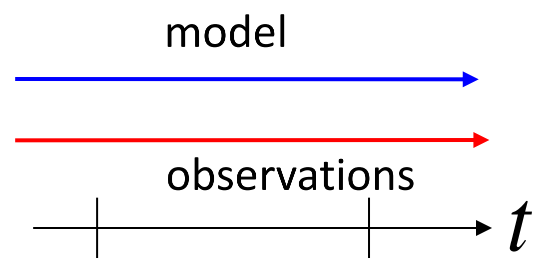
Ensemble Kalman-Bucy filters

Javier Amezcua, Sebastian Reich, Kayo Ide, Eugenia Kalnay, Kay Bergemann

Kalman-Bucy filter

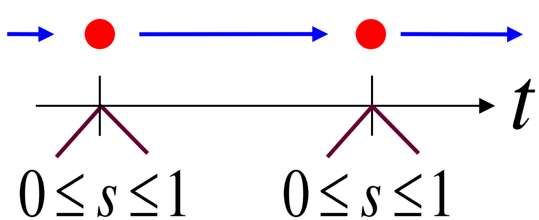
The Kalman-Bucy filter considers both the **model** and **observations** to be **time-continuous**.

$$\frac{d\mathbf{P}}{dt} = -\mathbf{P}\mathbf{H}^T\mathbf{R}_c^{-1}\mathbf{H}\mathbf{P} + \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T + \mathbf{Q}_c$$



$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{F}\hat{\mathbf{x}} - \mathbf{P}\mathbf{H}^{\mathrm{T}}\mathbf{R}_c^{-1}\left(\mathbf{H}\hat{\mathbf{x}} - \mathbf{y}\right)$$

It can also be used for discrete observations by spanning a **pseudotime** *s* (BGR09, BR10).



KBF in pseudotime:

$$\frac{d\mathbf{P}}{ds} = -\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P}$$

$$\frac{d\hat{\mathbf{x}}}{ds} = -\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\left(\mathbf{H}\hat{\mathbf{x}} - \mathbf{y}\right)$$

Ensemble Kalman-Bucy filter

Ensemble version:

$$\frac{d\mathbf{X}}{ds} = -\frac{1}{2(M-1)}\mathbf{X}\mathbf{X}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X}$$ BGR09

$$+ \frac{d\bar{\mathbf{x}}}{ds} = -\frac{1}{M-1}\mathbf{X}\mathbf{X}^T\mathbf{H}^T\mathbf{R}^{-1}\left(\mathbf{H}\bar{\mathbf{x}} - \mathbf{y}\right) \quad =$$

$$\frac{d\overline{\overline{\mathbf{X}}}}{ds} = -\frac{1}{2(M-1)}\overline{\overline{\mathbf{X}}}[\mathbf{I}-\mathbf{U}]\overline{\overline{\mathbf{X}}}^T\mathbf{H}^T\mathbf{R}^{-1}\left[\mathbf{H}\overline{\overline{\mathbf{X}}}[\mathbf{I}+\mathbf{U}] - 2\mathbf{y}\mathbf{1}^T\right]$$ BR10

Observations participate in updating both the mean and the perturbations.

The EKBF allows for **mollification** of analysis increments, **non-Gaussian** unimodal and multimodal **extensions**, assimilating **quasi-continuous observations**, etc.

Numerical integration in the EKBF

The ODE for perturbations:

$$\frac{d\mathbf{X}}{ds} = -\frac{1}{2(M-1)}\mathbf{X}\mathbf{X}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X} \quad \xrightarrow{\mathbf{R}=\sigma^2\mathbf{I} \qquad \mathbf{H}=\mathbf{I}} \quad \frac{d}{ds}\mathbf{X} = -\frac{1}{2\sigma^2(M-1)}\mathbf{X}\mathbf{X}^{\mathrm{T}}\mathbf{X}$$

Writing it explicitly:

$$\frac{d}{ds}\begin{bmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_M^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_M^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_M^{(N)} \end{bmatrix} = -\frac{1}{2\sigma^2(M-1)}\begin{bmatrix} \sum_{m=1}^{M}\left(x_m^{(1)}\sum_{n=1}^{N}x_1^{(n)}x_m^{(n)}\right) & \sum_{m=1}^{M}\left(x_m^{(1)}\sum_{n=1}^{N}x_2^{(n)}x_m^{(n)}\right) & \cdots & \sum_{m=1}^{M}\left(x_m^{(1)}\sum_{n=1}^{N}x_M^{(n)}x_m^{(n)}\right) \\ \sum_{m=1}^{M}\left(x_m^{(2)}\sum_{n=1}^{N}x_1^{(n)}x_m^{(n)}\right) & \sum_{m=1}^{M}\left(x_m^{(2)}\sum_{n=1}^{N}x_2^{(n)}x_m^{(n)}\right) & \cdots & \sum_{m=1}^{M}\left(x_m^{(2)}\sum_{n=1}^{N}x_M^{(n)}x_m^{(n)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{m=1}^{M}\left(x_m^{(N)}\sum_{n=1}^{N}x_1^{(n)}x_m^{(n)}\right) & \sum_{m=1}^{M}\left(x_m^{(N)}\sum_{n=1}^{N}x_2^{(n)}x_m^{(n)}\right) & \cdots & \sum_{m=1}^{M}\left(x_m^{(N)}\sum_{n=1}^{N}x_M^{(n)}x_m^{(n)}\right) \end{bmatrix}$$

This system must be solved with explicit integration methods.

BGR09 and BR10 used **Euler forward** with 4 steps over $0 \le s \le 1$

$$\frac{d}{ds}\mathbf{x} = f(\mathbf{x}) \rightarrow \mathbf{x}_{j+1} = \mathbf{x}_j + \Delta s\, f(\mathbf{x}_j)$$

Does this scheme guarantee stability?

Numerical integration in the EKBF

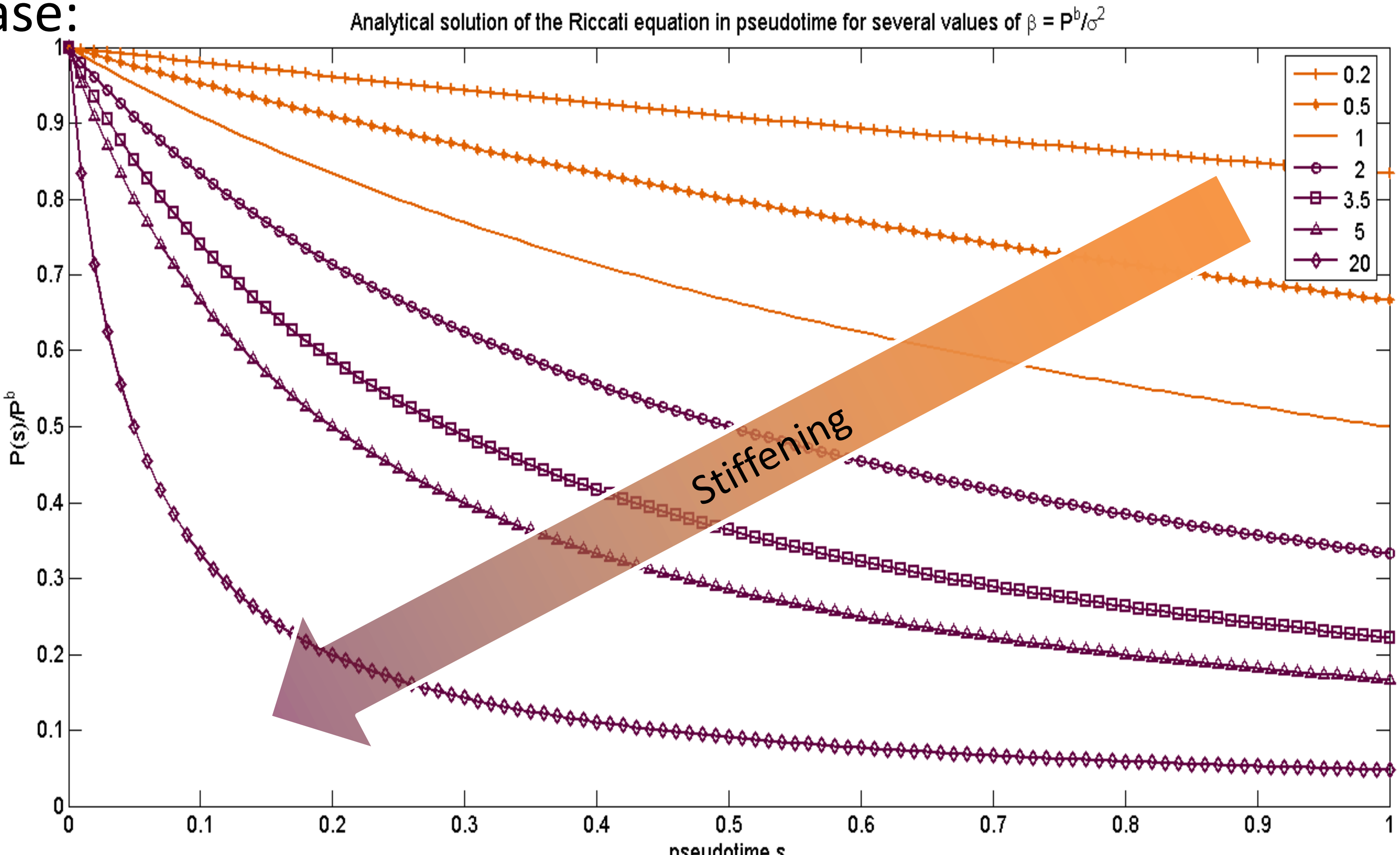
Consider the KBF equation for covariance and its solution:

