Enhancement of near-cloaking. Part II: the Helmholtz equation*

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Abstract

The aim of this paper is to extend the method of [6] to scattering problems. We construct very effective near-cloaking structures for the scattering problem at a fixed frequency. These new structures are, before using the transformation optics, layered structures and are designed so that their first scattering coefficients vanish. Inside the cloaking region, any target has near-zero scattering cross section for a band of frequencies. We analytically show that our new construction significantly enhances the cloaking effect for the Helmholtz equation.

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1 Introduction

The cloaking problem in electromagnetic wave scattering is to make a target invisible from far-field measurements. The difficulty is to construct permeability and permittivity distributions of a cloaking structure such that any target placed inside the structure has zero scattering cross section. Since the pioneer works [11], [14], and [19], extensive research has been done on cloaking in electromagnetic scattering. We refer to [10], [9], and [21] for recent development on electromagnetic cloaking. One of the main tools to obtain cloaking is to use a change of variables scheme (also called transformation optics). The change of variables based cloaking method uses a singular transformation to boost the material property so that it makes a cloaking region look like a point to outside measurements. However, this transformation induces the singularity of material constants in the transversal direction (also in the tangential direction in two dimensions), which causes difficulty both in the theory and applications. To overcome this weakness, so called 'near cloaking' is naturally considered, which is a regularization or an approximation of singular cloaking. In [13], instead of the singular transformation, the authors use a regular one to push forward the material constant in the conductivity equation, in which a small ball is blown up to the cloaking region. In [12], this regularization point of view is adopted for the Helmholtz equation. See also [15, 18]. It is worth mentioning that there is yet another kind of cloaking in which the cloaking region is outside the cloaking device, for instance, anomalous localized resonance [16, 17, 3]. See also [2].

The purpose of this paper is to propose a new cancellation technique in order to achieve enhanced invisibility from scattering cross section.

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Our approach extends to scattering problems the method first provided in [6] to achieve near-cloaking for the conductivity problem from Dirichlet-to-Neumann measurements. It is based on the multi-coating which cancels the scattering coefficients of the cloak. We first design a structure coated around a perfect insulator to have vanishing scattering coefficients of lower orders and show that the order of the scattering cross section of a small perfect insulator can be reduced significantly. We then obtain near-cloaking structure by pushing forward the multi-coated structure around a small object via the standard blow-up transformation. Note that such structure achieves near-cloaking for a band of frequencies. In order to illustrate the viability of our method, we give numerical values for the permittivity and permeability parameters and the thickness of the layers of scattering coefficients vanishing structures.

This paper is organized as follows. In the next section we derive the multi-polar expansion of the solution to the Helmholtz equation, and define the scattering coefficients. In Section 3 we characterize the scattering coefficients vanishing structures. In Section 4 we show that the near-cloaking is enhanced if a scattering coefficients vanishing structure is used. In Section 5 we present some numerical examples of the scattering coefficients vanishing structures.

Even though we consider only two dimensional Helmholtz equation in this paper, the same argument can be applied to the equation in three dimensions. The multi-coating technique developed in this paper can be extended to the full Maxwell equations, which will be a subject of a forthcoming paper.

2 Scattering coefficients

For $k > 0$, the fundamental solution to the Helmholtz operator $(\Delta + k^2)$ in two dimensions satisfying

$$(\Delta + k^2)\Gamma_k(\mathbf{x}) = \delta_0$$

with the outgoing radiation condition is given by

$$\Gamma_k(\mathbf{x}) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{x}|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. For $|\mathbf{x}| > |\mathbf{y}|$, we recall Graf's addition formula [22],

$$H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) = \sum_{n\in\mathbb{Z}} H_n^{(1)}(k|\mathbf{x}|)e^{in\theta_{\mathbf{x}}} J_n(k|\mathbf{y}|)e^{-in\theta_{\mathbf{y}}}, \tag{2.1}$$

where $\mathbf{x} = (|\mathbf{x}|, \theta_{\mathbf{x}})$ and $\mathbf{y} = (|\mathbf{y}|, \theta_{\mathbf{y}})$ in polar coordinates. Here $H_n^{(1)}$ is the Hankel function of the first kind of order n and J_n is the Bessel function of order n.

We first define the scattering coefficients of an inclusion (with two phase electromagnetic materials) and derive some important properties of them.

Let D be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary ∂D, and let (ϵ_0, μ_0) be the pair of electromagnetic parameters (permittivity and permeability) of $\mathbb{R}^2 \setminus \bar{D}$ and (ϵ_1, μ_1) be that of D. Then the permittivity and permeability distributions are given by

$$\epsilon = \epsilon_0 \chi(\mathbb{R}^2 \setminus \bar{D}) + \epsilon_1 \chi(D) \quad \text{and} \quad \mu = \mu_0 \chi(\mathbb{R}^2 \setminus \bar{D}) + \mu_1 \chi(D). \tag{2.2}$$

Here and throughout this paper $\chi(D)$ is the characteristic function of D.

Given a frequency ω, set $k = \omega\sqrt{\epsilon_1\mu_1}$ and $k_0 = \omega\sqrt{\epsilon_0\mu_0}$. For a function U satisfying $(\Delta + k_0^2)U = 0$ in $\mathbb{R}^2$, we consider the scattered wave u, *i.e.*, the solution to the following equation:

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu}\nabla u + \omega^2 \epsilon u = 0 \quad \text{in } \mathbb{R}^2, \\ (u - U) \text{ satisfies the outgoing radiation condition.} \end{cases} \tag{2.3}$$

Let $\mathcal{S}_D^k[\varphi]$ be the single layer potential defined by the kernel Γ_k, *i.e.*,

$$\mathcal{S}_D^k[\varphi](\mathbf{x}) = \int_{\partial D} \Gamma_k(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})d\sigma(\mathbf{y}).$$

Let $\mathcal{S}_D^{k_0}$ be the single layer potential associated with the kernel Γ_{k_0}. Then, from, for example, [5], we know that the solution to (2.3) can be represented using the single layer potentials $\mathcal{S}_D^{k_0}$ and $\mathcal{S}_D^k$ as follows

$$u(\mathbf{x}) = \begin{cases} U(\mathbf{x}) + \mathcal{S}_D^{k_0}[\psi](\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2 \setminus \bar{D}, \\ \mathcal{S}_D^k[\varphi](\mathbf{x}), & \mathbf{x} \in D, \end{cases} \tag{2.4}$$

where the pair $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ is the unique solution to

$$\begin{cases} \mathcal{S}_D^k[\varphi] - \mathcal{S}_D^{k_0}[\psi] = U \\ \dfrac{1}{\mu}\dfrac{\partial(\mathcal{S}_D^k[\varphi])}{\partial \nu}\bigg|_- - \dfrac{1}{\mu_0}\dfrac{\partial(\mathcal{S}_D^{k_0}[\psi])}{\partial \nu}\bigg|_+ = \dfrac{1}{\mu_0}\dfrac{\partial U}{\partial \nu} \end{cases} \quad \text{on } \partial D. \tag{2.5}$$

Here, $+$ and $-$ in the subscripts respectively indicate the limit from outside D and inside D to ∂D along the normal direction and $\partial/\partial\nu$ denotes the normal derivative. It is proved in [5] that there exists a constant $C = C(k, k_0, D)$ such that

$$\|\varphi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \le C(\|U\|_{L^2(\partial D)} + \|\nabla U\|_{L^2(\partial D)}). \tag{2.6}$$

It is also proved in the same paper that there are constants ρ_0 and $C = C(k, k_0, D)$ independent of ρ as long as $\rho \le \rho_0$ such that

$$\|\varphi_\rho\|_{L^2(\partial D)} + \|\psi_\rho\|_{L^2(\partial D)} \le C(\|U\|_{L^2(\partial D)} + \|\nabla U\|_{L^2(\partial D)}), \tag{2.7}$$

where $(\varphi_\rho, \psi_\rho)$ is the solution of (2.5) with k and k_0 respectively replaced by ρk and ρk_0.

