

Enhancement of Near Cloaking Using Generalized Polarization Tensors Vanishing Structures. Part I: The Conductivity Problem*

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Abstract

The aim of this paper is to provide an original method of constructing very effective near-cloaking structures for the conductivity problem. These new structures are such that their first Generalized Polarization Tensors vanish. We show that this in particular significantly enhances the cloaking effect. We then present some numerical examples of Generalized Polarization Tensors vanishing structures.

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1 Introduction

The central problem in the electrical impedance tomography is to reconstruct the unknown conductivity distribution of a conductor using the boundary measurements, the Dirichlet-to-Neumann (DtN) map. Even if unique identifiability by the DtN map holds for wide class of conductivity distribution ([2, 15, 17] to cite only a few), Greenleaf *et al.* [5] found a (singular) conductivity distribution whose DtN map is exactly the same as the one associated to the constant conductivity distribution. They use a change of variables scheme to create the desired conductivity distribution. They push forward the material constant by the transformation blowing up a point to a ball or a disk. It turned out that the change of variables scheme can be applied to cloaking: Pendry *et al.* [16] and Leonhardt [9] used similar idea to initiate the research on cloaking. Cloaking is to make a target invisible with respect to probing by electromagnetic waves. Since then, extensive work has been produced on cloaking in the context of conductivity and electromagnetism. We refer to [4] (also [3]) for recent development on the cloaking. It is worth mentioning that there is yet another kind of cloaking in which the cloaking region is outside the cloaking device, for instance, anomalous localized resonance [13, 14].

The change of variables based cloaking method uses the singular transformation to boost the material property so that it makes a cloaking region look like a point to outside measurements. However, this transformation induces the singularity of material constants in the transversal direction (also in the tangential direction in two dimensions), which invokes the difficulty both in

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the theory and applications. To overcome this weakness, so called ‘near cloaking’ is naturally considered, which is the regularization or the approximation of singular cloaking. In [8], instead of the singular transformation, Kohn *et al.* use a regular one to push forward the material constant in the conductivity equation, in which a small ball (of radius ρ) is blew up to the cloaking region. They estimate that this near-cloaking can be approximated to the perfect one with the order of ρ^d in the space of dimension d .

The purpose of this paper is to propose a new cancelation technique in order to achieve enhanced near-invisibility. Our approach is based on the multi-coating which cancels the generalized polarization tensors (GPTs) of the cloaking device. We first design a structure coated around an inclusion to have vanishing GPTs of lower orders and show that the order of perturbation due to a small inclusion can be reduced significantly. We then obtain near-cloaking structure by pushing forward the multi-coated structure around a small object via the usual blow-up transformation. For the conductivity equation, we show that the order of near-cloaking is ρ^{2N} using N coatings, which is a significant improvement over ρ^2 approximation obtained in [8]. We give numerical examples for the material parameters and the thickness of the layers for the GPT vanishing structures.

This paper is organized as follows. In the next section we derive the multi-polar expansion of the solution to the conductivity equation which is slightly different from the usual one, and define the (contracted) generalized polarization tensors. In section 3 we characterize the GPT vanishing structures. In section 4 we show that the near-cloaking is enhanced (to ρ^{2N}) if the GPT vanishing structure is used. In section 5 we present some numerical examples of the GPT vanishing structures. We end this paper with a brief conclusion.

Even though we consider only two dimensional conductivity equation in this paper, the same argument can be applied to the equation in three dimensions. The multi-coating technique developed in this paper can be applied to the Helmholtz equation to enhance the near cloaking obtained in [6, 7, 11]. The results for the Helmholtz equation will be presented in the forthcoming paper.

2 Far-field behavior of the solution

Let Ω be a domain in \mathbb{R}^2 containing 0 possibly with multiple components with Lipschitz boundary. For a given harmonic function H in \mathbb{R}^2 , consider

$$\begin{cases} \nabla \cdot (\sigma_0 \chi(\mathbb{R}^2 \setminus \bar{\Omega}) + \sigma \chi(\Omega)) \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \rightarrow \infty, \end{cases} \quad (2.1)$$

where σ_0 and σ are conductivities (positive constants) of $\mathbb{R}^2 \setminus \Omega$ and Ω , respectively. The solution u to (2.1) admits the multipolar expansion [1]

$$(u - H)(\mathbf{x}) = \sum_{|\alpha|, |\beta|=1}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha! \beta!} \partial_{\mathbf{x}}^{\alpha} \Gamma(\mathbf{x}) M_{\alpha\beta} \partial^{\beta} H(0), \quad |\mathbf{x}| \rightarrow \infty, \quad (2.2)$$

where $M_{\alpha\beta} = M_{\alpha\beta}(\Omega, \frac{\sigma}{\sigma_0})$ are the Generalized Polarization Tensors (GPTs) associated with the inclusion Ω and the conductivity contrast $\frac{\sigma}{\sigma_0}$ and $\Gamma(\mathbf{x})$ is the fundamental solution of the Laplacian, *i.e.*,

$$\Gamma(\mathbf{x}) = \frac{1}{2\pi} \ln |\mathbf{x}|.$$

Here and throughout this paper $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are multi-indices and $|\alpha| = \alpha_1 + \alpha_2$.