$$\frac{d\mathbf{P}}{ds} = -\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P} \longrightarrow \mathbf{P}(s) = \mathbf{P}^b\left(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P}^b s + \mathbf{I}\right)^{-1}, 0 \le s \le 1$$

In the scalar case:

$$\frac{P_1(s)}{P_1^b} = \frac{1}{\beta s + 1} \approx 1 - \beta s + (\beta s)^2 - \ldots$$

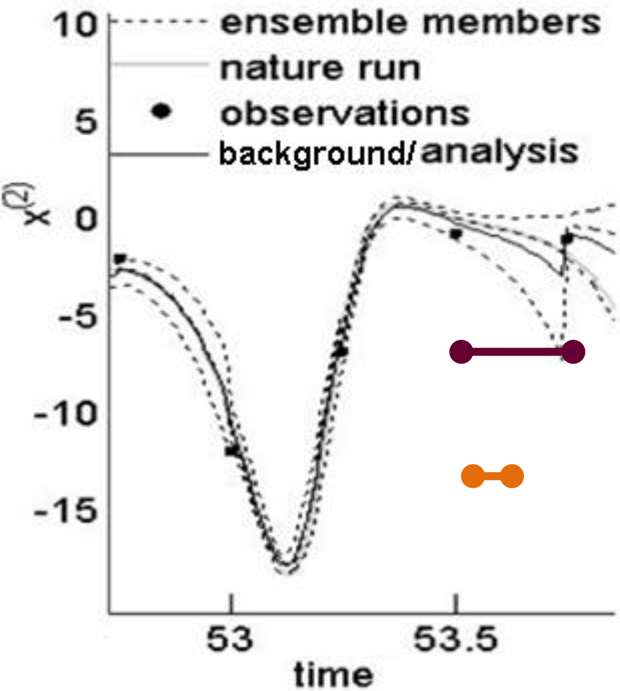
$$\beta = \frac{P_1^b}{\sigma_1^2}$$



Numerical integration in the EKBF

For the general case:

$$\beta = \frac{\left|\mathbf{Y}^{b^T}\mathbf{R}^{-1}\mathbf{Y}^{b}\right|}{M-1}, \mathbf{Y}^{b} = \mathbf{H}\mathbf{X}^{b}$$

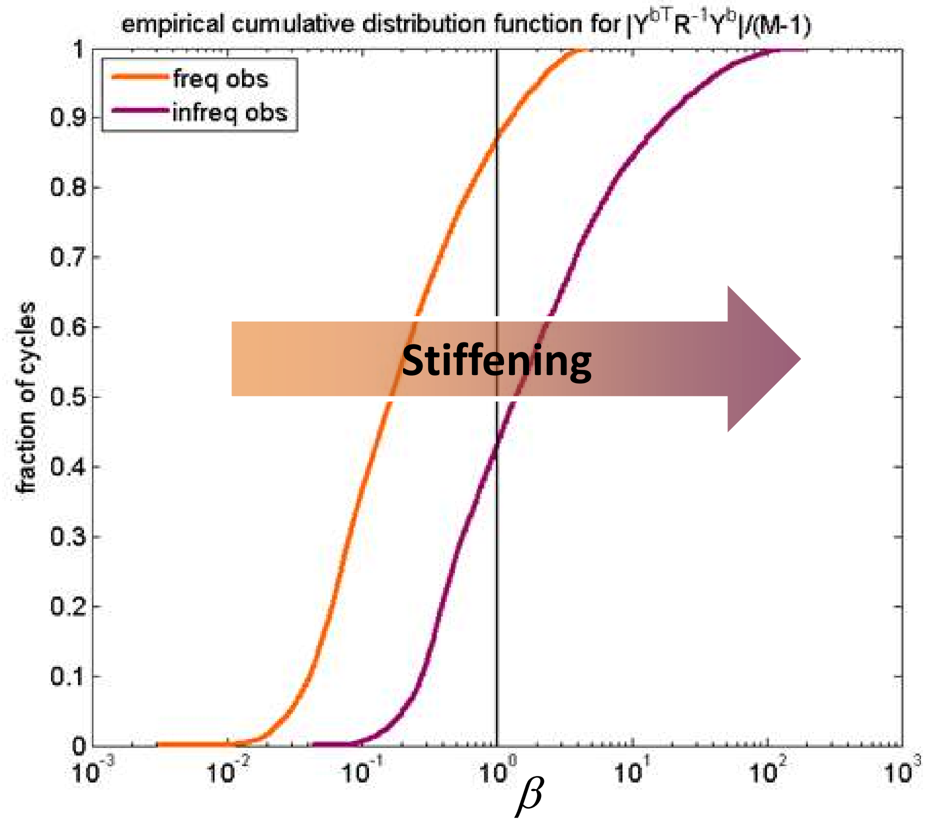


Frequent obs (8 steps), linear growth

Infrequent obs (25 steps), nonlinear growth

This ratio depends upon:

- Frequency of observations/length of assimilation window
- Density of observations in an area
- Nonlinearity in the forecast model



Numerical integration in the EKBF

To avoid stiffening, a linearly semi-implicit method is proposed:

$$\frac{d\mathbf{X}}{ds} = -\frac{1}{2}\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X} \quad \Longrightarrow \quad \frac{\mathbf{X}_{k+1}-\mathbf{X}_k}{\Delta s} = -\frac{1}{2}\left(\alpha \mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X}_k + (1-\alpha)\mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{X}_{k+1}\right)$$

$\alpha = 1$ recovers Euler Forward.

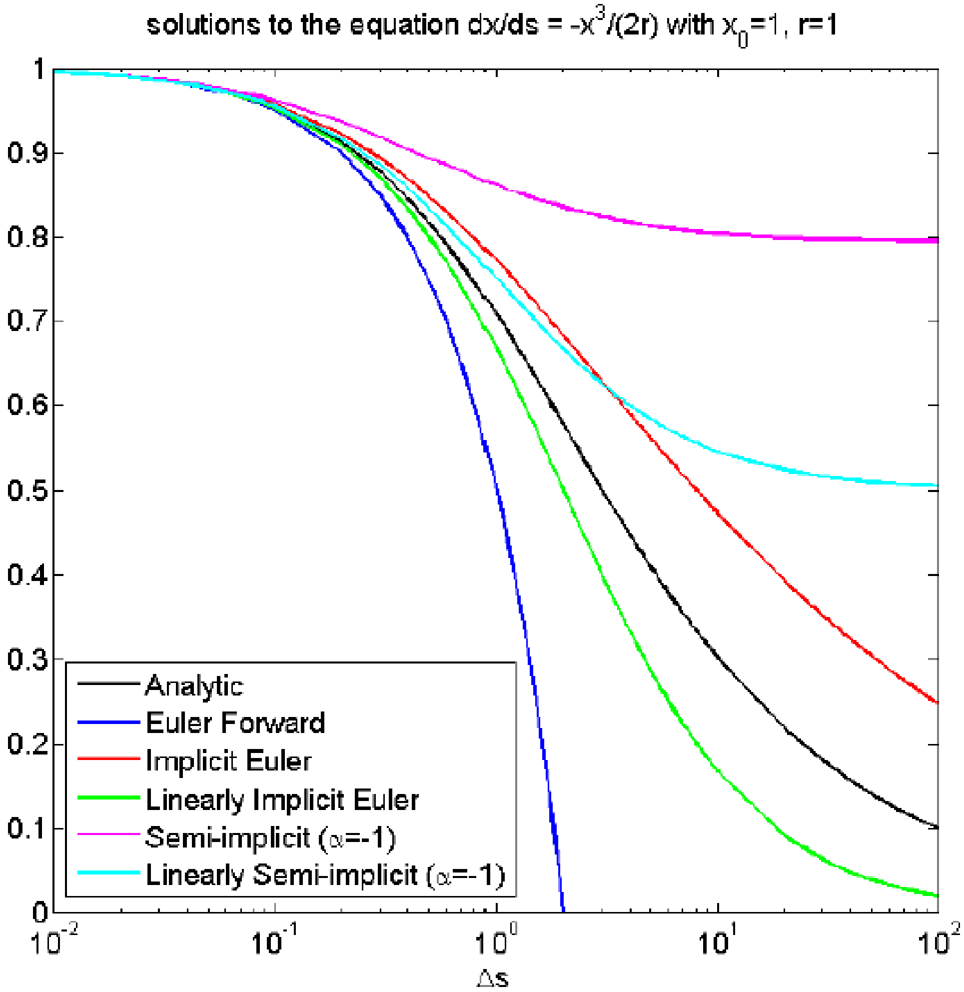
$\alpha = 0$ gives a linearly implicit Euler.