Note that the following asymptotic formula holds as $|\mathbf{x}| \to \infty$, which can be seen from (2.4) and Graf's formula (2.1):

$$u(\mathbf{x}) - U(\mathbf{x}) = -\frac{i}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k_0|\mathbf{x}|)e^{in\theta_{\mathbf{x}}}\int_{\partial D} J_n(k_0|\mathbf{y}|)e^{-in\theta_{\mathbf{y}}}\psi(\mathbf{y})d\sigma(\mathbf{y}). \tag{2.8}$$

Let (φ_m, ψ_m) be the solution to (2.5) with $J_m(k_0|\mathbf{x}|)e^{im\theta_{\mathbf{x}}}$ in the place of $U(\mathbf{x})$. We define the *scattering coefficient* as follows.

Definition 1. *The scattering coefficients W_{nm}, $m, n \in \mathbb{Z}$, associated with the permittivity and permeability distributions ϵ, μ and the frequency ω (or k, k_0, D) are defined by*

$$W_{nm} = W_{nm}[\epsilon, \mu, \omega] := \int_{\partial D} J_n(k_0|\mathbf{y}|)e^{-in\theta_{\mathbf{y}}}\psi_m(\mathbf{y})d\sigma(\mathbf{y}). \tag{2.9}$$

We first obtain the following lemma for the size of $|W_{nm}|$.

Lemma 2.1. *There is a constant C depending on (ϵ, μ, ω) such that*

$$|W_{nm}[\epsilon, \mu, \omega]| \le \frac{C^{|n|+|m|}}{|n|^{|n|}|m|^{|m|}} \quad \text{for all } n, m \in \mathbb{Z}. \tag{2.10}$$

Moreover, there exists ρ_0 such that, for all $\rho \le \rho_0$,

$$|W_{nm}[\epsilon, \mu, \rho\omega]| \le \frac{C^{|n|+|m|}}{|n|^{|n|}|m|^{|m|}}\rho^{|n|+|m|} \quad \text{for all } n, m \in \mathbb{Z}, \tag{2.11}$$

where the constant C depends on (ϵ, μ, ω) but is independent of ρ.

Proof. Let $U(\mathbf{x}) = J_m(k_0|\mathbf{x}|)e^{im\theta_\mathbf{x}}$ and (φ_m, ψ_m) be the solution to (2.5). Since

$$J_m(t) \sim \frac{1}{\sqrt{2\pi|m|}}\Big(\frac{et}{2|m|}\Big)^{|m|} \tag{2.12}$$

as $m \to \infty$ (see [1]), we have

$$\|U\|_{L^2(\partial D)} + \|\nabla U\|_{L^2(\partial D)} \le \frac{C^{|m|}}{|m|^{|m|}}$$

for some constant C. Thus it follows from (2.6) that

$$\|\psi_m\|_{L^2(\partial D)} \le \frac{C^{|m|}}{|m|^{|m|}} \tag{2.13}$$

for another constant C. So we get (2.10) from (2.9).

Now let (φ_m, ψ_m) be the solution to (2.5) with k and k_0 replaced by ρk and ρk_0. One can see from (2.7) that (2.13) still holds for some C independent of ρ as long as $\rho \le \rho_0$ for some ρ_0. Note that

$$W_{nm}[\epsilon, \mu, \rho\omega] = \int_{\partial D} J_n(\rho k_0|\mathbf{y}|)e^{-in\theta_\mathbf{y}}\psi_{m,\rho}(\mathbf{y})d\sigma(\mathbf{y}),$$

where $(\varphi_{m,\rho}, \psi_{m,\rho})$ is the solution to (2.5) with k and k_0 respectively replaced by ρk and ρk_0 and $J_m(k_0\rho|\mathbf{x}|)e^{im\theta_\mathbf{x}}$ in the place of $U(\mathbf{x})$. So one can use (2.12) to obtain (2.11). This completes the proof. □

If U is given as

$$U(\mathbf{x}) = \sum_{m\in\mathbb{Z}} a_m(U)J_m(k_0|\mathbf{x}|)e^{im\theta_\mathbf{x}} \tag{2.14}$$

where $a_m(U)$ are constants, it follows from the principle of superposition that the solution (φ, ψ) to (2.5) is given by

$$\psi = \sum_{m\in\mathbb{Z}} a_m(U)\psi_m.$$

Then one can see from (2.8) that the solution u to (2.3) can be represented as

$$u(\mathbf{x}) - U(\mathbf{x}) = -\frac{i}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k_0|\mathbf{x}|)e^{in\theta_\mathbf{x}} \sum_{m\in\mathbb{Z}} W_{nm}a_m(U) \quad \text{as } |\mathbf{x}| \to \infty. \tag{2.15}$$

In particular, if U is given by a plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k}\cdot\mathbf{k} = k_0^2$, then

$$u(\mathbf{x}) - e^{i\mathbf{k}\cdot\mathbf{x}} = -\frac{i}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k_0|\mathbf{x}|)e^{in\theta_\mathbf{x}} \sum_{m\in\mathbb{Z}} W_{nm}e^{im(\frac{\pi}{2}-\theta_\mathbf{k})} \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.16}$$

where $\mathbf{k} = k_0(\cos\theta_\mathbf{k}, \sin\theta_\mathbf{k})$ and $\mathbf{x} = (|\mathbf{x}|, \theta_\mathbf{x})$. In fact, since

$$e^{ik_0 r\sin\theta} = \sum_{m\in\mathbb{Z}} J_m(k_0 r)e^{im\theta},$$

we have

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\mathbf{k})}J_m(k_0|\mathbf{x}|)e^{im\theta_\mathbf{x}}, \tag{2.17}$$

and

$$\psi = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\mathbf{k})}\psi_m. \tag{2.18}$$

Thus (2.16) holds. It is worth emphasizing that the expansion formula (2.15) or (2.16) determines uniquely the scattering coefficients W_{nm}, for $n, m \in \mathbb{Z}$.

