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Research Article

Enhancing Steady-State Entanglement Generated by a Nondegenerate Three-Level Laser with Thermal Reservoir

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We analyze a nondegenerate three-level cascade laser with an open cavity and coupled to a two-mode thermal reservoir employing the stochastic differential equations for atomic operators associated with the normal ordering. Applying the large-time approximation scheme, we obtain the solutions for the corresponding equations of evolution of the expectation values of atomic operators. Furthermore, employing the resulting solutions, we studied the photon as well as cavity atomic-state entanglement amplification of the cavity radiation.

1. Introduction

Three-level cascade lasers have received considerable attention over the years in connection with the strong correlation between the modes of the generated radiation that leads to a substantial degree of nonclassical features [1–24]. When a three-level atom in a cascade configuration makes a transition from top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called a degenerate three-level atom; otherwise, it is called nondegenerate. The squeezing and statistical properties of the light produced by three-level lasers when the atoms are initially prepared in a coherent superposition of the top and bottom levels or when these levels are coupled by a strong coherent light have been studied by several authors [10, 25–33]. The authors have found that these quantum optical systems can generate squeezed light under certain conditions.

Moreover, Abebe [9] has studied the squeezing and entanglement properties of the light generated by a coherently driven nondegenerate three-level laser possessing an open cavity and coupled to a two-mode vacuum reservoir. He has obtained that the maximum quadrature squeezing of the light generated by the laser, operating below threshold, and found to be 50% below the vacuum-state level. Similarly, Abebe [24] also studied the quantum properties of the light produced by a coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir. In this study, he found that the maximum quadrature squeezing is 43% below the vacuum-state level, which is slightly less than the result found on open cavity [9]. He also found that the photon numbers of a two-mode light beams are correlated. The analysis of [24] showed that the intracavity quadrature squeezing is enhanced due to the driven coherent light. It is found that the squeezing and entanglement in the two-mode light are directly related. As a result, an increase in the degree of squeezing directly implies an increase in the degree of entanglement. This shows that whenever there is squeezing in the two-mode light, there exists entanglement in the system.

Continuous variable entanglement is the quantum feature that can be generated by the correlated photons produced in the quantum optical system as studied by [13–16, 20, 21]. Quantum entanglement, which has no classical counterpart, is the key ingredient for quantum information as a resource of tasks like quantum memories for quantum computers, quantum information and communication, quantum dense coding, quantum teleportation for secure communication, and atom clocks and interferometers for quantum sensing and metrology to name a few of its only fascinating applications [17, 19, 22, 34, 35]. In general, degree of entanglement degrades as it interacts with the environment. On the other hand, the efficiency of quantum information processing highly depends on the degree of entanglement. As a consequence, it is desirable to generate strongly entangled continuous variable states which can survive from environmental noise. Moreover, many schemes have been proposed to produce a strong entangled light from three-level laser using different techniques theoretically [12–16, 24]. Some authors have studied the effect of thermal reservoir on the quantum properties of light generated by the three-level laser [15]. Gashu et al. have found that the effect of thermal light in the laser cavity decreases the degree of entanglement [13].

In this paper, we study the quantum properties of the cavity light beams produced by a coherently driven nondegenerate three-level laser with an open cavity and coupled to a two-mode thermal reservoir via a single-port mirror. First, we obtain the master equation for a coherently driven nondegenerate three-level atom with the cavity modes and the quantum Langevin equations for the cavity mode operators. Employing the master equation and the large-time approximation scheme, we drive the equations of evolution of the expectation values of the atomic operators. Hence, we determine the steady-state solutions of the resulting equations of evolution. Here, we carry out our calculation by putting the noise operators associated with the two-mode thermal reservoir in normal order. Applying the steady-state solutions of the resulting equations of evolution along with the quantum Langevin equations, we obtain the photon and cavity atomic-state entanglement.

2. The Model

We consider here the case in which N nondegenerate three-level atoms in cascade configuration are available in an open cavity. We denote the top, intermediate, and bottom levels of the three-level atom by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively. As shown in Figure 1 for nondegenerate cascade configuration, when the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b_k$ to $|c\rangle_k$, two photons with different frequencies are emitted. The emission of light when the atoms make the transition from the top level to the intermediate level is light mode a and the emission of light when the atoms make the transition from the intermediate level to the bottom level is light mode b. We assume that the cavity mode a is at resonance with transition $|a\rangle_k \rightarrow |b\rangle_k$ and the cavity mode b is at resonance with the transition $|b\rangle_k \rightarrow |c\rangle_k$, with top and bottom levels of the three-level atom coupled by coherent light. The interaction of the three-level atoms with the cavity mode radiation and classical field is described by the Hamiltonian of the following form [1, 25].

$$\hat{H} = \hat{H}_0 + \hat{H}_I, \tag{1}$$

where the first term can be rewritten, assuming $\hbar = 1$ for simplicity sake only, as follows:

$$\hat{H}_0 = \sum_{j=a,b,c} \omega_j |j\rangle\langle j| + \omega_{ab} a\wedge^{\dagger} \hat{a} + \omega_{bc} b\wedge^{\dagger} \hat{b}, \tag{2}$$

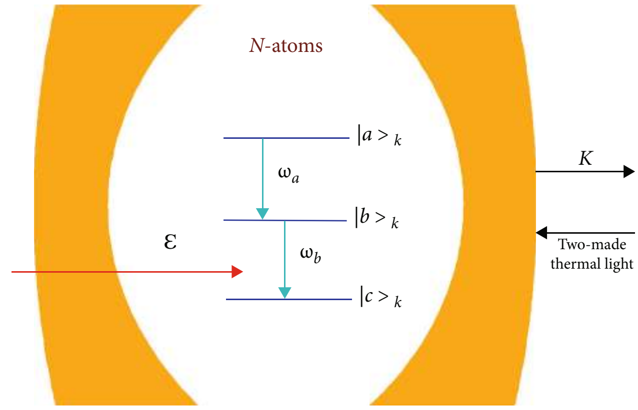

Figure 1: Schematic representation of a nondegenerate three-level laser coupled to a two-mode thermal reservoir. Here, ε is the amplitude of the driving coherent light that couples the top and bottom levels of the atom. And also κ is the cavity damping constant and it is assumed the same for both transitions. The top, intermediate, and bottom levels of the three-level atom are denoted by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively, where $k = 1, 2, \cdots, N$ are the number of atoms inside the cavity. When the atom makes a transition from level $|a\rangle_k \rightarrow |b\rangle_k$ and from levels $|b\rangle_k \rightarrow |c\rangle_k$, two photons with different frequencies (ω_a and ω_b) are emitted (color online).

in which ω_a, ω_b, and ω_c indicate the frequencies of the three states of the atom. On the other hand, $\omega_{ab} = \omega_a - \omega_b$ and $\omega_{bc} = \omega_b - \omega_c$ represent the frequencies of the allowed dipole atomic transitions and are resonant with the annihilation operators of the two cavity modes $\hat{a}$ and $\hat{b}$.