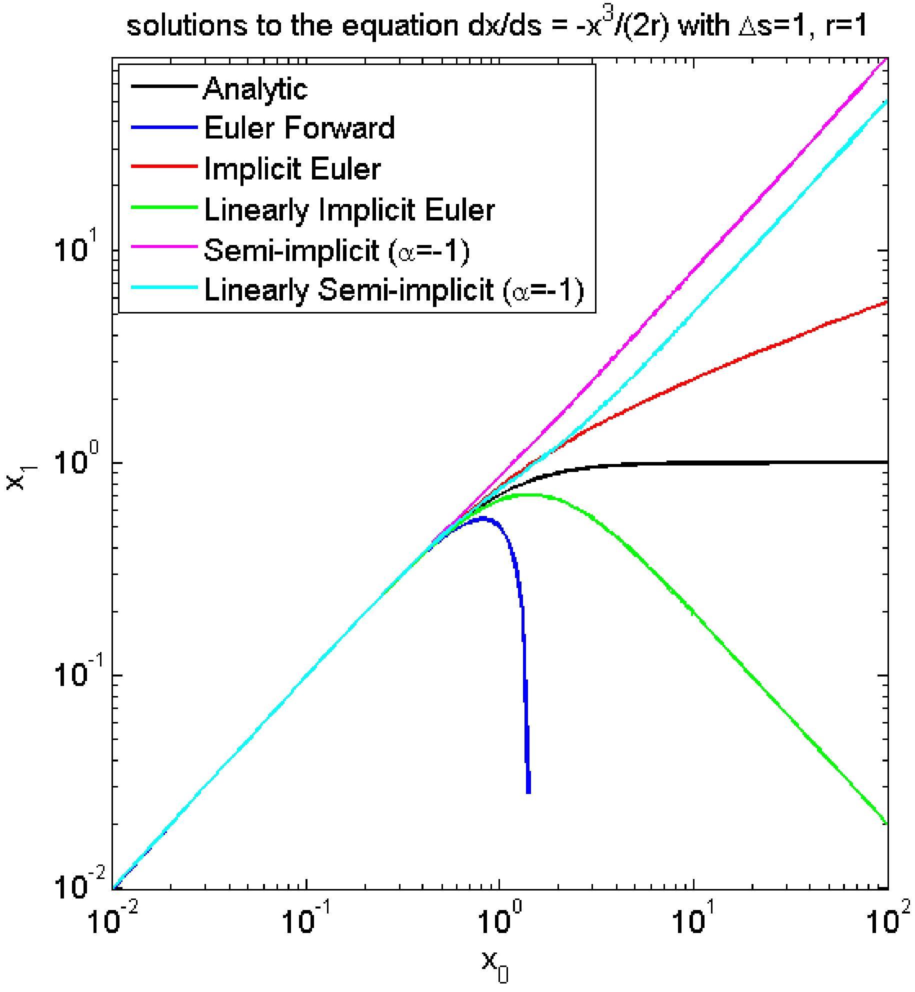
$\alpha = -1$ gives:

$$\mathbf{X}_{k+1} = \mathbf{X}_k - \frac{\Delta s}{2}\mathbf{P}_k\mathbf{H}^T\left(\mathbf{I} + \Delta s\mathbf{H}\mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\right)^{-1}\mathbf{R}^{-1}\mathbf{H}\mathbf{X}_k$$

when $\Delta s\mathbf{P}_k >> \mathbf{R}$ $\Longrightarrow$ $\mathbf{X}_{k+1} \rightarrow 1/2\,\mathbf{X}_k$

Numerical integration in the EKBF





Numerical integration in the EKBF

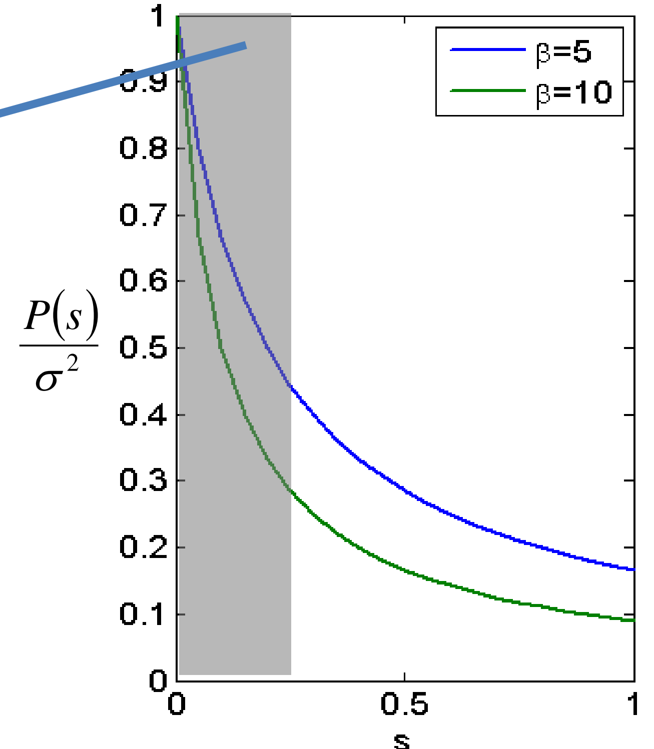
A Diagonal Semi-Implicit (DSI) approximation:

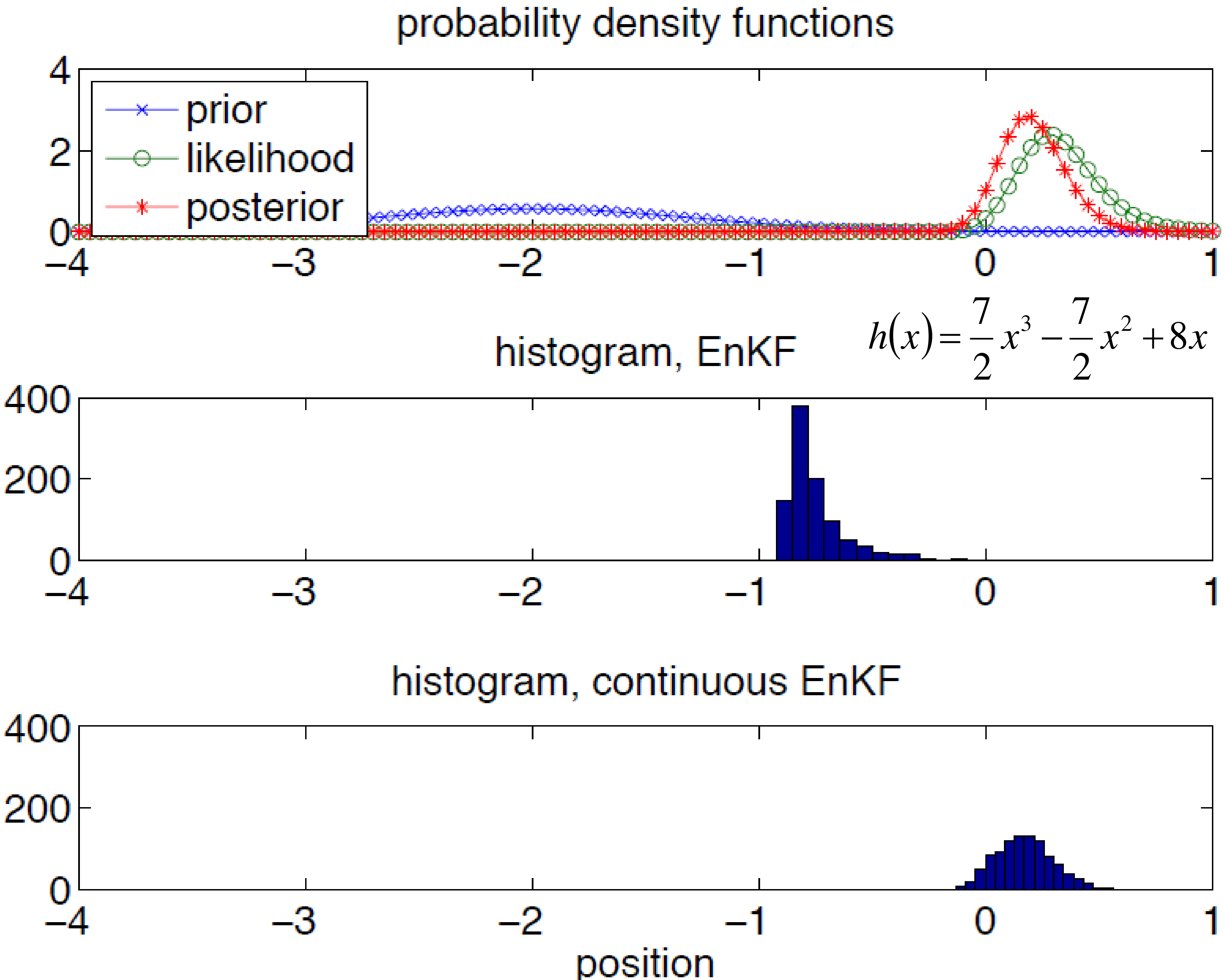
$$\mathbf{X}_{k+1} = \mathbf{X}_k - \frac{\Delta s}{2}\mathbf{P}_k\mathbf{H}^T\left(diag\left(\mathbf{I} + \Delta s\mathbf{H}\mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\right)\right)^{-1}\mathbf{R}^{-1}\mathbf{H}\mathbf{X}_k$$

