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Enhancing complex-network synchronization

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Abstract. – Heterogeneity in the degree (connectivity) distribution has been shown to suppress synchronization in networks of symmetrically coupled oscillators with uniform coupling strength (unweighted coupling). Here we uncover a condition for enhanced synchronization in weighted networks with asymmetric coupling. We show that, in the optimum regime, synchronizability is solely determined by the average degree and does not depend on the system size and the details of the degree distribution. In scale-free networks, where the average degree may increase with heterogeneity, synchronizability is drastically enhanced and may become positively correlated with heterogeneity, while the overall cost involved in the network coupling is significantly reduced as compared to the case of unweighted coupling.

Networks of dynamical elements serve as natural models for a variety of systems, with examples ranging from cell biology to epidemiology to the Internet [1]. Many of these complex networks display common structural features, such as the small-world [2] and scale-free properties [3]. Small-world networks (SWNs) exhibit short average distance between nodes and high clustering [2], while scale-free networks (SFNs) are characterized by an algebraic, highly heterogeneous distribution of degrees (number of links per node) [3]. The interplay between structure and dynamics has attracted a great deal of attention, especially in connection with the problem of synchronization of coupled oscillators [4–10]. The ability of a network to synchronize is generally enhanced in both SWNs and random SFNs as compared to regular lattices [11]. This enhancement was previously believed to be due to the decrease of the average distance between oscillators. Recently, it was shown that random networks with strong heterogeneity in the degree distribution, such as SFNs, are much more difficult to synchronize than random homogeneous networks [7], even though the former display smaller average path length [12]. This suggests that, although structurally advantageous [13], the scale-free property may be dynamically detrimental. Here we present a solution to this problem.

A basic assumption of most previous works is that the oscillators are coupled symmetrically and with the same coupling strength. Under the assumption of symmetric coupling, the maximum synchronizability may be indeed achieved when the coupling strength is uniform [14].

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But to get better synchronizability the couplings are not necessarily symmetrical. Many realistic networks are actually directed [1] and weighted [15]. In particular, the communication capacity of a node is likely to saturate when the degree becomes large.

In this letter, we study the impact that asymmetry and saturation of connection strength have on the synchronization dynamics on complex networks. As a prime example, we consider complete synchronization of linearly coupled identical oscillators, namely

$$\dot{x_i} = f(x_i) - \sigma \sum_{j=1}^{N} A_{ij} \big[h(x_i) - h\big(x_j\big)\big], \quad i = 1, \ldots N, \tag{1}$$

where $f = f(x)$ describes the dynamics of each individual oscillator, $h = h(x)$ is the output function, and σ is the overall coupling strength. Matrix $A = (A_{ij})$ is the adjacency matrix of the underlying network of couplings, where $A_{ij} = w_{ij}$ if there is a link of strength $w_{ij} > 0$ from node j to node i, and 0 otherwise. The network is unweighted (weighted) if all (not all) the nonzero elements of A are equal to each other, and the network is undirected (directed) if matrix A is symmetric (asymmetric).

Equation (1) can be written as $\dot{x_i} = f(x_i) - \sigma \sum_{j=1}^{N} G_{ij} h(x_j)$, where $G_{ij} = \delta_{ij} \sum_{j=1}^{N} A_{ij} - A_{ij}$ are the elements of the coupling matrix $G = (G_{ij})$. In the case of symmetrically coupled oscillators with uniform coupling strength, A is a symmetric binary matrix and G is the usual (symmetric) Laplacian matrix $L = (L_{ij})$ [16]. For $G_{ij} = L_{ij}$, heterogeneity in the degree distribution suppresses synchronization in various classes of complex networks [7]. In order to enhance the synchronizability of heterogeneous networks, we propose to scale the coupling strength by the degrees of the nodes. For specificity, we consider

$$G_{ij} = L_{ij}/k_i^{\beta}, \tag{2}$$

where k_i is the degree of node i and β is a tunable parameter. The underlying network associated with the Laplacian matrix L is undirected and unweighted. But with the introduction of the weights in eq. (2), the network of couplings becomes not only weighted but also directed because the resulting matrices G and A are in general asymmetric. This is a special kind of directed network where the number of *in*-links is equal to the number of *out*-links in each node, and the directions are encoded in the strengths of in- and out-links. These networks are, nevertheless, more general than the unweighted networks considered in refs. [6,7]. (Although beyond the scope of this work, even more general networks can be considered within the same framework.) We say that the network or coupling is weighted when $\beta \neq 0$ and unweighted when $\beta = 0$.