The interaction of a nondegenerate three-level atom with two-mode cavity radiation can be expressed in the interaction picture with the rotating wave approximation (RWA) by the Hamiltonian of the following form [1, 33].

$$\hat{H}_S = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a} - a\wedge^{\dagger}\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - b\wedge^{\dagger}\hat{\sigma}_b^k\right] + \frac{i\Omega}{2}\left[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\right], \tag{3}$$

where g is the coupling constant between the atom and cavity mode a or b; $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b, and

$$\begin{aligned} \hat{\sigma}_a^k &= |b\rangle_{k\,k}\langle a|, \\ \hat{\sigma}_b^k &= |c\rangle_{k\,k}\langle b|, \\ \hat{\sigma}_c^k &= |c\rangle_{k\,k}\langle a| \end{aligned} \tag{4}$$

are the lowering atomic operators. Here, $\Omega = 2\varepsilon\lambda$, in which ε, considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light and the three-level atom.

2.1. The Master Equation. The quantum analysis of the interaction of a system such as a cavity mode or a three-level atom with the external environment is a relatively complex problem. The external environment, usually referred to as a reservoir, can be thermal light, ordinary, or squeezed vacuum. We are interested in the dynamics of the system and this is describable by the master equation or quantum Langevin equations. Here, we obtain the above set of dynamical equations for a cavity mode coupled to a thermal reservoir via a single-port mirror. The resulting equations are easily adaptable to the case when the external environment is either a thermal or a vacuum reservoir. We then focus our study when the cavity mode is coupled to a thermal reservoir. A system coupled with a thermal reservoir can be described by the Hamiltonian [2, 25].

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{5}$$

where $\hat{H}_S$ is the Hamiltonian of the system and $\hat{H}_{SR}$ describes the interaction between the system and the reservoir. Suppose $\hat{\chi}(t)$ is the density operator for the system and the reservoir, then the equation of evolution of this density operator is given by the following [2, 25]:

$$\frac{d}{dt}\hat{\chi}(t) = -i\left[\hat{H}_S(t) + \hat{H}_{SR}, \hat{\chi}(t)\right]. \tag{6}$$

We are interested in the quantum dynamics of the system alone. Hence, taking into account Equation (6), we see that the density operator for the system, also known as the reduced density operator

$$\hat{\rho}(t) = Tr_R\hat{\chi}(t), \tag{7}$$

evolves in time according to

$$\frac{d}{dt}\hat{\rho}(t) = -i\left[\hat{H}(t), \hat{\rho}(t)\right] - iTr\left[\hat{H}_{SR}(t), \hat{\chi}(t)\right], \tag{8}$$

in which Tr_R indicates the trace over the reservoir variables only. On the other hand, a formal solution of Equation (6) can be written as follows [2, 25]:

$$\hat{\chi}(t) = \hat{\chi}(0) - i\int_0^t \left[\hat{H}_S\left(t'\right) + \hat{H}_{SR}\left(t'\right), \hat{\chi}\left(t'\right)\right]dt'. \tag{9}$$

In order to obtain mathematically manageable $\hat{\chi}(t')$ by some approximately valid expression, then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. This can be achieved by letting a beam of thermal light (or light in a vacuum state) of constant intensity fall continuously on the system. Moreover, we decouple the system and reservoir density operators, so that

$$\hat{\chi}\left(t'\right) = \hat{\rho}\left(t'\right)\hat{R}. \tag{10}$$

Therefore, with the aid of this, one can rewrite Equation (9) as follows:

$$\hat{\chi}(t) = \hat{\rho}(0)\hat{R} - \int_0^t \left[\hat{H}_S\left(t'\right) + \hat{H}_{SR}\left(t'\right), \hat{\rho}\left(t'\right)\hat{R}\right]dt'. \tag{11}$$

Now on substituting Equation (11) into Equation (8), there follows

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = &-i\left[\hat{H}_S(t), \hat{\rho}(t)\right] - i\left[\left\langle\hat{H}_{SR}(t)\right\rangle_R, \hat{\rho}(0)\right]\\ &-\int_0^t \left[\left\langle\hat{H}_{SR}(t)\right\rangle_R, \left[\hat{H}_S\left(t'\right), \hat{\rho}\left(t'\right)\right]\right]dt'\\ &-\int_0^t Tr_R\left[\hat{H}_{SR}(t), \left[\hat{H}_{SR}\left(t'\right), \hat{\rho}\left(t'\right)\hat{R}\right]\right]dt',\end{aligned} \tag{12}$$

where S and R refer to the system and reservoir variables and $\hat{\rho}(0)$ represents the radiation initially in the cavity. The interaction of Hamiltonian for N nondegenerate three-level atoms coupled to thermal reservoir is as follows [25]:

$$\hat{H}_{SR}(t) = i\sum_j \lambda_j\left[\hat{\sigma}_a^{\dagger k}\hat{a}_j - \hat{a}_j^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b}_j - \hat{b}_j^\dagger\hat{\sigma}_b^k\right], \tag{13}$$

where λ_j is the coupling constant for the j^{th} mode of the reservoir and $(\hat{a}_j, \hat{b}_j)$ are the annihilation operators of the two-mode thermal reservoir. Now, using the density operator of the thermal reservoir [32],

$$\hat{R} = \sum_{n=0}^{\infty}\frac{\bar{n}_{\text{th}}^n}{\left(1+\bar{n}_{\text{th}}\right)^{n+1}}|n\rangle\langle n|, \tag{14}$$

one can easily check that

$$\begin{aligned}\left\langle\hat{a}_j\right\rangle_R &= \sum_{n=0}^{\infty}\frac{\bar{n}_{\text{th}}^n}{\left(1+\bar{n}_{\text{th}}\right)^{n+1}}Tr_R\left(|n\rangle\langle n|\hat{a}_j\right)\\ &= \sum_{n=0}^{\infty}\frac{\bar{n}_{\text{th}}^n}{\left(1+\bar{n}_{\text{th}}\right)^{n+1}}\langle n \mid n-2\rangle = 0.\end{aligned} \tag{15}$$