And for the full ensemble:

$$\overline{\overline{\mathbf{X}}}_{k+1} = \overline{\overline{\mathbf{X}}}_k - \frac{\Delta s}{2}\mathbf{P}_k\mathbf{H}^T\left(diag\left(\mathbf{H}\mathbf{P}_k\mathbf{H}^T\mathbf{R}^{-1}\Delta s + \mathbf{I}\right)\right)^{-1}\mathbf{R}^{-1}\left[\mathbf{H}\overline{\overline{\mathbf{X}}}_k\left(\mathbf{I} + \mathbf{U}\right) - 2\mathbf{y}\mathbf{1}^T\right]$$

More resolution is needed at the beginning of pseudotimewhen integrating for large β.




probability density functions
prior
likelihood
posterior
4
2
0
−4
−3
−2
−1
0
1
$h(x)=\frac{7}{2}x^3-\frac{7}{2}x^2+8x$
histogram, EnKF
400
200
0
−4
−3
−2
−1
0
1
histogram, continuous EnKF
400
200
0
−4
−3
−2
−1
0
1
position

Modified Lorenz 40var model

- A model with the usual [slow] variables and coupled fast variables.

$$\frac{d}{dt}x_i = (x_{i+1} - x_{i-2})x_{i-1} - x_i + 8 \quad + \quad \frac{d^2}{dt^2}h_i = \frac{1}{\varepsilon^2}\left[-h_i + \alpha^2\left[h_{i+1} - 2h_i + h_{i-1}\right]\right]$$

$\alpha > 0$ Controls de dispersion in the grid.

$0 < \varepsilon << 1$ Controls the faster evolution (wrt the original variables).

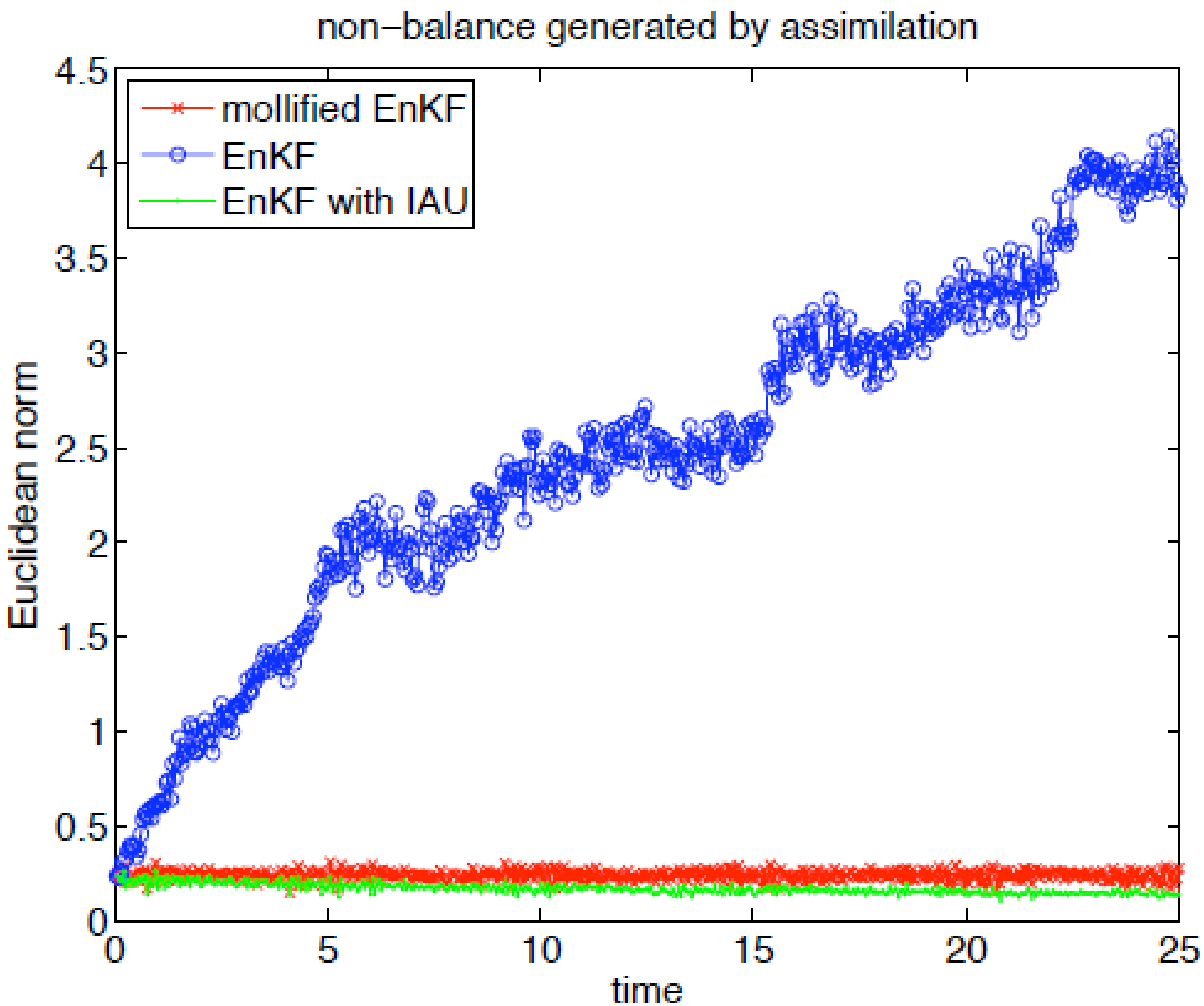
- Variables coupled through an energy exchange. $E_{coupling} = -\delta \sum_{i=1}^{40} h_i x_i$
- The new system

$$\frac{d}{dt}x_i = (1-\delta)(x_{i+1} - x_{i-2})x_{i-1} + \delta(x_{i-1}h_{i+1} - x_{i-2}h_{i-1}) - x_i + 8$$

$$\varepsilon^2 \frac{d^2}{dt^2}h_i = -h_i + \alpha^2\left[h_{i+1} - 2h_i + h_{i-1}\right] + x_i$$

$\delta > 0$ Coupling strength

Results of the MEnKF in the modified Lorenz 40var model



Transform formulations: Ensemble Transform Kalman Bucy Filters (ETKBFs)

For the perturbations:

$$\mathbf{X}^a = \mathbf{X}^b \mathbf{W}^a \qquad \mathbf{X} \in \Re^{N \times M} \quad M << N$$

$$\mathbf{W} \in \Re^{M \times M}$$

$$\frac{d\mathbf{W}}{ds} = -\frac{1}{2(M-1)} \mathbf{W}\mathbf{W}^T \mathbf{Y}^{b^T} \mathbf{R}^{-1} \mathbf{Y}^b \mathbf{W}$$

$$\mathbf{W}(0) = \mathbf{I} \qquad \mathbf{W}^a = \mathbf{W}(1)$$

Similar scheme for the full ensemble (Direct Ensemble Transform Kalman-Bucy Filter, DETKBF).