We seek an expression of the multipolar expansion which is slightly different from (2.2). For multi-indices α, β with $|\alpha| = n, |\beta| = n$, define (a_{α}^n) and (b_{β}^n) by

$$\sum_{|\alpha|=n} a_{\alpha}^n \mathbf{x}^{\alpha} = r^n \cos n\theta \quad \text{and} \quad \sum_{|\beta|=n} b_{\beta}^n \mathbf{x}^{\beta} = r^n \sin n\theta, \quad (2.3)$$

and define

$$M_{mn}^{cc} = \sum_{|\alpha|=m} \sum_{|\beta|=n} a_\alpha^m a_\beta^n M_{\alpha\beta}, \quad (2.4)$$

$$M_{mn}^{cs} = \sum_{|\alpha|=m} \sum_{|\beta|=n} a_\alpha^m b_\beta^n M_{\alpha\beta}, \quad (2.5)$$

$$M_{mn}^{sc} = \sum_{|\alpha|=m} \sum_{|\beta|=n} b_\alpha^m a_\beta^n M_{\alpha\beta}, \quad (2.6)$$

$$M_{mn}^{ss} = \sum_{|\alpha|=m} \sum_{|\beta|=n} b_\alpha^m b_\beta^n M_{\alpha\beta}. \quad (2.7)$$

We call these coefficients the contracted GPTs.

Using the expansion of $\ln |\mathbf{x} - \mathbf{y}|$, we have

$$\sum_{|\alpha|=n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\mathbf{x}}^\alpha \Gamma(\mathbf{x}) \mathbf{y}^\alpha = \frac{-1}{2\pi n} \left[\frac{\cos n\theta_{\mathbf{x}}}{r_{\mathbf{x}}^n} r_{\mathbf{y}}^n \cos n\theta_{\mathbf{y}} + \frac{\sin n\theta_{\mathbf{x}}}{r_{\mathbf{x}}^n} r_{\mathbf{y}}^n \sin n\theta_{\mathbf{y}} \right], \quad n > 0, \quad (2.8)$$

where $\mathbf{x} = r_{\mathbf{x}}(\cos \theta_{\mathbf{x}}, \sin \theta_{\mathbf{x}})$ and $\mathbf{y} = r_{\mathbf{y}}(\cos \theta_{\mathbf{y}}, \sin \theta_{\mathbf{y}})$, which is valid if $|\mathbf{x}| \gg 1$ and $\mathbf{y} \in \partial\Omega$, and hence

$$\frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\mathbf{x}}^\alpha \Gamma(\mathbf{x}) = \frac{-1}{2\pi n} \left[a_\alpha^n \frac{\cos n\theta_{\mathbf{x}}}{r_{\mathbf{x}}^n} + b_\alpha^n \frac{\sin n\theta_{\mathbf{x}}}{r_{\mathbf{x}}^n} \right]. \quad (2.9)$$

If the harmonic function H admits the expansion

$$H(\mathbf{x}) = H(0) + \sum_{n=1}^{\infty} r^n (a_n^c(H) \cos n\theta + a_n^s(H) \sin n\theta)$$

with $\mathbf{x} = (r \cos \theta, r \sin \theta)$, then we have

$$\frac{\partial^\alpha H(0)}{\alpha!} = a_n^c(H) a_\alpha^n + a_n^s(H) b_\alpha^n. \quad (2.10)$$

Plugging (2.9) and (2.10) into (2.2) we have the following formula

$$\begin{aligned} (u - H)(\mathbf{x}) &= - \sum_{m=1}^{\infty} \frac{\cos m\theta}{2\pi m r^m} \sum_{n=1}^{\infty} (M_{mn}^{cc} a_n^c(H) + M_{mn}^{cs} a_n^s(H)) \\ &\quad - \sum_{m=1}^{\infty} \frac{\sin m\theta}{2\pi m r^m} \sum_{n=1}^{\infty} (M_{mn}^{sc} a_n^c(H) + M_{mn}^{ss} a_n^s(H)) \quad \text{as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \quad (2.11)$$

We emphasize that (2.11) is valid even if σ (the conductivity of Ω) is not a constant but a variable. In the next section we will design a conductivity distribution σ of Ω so that $M_{mn}^{cc} = M_{mn}^{cs} = M_{mn}^{sc} = M_{mn}^{ss} = 0$ for all $m, n \leq N$ for a given integer N . We call such a conductivity distribution *GPT-vanishing structure* or *coating* of order N .