We now show that the scattering coefficients are basically the Fourier coefficients of the far-field pattern (the scattering amplitude) which is 2π-periodic function in two dimensions. The far-field pattern $A_\infty[\epsilon, \mu, \omega]$, when the incident field is given by $e^{i\mathbf{k}\cdot\mathbf{x}}$, is defined to be

$$u(\mathbf{x}) - e^{i\mathbf{k}\cdot\mathbf{x}} = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_0|\mathbf{x}|}}{\sqrt{8\pi k_0|\mathbf{x}|}} A_\infty[\epsilon, \mu, \omega](\theta_\mathbf{k}, \theta_\mathbf{x}) + o(|\mathbf{x}|^{-\frac{1}{2}}) \quad \text{as } |\mathbf{x}| \to \infty. \tag{2.19}$$

Recall that

$$H_0^{(1)}(t) \sim \sqrt{\frac{2}{\pi t}} e^{i(t-\frac{\pi}{4})} \quad \text{as } t \to \infty, \tag{2.20}$$

where $\sim$ indicates that the difference between the right-hand and left-hand side is $O(t^{-1})$. If $|\mathbf{x}|$ is large while $|\mathbf{y}|$ is bounded, then we have

$$|\mathbf{x} - \mathbf{y}| = |\mathbf{x}| - |\mathbf{y}| \cos(\theta_\mathbf{x} - \theta_\mathbf{y}) + O(\frac{1}{|\mathbf{x}|}),$$

and hence

$$H_0^{(1)}(k_0|\mathbf{x} - \mathbf{y}|) \sim e^{-\frac{\pi i}{4}} \sqrt{\frac{2}{\pi k_0|\mathbf{x}|}} e^{ik_0(|\mathbf{x}| - |\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y}))} \quad \text{as } |\mathbf{x}| \to \infty.$$

Thus, from (2.4), we get

$$u(\mathbf{x}) - e^{i\mathbf{k}\cdot\mathbf{x}} \sim -ie^{-\frac{\pi i}{4}} \frac{e^{ik_0|\mathbf{x}|}}{\sqrt{8\pi k_0|\mathbf{x}|}} \int_{\partial D} e^{-ik_0|\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y})} \psi(\mathbf{y})\, d\sigma(\mathbf{y}) \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.21}$$

and infer that the far-field pattern is given by

$$A_\infty(\theta_\mathbf{k}, \theta_\mathbf{x}) = \int_{\partial D} e^{-ik_0|\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y})} \psi(\mathbf{y})\, d\sigma(\mathbf{y}), \tag{2.22}$$

where ψ is given by (2.18).

Let

$$A_\infty(\theta_\mathbf{k}, \theta_\mathbf{x}) = \sum_{n\in\mathbb{Z}} b_n(\theta_\mathbf{k}) e^{in\theta_\mathbf{x}}$$

be the Fourier series of $A_\infty(\theta_\mathbf{k}, \cdot)$. From (2.22) it follows that

$$\begin{aligned} b_n(\theta_\mathbf{k}) &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\partial D} e^{-ik_0|\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y})} \psi(\mathbf{y})\, d\sigma(\mathbf{y})\, e^{-in\theta_\mathbf{x}}\, d\theta_\mathbf{x} \\ &= \frac{1}{2\pi} \int_{\partial D} \int_0^{2\pi} e^{-ik_0|\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y})} e^{-in\theta_\mathbf{x}}\, d\theta_\mathbf{x}\, \psi(\mathbf{y})\, d\sigma(\theta_\mathbf{y}). \end{aligned}$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ik_0|\mathbf{y}|\cos(\theta_\mathbf{x} - \theta_\mathbf{y})} e^{-in\theta_\mathbf{x}}\, d\theta_\mathbf{x} = J_n(k_0|\mathbf{y}|) e^{-in(\theta_\mathbf{y} + \frac{\pi}{2})},$$

we deduce that

$$b_n(\theta_\mathbf{k}) = \int_{\partial D} J_n(k_0|\mathbf{y}|) e^{-in(\theta_\mathbf{y} + \frac{\pi}{2})} \psi(\mathbf{y})\, d\sigma(\theta_\mathbf{y}).$$

Using (2.18) we now arrive at the following theorem.

Theorem 2.2. *Let θ and θ' be respectively the incident and scattered direction. Then we have*

$$A_\infty[\epsilon, \mu, \omega](\theta, \theta') = \sum_{n,m\in\mathbb{Z}} i^{(m-n)} e^{in\theta'} W_{nm}[\epsilon, \mu, \omega] e^{-im\theta}. \tag{2.23}$$

We emphasize that the series in (2.23) converges uniformly thanks to (2.10). We also note that if U is given by (2.14) then the scattering amplitude, which we denote by $A_\infty[\epsilon,\mu,\omega](U,\theta')$, is given by

$$A_\infty[\epsilon,\mu,\omega](U,\theta') = \sum_{n\in\mathbb{Z}} i^{-n} e^{in\theta'} \sum_{m\in\mathbb{Z}} W_{nm} a_m(U). \tag{2.24}$$

The conversion of the far-field to the near field is achieved via formula (2.16).

The scattering cross section $S[\epsilon,\mu,\omega]$ is defined by

$$S[\epsilon,\mu,\omega](\theta') := \int_0^{2\pi} \left| A_\infty[\epsilon,\mu,\omega](\theta,\theta') \right|^2 d\theta. \tag{2.25}$$

See [7, 20]. As an immediate consequence of Theorem 2.2 we obtain the following corollary.

Corollary 2.3. *We have*

$$S[\epsilon,\mu,\omega](\theta') = 2\pi \sum_{m\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} i^{-n} W_{nm}[\epsilon,\mu,\omega] e^{in\theta'} \right|^2. \tag{2.26}$$

It is worth mentioning that the optical theorem [7, 20] leads to a natural constraint on W_{nm}. In fact, we have

$$\Im m\, A_\infty[\epsilon,\mu,\omega](\theta',\theta') = -\sqrt{\frac{\omega}{8\pi}} S[\epsilon,\mu,\omega](\theta'), \quad \forall\, \theta' \in [0,2\pi], \tag{2.27}$$

or equivalently,

$$\Im m \sum_{n,m\in\mathbb{Z}} i^{m-n} e^{i(n-m)\theta'} W_{nm}[\epsilon,\mu,\omega] = -\sqrt{\frac{\pi\omega}{2}} \sum_{m\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} i^{-n} W_{nm}[\epsilon,\mu,\omega] e^{in\theta'} \right|^2, \quad \forall\, \theta' \in [0,2\pi]. \tag{2.28}$$

In the next section, we compute the scattering coefficients of multiply coated inclusions and provide structures whose scattering coefficients vanish. Such structures will be used in Section 4 to enhance near cloaking. Any target placed inside such structures will have nearly vanishing scattering cross section S, uniformly in the direction θ'.

3 S-vanishing structures

The purpose of this section is to construct multiply layered structures whose scattering coefficients vanish. We call such structures *S-vanishing structures*. We design a multi-coating around an insulated inclusion D, for which the scattering coefficients vanish. The computations of the scattering coefficients of multi-layered structures (with multiple phase electromagnetic materials) follow in exactly the same way as in Section 2. The system of two equations (2.5) should be replaced by a system of $2\times$ the number of phase interfaces (-1 if the core is perfectly insulating).