Following the same procedure, we obtain the following:

$$\begin{aligned}\left\langle\hat{b}_j\right\rangle_R &= \left\langle\hat{a}_j^2\right\rangle = \left\langle\hat{b}_j^2\right\rangle = \left\langle\hat{a}_j^\dagger\hat{b}_j\right\rangle_R\\ &= 0, \left\langle\hat{b}_j^\dagger\hat{a}_j\right\rangle_R = \left\langle\hat{a}_j\hat{b}_j\right\rangle_R = \left\langle\hat{b}_j\hat{a}_j\right\rangle_R = 0.\end{aligned} \tag{16}$$

In view of these results, we see that

$$\left\langle\hat{H}_{SR}(t)\right\rangle_R = 0. \tag{17}$$

Therefore, the second and the third commutation relations in Equation (12) are zero. This confirms that the thermal light in the cavity does not directly contribute to the master equation. In addition, applying the commutation relation $[\hat{a}_j, \hat{a}_j^\dagger] = 1$, we then note that $\langle\hat{a}_j\hat{a}_j^\dagger\rangle = \bar{n}_{\text{th}} + 1$ and $\langle\hat{a}_j^\dagger\hat{a}_j\rangle = \bar{n}_{\text{th}}$, where $\bar{n}_a = \bar{n}_b = \bar{n}_{\text{th}}$ is the mean photon number for the thermal reservoir. As a result, solving the remaining terms by following the standard approach yields the following equation [2]:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) &= g\Big[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - a\wedge^{\dagger}\hat{\sigma}_a^{k}\hat{\rho} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - b\wedge^{\dagger}\hat{\sigma}_b^{k}\hat{\rho} \\ &\quad - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\rho}a\wedge^{\dagger}\hat{\sigma}_a^{k} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} + \hat{\rho}b\wedge^{\dagger}\hat{\sigma}_b^{k}\Big] \\ &\quad + \frac{\Omega}{2}\Big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\sigma}_c^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^{k}\Big] \\ &\quad + \frac{\gamma}{2}\bar{n}_{\mathrm{th}}\Big[2\hat{\sigma}_a^{\dagger k}\hat{\rho}\hat{\sigma}_a^{k} - \hat{\sigma}_a^{k}\hat{\sigma}_a^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{k}\hat{\sigma}_a^{\dagger k}\Big] \\ &\quad + \frac{\gamma}{2}(\bar{n}_{\mathrm{th}}+1)\Big[2\hat{\sigma}_a^{k}\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^{k}\Big] \\ &\quad + \frac{\gamma}{2}\bar{n}_{\mathrm{th}}\Big[2\hat{\sigma}_b^{\dagger k}\hat{\rho}\hat{\sigma}_b^{k} - \hat{\sigma}_b^{k}\hat{\sigma}_b^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{k}\hat{\sigma}_b^{\dagger k}\Big] \\ &\quad + \frac{\gamma}{2}(\bar{n}_{\mathrm{th}}+1)\Big[2\hat{\sigma}_b^{k}\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^{k}\Big],\end{aligned} \tag{18}$$

where $\gamma_a = \gamma_b = \gamma = 2h\lambda^2$, considered to be the same for levels $|a\rangle$ and $|b\rangle$, is the spontaneous emission decay constant.

2.2. Quantum Langevin Equations. We recall that the laser cavity is coupled to a two-mode thermal reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the thermal reservoir in normal order. Thus, the noise operators will not have any effect on the dynamics of the cavity mode operators [2, 9, 24]. We can therefore drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as follows:

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i\big[\hat{a}, \hat{H}\big], \tag{19a}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i\big[\hat{b}, \hat{H}\big], \tag{19b}$$

where κ is the cavity damping constant. Then, in view of Equation (3), the quantum Langevin equations for cavity mode operators $\hat{a}$ and $\hat{b}$ turn out to be

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^{k}, \tag{20a}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^{k}. \tag{20b}$$

Following the procedure described in Reference [9, 24], for N number of atoms, we have the following:

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a, \tag{21a}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b. \tag{21b}$$

The sum of Equations (21a) and (21b) yields

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m}. \tag{22}$$

2.3. Stochastic Differential Equations. Here, we seek to obtain the equations of evolution of the expectation values of the atomic operators by applying the master equation and the large-time approximation scheme. To this end, making use of the master equation described by Equation (18) for any operator $\hat{A}$ and the fact that