The variational equations governing the linear stability of a synchronized state $\{x_i(t) = s(t), \forall i\}$ can be diagonalized into N blocks of the form $\dot{\eta} = [Df(s) - \alpha Dh(s)]\eta$, where $\alpha = \sigma\lambda_i$, and λ_i are the eigenvalues of the coupling matrix G, ordered as $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$ (see below). The largest Lyapunov exponent $\Lambda(\alpha)$ of this equation can be regarded as a master stability function, which determines the linear stability of the synchronized state [17]: the synchronized state is stable if $\Lambda(\sigma\lambda_i) < 0$ for $i = 2, \ldots N$. (The eigenvalue λ_1 corresponds to a mode parallel to the synchronization manifold.) For many widely studied oscillatory systems [6, 17], the master stability function $\Lambda(\alpha)$ is negative in a finite interval (α_1, α_2). Therefore, the network is synchronizable for some σ when the eigenratio $R = \lambda_N/\lambda_2$ satisfies $R < \alpha_2/\alpha_1$. The ratio α_2/α_1 depends only on the dynamics (f, h, and s), while the eigenratio R depends only on the coupling matrix G. The problem of synchronization is then reduced to the analysis of eigenvalues of the coupling matrix [6]: the smaller the eigenratio R the more synchronizable the network.

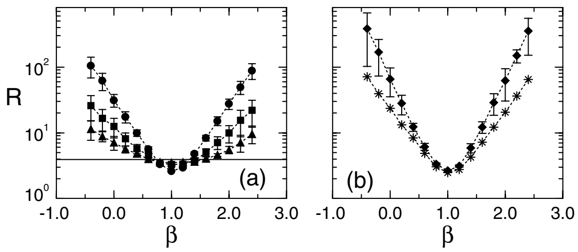

Fig. 1 – Eigenratio R as a function of β: (a) random SFNs with $\gamma = 3$ (●), $\gamma = 5$ (■), $\gamma = 7$ (▲), and $\gamma = \infty$ (solid line), for $k_{\min} = 10$; (b) networks with expected scale-free sequence (♦) for $\gamma = 3$ and $\tilde{k}_{\min} = 10$, and growing SFNs (★) for $\gamma = 3$ and $m = 10$. Each curve is the result of an average over 50 realizations for $N = 1024$. The error bars for growing SFNs are smaller than the size of the symbols.

Here we show that, as a function of β, the eigenratio R has a global minimum at $\beta = 1$. In large sufficiently random networks, our analysis shows that the eigenratio at $\beta = 1$ is primarily determined by the average degree $\bar{k}$ of the network and *does not* depend on the degree distribution and system size, in sharp contrast with the case of unweighted coupling ($\beta = 0$), where synchronization is strongly suppressed as the heterogeneity or number of oscillators is increased. Furthermore, we show that the total cost involved in the network coupling is significantly reduced for $\beta = 1$ when compared to $\beta = 0$. As a result, structural robustness [13] and improved synchronizability can coexist in scale-free and other heterogeneous networks. We observe that the case $\beta = 1$ has been considered previously in the context of pulse-coupled oscillators because, in the case of pulse oscillators, heterogeneity in the incoming signal desynchronizes the system [8, 9]. However, in the context of this work, where complete synchronization is always a solution of eq. (1), previous works on the stability of completely synchronized states have focused on the case of unweighted coupling (see ref. [10] for an exception). To the best of our knowledge, this is the first systematic study of complete synchronization as a function of β.