3 GPT vanishing structures

To obtain GPT-vanishing structures, we use a disc with multiple coatings. The idea comes from Hashin's neutral inclusion which is a disc with a single coating [12]. The special property of the neutral inclusion is that it does not perturb the uniform fields outside the inclusion, which is equivalent to the first order polarization tensors of the inclusion vanishing. In other words, Hashine's neutral inclusion is a GPT-vanishing structure of order 1.

Let Ω be a disk of radius r_1 . For a positive integer N , let $0 < r_{N+1} < r_N < \dots < r_1$ and define

$$A_j := \{r_{j+1} < r \leq r_j\}, \quad j = 1, 2, \dots, N. \quad (3.1)$$

Let $A_0 = \mathbb{R}^2 \setminus \Omega$ and $A_{N+1} = \{r \leq r_{N+1}\}$. Set σ_j to be the conductivity of A_j for $j = 1, 2, \dots, N+1$, and $\sigma_0 = 1$. Let

$$\sigma = \sum_{j=0}^{N+1} \sigma_j \chi(A_j). \quad (3.2)$$

Let $M_{mn}^{cc}[\sigma]$, etc, denote the (contracted) GPTs associated with σ (and Ω). Because of the symmetry of the disc, one can easily see that

$$M_{mn}^{cs}[\sigma] = M_{mn}^{sc}[\sigma] = 0 \quad \text{for all } m, n, \quad (3.3)$$

$$M_{mn}^{cc}[\sigma] = M_{mn}^{ss}[\sigma] = 0 \quad \text{if } m \neq n, \quad (3.4)$$

and

$$M_{nn}^{cc}[\sigma] = M_{nn}^{ss}[\sigma] \quad \text{for all } n. \quad (3.5)$$

Let $M_n = M_{nn}^{cc}$, $n = 1, 2, \dots$, for the simplicity of notation.

To compute M_k , we look for solutions u_k to

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \mathbb{R}^2 \quad (3.6)$$

of the form

$$u_k(\mathbf{x}) = a_j^{(k)} r^k \cos k\theta + \frac{b_j^{(k)}}{r^k} \cos k\theta \quad \text{in } A_j, \quad j = 0, 1, \dots, N+1, \quad (3.7)$$

with $a_0^{(k)} = 1$ and $b_{N+1}^{(k)} = 0$. Then u_k is the solution to (2.1) with $H(\mathbf{x}) = r^k \cos k\theta$, and satisfies

$$(u - H)(\mathbf{x}) = \frac{b_0^{(k)}}{r^k} \cos k\theta \quad \text{as } |\mathbf{x}| \rightarrow \infty. \quad (3.8)$$

Hence, we have

$$M_k = -2\pi k b_0^{(k)}. \quad (3.9)$$

We observe that the solution to (3.6) is given by

$$u_k(r e^{i\theta}) = \left(r^k - \frac{M_k}{2\pi k r^k} \right) \cos k\theta \quad \text{in } A_0. \quad (3.10)$$

Thus

$$r^k \pm \frac{M_k}{2\pi k r^k} \neq 0 \quad \text{for any } r \geq 2,$$

since otherwise $u_k(\mathbf{x}) = 0$ or $u_k(\mathbf{x}) = \text{constant}$ in $|\mathbf{x}| \leq r$ for some $r \geq 2$. Thus we have

$$r^k - \frac{M_k}{2\pi k r^k} > 0 \quad \text{and} \quad r^k + \frac{M_k}{2\pi k r^k} > 0 \quad \text{for any } r \geq 2,$$

and hence

$$|M_k| \leq 2\pi k 2^{2k} \quad \text{for all } k. \quad (3.11)$$

The transmission conditions on the interface $\{r = r_j\}$ are given by

$$a_j^{(k)} r_j^k + \frac{b_j^{(k)}}{r_j^k} = a_{j-1}^{(k)} r_j^k + \frac{b_{j-1}^{(k)}}{r_j^k}, \quad (3.12)$$

$$\sigma_j \left(a_j^{(k)} r_j^{k-1} - \frac{b_j^{(k)}}{r_j^{k+1}} \right) = \sigma_{j-1} \left(a_{j-1}^{(k)} r_j^{k-1} - \frac{b_{j-1}^{(k)}}{r_j^{k+1}} \right). \quad (3.13)$$

Thus we have

$$\begin{bmatrix} a_j^{(k)} \\ b_j^{(k)} \end{bmatrix} = \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\ (\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} a_{j-1}^{(k)} \\ b_{j-1}^{(k)} \end{bmatrix}, \quad (3.14)$$

and hence

$$\begin{bmatrix} a_{N+1}^{(k)} \\ 0 \end{bmatrix} = \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\ (\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} 1 \\ b_0^{(k)} \end{bmatrix}. \quad (3.15)$$

Let

$$P^{(k)} = \begin{bmatrix} p_{11}^{(k)} & p_{12}^{(k)} \\ p_{21}^{(k)} & p_{22}^{(k)} \end{bmatrix} := \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1})r_j^{-2k} \\ (\sigma_j - \sigma_{j-1})r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix}. \quad (3.16)$$

Then, we have

$$b_0^{(k)} = -\frac{p_{21}^{(k)}}{p_{22}^{(k)}}. \quad (3.17)$$

Note that $M_k = 0$ if and only if $P^{(k)}$ is an upper triangular matrix, *i.e.*, $p_{21}^{(k)} = 0$. Let

$$\lambda_j := \frac{\sigma_j - \sigma_{j-1}}{\sigma_j + \sigma_{j-1}}, \quad j = 1, \dots, N. \quad (3.18)$$

We now characterize GPT-vanishing structures.