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}(t)}{dt}\hat{A}\right), \tag{23}$$

along with the master (Equation (18)), one can readily establish that

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{\sigma}_a^{k}\right\rangle &= g\left[\left\langle\hat{\eta}_b^{k}\hat{a}\right\rangle - \left\langle\hat{\eta}_a^{k}\hat{a}\right\rangle + \left\langle b\wedge^{\dagger}\hat{\sigma}_c^{k}\right\rangle\right] + \frac{\Omega}{2}\left\langle\hat{\sigma}_b^{\dagger k}\right\rangle \\ &\quad - \frac{\gamma}{2}(3\bar{n}_{\mathrm{th}}+2)\left\langle\hat{\sigma}_a^{k}\right\rangle,\end{aligned} \tag{24a}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{\sigma}_b^{k}\right\rangle &= g\left[\left\langle\hat{\eta}_c^{k}\hat{b}\right\rangle - \left\langle a\wedge^{\dagger}\hat{\sigma}_c^{k}\right\rangle - \left\langle\hat{\eta}_b^{k}\hat{b}\right\rangle\right] - \frac{\Omega}{2}\left\langle\hat{\sigma}_a^{\dagger k}\right\rangle \\ &\quad - \frac{\gamma}{2}(3\bar{n}_{\mathrm{th}}+2)\left\langle\hat{\sigma}_b^{k}\right\rangle,\end{aligned} \tag{24b}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{\sigma}_c^{k}\right\rangle &= g\left[\left\langle\hat{\sigma}_b^{k}\hat{a}\right\rangle - \left\langle\hat{\sigma}_a^{k}\hat{b}\right\rangle\right] + \frac{\Omega}{2}\left[\left\langle\hat{\eta}_c^{k}\right\rangle - \left\langle\hat{\eta}_a^{k}\right\rangle\right] \\ &\quad - \gamma[\bar{n}_{\mathrm{th}}+1]\left\langle\hat{\sigma}_c^{k}\right\rangle,\end{aligned} \tag{24c}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{\eta}_a^{k}\right\rangle &= g\left[\left\langle\hat{\sigma}_a^{\dagger k}\hat{a}\right\rangle + \left\langle a\wedge^{\dagger}\hat{\sigma}_a^{k}\right\rangle\right] + \frac{\Omega}{2}\left[\left\langle\hat{\sigma}_c^{k}\right\rangle + \left\langle\hat{\sigma}_c^{\dagger k}\right\rangle\right] \\ &\quad - \frac{\gamma}{2}[\bar{n}_{\mathrm{th}}+1]\left\langle\hat{\eta}_a^{k}\right\rangle,\end{aligned} \tag{24d}$$

$$\begin{aligned}\frac{d}{dt}\left\langle\hat{\eta}_b^{k}\right\rangle &= g\left[\left\langle\hat{\sigma}_b^{\dagger}\hat{b}\right\rangle + \left\langle b\wedge^{\dagger}\hat{\sigma}_b^{k}\right\rangle - \left\langle\hat{\sigma}_a^{\dagger k}\hat{a}\right\rangle - \left\langle a\wedge^{\dagger}\hat{\sigma}_a^{k}\right\rangle\right] \\ &\quad - \gamma(2\bar{n}_{\mathrm{th}}+1)\left[\left\langle\hat{\eta}_a^{k}\right\rangle - \left\langle\hat{\eta}_b^{k}\right\rangle\right],\end{aligned} \tag{24e}$$

where

$$\hat{\eta}_a^{k} = |a\rangle_{k\,k}\langle a|, \tag{25a}$$

$$\hat{\eta}_b^{k} = |b\rangle_{k\,k}\langle b|, \tag{25b}$$

$$\hat{\eta}_c^{k} = |c\rangle_{k\,k}\langle c|. \tag{25c}$$

We see that Equations (24a)–(24e) are the nonlinear differential equations, and hence, it is not possible to find exact time-dependent solutions of these equations. We intend to overcome this problem by applying the large-time approximation [9, 24]. Therefore, employing this approximation scheme, we get from Equations (21a) and (21b) the approximately valid relations:

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{26a}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{26b}$$

Now introducing Equations (26a) and (26b) into Equations (24a)–(24e) and summing over N three-level atoms, we get the following:

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\frac{1}{2}(\gamma+\gamma_c)(3\bar{n}_{\mathrm{th}}+2)\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger\rangle, \tag{27a}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{1}{2}(\gamma+\gamma_c)(3\bar{n}_{\mathrm{th}}+1)\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^\dagger\rangle, \tag{27b}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{1}{2}(\gamma+\gamma_c)(2\bar{n}_{\mathrm{th}}+1)\langle\hat{m}_c\rangle + \frac{\Omega}{2}\left[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\right], \tag{27c}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -(\gamma+\gamma_c)(\bar{n}_{\mathrm{th}}+1)\langle\hat{N}_a\rangle + \frac{\Omega}{2}\left[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\right], \tag{27d}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = (\gamma+\gamma_c)(2\bar{n}_{\mathrm{th}}+1)\left[\langle\hat{N}_b\rangle - \langle\hat{N}_a\rangle\right], \tag{27e}$$

where $\gamma_c = 4g^2/\kappa$ is the stimulated emission decay constant. For N number of atoms, we see that $\hat{m}_j = \sum_{k=1}^{N}\hat{\sigma}_j^k$ and $\hat{N}_j = \sum_{k=1}^{N}\hat{\eta}_j^k$, where $j = a, b, c$. Hence, the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ represent the number of atoms in the top, middle, and bottom levels, respectively. In addition, employing the completeness relation,

$$\hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k = \hat{I}, \tag{28}$$

we easily arrive at

$$\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle = N. \tag{29}$$

Furthermore, using the definition $\hat{\sigma}_a^k = |b\rangle_{kk}\langle a|$ and setting for any k, $\hat{\sigma}_a^k = |b\rangle\langle a|$, we have $\hat{m}_a = N|b\rangle\langle a|$. Following the same procedure, one can also easily establish that $\hat{m}_b = N|c\rangle\langle b|$, $\hat{m}_c = N|c\rangle\langle a|$, $\hat{N}_a = N|a\rangle\langle a|$, $\hat{N}_b = N|b\rangle\langle b|$, and $\hat{N}_c = N|c\rangle\langle c|$. Using the definition $\hat{m} = \hat{m}_a + \hat{m}_b$ [9], it can be readily established that

$$m\wedge^\dagger\hat{m} = N\left(\hat{N}_a + \hat{N}_b\right), \tag{30a}$$

$$\hat{m}m\wedge^\dagger = N\left(\hat{N}_b + \hat{N}_c\right), \tag{30b}$$

$$m\wedge^2 = N\hat{m}_c. \tag{30c}$$

Following a straightforward algebra described by [9, 24], it is possible to obtain the steady-state solution of the stochastic differential equation of the atomic operators:

