segmentation into concentric isotropic coatings, Optics Express, 17 (2009), 13467–13478.

[31] M. E. Taylor, *Partial Differential Equations II. Qualitative Studies of Linear Equations*, Appl. Math. Sci., Vol. 116, Springer-Verlag, New York, 1996.

[32] R. H. Torres, Maxwell's equations and dielectric obstacles with Lipschitz boundaries, J. London Math. Soc. (2) 57 (1998), 157–169.

[33] Y. A. Urzhumov, N. B. Kundtz, D. R. Smith, and J. B. Pendry, Cross-section comparisons of cloaks designed by transformation optics and optical conformal mapping approaches, J. Opt., 13 (2011), 024002.