For positive numbers $r_1,\ldots,r_{L+1}$ with $2 = r_1 > r_2 > \cdots > r_{L+1} = 1$, let

$$A_j := \{x : r_{j+1} \le |x| < r_j\}, \quad j = 1,\ldots,L, \quad A_0 := \mathbb{R}^2 \setminus \overline{A_1}, \quad A_{L+1}(=D) := \{x : |x| < 1\}.$$

Let (μ_j,ϵ_j) be the pair of permeability and permittivity of A_j for $j = 0,1,\ldots,L+1$. Set $\mu_0 = 1$ and $\epsilon_0 = 1$. Let

$$\mu = \sum_{j=0}^{L+1} \mu_j \chi(A_j) \quad \text{and} \quad \epsilon = \sum_{j=0}^{L+1} \epsilon_j \chi(A_j). \tag{3.1}$$

In this case the scattering coefficient $W_{nm} = W_{nm}[\mu, \epsilon, \omega]$ can be defined using (2.15). In fact, if u is the solution to

$$\nabla \cdot \frac{1}{\mu} \nabla u + \omega^2 \epsilon u = 0 \qquad \text{in } \mathbb{R}^2 \tag{3.2}$$

with the outgoing radiation condition on $u - U$ where U is given by (2.14), then $u - U$ admits the asymptotic expansion (2.15) with $k_0 = \omega\sqrt{\epsilon_0 \mu_0}$.

Exactly like the conductivity case [6] one can show using symmetry that

$$W_{nm} = 0 \quad \text{if } m \neq n. \tag{3.3}$$

Let us define W_n by

$$W_n := W_{nn}. \tag{3.4}$$

Our purpose is to design, given N and ω, μ and ϵ so that $W_n[\mu, \epsilon, \omega] = 0$ for $|n| \leq N$. We call such a structure (μ, ϵ) an *S-vanishing structure of order N at frequency ω*. Since $H_{-n}^{(1)} = (-1)^n H_n^{(1)}$ and $J_{-n} = (-1)^n J_n$, we have

$$W_{-n,-n} = W_{nn}, \tag{3.5}$$

and hence it suffices to consider W_{nn} only for $n \geq 0$.

Note that (2.28) leads to

$$\Im m \sum_{n \in \mathbb{Z}} W_n[\epsilon, \mu, \omega] = -\sqrt{\frac{\pi\omega}{2}} \sum_{n \in \mathbb{Z}} \Big| W_n[\epsilon, \mu, \omega] \Big|^2. \tag{3.6}$$

Let $k_j := \omega\sqrt{\mu_j \epsilon_j}$ for $j = 0, 1, \ldots, L$. We assume that $\mu_{L+1} = \infty$, which amounts to that the solution satisfies the zero Neumann condition on $|\mathbf{x}| = r_{L+1} (= 1)$. To compute W_n for $n \geq 0$, we look for solutions u_n to (3.2) of the form

$$u_n(\mathbf{x}) = a_j^{(n)} J_n(k_j r) e^{in\theta} + b_j^{(n)} H_n^{(1)}(k_j r) e^{in\theta}, \quad \mathbf{x} \in A_j, \quad j = 0, \ldots, L, \tag{3.7}$$

with $a_0^{(n)} = 1$. Note that

$$W_n = 4ib_0^{(n)}. \tag{3.8}$$

The solution u_n satisfies the transmission conditions

$$u_n|_+ = u_n|_- \quad \text{and} \quad \frac{1}{\mu_{j-1}} \frac{\partial u_n}{\partial \nu}\Big|_+ = \frac{1}{\mu_j} \frac{\partial u_n}{\partial \nu}\Big|_- \quad \text{on } |\mathbf{x}| = r_j$$

for $j = 1, \ldots, L$, which reads

$$\begin{aligned} &\begin{bmatrix} J_n(k_j r_j) & H_n^{(1)}(k_j r_j) \\ \sqrt{\dfrac{\epsilon_j}{\mu_j}} J_n'(k_j r_j) & \sqrt{\dfrac{\epsilon_j}{\mu_j}} \left(H_n^{(1)}\right)'(k_j r_j) \end{bmatrix} \begin{bmatrix} a_j^{(n)} \\ b_j^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} J_n(k_{j-1} r_j) & H_n^{(1)}(k_{j-1} r_j) \\ \sqrt{\dfrac{\epsilon_{j-1}}{\mu_{j-1}}} J_n'(k_{j-1} r_j) & \sqrt{\dfrac{\epsilon_{j-1}}{\mu_{j-1}}} \left(H_n^{(1)}\right)'(k_{j-1} r_j) \end{bmatrix} \begin{bmatrix} a_{j-1}^{(n)} \\ b_{j-1}^{(n)} \end{bmatrix}. \end{aligned} \tag{3.9}$$

The Neumann condition $\frac{\partial u_n}{\partial \nu}|_+ = 0$ on $|\mathbf{x}| = r_{L+1}$ amounts to

$$\begin{bmatrix} 0 & 0 \\ J_n'(k_L) & \left(H_n^{(1)}\right)'(k_L) \end{bmatrix} \begin{bmatrix} a_L^{(n)} \\ b_L^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P^{(n)}[\epsilon, \mu, \omega] \begin{bmatrix} a_0^{(n)} \\ b_0^{(n)} \end{bmatrix}, \tag{3.11}$$

where

$$P^{(n)}[\epsilon,\mu,\omega] := \begin{bmatrix} 0 & 0 \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = (-\frac{\pi}{2}i\omega)^L \left(\prod_{j=1}^{L}\mu_j r_j\right) \begin{bmatrix} 0 & 0 \\ J_n'(k_L) & \left(H_n^{(1)}\right)'(k_L) \end{bmatrix}$$

$$\times \prod_{j=1}^{L} \begin{bmatrix} \sqrt{\frac{\epsilon_j}{\mu_j}}\left(H_n^{(1)}\right)'(k_j r_j) & -H_n^{(1)}(k_j r_j) \\ -\sqrt{\frac{\epsilon_j}{\mu_j}} J_n'(k_j r_j) & J_n(k_j r_j) \end{bmatrix} \begin{bmatrix} J_n(k_{j-1} r_j) & H_n^{(1)}(k_{j-1} r_j) \\ \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}} J_n'(k_{j-1} r_j) & \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}}\left(H_n^{(1)}\right)'(k_{j-1} r_j) \end{bmatrix}.$$

In order to have a structure whose scattering coefficients W_n vanishes up to the N-order, we need to have $b_0^{(n)} = 0$ (when $a_0^{(n)} = 1$) for $n = 0, \ldots, N$, which amounts to

$$p_{21}^{(n)} = 0 \quad \text{for } n = 0, \ldots, N \tag{3.12}$$

because of (3.11). We emphasize that $p_{22}^{(n)} \neq 0$. In fact, if $p_{22}^{(n)} = 0$, then (3.11) can be fulfilled with $a_0^{(n)} = 0$ and $b_0^{(n)} = 1$. It means that there exists (μ, ϵ) on $\mathbb{R}^2 \setminus D$ such that the following problem has a solution:

$$\begin{cases} \nabla \cdot \frac{1}{\mu}\nabla u + \omega^2 \epsilon u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu}\Big|_+ = 0 \quad \text{on } \partial D, \\ u(\mathbf{x}) = H_n^{(1)}(k_0 r)e^{in\theta} \quad \text{for } |\mathbf{x}| = r > 2, \end{cases} \tag{3.13}$$

which is not possible.

We note that (3.12) is a set of conditions on (μ_j, ϵ_j) and r_j for $j = 1, \ldots, L$. In fact, $p_{21}^{(n)}$ is a nonlinear algebraic function of μ_j, ϵ_j and r_j, $j = 1, \ldots, L$. We are not able to show existence of (μ_j, ϵ_j) and r_j, $j = 1, \ldots, L$, satisfying (3.12) even if it is quite important to do so. But the solutions (at fixed frequency) can be computed numerically in the same way as in the conductivity case [6].