$$\langle\hat{N}_a\rangle = \left[\frac{\Omega^2 N}{(\gamma_c+\gamma)^2(\bar{n}_{\mathrm{th}}+1)(2\bar{n}_{\mathrm{th}}+1)+3\Omega^2}\right], \tag{31a}$$

$$\langle\hat{N}_b\rangle = \left[\frac{\Omega^2 N}{(\gamma_c+\gamma)^2(\bar{n}_{\mathrm{th}}+1)(2\bar{n}_{\mathrm{th}}+1)+3\Omega^2}\right], \tag{31b}$$

$$\langle\hat{N}_c\rangle = \left[\frac{\left[(\gamma_c+\gamma)^2(\bar{n}_{\mathrm{th}}+1)(2\bar{n}_{\mathrm{th}}+1)+\Omega^2\right]N}{(\gamma_c+\gamma)^2(\bar{n}_{\mathrm{th}}+1)(2\bar{n}_{\mathrm{th}}+1)+3\Omega^2}\right], \tag{31c}$$

$$\langle\hat{m}_c\rangle = \left[\frac{\Omega(\gamma_c+\gamma)(\bar{n}_{\mathrm{th}}+1)N}{(\gamma_c+\gamma)^2(\bar{n}_{\mathrm{th}}+1)(2\bar{n}_{\mathrm{th}}+1)+3\Omega^2}\right]. \tag{31d}$$

3. Entanglement

In this section, we seek to study the degree of entanglement of photon states and the atom entanglement of the two-mode cavity light produced by a system under consideration.

3.1. Photon Entanglement. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently; instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, and polarization performed on entangled particles are found to be appropriately correlated. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. It is in this spirit that this section is devoted to the analysis of the entanglement of the two-mode photon states. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents,

$$\hat{\rho} \neq \sum_k p_k\hat{\rho}_k^{(1)}\otimes\hat{\rho}_k^{(2)}, \tag{32}$$

in which $p_k \gg 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states. On the other hand, a maximally entangled CV state can be expressed as a co-eigenstate of a pair of EPR-type operators [34] such as $\hat{x}_a - \hat{x}_b$ and $\hat{P}_a - \hat{P}_b$. The total variance of these two operators reduces to zero for maximally entangled CV states. According to the inseparable criteria given by Duan et al. [35], cavity photon states of a system are entangled, if the sum of the variance of a pair of EPR-like operators,

$$\hat{s} = \hat{x}_a - \hat{x}_b, \tag{33a}$$

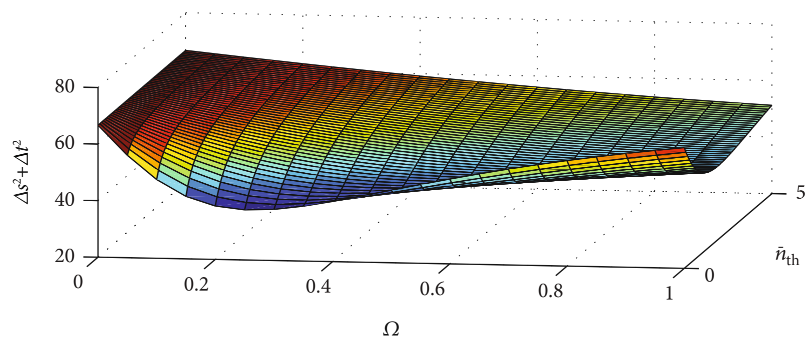

FIGURE 2: A plot of the photon entanglement of the two-mode cavity light (Equation (37)) versus Ω and $\bar{n}_{th}$ for $\gamma_c = 0.4$ and $\gamma = 0.2$ (color online) $N = 50$.

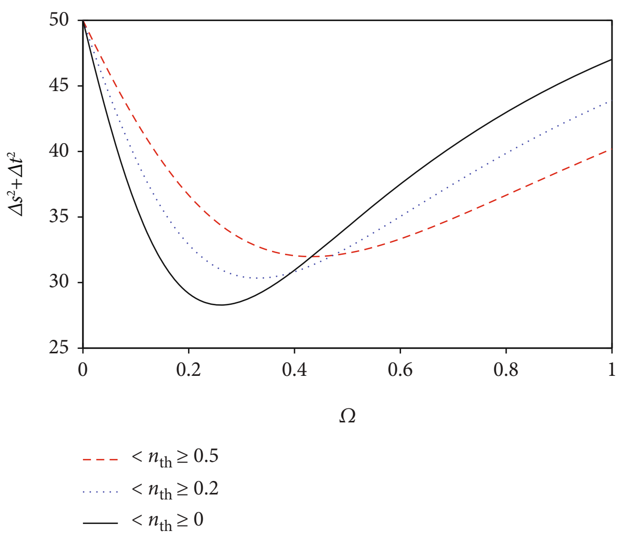

FIGURE 3: Plots of the photon entanglement of the two-mode cavity light (Equation (37)) versus Ω for $\gamma_c = 0.4$ and $\gamma = 0.2$, and for different values of $\bar{n}_{th}$ (color online) $N = 50$.

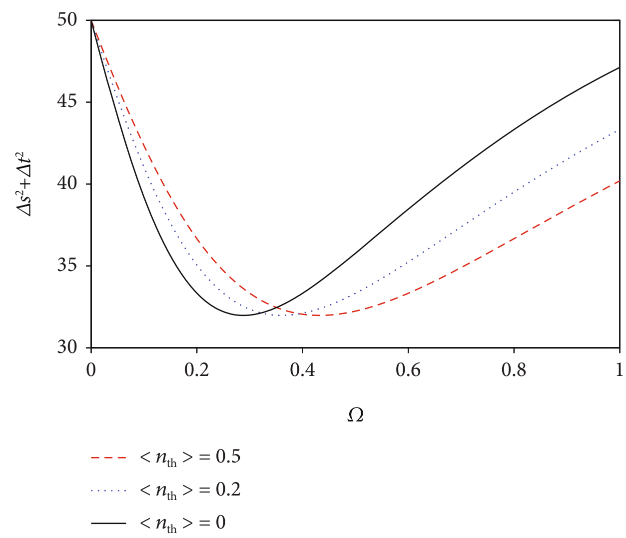

FIGURE 4: A plot of the photon entanglement of the two-mode cavity light (Equation (37)) versus Ω and $\bar{n}_{th}$ for $\gamma_c = 0.4$ and $\gamma = 0.2$, and for different values of $\bar{n}_{th}$.