Models used

Lorenz 1963 model

Strongly nonlinear

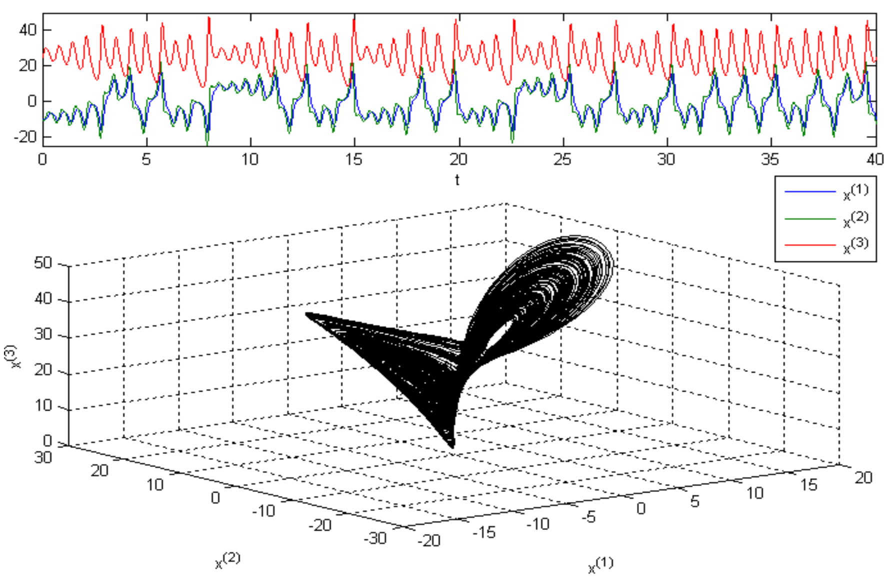
3-variable model

$$\dot{x}^{(1)} = \sigma\left(x^{(2)} - x^{(1)}\right) \qquad \sigma = 10$$

$$\dot{x}^{(2)} = x^{(1)}\left(r - x^{(3)}\right) - x^{(2)} \qquad r = 8/3$$

$$\dot{x}^{(3)} = x^{(1)}x^{(2)} - bx^{(3)} \qquad b = 28$$

Integrated using RK4 with $\Delta t = 0.01$



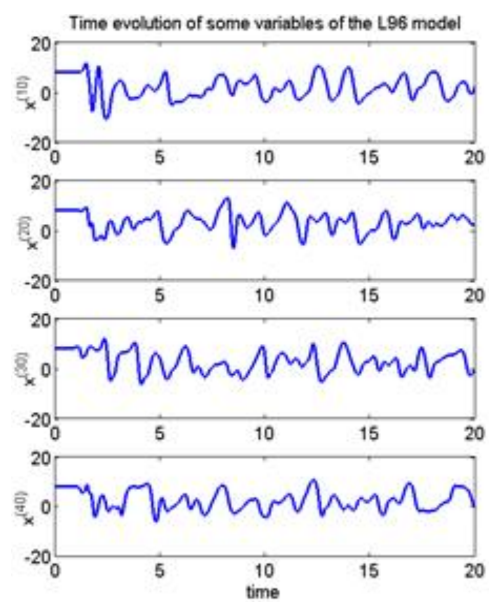
Lorenz 1996 model

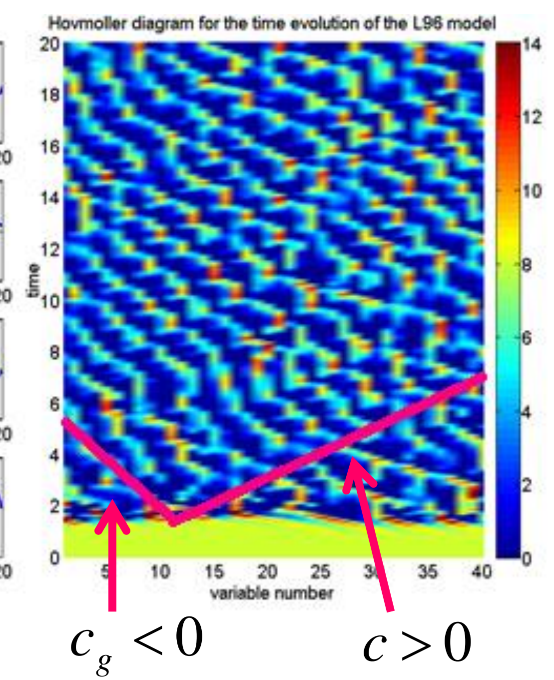
40-variable nonlinear cyclic model

$$\dot{x}^{(q)} = \left(x^{(q+1)} - x^{(q-2)}\right)x^{(q-1)} - x^{(q)} + F$$

$$q = 1,2,\ldots,40 \quad x^{(j)} \equiv x^{(\mathrm{mod}(j,40))} \quad F = 8$$

Integrated using RK4 with $\Delta t = 0.025$

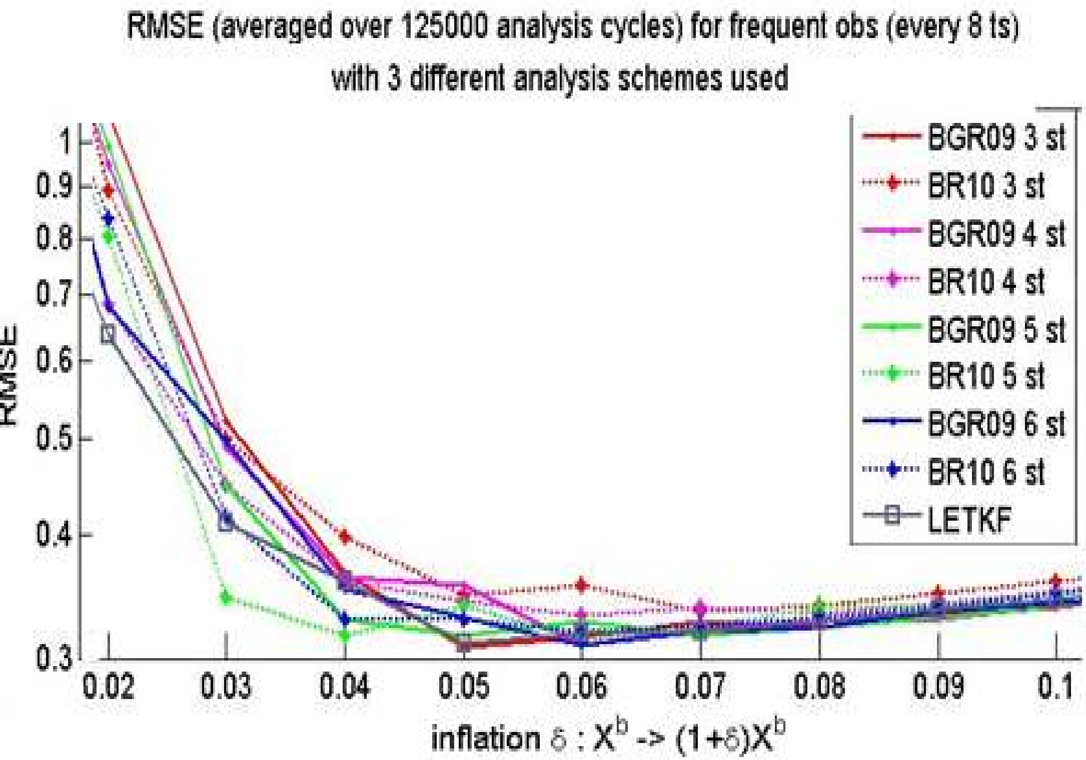