Proposition 3.1. *If there are non-zero constants $\lambda_1, \dots, \lambda_{N+1}$ ($|\lambda_j| < 1$) and $r_1 > \dots > r_{N+1} > 0$ such that*

$$\prod_{j=1}^{N+1} \begin{bmatrix} 1 & \lambda_j r_j^{-2k} \\ \lambda_j r_j^{2k} & 1 \end{bmatrix} \text{ is an upper triangular matrix for } k = 1, 2, \dots, N, \quad (3.19)$$

then (Ω, σ) , given by (3.1), (3.2), and (3.18), is a GPT-vanishing structure with $M_k = 0$ for $k \leq N$. More generally, if that there are non-zero constants $\lambda_1, \lambda_2, \lambda_3, \dots$ ($|\lambda_j| < 1$) and $r_1 > r_2 > r_3 > \dots$ such that r_n converges to a positive number, say $r_\infty > 0$, and

$$\prod_{j=1}^{\infty} \begin{bmatrix} 1 & \lambda_j r_j^{-2k} \\ \lambda_j r_j^{2k} & 1 \end{bmatrix} \text{ is an upper triangular matrix for every } k, \quad (3.20)$$

then (Ω, σ) , given by (3.1), (3.2), and (3.18), is a GPT-vanishing structure with $M_k = 0$ for all k .

Note that (3.19) and (3.20) are nonlinear equations. For example, if $N = 3$, equation (3.19) is simple and reduces to

$$\lambda_1 r_1^{2k} + \lambda_2 r_2^{2k} + \lambda_3 r_3^{2k} + \lambda_1 \lambda_2 \lambda_3 r_1^{2k} r_2^{-2k} r_3^{2k} = 0, \quad k = 1, 2, 3. \quad (3.21)$$

It is quite easy to show that it admits infinitely many solutions $\lambda_1, \lambda_2, \lambda_3$ ($|\lambda_j| < 1$) and $r_1 > r_2 > r_3 > 0$. However, as N gets larger, solving analytically equation (3.19) seems too complicated, and even proving existence of solutions to (3.19) or (3.20) seems to be quite challenging. We present a simple numerical method to find the GPT-vanishing structures in Section 5, which is also important from a practical point of view. These numerical evidences show us that (3.19) has solutions, even though we are not able to prove it.

4 Near-cloaking using GPT-vanishing structures

In this section we achieve enhanced near-cloaking by using GPT-vanishing structures.

Let B_r be the disk centered at the origin with radius r and $B = B_1$. Let A_j , $j = 0, 1, \dots, N+1$ be defined by (3.1) with $r_1 = 2$ and $r_{N+1} \geq 1$. Let σ be the conductivity distribution defined by (3.2). For a given domain Ω , we denote the DtN map of Ω with the conductivity σ as $\Lambda_\Omega[\sigma]$, which is given by

$$\Lambda_\Omega[\sigma](f) = \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} \quad (4.1)$$

where u is the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

Proposition 4.1. *Let σ be a conductivity profile on \mathbb{R}^2 defined as (3.2) and for a small positive constant let*

$$\Psi_{\frac{1}{\rho}}(\mathbf{x}) = \frac{1}{\rho} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2. \quad (4.3)$$

Then the following holds for $k = 0, 1, 2, \dots$ and $s > 0$

$$\left(\Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_s}[1] \right) (e^{\pm ik\theta}) = \frac{2ks^{-1}\rho^{2k}M_k[\sigma]}{2\pi ks^{2k} - M_k[\sigma]\rho^{2k}} e^{\pm ik\theta}. \quad (4.4)$$

Before proving Proposition 4.1, let us make a few remarks. If σ is a GPT-vanishing structure of order N , i.e., $M_k = 0$ for all $k \leq N$, then (4.4) shows that

$$\left(\Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_s}[1] \right) (e^{\pm ik\theta}) = 0 \quad \text{for all } |k| \leq N. \quad (4.5)$$

In other words, $\Lambda_B[\sigma \circ \Psi_{\frac{1}{\rho}}]$ and $\Lambda_B[1]$ cannot be distinguished by slowly oscillating Dirichlet data. Moreover, the complete GPT-vanishing structure (3.20) is achieved, then

$$\left(\Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_s}[1] \right) (e^{\pm ik\theta}) = 0 \quad \text{for all } k, \quad (4.6)$$

which would yield the perfect cloaking.