TABLE 1: Numerical value of the degree of entanglement from Figure 3 for values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\bar{n}_{th} = 0.5$, and $N = 50$.

Spontaneous emission	$\Delta s^2 + \Delta t^2$	Occurs at
$\gamma = 0$	33.5%	$\Omega = 0.4343$
$\gamma = 0.1$	33.5%	$\Omega = 0.3636$
$\gamma = 0.2$	33.5%	$\Omega = 0.2929$

TABLE 2: Numerical value of the degree of entanglement from Figure 4 for values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 50$.

$\bar{n}_{th}$	$\Delta s^2 + \Delta t^2$	Occurs at
$\bar{n}_{th} = 0$	36%	$\Omega = 0.4343$
$\bar{n}_{th} = 0.2$	34.2%	$\Omega = 0.3636$
$\bar{n}_{th} = 0.5$	33.5%	$\Omega = 0.2929$

$$\hat{t} = \hat{p}_a + \hat{p}_b, \tag{33b}$$

with

$$\hat{x}_a = \frac{1}{\sqrt{2}}\left(\hat{a} + a\wedge^{\dagger}\right), \quad \hat{x}_b = \frac{1}{\sqrt{2}}\left(\hat{b} + b\wedge^{\dagger}\right), \tag{34a}$$

$$\hat{p}_a = \frac{i}{\sqrt{2}}\left(a\wedge^{\dagger} - \hat{a}\right), \quad \hat{p}_b = \frac{i}{\sqrt{2}}\left(b\wedge^{\dagger} - \hat{b}\right) \tag{34b}$$

are quadrature operators for modes a and b, satisfy

$$\Delta s^2 + \Delta t^2 < 2N, \tag{35}$$

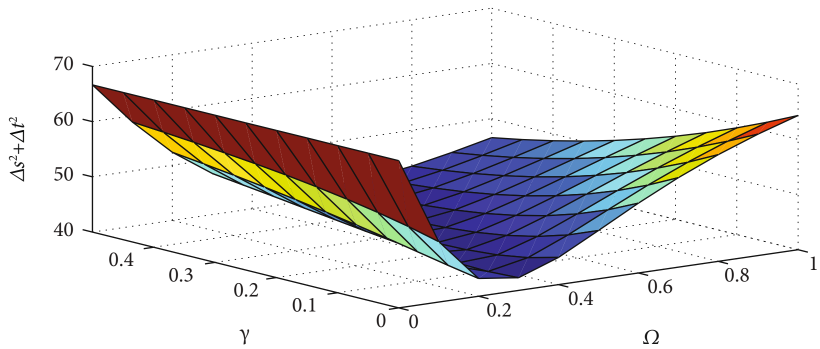

FIGURE 5: A plot of the photon entanglement of the two-mode cavity light (Equation (37)) versus Ω and γ for $\gamma_c = 0.4$ and $\bar{n}_{\text{th}} = 0.5$ (color online) $N = 50$.

and recalling the cavity mode operators $\hat{a}$ and $\hat{a}$ are Gaussian variables with zero mean, we readily get

$$\Delta s^2 + \Delta t^2 = \frac{2\gamma_c}{\kappa}\left[N + \left\langle \hat{N}_b \right\rangle \pm 2\langle \hat{m}_c \rangle\right]. \quad (36)$$

On the account of Equations (31b) and (31d), the photon entanglement of the two-mode cavity light takes, at steady state, the form

$$\Delta s^2 + \Delta t^2 = \left(2\frac{\gamma_c}{\kappa}N\right)\left[\frac{(\gamma_c+\gamma)^2(\bar{n}_{\text{th}}+1)(2\bar{n}_{\text{th}}+1)}{\Delta} + \frac{4\Omega^2 - 2\Omega(\gamma_c+\gamma)(\bar{n}_{\text{th}}+1)}{\Delta}\right], \quad (37)$$

where

$$\Delta = \left[(\gamma_c+\gamma)^2(\bar{n}_{\text{th}}+1)(2\bar{n}_{\text{th}}+1) + 3\Omega^2\right]. \quad (38)$$

This is with the help of the criterion (Equation ((35))) that a significant entanglement between the states of the light generated in the cavity. This is due to the strong correlation between the radiations emitted when the atoms decay from the upper energy level to the lower via the intermediate level.

It is clearly indicated in Figure 2 the cavity radiation is found to be entangled for all parameters under consideration. It can be observed that the degree of entanglement increases for smaller values of the driving coherent light Ω, but decreases for the larger values. Moreover, when we see the plots in Figure 3 as well as from the data in Table 1 that as the spontaneous emission decay constant increases, the photon entanglement also decreases for a fixed value of pumping. Furthermore, it is not difficult to see from the plots in Figure 4 that as a thermally seeded light mean photon number value increases, the photon entanglement also decreases. From this plot, the thermal light significantly damages the photon entanglement in the earlier stages (at $\Omega = 0$) of the lasing process.

On the other hand, Figure 4 shows the degree of entanglement of the cavity radiation versus the thermal light, $\bar{n}_{\text{th}}$, and the driving coherent light, Ω, for values of $\kappa = 0.8$, $\gamma_c =$

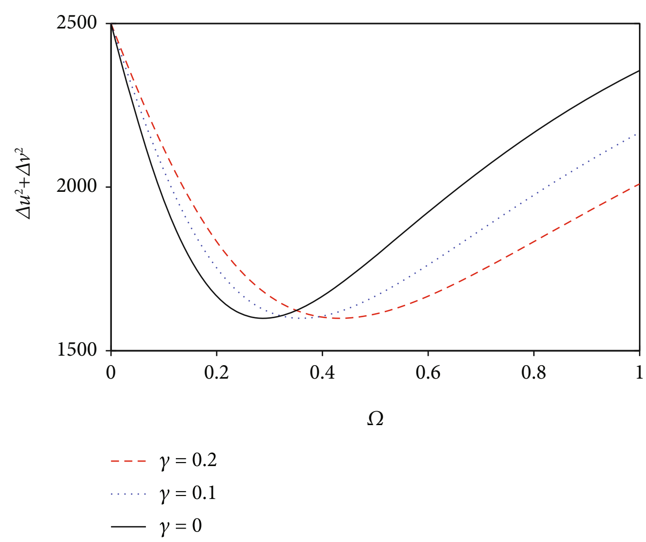

FIGURE 6: Plots of the atom entanglement of the two-mode cavity light (Equation (44)) versus Ω for $\gamma_c = 0.4$ and $\gamma = 0.2$, and for different values of $\bar{n}_{\text{th}}$ (color online) $N = 50$.