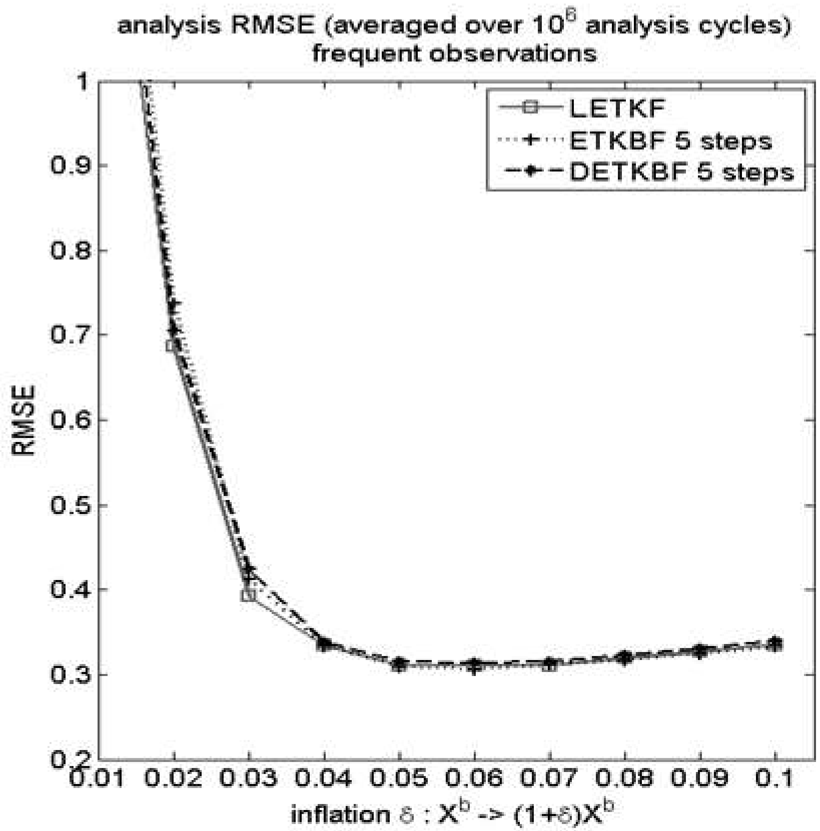


Experiments

Lorenz 1963, frequent observations

$$\mathbf{H} = \mathbf{I}, \mathbf{R} = 2\mathbf{I}, M = 3$$





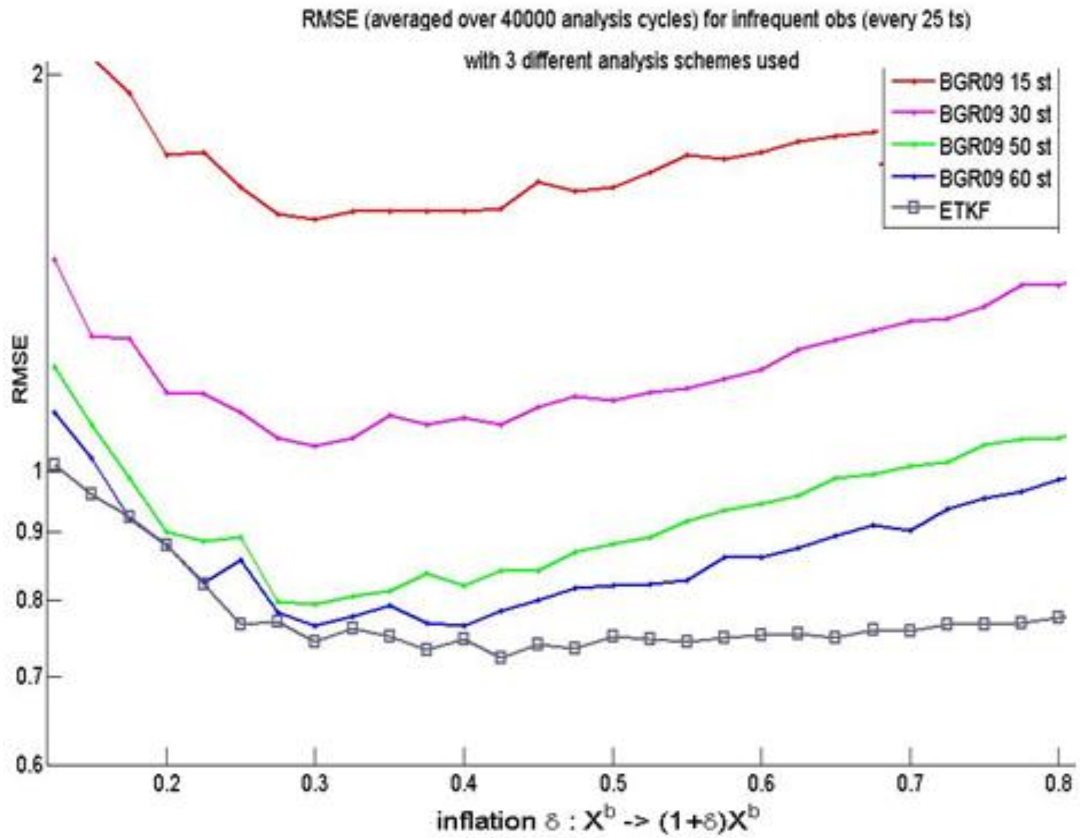
Euler Forward **Diagonal Semi-Implicit**

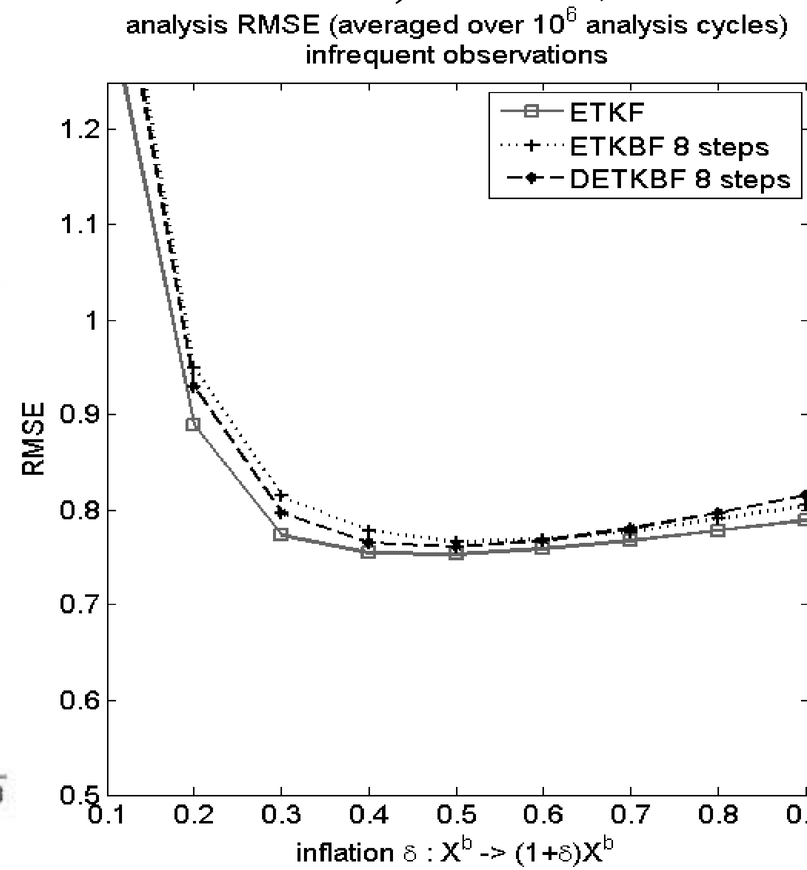
For frequent observations the performance is the same starting with 3 steps and regardless the integration scheme.

Experiments

Lorenz 1963, infrequent observations

$$\mathbf{H} = \mathbf{I}, \mathbf{R} = 2\mathbf{I}, M = 3$$





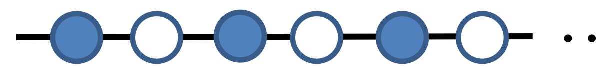
Euler Forward **Diagonal Semi-Implicit**

For infrequent observations, the EKBFs would not be affordable if it were not for the DSI method. The steps are reduced from ~60 to ~8.