In matrix notation, eq. (2) can be written as $G = D^{-\beta}L$, where $D = \mathrm{diag}\{k_1, k_2, \dots k_N\}$ is the diagonal matrix of degrees. From the identity $\det(D^{-\beta}L - \lambda I) = \det(D^{-\beta/2}LD^{-\beta/2} - \lambda I)$, valid for any λ, we have that the spectrum of eigenvalues of matrix G is equal to the spectrum of a symmetric matrix defined as $H = D^{-\beta/2}LD^{-\beta/2}$. As a result, all the eigenvalues of matrix G are real. Moreover, because H is positive semidefinite, all the eigenvalues are nonnegative and, because the rows of G have zero sum, the smallest eigenvalue λ_1 is always zero, as assumed above. If the network is connected, $\lambda_2 > 0$ for any finite β. For spectral properties of unweighted complex networks, see refs. [18–21].

We first examine the dependence on β. Physically, we expect the synchronizability to be strongly influenced by the strength of the input coupling at each oscillator. When $\beta < 1$, oscillators with larger degree are more strongly coupled than oscillators with smaller degree. When $\beta > 1$, the opposite happens. Because (α_1, α_2) is finite, for the network to synchronize, the overall coupling strength σ must be large enough to synchronize the least coupled oscillators and small enough to synchronize the most coupled ones (*i.e.*, the synchronizability of these oscillators is expected to be primarily determined by the modes associated with the eigenvalues λ_2 and λ_N, respectively). Therefore, for both $\beta < 1$ and $\beta > 1$, some oscillators are more strongly coupled than others, and the ability of the network to synchronize is limited by those oscillators that are least and most strongly coupled. We then expect the network to achieve maximum synchronizability at $\beta = 1$. In fig. 1, we show the numerical verification of this hypothesis on three different models of SFNs, defined as follows:

A) *Random SFNs* [22]. Each node is assigned to have a number k_i of "half-links" according to the probability distribution $P(k) \sim k^{-\gamma}$, where γ is a scaling exponent and $k \geq k_{\min}$. The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. In the limit $\gamma = \infty$, all the nodes have the same degree $k = k_{\min}$.

B) *Networks with expected scale-free sequence* [20]. The network is generated from a sequence $\tilde{k}_1, \tilde{k}_2, \ldots \tilde{k}_N$, where $\max_i \tilde{k}_i^2 < \sum_i \tilde{k}_i$, so that links are independently assigned to each pair of nodes (i, j) with probability $p_{ij} = \tilde{k}_i \tilde{k}_j / \sum_i \tilde{k}_i$. When the expected degrees $\tilde{k}_i \geq \tilde{k}_{\min}$ follow the distribution $P(\tilde{k}) \sim \tilde{k}^{-\gamma}$, we have a network with expected scale-free sequence.

C) *Growing SFNs* [23]. We start with a fully connected network with m nodes and at each time step a new node with m links is added to the network. Each new link is connected to a node i in the network with probability $\Pi_i \sim (1-p)k_i + p$, where $0 \leq p \leq 1$ is a tunable parameter. For large degrees, the scaling exponent of the resulting network is $\gamma = 3 + p[m(1-p)]^{-1}$. For $p = 0$, we recover the Barabási-Albert model [3].

As shown in fig. 1, a pronounced minimum for the eigenratio R at $\beta = 1$ is observed in each case. A similar minimum for R at $\beta = 1$ is also observed in many other models of complex networks, including the Watts-Strogatz model [2] of SWNs. The only exception is the class of homogeneous networks, where all the nodes have the same degree k. In this case, the weights can be factored out and R is independent of β, as shown in fig. 1(a) for random homogeneous networks with $k = 10$ (solid line).