0.4, $\gamma = 0.2$, and $N = 50$ (Table 2). As we see from the plot, the degree of entanglement increases for small values of $\bar{n}_{\text{th}}$ and for small values of the pumping light which couples the top and bottom levels of the atoms. The effect of the absence of thermal reservoir, $\bar{n}_{\text{th}} = 0$, is to increase the intracavity degree of entanglement for small values of Ω. In addition, it is clearly shown in Figure 5 that the photon entanglement increases for small values of γ and the driving coherent light, Ω. On both figures, the maximum degree of entanglement occurs at which $\Omega = 0.2929$.

3.2. Cavity Atomic-State Entanglement. The quantum entanglement between the two cavity modes a and b is proposed by Duan-Giedke-Cirac-Zoller (DGCZ) [35], which is a sufficient condition for entangled quantum states. According to DGCZ, a quantum state of a system is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality

$$\Delta u^2 + \Delta v^2 < 2N^2. \quad (39)$$

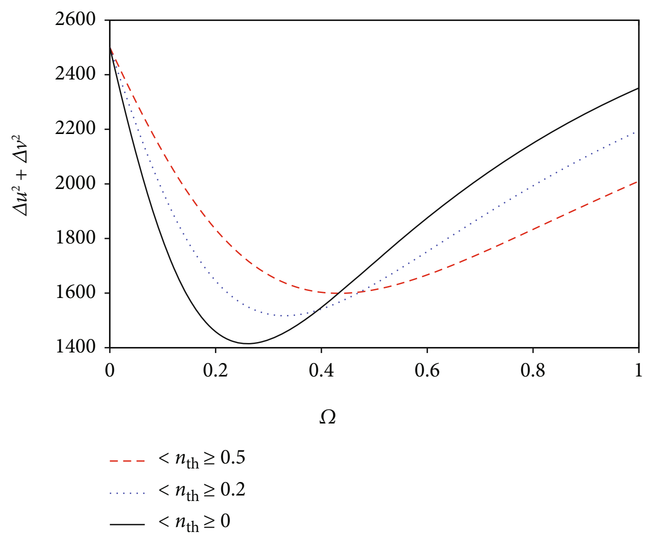

FIGURE 7: Plots of the atom entanglement of the two-mode cavity light (Equation (44)) versus Ω for $\gamma_c = 0.4$ and $\gamma = 0.2$, and for different values of $\bar{n}_{\text{th}}$ (color online) $N = 50$.

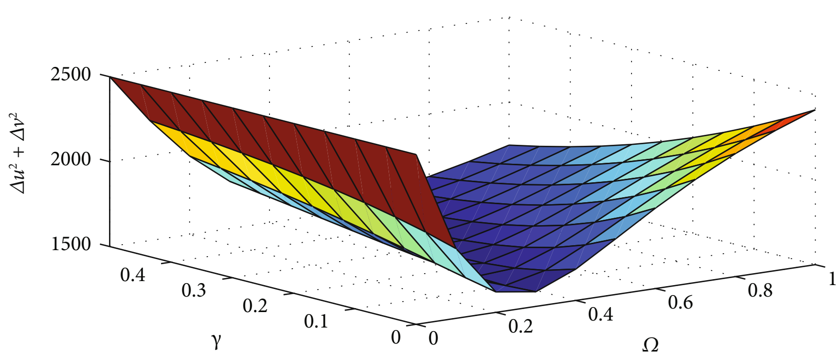

FIGURE 8: A plot of the atom entanglement of the two-mode cavity light (Equation (44)) versus γ and Ω for $\gamma_c = 0.4$ and $\bar{n}_{\text{th}} = 0.5$ (color online) $N = 50$.

On the other hand, cavity atomic states of a system are entangled, if the sum of the variance of a pair of EPR-like operators

$$\hat{u} = \hat{x}'_a - \hat{x}'_b, \tag{40a}$$

$$\hat{v} = \hat{p}'_a + \hat{p}'_b, \tag{40b}$$

where

$$\hat{x}'_a = \frac{1}{\sqrt{2}}\left(\hat{m}_a + \hat{m}_a^\dagger\right), \quad \hat{x}'_b = \frac{1}{\sqrt{2}}\left(\hat{m}_b + \hat{m}_b^\dagger\right), \tag{41a}$$

$$\hat{p}'_a = \frac{i}{\sqrt{2}}\left(\hat{m}_a^\dagger - \hat{m}_a\right), \quad \hat{p}'_b = \frac{i}{\sqrt{2}}\left(\hat{m}_b^\dagger - \hat{m}_b\right), \tag{41b}$$

are the atomic quadrature operators. Since $\hat{m}_a$ and $\hat{m}_b$ are Gaussian variables with zero means, so one can easily verify that,

$$\begin{aligned}\Delta u^2 + \Delta v^2 = {} & \left[\left\langle \hat{m}_a^\dagger \hat{m}_a \right\rangle + \left\langle \hat{m}_a \hat{m}_a^\dagger \right\rangle + \left\langle \hat{m}_b^\dagger \hat{m}_b \right\rangle \right. \\ & \left. + \left\langle \hat{m}_b \hat{m}_b^\dagger \right\rangle - \left\langle \hat{m}_b^\dagger \hat{m}_a^\dagger \right\rangle - \left\langle \hat{m}_a \hat{m}_b \right\rangle \right].\end{aligned} \tag{42}$$