Proof of Proposition 4.1. If u is the solution to

$$\begin{cases} \nabla \cdot (\sigma \circ \Psi_{\frac{1}{\rho}}) \nabla u = 0 & \text{in } B_s, \\ u = f & \text{on } |\mathbf{x}| = s, \end{cases} \quad (4.7)$$

then $\tilde{u} := u \circ \Psi_\rho$ satisfies

$$\begin{cases} \nabla \cdot \sigma \nabla \tilde{u} = 0 & \text{in } B_{\frac{s}{\rho}}, \\ \tilde{u} = f \circ \Psi_\rho & \text{on } |\mathbf{y}| = \frac{s}{\rho}. \end{cases} \quad (4.8)$$

Moreover, we have

$$\frac{\partial \tilde{u}}{\partial \nu} \Big|_{|\mathbf{y}| = \frac{s}{\rho}}(\mathbf{y}) = \rho \frac{\partial u}{\partial \nu} \Big|_{|\mathbf{x}| = s}(\rho \mathbf{y}).$$

Therefore,

$$\Lambda_{B_{\frac{s}{\rho}}}[\sigma](f \circ \Psi_\rho) = \rho \Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}](f) \circ \Psi_\rho. \quad (4.9)$$

To compute $\Lambda_{B_{\frac{s}{\rho}}}[\sigma]$, set $f(e^{i\theta}) = e^{ik\theta}$ for a fixed $k \in \mathbb{N}$ ($k \geq 0$). Then, the solution \tilde{u}_k to (4.8) is given by

$$\tilde{u}_k(\mathbf{x}) = a_j^{(k)} r^k e^{ik\theta} + \frac{b_j^{(k)}}{r^k} e^{ik\theta} \quad \text{in } A_j, \quad j = 0, 1, \dots, N+1,$$

with

$$a_0^{(k)} s^k \rho^{-k} + b_0^{(k)} s^{-k} \rho^k = 1 \quad (4.10)$$

and $b_{N+1}^{(k)} = 0$. From (3.14) and (3.16), we get

$$\begin{bmatrix} a_{N+1}^{(k)} \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11}^{(k)} & p_{12}^{(k)} \\ p_{21}^{(k)} & p_{22}^{(k)} \end{bmatrix} \begin{bmatrix} a_0^{(k)} \\ b_0^{(k)} \end{bmatrix},$$

and hence

$$b_0^{(k)} - \frac{p_{21}^{(k)}}{p_{22}^{(k)}} a_0^{(k)} = -\frac{M_k[\sigma]}{2\pi k} a_0^{(k)}.$$

Substituting it into (4.10), we have

$$b_0^{(k)} = \frac{-M_k[\sigma] \rho^k s^k}{2\pi k s^{2k} - M_k[\sigma] \rho^{2k}}.$$

and we obtain

$$\begin{aligned} \Lambda_{B_{\frac{s}{\rho}}}[\sigma](f \circ \Psi_\rho) &= \frac{\partial \tilde{u}_k}{\partial \nu} \Big|_{|y|=\frac{s}{\rho}} \\ &= k \frac{\rho}{s} (a_0^{(k)} s^k \rho^{-k} - b_0^{(k)} s^{-k} \rho^k) e^{ik\theta} \\ &= k \frac{\rho}{s} (a_0^{(k)} s^k \rho^{-k} + b_0^{(k)} s^{-k} \rho^k) e^{ik\theta} - 2k b_0^{(k)} \frac{\rho^{k+1}}{s^{k+1}} e^{ik\theta} \\ &= k \frac{\rho}{s} e^{ik\theta} - 2k b_0^{(k)} \frac{\rho^{k+1}}{s^{k+1}} e^{ik\theta}. \end{aligned}$$

From (4.9) we have

$$\begin{aligned} \Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}](f) &= \frac{k}{s} e^{ik\theta} - 2k b_0^{(k)} \frac{\rho^k}{s^{k+1}} e^{ik\theta} \\ &= \Lambda_{B_s}[1](e^{ik\theta}) + \frac{2k s^{-1} \rho^{2k} M_k[\sigma]}{2\pi k s^{2k} - M_k[\sigma] \rho^{2k}} e^{ik\theta}. \end{aligned}$$

Thus we get (4.4). The same argument works for $k < 0$ as well. The proof is complete. \square

If f admits Fourier expansion $f(e^{i\theta}) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}$, then we have from (4.4)

$$\left(\Lambda_B[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_B[1] \right)(f) = \sum_{k \in \mathbb{Z}} \frac{2|k| \rho^{2|k|} s^{-1} M_{|k|}[\sigma]}{2\pi |k| s^{2|k|} - \rho^{2|k|} M_{|k|}[\sigma]} f_k e^{ik\theta}. \quad (4.11)$$