In heterogeneous networks, the synchronizability is significantly enhanced when the coupling is suitably weighted, as shown in fig. 2 for SFNs with $\beta = 1$. In SFNs, the heterogeneity (variance) of the degree distribution increases as the scaling exponent γ is reduced. When the coupling is unweighted ($\beta = 0$), the eigenratio R increases with heterogeneity, but the eigenratio does not increase and may even decrease with heterogeneity when the coupling is weighted ($\beta = 1$), as shown in figs. 2(a)-(c). The enhancement is particularly large for small γ, where the networks are highly heterogeneous (note the logarithmic scale in fig. 2). The networks become more homogeneous as γ is increased. In the limit $\gamma = \infty$, random SFNs converge to random homogeneous networks with the same degree $k_{\min}$ for all nodes (fig. 2(a)), while networks with expected scale-free sequence converge to Erdős-Rényi random networks [24], which have links assigned with the same probability between each pair of nodes (fig. 2(b)). As one can see from fig. 2(b), the synchronizability is strongly enhanced even in the relatively homogeneous Erdős-Rényi model; such an enhancement occurs also in growing networks (fig. 2(c)). (In SWNs of pulse-coupled oscillators, the speed for effective synchronization to be achieved is also enhanced at $\beta = 1$ [9].) Surprisingly, for $\beta = 1$, the eigenratio R turns out to be well approximated by the corresponding eigenratio of random homogeneous networks with the same average degree (figs. 2(a)-(c)). Therefore, for $\beta = 1$, the variation of the eigenratio R with the heterogeneity in figs. 2(a) and (b) is mainly due to the variation of the average degree of the networks, which increases as the scaling exponent γ is reduced. Moreover, the eigenratio R appears to be independent of the system size for $\beta = 1$ in large SFNs, in contrast to the unweighted case, where R increases strongly with the number of oscillators (figs. 2(d)-(f)).

We now present an approximation for the eigenratio R that supports and extends our numerical observations. In what follows we focus on the case $\beta = 1$. Based on results of ref. [20] for random networks with arbitrary expected degrees, which includes important

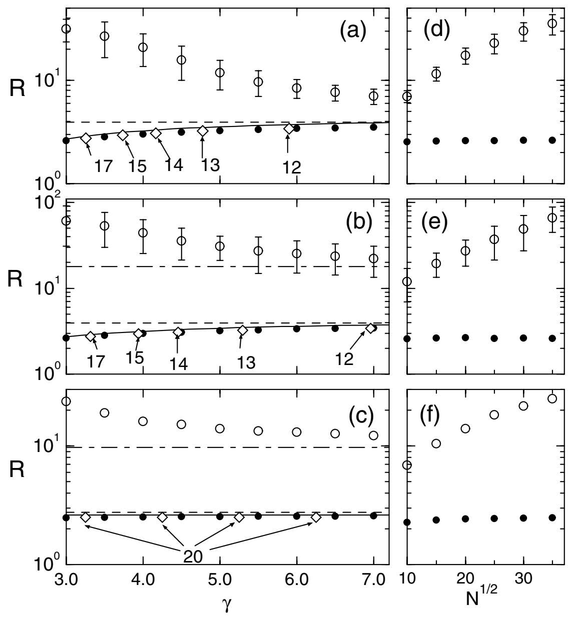

Fig. 2 – (a)-(c) Eigenratio R as a function of the scaling exponent γ: (a) random SFNs, (b) networks with expected scale-free sequence, and (c) growing SFNs, for $\beta = 1$ (•) and $\beta = 0$ (∘). The other parameters are the same as in fig. 1. Also plotted are the bound in eq. (4) (solid line) and R at $\gamma = \infty$ for $\beta = 1$ (dashed line) and $\beta = 0$ (dot-dashed line). The ⋄ symbols correspond to random homogeneous networks with the same average degree of the corresponding SFNs, as indicated in the figure. (d)-(f) R as a function of the system size for $\gamma = 3$ and the models in (a)-(c), respectively. The legend is the same as in (a)-(c). The error bars for growing SFNs and for $\beta = 1$ are smaller than the size of the symbols.

SFNs, we get

$$\max\{1-\lambda_2, \lambda_N - 1\} = [1 + \mathrm{o}(1)]\frac{2}{\sqrt{\tilde{k}}}\,, \tag{3}$$

where $\tilde{k}$ is the average expected degree. This result is rigorous for networks with a given expected degree sequence $\tilde{k}_1, \tilde{k}_2, \dots \tilde{k}_N$, as defined in the model B) above. The assumption for this result is $\tilde{k}_{\min} \equiv \min_i \tilde{k}_i$ to be large as compared to $\sqrt{\tilde{k}}\ln^3 N$, but our numerical simulations suggest that this assumption can be released considerably because eq. (3) is observed to hold for $\tilde{k}_{\min}$ as small as $2\sqrt{\tilde{k}}$. Having released this assumption, from eq. (3) we have the following explicit upper bound for the eigenratio in large networks:

$$R \le \frac{1 + 2/\sqrt{\tilde{k}}}{1 - 2/\sqrt{\tilde{k}}}\,. \tag{4}$$

Therefore, the eigenratio is bounded by a function of the average degree, which does not depend on the system size, in agreement with the results in figs. 2(d)-(f). (This also agrees with the apparent size independence of the synchronization threshold observed in ref. [10] for simulations on SFNs of coupled quadratic maps.) Moreover, we expect R to approach the upper bound in eq. (4) because the semicircle law holds and the spectrum is symmetric around 1 for $\tilde{k}_{\min} \gg \sqrt{\tilde{k}}$ in the thermodynamical limit [20, 21]. This prediction is confirmed