It then follows that

$$\Delta u^2 + \Delta v^2 = N\left[N + \left\langle \hat{N}_a \right\rangle - 2\langle \hat{m}_c \rangle\right]. \tag{43}$$

On the account of Equations (31a) and (31d), the cavity atomic-state entanglement of the two-mode cavity light takes, at steady state, the form

$$\begin{aligned}\Delta u^2 + \Delta v^2 = N^2 & \left[\frac{(\gamma_c + \gamma)^2 (\bar{n}_{\text{th}} + 1)[(2\bar{n}_{\text{th}} + 1)}{\Delta} \right. \\ & \left. + \frac{4\Omega^2 - 2\Omega(\gamma_c + \gamma)(\bar{n}_{\text{th}} + 1)]}{\Delta}\right].\end{aligned} \tag{44}$$

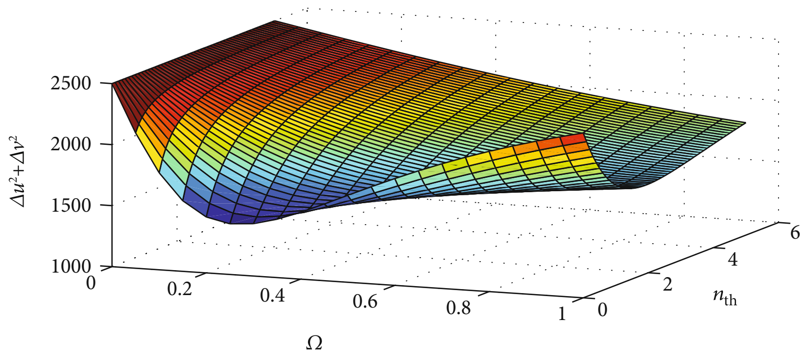

FIGURE 9: A plot of the atom entanglement of the two-mode cavity light (Equation (44)) versus Ω and $\bar{n}_{th}$ for $\gamma_c = 0.4$ and $\gamma = 0.2$ (color online) $N = 50$.

TABLE 3: Numerical value of the degree of atom entanglement from Figure 6 for values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\bar{n}_{th} = 0.5$, and $N = 50$.

Spontaneous emission	$\Delta u^2 + \Delta v^2$	Occurs at
$\gamma = 0$	36%	$\Omega = 0.2929$
$\gamma = 0.1$	36%	$\Omega = 0.3636$
$\gamma = 0.2$	36%	$\Omega = 0.4343$

TABLE 4: Numerical value of the degree of entanglement from Figure 7 for values of $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 50$.

$\bar{n}_{th}$	$\Delta u^2 + \Delta v^2$	Occurs at
$\bar{n}_{th} = 0$	43.44%	$\Omega = 0.2626$
$\bar{n}_{th} = 0.2$	39.32%	$\Omega = 0.3232$
$\bar{n}_{th} = 0.5$	36%	$\Omega = 0.4343$

When we observe Equation (44), the cavity atomic-state entanglement of the two-mode cavity light highly depends on the number of atoms.

On the other hand, in the absence of the driving coherent light, $\Omega = 0$, Equation (44) turns out to be

$$\Delta u^2 + \Delta v^2 = N^2. \tag{45}$$

On the basis of the criteria (Equation (45)), we clearly see from Figure 6 that the two states of the generated light are strongly entangled at steady state. It is not difficult to observe that entanglement disappears when there is no pumping, and it would be stronger for certain values of a driving coherent light for each value of the spontaneous emission decay constant, γ. Moreover, when we see the plots in Figure 6 that as the spontaneous emission decay constant increases, the atom entanglement also decreases when it occurs at the same value of Ω.

When we see the plots in Figure 7 that as a thermally seeded light mean photon number value increases, the atom entanglement also decreases. Similarly, as we observe the data on the plots from this figure, the photon entanglement increased with increasing of a thermally seeded light mean photon number value. In addition, the comparison of Figures 8 and 9 as well as Tables 3 and 4 unfortunately reveals that the degree of entanglement increases with the decrease in spontaneous emission and thermal light, respectively. This indicates that the degree of entanglement enhanced for small values of γ and $\bar{n}_{th}$ with other parameters is fixed.

It is clearly shown in Figures 8 and 9 that the atom entanglement increases with the decrease in $\bar{n}_{th}$ and γ, respectively. On both figures, the maximum degree of entanglement occurs at which $\Omega = 0.2929$ is the maximum degree of a cavity light entanglement occurring point as we observe from the plots of Figures 3 and 5. This indicates that similar to the photon entanglement, the atom entanglement is also directly related to the two-mode quadrature squeezing [9].

On the basis of the criteria (Equation (35) and Equation (39)), we clearly see that the two states of the generated light are strongly entangled at steady state. Moreover, the absence of thermal light leads to an increase in the degree of entanglement.

4. Conclusion

In this study, the quantum properties of a nondegenerate three-level laser driven by coherent light and coupled to a two-mode thermal reservoir via a single-port mirror whose open cavity contains N nondegenerate three-level atoms are thoroughly analyzed. It is carried out the analysis by putting the noise operators associated with the thermal reservoir in normal order and by considering the interaction of the three-level atoms with the thermal reservoir outside the cavity. The master equation and the quantum Langevin equations for the cavity light are obtained. Applying these equations, the stochastic differential equations for the atomic operators are determined. Making use of the large-time approximation scheme, we found the solutions of atomic and cavity mode operators. Applying these results, the entanglement photons as well as atoms, at steady state, are calculated.

The analysis showed that the intracavity photon entanglement is enhanced due to in the absence of spontaneous emission as well as initially seeded thermal light. Hence, the presence of the spontaneous emission decreases the degree of photon entanglement for fixed values of driving coherent light Ω which couples the top and bottom levels of a three-level atom. Moreover, when the spontaneous emission decay constant increases, the atom entanglement also decreases when it occurs at the same value of Ω. Both the photon entanglement and the atom entanglement increased with the decreasing in a thermally seeded light mean photon number value and for small values of a spontaneous emission decay constant.

Data Availability

The article contains theoretical material. This paper has links to articles and textbooks from other researchers and does not use any data.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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