For the GPT-vanishing structure of order N , we have $M_k = 0$ for all $k \leq N$ and from (3.11) that

$$\rho^{2k} |M_k| \leq 2\pi k (2\rho)^{2k} \quad \text{for all } k > N.$$

Since $\rho \ll 1$, we obtain

$$\rho^{2k} |M_k| \leq C \rho^{2N+2} \quad \text{for all } k \in \mathbb{N} \quad (4.12)$$

for some constant C independent of k , and hence

$$\left\| \Lambda_{B_s}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_s}[1] \right\| \leq C \rho^{2N+2}, \quad (4.13)$$

where the norm is the operator norm of $H^{1/2}(\partial B_s)$ into $H^{-1/2}(\partial B_s)$. Here $H^{1/2}(\partial B_s)$ is the usual Sobolev space of order $1/2$ on ∂B_s and $H^{-1/2}(\partial B_s)$ is its dual.

We mention that the solutions for GPT-vanishing structure exist numerically even when $\sigma_{N+1} = 0$, which is equivalent to prescribe the insulating condition on the boundary of the inner core. Some examples with $\sigma_{N+1} = 0$ and $r_{N+1} = 1$ are given in the following section. In such structures, the conductivity $\sigma \circ \Psi_{\frac{1}{\rho}}$ is 0 in $|\mathbf{x}| \leq \rho$, fluctuates in $\rho < |\mathbf{x}| < 2\rho$, and is 1 in $|\mathbf{x}| > 2\rho$. In order to have near-cloaking device in $1 < |\mathbf{x}| < 2$ with the core part $|\mathbf{x}| \leq 1$ insulated as was considered in [8], one may use transformation as was done in [4, 16].

For a given domain Ω and a subdomain $B \subset \Omega$, we denote the DtN map of Ω with the conductivity σ as $\Lambda_{\Omega, B}[\sigma]$, which is given by

$$\Lambda_{\Omega, B}[\sigma](f) = \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} \quad (4.14)$$

where u is the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \Omega \setminus \overline{B}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \\ u = f & \text{on } \partial \Omega. \end{cases} \quad (4.15)$$

By (4.4), we have

$$\left(\Lambda_{B_2, B_\rho}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_2, \emptyset}[1] \right)(f) = \sum_{k \in \mathbb{Z}} \frac{|k| \left(\frac{\rho}{2}\right)^{2|k|} M_{|k|}}{2\pi |k| - \left(\frac{\rho}{2}\right)^{2|k|} M_{|k|}} f_k e^{ik\theta},$$

and hence by (4.12)

$$\left\| \Lambda_{B_2, B_\rho}[\sigma \circ \Psi_{\frac{1}{\rho}}] - \Lambda_{B_2, \emptyset}[1] \right\| \leq C \rho^{2N+2}. \quad (4.16)$$

We define the transformation $F_\rho : B_2 \rightarrow B_2$ by

$$F_\rho(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{for } \frac{3}{2} \leq |\mathbf{x}| \leq 2, \\ \left(\frac{3-3\rho}{3-2\rho} + \frac{1}{3-2\rho} |\mathbf{x}| \right) \frac{\mathbf{x}}{|\mathbf{x}|} & \text{for } \rho \leq |\mathbf{x}| \leq \frac{3}{2}, \\ \frac{\mathbf{x}}{\rho} & \text{for } |\mathbf{x}| \leq \rho. \end{cases} \quad (4.17)$$

Then one can easily see that

$$\Lambda_{B_2, B_\rho}[\sigma \circ \Psi_{\frac{1}{\rho}}] = \Lambda_{B_2, B_1} \left[(F_\rho)_*(\sigma \circ \Psi_{\frac{1}{\rho}}) \right], \quad (4.18)$$

where

$$(F_\rho)_*(\sigma \circ \Psi_{\frac{1}{\rho}}) = \frac{(DF_\rho)(\sigma \circ \Psi_{\frac{1}{\rho}})(DF_\rho)^T}{|\det(DF_\rho)|} \circ F_\rho^{-1}.$$

We then obtain the following theorem, which is the main result of this paper, from (4.16).

Theorem 4.2. *There exists a constant C independent of ρ such that*

$$\left\| \Lambda_{B_2, B_1} \left[(F_\rho)_*(\sigma \circ \Psi_{\frac{1}{\rho}}) \right] - \Lambda_{B_2, \emptyset}[1] \right\| \leq C \rho^{2N+2}. \quad (4.19)$$

5 Numerical examples

In this section we present some numerical examples of $\sigma_1, \dots, \sigma_{N+1}$ and $r_1 > \dots > r_{N+1}$ satisfying (3.19).