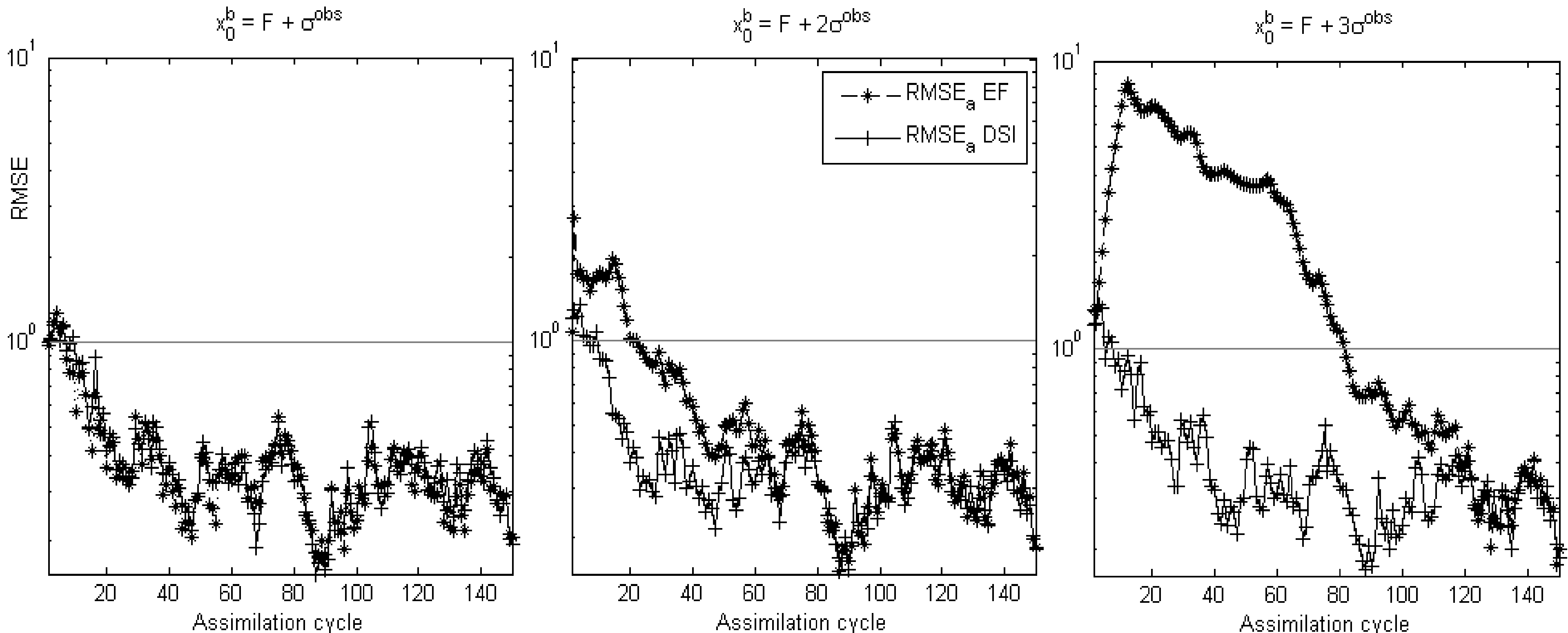
Experiments

Lorenz 1996, obs every 2 time steps, observing every other gridpoint:

$$\mathbf{R} = 2\mathbf{I}, M = 10$$



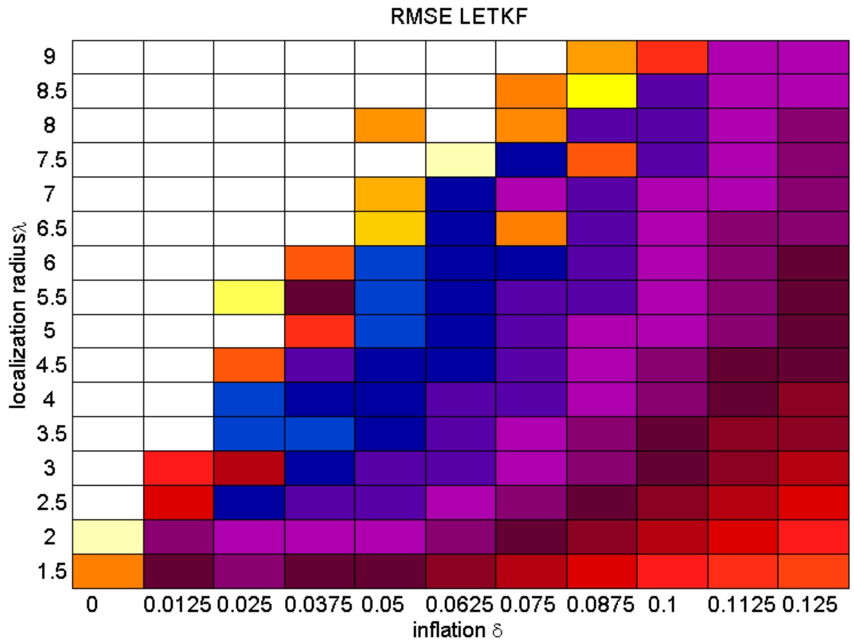
A 'cold' ensemble initialization benefits from the DSI method: if the initial conditions are very inaccurate, the EF integration will fail.

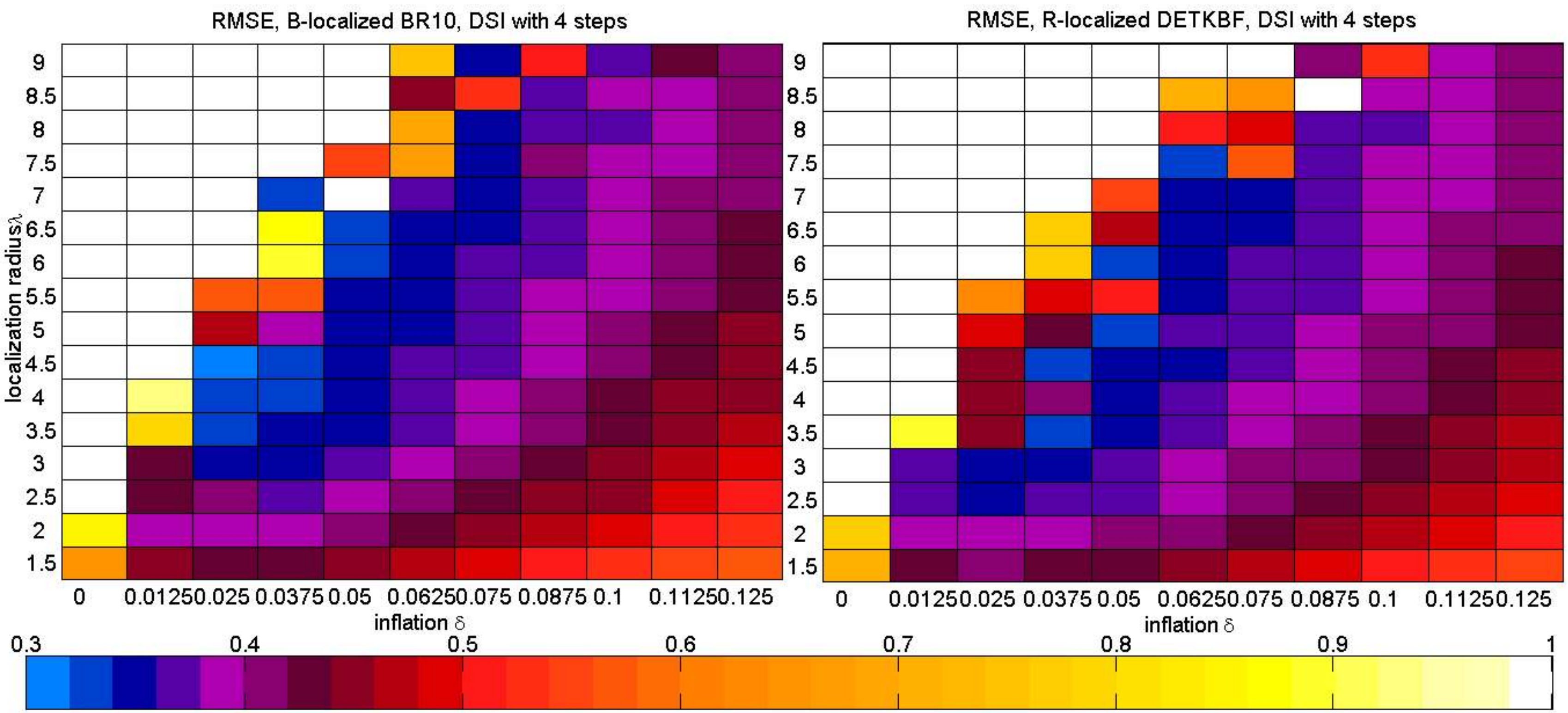


Experiments

Lorenz 1996: Experimenting with localization and inflation. $M = 10$

4 steps are enough for the ETKBFs to have comparable results to LETKF.