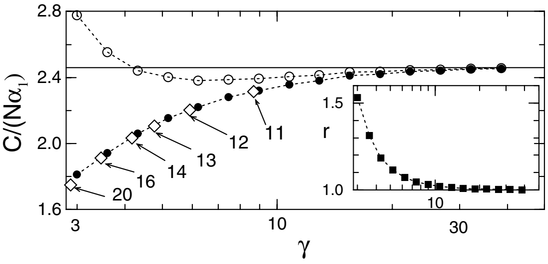

Fig. 3 – Normalized cost as a function of the scaling exponent γ for random SFNs with $\beta = 1$ (•) and $\beta = 0$ (∘), and for random homogeneous networks with the same average degree (◊). The solid line corresponds to $\gamma = \infty$. Inset: ratio $r = C_0/C_1$ of the cost for $\beta = 0$ (C_0) and $\beta = 1$ (C_1) as a function of γ. The other parameters are the same as in fig. 1. The error bars are smaller than the size of the symbols.

numerically under much weaker conditions, as shown in figs. 2(a)-(c), where one can see a remarkable agreement between the approximate and exact values of R for all three models of SFNs. Since the bound in eq. (4) does not depend on the degree distribution, this result also explains the agreement between the eigenratio for weighted SFNs (figs. 2(a)-(c), •) and the eigenratio for random homogeneous networks with the same average degree (figs. 2(a)-(c), ◊). A similar agreement is observed in many other complex networks.

We also consider the influence of degree correlation and clustering. Our extensive numerical computation on the models of refs. [25, 26] shows that the eigenratio R generally increases with increasing clustering and assortativity in correlated networks. However, a pronounced global minimum for R as a function of β is always observed at $\beta = 1$. In addition, weighted networks at $\beta = 1$ are much more insensitive to the effects of correlation than their unweighted counterparts. The same tendency is observed in the growing model with aging of ref. [27], which has nontrivial clustering and correlation. All together, these suggest that our results are quite robust and expected to hold on real-world networks as well.

Now we address the important problem of the *cost* involved in the connections of the network. The cost C is naturally defined as the total strength of all directed links, *i.e.*, $C = \sigma_{\min} \sum_i k_i^{1-\beta}$, where $\sigma_{\min} = \alpha_1/\lambda_2$ is the minimum overall coupling strength for the network to synchronize. Strikingly, in heterogeneous networks, the cost for $\beta = 1$ is considerably smaller than the cost for $\beta = 0$ (fig. 3). Therefore, cost reduction is another important advantage of the weighted coupling. Moreover, the cost for $\beta = 1$ is well approximated by the cost for random homogeneous networks with the same average degree k, as indicated in fig. 3. In this case, for large k we have $C/(N\alpha_1) = 1/\lambda_2 \approx 1/(1 - 2/\sqrt{k})$ and the cost is reduced as k is increased.

In summary, we have introduced a model of weighted networks with asymmetric coupling which, we believe, can serve as a paradigm to address various issues regarding dynamics on complex networks. Within this model, we have shown that suitably weighted networks display significantly improved synchronizability and lower cost. As compared to the unweighted case ($\beta = 0$), synchronizability is significantly enhanced not only for $\beta = 1$ but also for a wide interval around $\beta = 1$ where the eigenratio R is nearly constant (see fig. 1). Similar enhancement is expected for networks with some degree of random heterogeneity in the connection strengths and, in particular, for networks where different nodes are normalized at different values of β according to a bounded distribution concentrated around $\beta = 1$. An important implication of our findings is that weighted SFNs can exhibit enhanced complete synchronization.

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