We fix N and $r_j = 2 - \frac{j-1}{N}$ for $j = 1, \dots, N+1$. We then solve the following equation for $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{N+1})$

$$\boldsymbol{\sigma} \mapsto M_k[\boldsymbol{\sigma}] = 0 \quad \text{for } k = 1, \dots, N. \quad (5.1)$$

Since (5.1) is a nonlinear equation, we solve it iteratively. Initially, $\sigma_j^{(0)}$ is set to be $2^{(-1)^j}$, $j = 1, \dots, N+1$. We iteratively modify $\boldsymbol{\sigma}^{(i)} = (\sigma_1^{(i)}, \dots, \sigma_{N+1}^{(i)})$ as

$$\boldsymbol{\sigma}^{(i+1)} = \boldsymbol{\sigma}^{(i)} - A_i^\dagger \mathbf{b}^{(i)},$$

where A_i^\dagger is the pseudoinverse of

$$A_i := \left. \frac{\partial(M_1, \dots, M_N)}{\partial \boldsymbol{\sigma}} \right|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(i)}},$$

and

$$\mathbf{b}^{(i)} = \left. \begin{bmatrix} M_1 \\ \vdots \\ M_N \end{bmatrix} \right|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(i)}}.$$

Example 1. Figure 5.1 shows computational results of the conductivity σ for $N = 3, 6, 9$. It clearly shows that the larger N is, the more σ fluctuates. One interesting thing to observe is that σ_{N+1} takes values of 1.9695, 0.9791, 1.0029 for $N = 3, 6, 9$, respectively: they are getting closer to 1 which is the conductivity of the exterior part.

Example 2. Here we see what happens if the conductivity σ_{N+1} of the core D is fixed with a value different from that of exterior, namely 1. We set N (the number of layers) to be 9 and $\sigma_{10} = 5$ and 0.2. The numerical results are illustrated in Figure 5.2. Conductivity profiles fluctuate more drastically and M_k takes greater values than those in Example 1.

Example 3. This example is for the near-cloaking for which the boundary of the core ($r = 1$) is insulated, and hence the conductivity of the core is set to be 0. Figure 5.3 shows the results of computation when $N = 3, 6$: the conductivity fluctuates on coatings near the core. When $N = 3$, the maximal conductivity is 5.5158 and the minimum conductivity is 0.4264. When $N = 6$, they are 11.6836 and 0.1706.

6 Conclusion

We have obtained new near-cloaking examples for the conductivity problem. We have shown that the GPT-vanishing structures can be used to enhance the near-cloaking. The GPTs up to the N -th order can be canceled using $(N+1)$ layers with different conductivity parameters. To make the numerical procedure simple, we have assumed that the layers are concentric disks centered at the origin with specific radii. The numerical simulations show that the GPT-vanishing structure exists with these radii. As mentioned in Introduction, the multi-coating technique can be applied to enhance near-cloaking for the Helmholtz equation.

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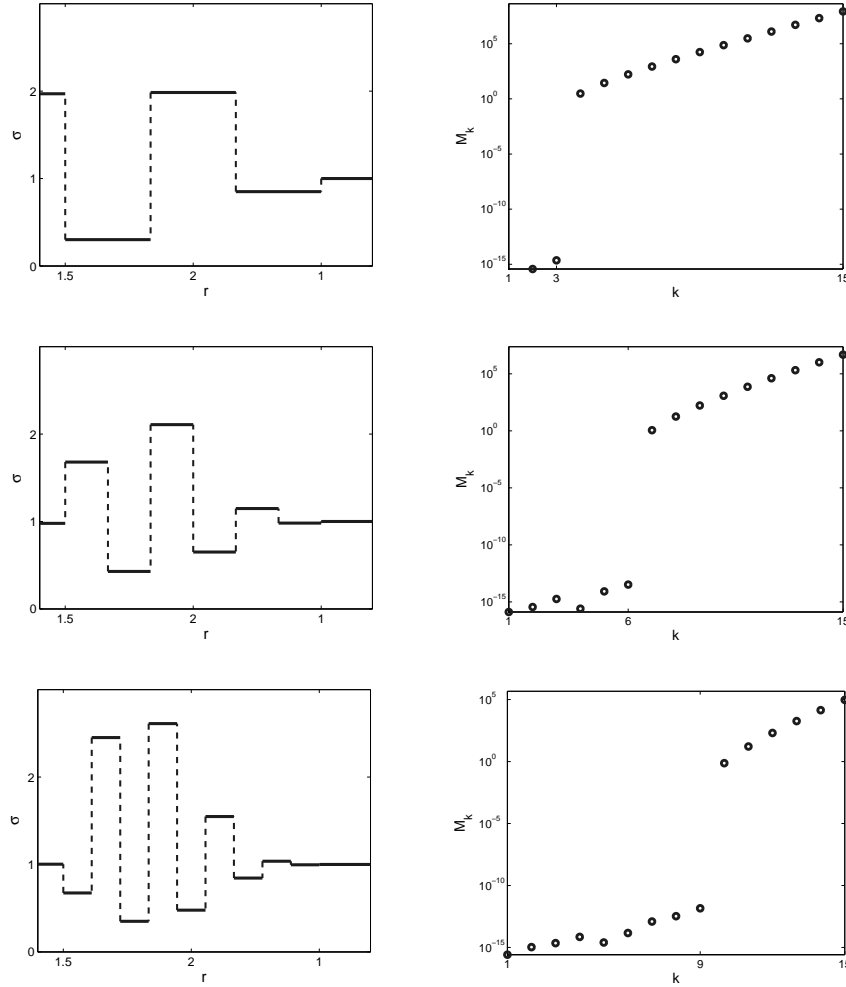