We now consider the S-vanishing structure for all (low) frequencies. Let ω be fixed and we look for a structure (μ, ϵ) such that

$$W_n[\mu, \epsilon, \rho\omega] = 0 \quad \text{for all } |n| \leq N \text{ and } \rho \leq \rho_0 \tag{3.14}$$

for some ρ_0. Such a structure may not exist. So instead we look for a structure such that

$$W_n[\mu, \epsilon, \rho\omega] = o(\rho^{2N}) \quad \text{for all } |n| \leq N \text{ and } \rho \to 0. \tag{3.15}$$

We call such a structure an *S-vanishing structure of order N at low frequencies.*

To investigate the behavior of $W_n[\mu, \epsilon, \rho\omega]$ as $\rho \to 0$, let us recall the behavior of Bessel functions for small arguments. As $t \to 0$, it is known that

$$J_n(t) = \frac{t^n}{2^n}\left(\frac{1}{\Gamma(n+1)} - \frac{\frac{1}{4}t^2}{\Gamma(n+2)} + \frac{(\frac{1}{4}t^2)^2}{2!\Gamma(n+3)} - \frac{(\frac{1}{4}t^2)^3}{3!\Gamma(n+4)} + \cdots\right),$$

$$Y_n(t) = -\frac{(\frac{1}{2}t)^{-n}}{\pi}\sum_{l=0}^{n-1}\frac{(n-l-1)!}{l!}(\frac{1}{4}t^2)^l + \frac{2}{\pi}\ln(\frac{1}{2}t)J_n(t)$$
$$- \frac{(\frac{1}{2}t)^n}{\pi}\sum_{l=0}^{\infty}(\psi(l+1) + \psi(n+l+1))\frac{(-\frac{1}{4}t^2)^l}{l!(n+l)!},$$

where $\psi(1) = -\gamma$ and $\psi(n) = -\gamma + \sum_{l=1}^{n-1} 1/l$ for $n \geq 2$ with γ being the Euler constant. In particular, if $n = 0$, we have

$$J_0(t) = 1 - \frac{1}{4}t^2 + \frac{1}{64}t^4 + O(t^6), \tag{3.16}$$

$$Y_0(t) = \frac{2}{\pi}\ln t + \frac{2}{\pi}(\gamma - \ln 2) - \frac{1}{2\pi}t^2 \ln t + \Big(\frac{1}{2\pi} - \frac{1}{2\pi}(\gamma - \ln 2)\Big)t^2 + O(t^4 \ln t). \tag{3.17}$$

Plugging these formulas into (3.11), we have

$$\begin{aligned}
P^{(0)}[\epsilon,\mu,\rho\omega] &= (-\frac{\pi}{2}i\rho\omega)^L \left(\prod_{j=1}^L \mu_j r_j\right) \begin{bmatrix} 0 & 0 \\ -\frac{k_L}{2}\rho + O(\rho^3) & \frac{2i}{\pi k_L}\rho^{-1} + O(\rho\ln\rho) \end{bmatrix} \\
&\times \prod_{j=1}^L \begin{bmatrix} \frac{2i}{\pi\omega\mu_j r_j}\rho^{-1} + O(\rho\ln\rho) & \frac{4}{\pi^2}\left(\frac{1}{\omega\mu_{j-1}r_j} - \frac{1}{\omega\mu_j r_j}\right)\frac{\ln\rho}{\rho} + O(\rho^{-1}) \\ \frac{r_j}{2}\omega\epsilon_j\left(1 - \frac{\epsilon_{j-1}}{\epsilon_j}\right)\rho + O(\rho^3) & \frac{2i}{\pi\omega\mu_{j-1}r_j}\rho^{-1} + O(\rho\ln\rho) \end{bmatrix} \\
&= \rho^{-1}\begin{bmatrix} 0 & 0 \\ O(\rho^2) & \frac{2i}{\pi k_L}\prod_{j=1}^L \frac{\mu_j}{\mu_{j-1}} + O(\rho) \end{bmatrix},
\end{aligned} \tag{3.18}$$

and, for $n \geq 1$,

$$\begin{aligned}
P^{(n)}[\epsilon,\mu,\rho\omega] &= (-i\frac{\pi}{2}\rho\omega)^L \left(\prod_{j=1}^L \mu_j r_j\right) \begin{bmatrix} 0 & 0 \\ \frac{nk_L^{n-1}}{2^n\Gamma(n+1)}\rho^{n-1} + O(\rho^n) & \frac{i2^n\Gamma(n+1)}{\pi k_L^{n+1}}\rho^{-n-1} + O(\rho^{-n}) \end{bmatrix} \\
&\times \prod_{j=1}^L \begin{bmatrix} \sqrt{\frac{\epsilon_j}{\mu_j}}\frac{i2^n\Gamma(n+1)}{\pi(k_j r_j)^{n+1}}\rho^{-n-1} + O(\rho^{-n}) & \frac{i2^n\Gamma(n)}{\pi(k_j r_j)^n}\rho^{-n} + O(\rho^{-n+1}) \\ -\sqrt{\frac{\epsilon_j}{\mu_j}}\frac{n(k_j r_j)^{n-1}}{2^n\Gamma(n+1)}\rho^{n-1} + O(\rho^n) & \frac{(k_j r_j)^n}{2^n\Gamma(n+1)}\rho^n + O(\rho^{n+1}) \end{bmatrix} \\
&\times \begin{bmatrix} \frac{(k_{j-1}r_j)^n}{2^n\Gamma(n+1)}\rho^n + O(\rho^{n+1}) & -\frac{i2^n\Gamma(n)}{\pi(k_{j-1}r_j)^n}\rho^{-n} + O(\rho^{-n+1}) \\ \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}}\frac{n(k_{j-1}r_j)^{n-1}}{2^n\Gamma(n+1)}\rho^{n-1} + O(\rho^n) & \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}}\frac{i2^n\Gamma(n+1)}{\pi(k_{j-1}r_j)^{n+1}}\rho^{-n-1} + O(\rho^{-n}) \end{bmatrix} \\
&= \frac{1}{2^L}\begin{bmatrix} 0 & 0 \\ \frac{nk_L^{n-1}}{2^n\Gamma(n+1)}\rho^{n-1} + O(\rho^n) & \frac{i2^n\Gamma(n+1)}{\pi k_L^{n+1}}\rho^{-n-1} + O(\rho^{-n}) \end{bmatrix} \\
&\times \prod_{j=1}^L \begin{bmatrix} a_j(b_j+1) + o(1) & c_j(b_j-1)\rho^{-2n} + o(\rho^{-2n}) \\ \frac{b_j-1}{c_j}\rho^{2n} + o(\rho^{2n}) & \frac{b_j+1}{a_j} + o(1) \end{bmatrix},
\end{aligned} \tag{3.19}$$

where

$$a_j := \left(\frac{k_{j-1}}{k_j}\right)^n, \quad b_j := \frac{\mu_j}{\mu_{j-1}}, \quad c_j := \frac{i2^{2n}\Gamma(n)\Gamma(n+1)}{\pi(k_{j-1}k_j r_j^2)^n},$$

with $k_j = \omega\sqrt{\epsilon_j\mu_j}$.

From the above calculations of the leading order terms of $P^{(n)}[\epsilon,\mu,\rho\omega]$ and the expansion formula of $J_n(t)$ and $Y_n(t)$, we see that $p_{21}^{(n)}$ and $p_{22}^{(n)}$ admit the following expansions:

$$p_{21}^{(n)}(\mu,\epsilon,t) = t^{n-1}\left(f_0^{(n)}(\mu,\epsilon) + \sum_{l=1}^{(N-n)}\sum_{j=0}^{L+1} f_{l,j}^{(n)}(\mu,\epsilon)t^{2l}(\ln t)^j + o(t^{2N-2n})\right) \tag{3.20}$$

and

$$p_{22}^{(n)}(\mu,\epsilon,t) = t^{-n-1}\left(g_0^{(n)}(\mu,\epsilon) + \sum_{l=1}^{(N-n)}\sum_{j=0}^{L+1} g_{l,j}^{(n)}(\mu,\epsilon)t^{2l}(\ln t)^j + o(t^{2N-2n})\right) \tag{3.21}$$

for $t = \rho\omega$ and some functions $f_0^{(n)}, g_0^{(n)}, f_{l,j}^{(n)}$, and $g_{l,j}^{(n)}$ independent of t.

Lemma 3.1. *For any pair of (μ,ϵ), we have*

$$g_0^{(n)}(\mu,\epsilon) \neq 0. \tag{3.22}$$

Proof. For $n = 0$, it follows from (3.18) that

$$g_0^{(0)}(\mu,\epsilon) = \frac{2i}{\pi\sqrt{\epsilon_L\mu_L}}\prod_{j=1}^{L}\frac{\mu_j}{\mu_{j-1}} \neq 0.$$

Suppose $n > 0$. Assume that there exists a pair of (μ,ϵ) such that $g_0^{(n)}(\mu,\epsilon) = 0$. Then the solution given by (3.7) with $a_0^{(n)} = 0$ and $b_0^{(n)} = 1$ satisfies

$$\begin{cases} \nabla\cdot\dfrac{1}{\mu}\nabla u + \rho^2\omega^2\epsilon u = 0 \qquad \text{in } \mathbb{R}^2\setminus\overline{D}, \\ \dfrac{\partial u}{\partial\nu}\Big|_+ = o(\rho^{-n}) \quad \text{on } \partial D, \\ u(\mathbf{x}) = H_n^{(1)}(\rho k_0 r)e^{in\theta} \quad \text{for } |\mathbf{x}| = r > 2. \end{cases} \tag{3.23}$$

Let $v(\mathbf{x}) := \lim_{\rho\to 0}\rho^n u(\mathbf{x})$. Then v satisfies

$$\begin{cases} \nabla\cdot\dfrac{1}{\mu}\nabla v = 0 \qquad \text{in } \mathbb{R}^2\setminus\overline{D}, \\ \dfrac{\partial v}{\partial\nu}\Big|_+ = 0 \quad \text{on } \partial D, \\ v(\mathbf{x}) = -\dfrac{i2^n\Gamma(n)}{\pi(\epsilon_0\mu_0)^{n/2}}r^{-n}e^{in\theta} \quad \text{for } |\mathbf{x}| = r > 2, \end{cases} \tag{3.24}$$

which is impossible. Thus $g_0^{(n)}(\mu,\epsilon) \neq 0$, as desired and the proof is complete. □

Equations (3.20) and (3.21) together with the above lemma give us the following proposition.

Proposition 3.2. *For $n \geq 1$, let W_n be defined by (3.4). We have*

$$W_n[\mu,\epsilon,t] = t^{2n}\left(W_n^0[\mu,\epsilon] + \sum_{l=1}^{(N-n)}\sum_{j=0}^{M_{n,l}} W_n^{l,j}[\mu,\epsilon]t^{2l}(\ln t)^j\right) + o(t^{2N}) \tag{3.25}$$

where $t = \rho\omega$, $M_{n,l} := (L+1)(N-n)$ (L being the number of layers), and the coefficients $W_n^0[\mu,\epsilon]$ and $W_n^{l,j}[\mu,\epsilon]$ are independent of t.

To construct an S-vanishing structure of order N at low frequencies, we need to have a pair (μ,ϵ) of the form (3.1) satisfying

$$W_n^0[\mu,\epsilon] = 0, \text{ and } W_n^{l,j}[\mu,\epsilon] = 0 \quad \text{for } 0 \leq n \leq N,\ 1 \leq l \leq (N-n),\ 1 \leq j \leq M_{n,l}. \tag{3.26}$$

We construct numerically such structures for small N in the last section.

4 Enhancement of near cloaking

In this section we show that the S-vanishing structures (after a transformation optics) enhance the near cloaking.

Let (μ, ϵ) be a S-vanishing structure of order N at low frequencies, *i.e.*, (3.26) holds, and it is of the form (3.1). It follows from (2.11), Theorem 2.2, and Proposition 3.2 that

$$A_\infty[\mu, \epsilon, \rho\omega](\theta, \theta') = o(\rho^{2N}) \tag{4.1}$$

uniformly in (θ, θ') if $\rho \le \rho_0$ for some ρ_0.

Let

$$\Psi_{\frac{1}{\rho}}(\mathbf{x}) = \frac{1}{\rho}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2. \tag{4.2}$$

Then we have

$$A_\infty\Big[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\Big] = A_\infty[\mu, \epsilon, \rho\omega]. \tag{4.3}$$

To see this, let u be the solution to

$$\begin{cases} \nabla \cdot \dfrac{1}{(\mu \circ \Psi_{\frac{1}{\rho}})(\mathbf{x})} \nabla u(\mathbf{x}) + \omega^2 (\epsilon \circ \Psi_{\frac{1}{\rho}})(\mathbf{x}) u(\mathbf{x}) = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{B_\rho}, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_\rho, \\ (u - U) \text{ satisfies the outgoing radiation condition,} \end{cases} \tag{4.4}$$

where $U(\mathbf{x}) = e^{ik_0(\cos\theta, \sin\theta)\cdot\mathbf{x}}$. Here B_ρ is the disk of radius ρ centered at 0. Define for $\mathbf{y} = \frac{1}{\rho}\mathbf{x}$

$$\tilde{u}(\mathbf{y}) := \Big(u \circ \Psi_{\frac{1}{\rho}}^{-1}\Big)(\mathbf{y}) = \Big(u \circ \Psi_\rho\Big)(\mathbf{y}) \quad \text{and} \quad \tilde{U}(\mathbf{y}) = \Big(U \circ \Psi_\rho\Big)(\mathbf{y}).$$

Then, we have

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu(\mathbf{y})} \nabla_{\mathbf{y}} \tilde{u}(\mathbf{y}) + \rho^2 \omega^2 \epsilon(\mathbf{y}) \tilde{u}(\mathbf{y}) = 0 \quad \text{in } \mathbb{R}^2, \\ \dfrac{\partial \tilde{u}}{\partial \nu} = 0 \quad \text{on } \partial B_1, \\ (\tilde{u} - \tilde{U}) \text{ satisfies the outgoing radiation condition.} \end{cases} \tag{4.5}$$