Figure 5.1: Graphs on the left column show the conductivity profile σ such that $M_{kk}^{cc} = 0$ for $k \leq N$ and plots on the right column show the values of $M_k = M_{kk}^{cc}$ for $N = 3, 6, 9$.

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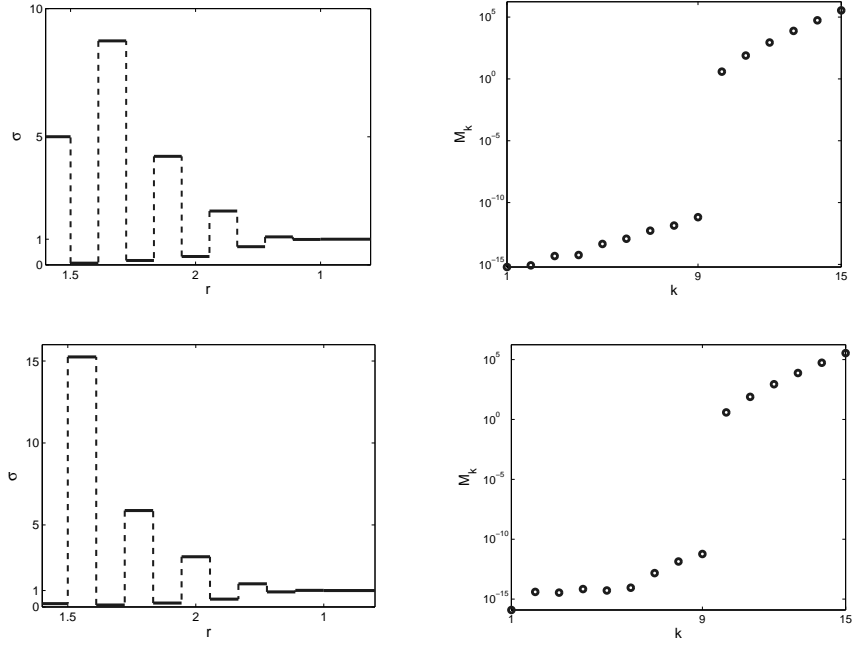


Figure 5.2: Conductivity profile for $N = 9$ when σ_{10} is fixed. The first row corresponds to $\sigma_{10} = 5$ and the second one to $\sigma_{10} = 0.2$.

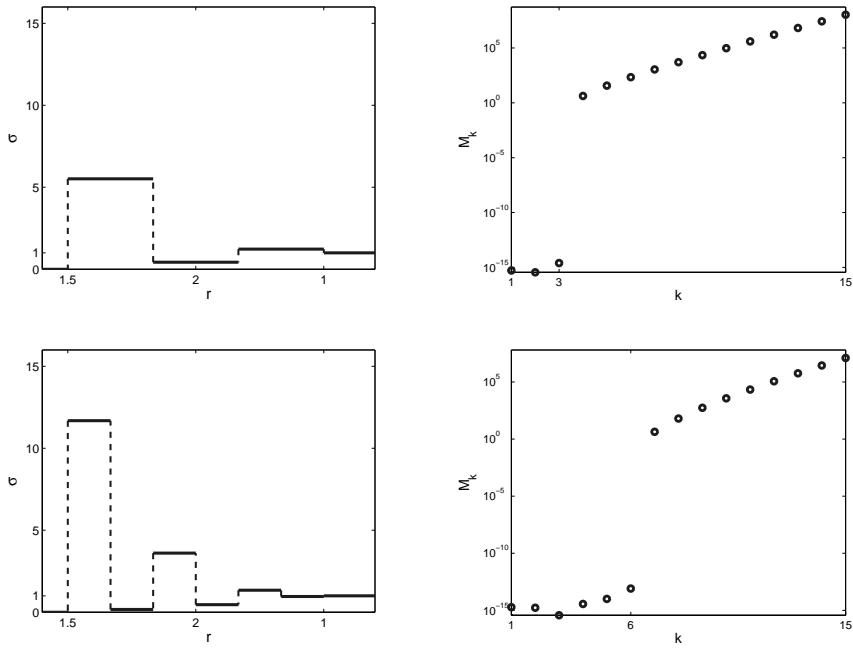


Figure 5.3: Conductivity profile for the near-cloaking with interior conductivity is 0. The first row is when $N = 3$ and the second one for $N = 6$.

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