From the definition of the far-field pattern A_∞, we get

$$(u - U)(\mathbf{x}) \sim -i e^{-\frac{\pi i}{4}} \frac{e^{ik_0|\mathbf{x}|}}{\sqrt{8\pi k_0 |\mathbf{x}|}} A_\infty\Big[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\Big](\theta, \theta') \quad \text{as } |\mathbf{x}| \to \infty,$$

and

$$(\tilde{u} - \tilde{U})(\mathbf{y}) \sim -i e^{-\frac{\pi i}{4}} \frac{e^{i\rho k_0|\mathbf{y}|}}{\sqrt{8\pi \rho k_0 |\mathbf{y}|}} A_\infty[\mu, \epsilon, \rho\omega](\theta, \theta') \quad \text{as } |\mathbf{y}| \to \infty,$$

where $\mathbf{x} = |\mathbf{x}|(\cos\theta', \sin\theta')$. So, we have (4.3). It then follows from (4.1) that

$$A_\infty\Big[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\Big](\theta, \theta') = o(\rho^{2N}). \tag{4.6}$$

We also obtain from (2.26)

$$S\Big[\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}}, \omega\Big](\theta') = o(\rho^{4N}). \tag{4.7}$$

It is worth emphasizing that $(\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}})$ is a multi-coated structure of radius 2ρ.

We now apply a transformation to the structure $(\mu \circ \Psi_{\frac{1}{\rho}}, \epsilon \circ \Psi_{\frac{1}{\rho}})$ to blow up the small disk of radius ρ. Let us recall the following well-known lemma (see, for instance, [10]).

Lemma 4.1. *Let F be a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ such that $F(\mathbf{x})$ is identity for $|\mathbf{x}|$ large enough. If v is a solution to*

$$\nabla \cdot A\nabla v + \omega^2 qv = 0 \quad in\ \mathbb{R}^2,$$

subject to the outgoing radiation condition, then w defined by $w(\mathbf{y}) = v(F^{-1}(\mathbf{y}))$ satisfies

$$\nabla \cdot (F_* A)\nabla w + \omega^2 (F_* q)w = 0 \quad in\ \mathbb{R}^2, \tag{4.8}$$

together with the outgoing radiation condition, where

$$(F_* A)(\mathbf{y}) = \frac{DF(\mathbf{x})A(\mathbf{x})DF^T(\mathbf{x})}{\det DF(\mathbf{x})} \quad and \quad (F_* q)(\mathbf{y}) = \frac{q(\mathbf{x})}{\det DF(\mathbf{x})} \tag{4.9}$$

with $\mathbf{x} = F^{-1}(\mathbf{y})$ and T being the transpose.

For a small number ρ, let F_ρ be the diffeomorphism defined by

$$F_\rho(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{for } |\mathbf{x}| \geq 2, \\ \Big(\dfrac{3-4\rho}{2(1-\rho)} + \dfrac{1}{4(1-\rho)}|\mathbf{x}|\Big)\dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{for } 2\rho \leq |\mathbf{x}| \leq 2, \\ \Big(\dfrac{1}{2} + \dfrac{1}{2\rho}|\mathbf{x}|\Big)\dfrac{\mathbf{x}}{|\mathbf{x}|} & \text{for } \rho \leq |\mathbf{x}| \leq 2\rho, \\ \dfrac{\mathbf{x}}{\rho} & \text{for } |\mathbf{x}| \leq \rho. \end{cases} \tag{4.10}$$

We then get from (4.6), (4.7), and Lemma 4.1 the main result of this paper.

Theorem 4.2. *If (μ, ϵ) is a S-vanishing structure of order N at low frequencies, then there exists ρ_0 such that*

$$A_\infty\Big[(F_\rho)_*(\mu \circ \Psi_{\frac{1}{\rho}}), (F_\rho)_*(\epsilon \circ \Psi_{\frac{1}{\rho}}), \omega\Big](\theta, \theta') = o(\rho^{2N}), \tag{4.11}$$

and

$$S\Big[(F_\rho)_*(\mu \circ \Psi_{\frac{1}{\rho}}), (F_\rho)_*(\epsilon \circ \Psi_{\frac{1}{\rho}}), \omega\Big](\theta') = o(\rho^{4N}), \tag{4.12}$$

for all $\rho \leq \rho_0$, uniformly in θ and θ'.

We first note the cloaking enhancement is achieved for all the frequencies smaller than ω. This is because (3.15) holds if we replace ω by $\omega' \leq \omega$. Then it is worth comparing (4.11) with (4.1). In (4.1), (μ, ϵ) is a multiply layered structure between radius 1 and 2 in which each layer is filled with an isotropic material, and enhanced near cloaking is achieved for low frequencies $\rho\omega$ with $\rho \leq \rho_0$. On the other hand, in (4.11) the frequency ω does not have to be small. In fact, (4.11) says that for any frequency ω there is a radius ρ which yields the enhanced near cloaking up to $o(\rho^{2N})$. We emphasize that $(F_\rho)_*(\mu \circ \Psi_{\frac{1}{\rho}})$ and $(F_\rho)_*(\epsilon \circ \Psi_{\frac{1}{\rho}})$ are anisotropic permittivity and permeability distributions. It is not clear whether we can achieve enhanced near cloaking at high frequencies by using isotropic layers as done in (4.1).

5 Numerical examples

In this section we provide numerical examples of S-vanishing structures of order N at low frequencies, *i.e.*, structures (μ, ϵ) of the form (3.1) satisfying (3.26). To do so, we use the gradient descent method to minimize over (μ, ϵ) the quantity

$$|W_n^0[\mu, \epsilon]|^2 + \sum_{l=1}^{(N-n)} \sum_{j=0}^{M_{n,l}} |W_n^{l,j}[\mu, \epsilon]|^2, \tag{5.1}$$

where $W_n^0[\mu,\epsilon]$ and $W_n^{l,j}[\mu,\epsilon]$ are the coefficients of $W_n[\mu,\epsilon,t]$ in (3.25). These coefficients are expressed in terms of Bessel functions and their derivatives. We use the symbolic toolbox of MATLAB in order to extract these coefficients.

It is quite challenging numerically to minimize the quantity in (5.1) for large N. In numerical examples we take $N = 2$. In this case one can show through tedious computations that the nonzero leading coefficients of $W_n[\mu,\epsilon,t]$ are as follows:

- $[t^2, t^4, t^4 \log t]$ for $n = 0$, *i.e.*, $W_0^{1,0}$, $W_0^{2,0}$, $W_0^{2,1}$;
- $[t^2, t^4, t^4 \log t]$ for $n = 1$, *i.e.*, W_1^0, $W_1^{1,0}$, $W_1^{1,1}$;
- $[t^4]$ for $n = 2$, *i.e.*, W_2^0.

Results of computations are given in Figure 5.1 when we use 0, 1, 2 layers ($L = 0, 1, 2$). The computed material parameters are $\mu_1 = 0.6$ and $\epsilon = 4/3$ when we use one layer, and $\mu = (1.4905, 0.27594)$, $\epsilon = (1.09271, 1.6702)$ when we use two layers. The first two columns show the distribution of μ and ϵ for $L = 0, 1, 2$. The last column shows the values of coefficients of $W_n[\mu,\epsilon,t]$: $W_0^{1,0}$, $W_0^{2,0}$, $W_0^{2,1}$ indicated by $(0,0,0)$; W_1^0, $W_1^{1,0}$, $W_1^{1,1}$ indicated by $(1,1,1)$; W_2^0 indicated by 2. The figures show that if $L = 0$ then none of the coefficients is zero; if $L = 1$, then the coefficient of t^2 is close to zero; if $L = 2$ all the coefficients are close to zero.

We then compute the scattering coefficients $W_n[\mu,\epsilon,t]$ using the computed (μ,ϵ) for $L = 0, 1, 2$. Figure 5.2 shows the results of computations for $n = 0, \ldots, 4$ and $t = 1, 0.1, 0.01$. It clearly shows that $W_n[\mu,\epsilon,t]$ for $n \leq 2$ gets smaller as the number of layers increases.

6 Conclusion

We have shown near-cloaking examples for the Helmholtz equation. We have designed a cloaking device that achieves enhanced cloaking effect. Any target placed inside the cloaking device has an approximately zero scattering cross section. Such cloaking device is obtained by the blow up using the transformation optics of a multi-coated insulating inclusion. In the numerical example, to minimize the scattering coefficients of the multi-coated insulating inclusion up to the second order, we set 2 layers with different permittivity and permeability properties. The technique presented in this paper can be extended to full Maxwell's equations. This will be the subject of a forthcoming paper.

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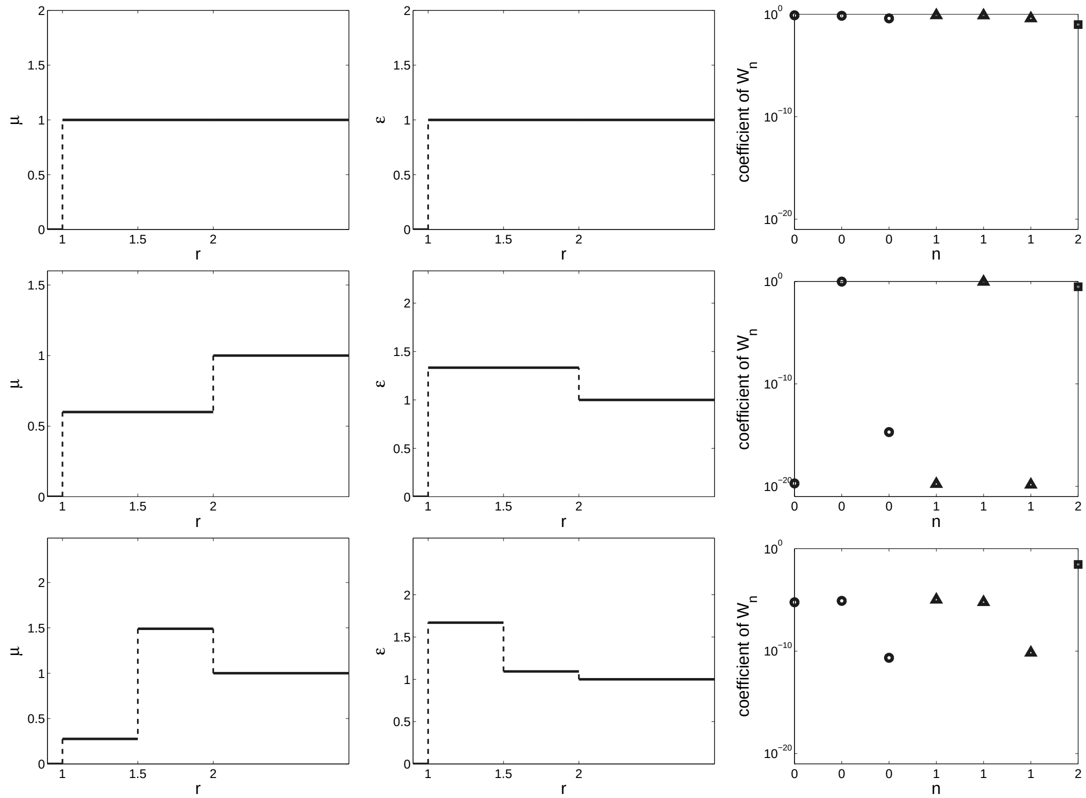

Figure 5.1: Graphs of the first and second columns show profiles of permeability μ and the permittivity ϵ. The third column shows the coefficients of $[t^2, t^4, t^4 \log t]$ in the expansion of W_0 (represented by $(0,0,0)$) and W_1 (represented by $(1,1,1)$), and the coefficient of $[t^4]$ in W_2 (represented by 2).

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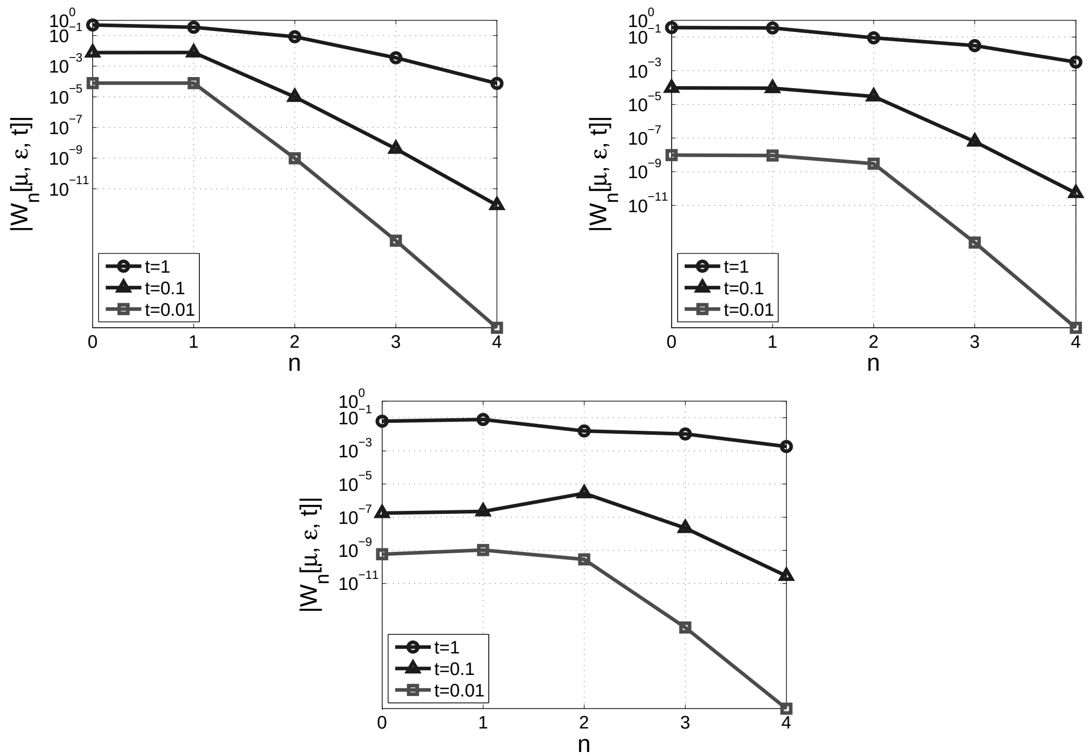

Figure 5.2: The scattering coefficient, $W_n[\mu, \epsilon, t]$, for $n = 0, \ldots, 4$ and $t = 1, 0.1, 0.01$ using the permeability and permittivity profiles computed in Figure 5.1.

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