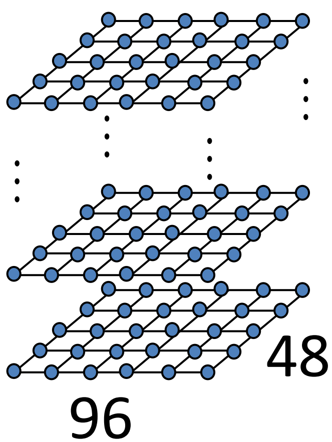


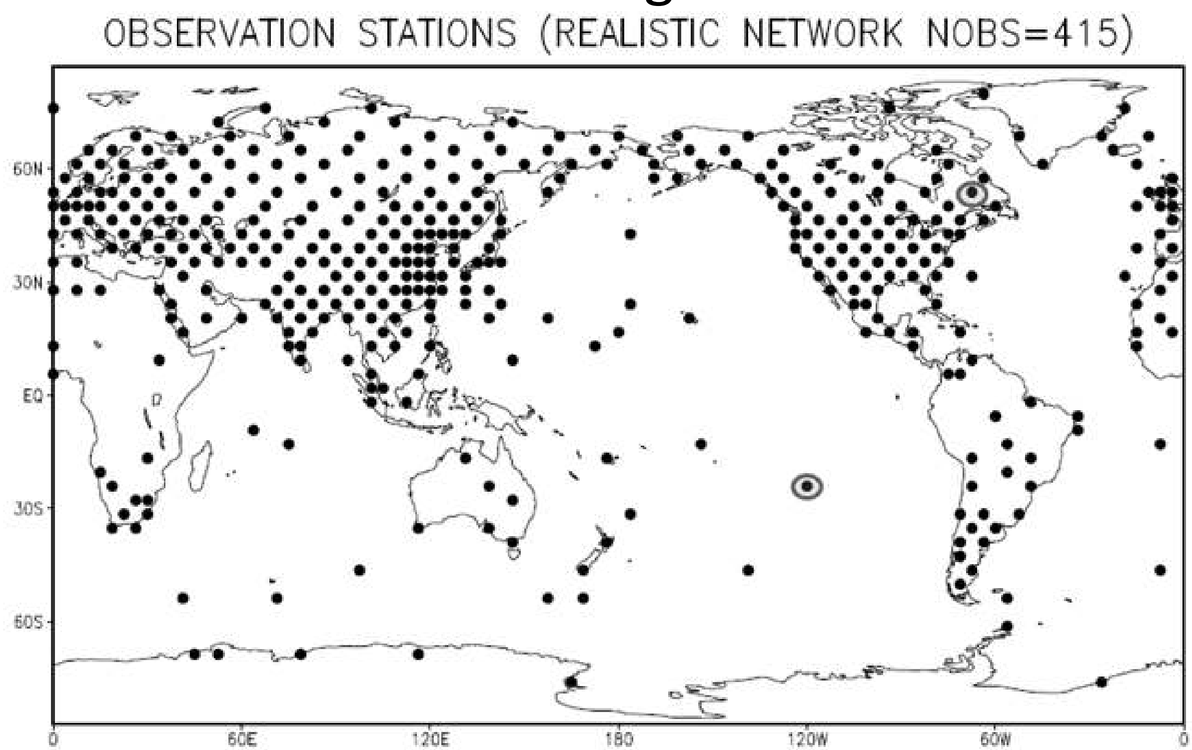


Models used

SPEEDY (Simplified Parameterizations, primitivE-Equations Dynamics, Molteni 2003)

- Time step is 40 minutes.
- Model variables: *u, v, T, q, ps*
- Spectral model with T30L7 resolution using σ-coordinates.





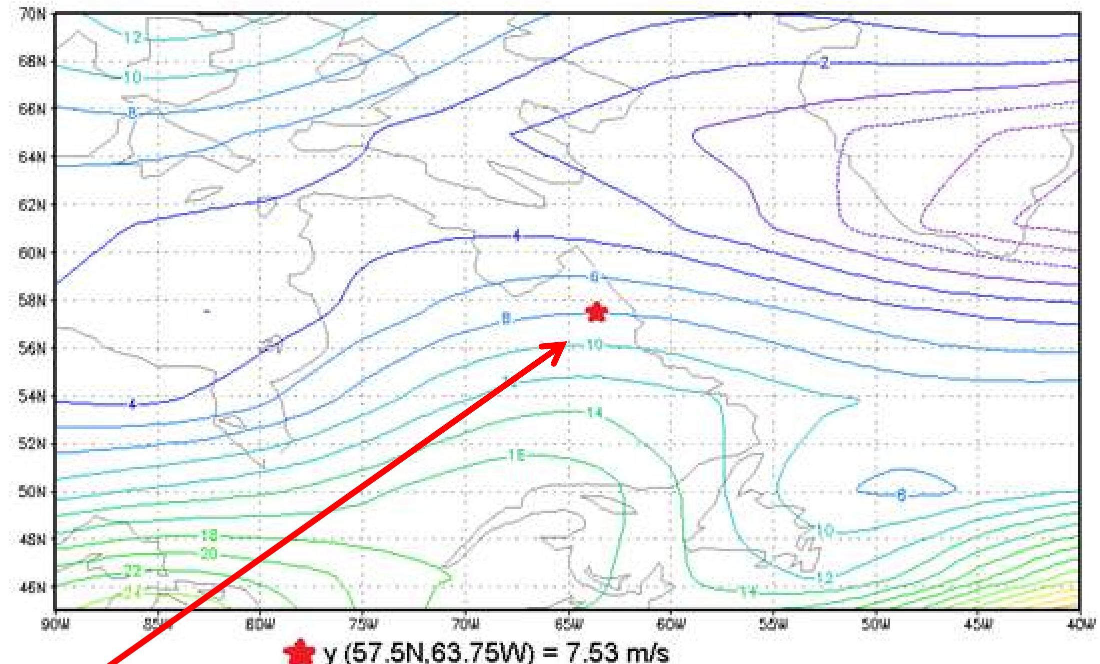
Experiments

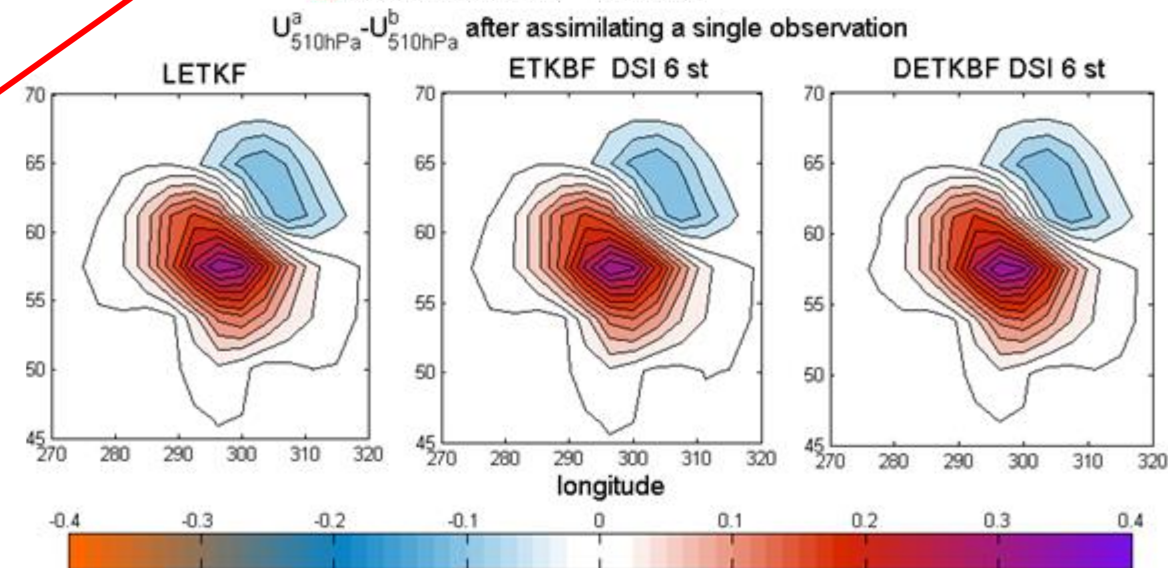
SPEEDY

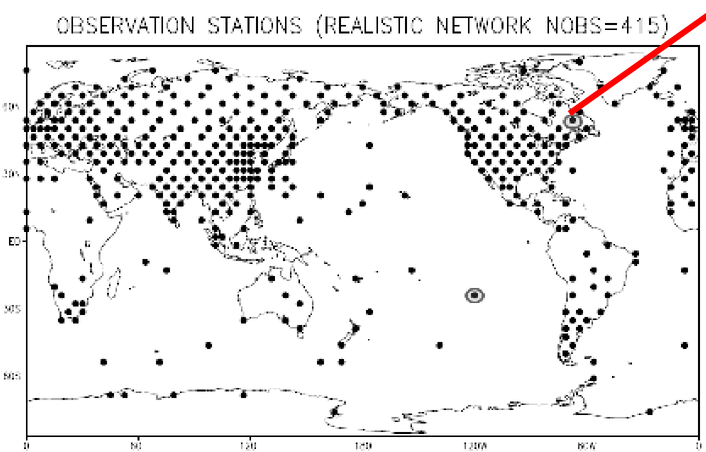
1 year of spin up, then 2 months of experiments.

Single obs experiment in a well-observed region.

4-6 steps guarantee good performance for ETKBFs.





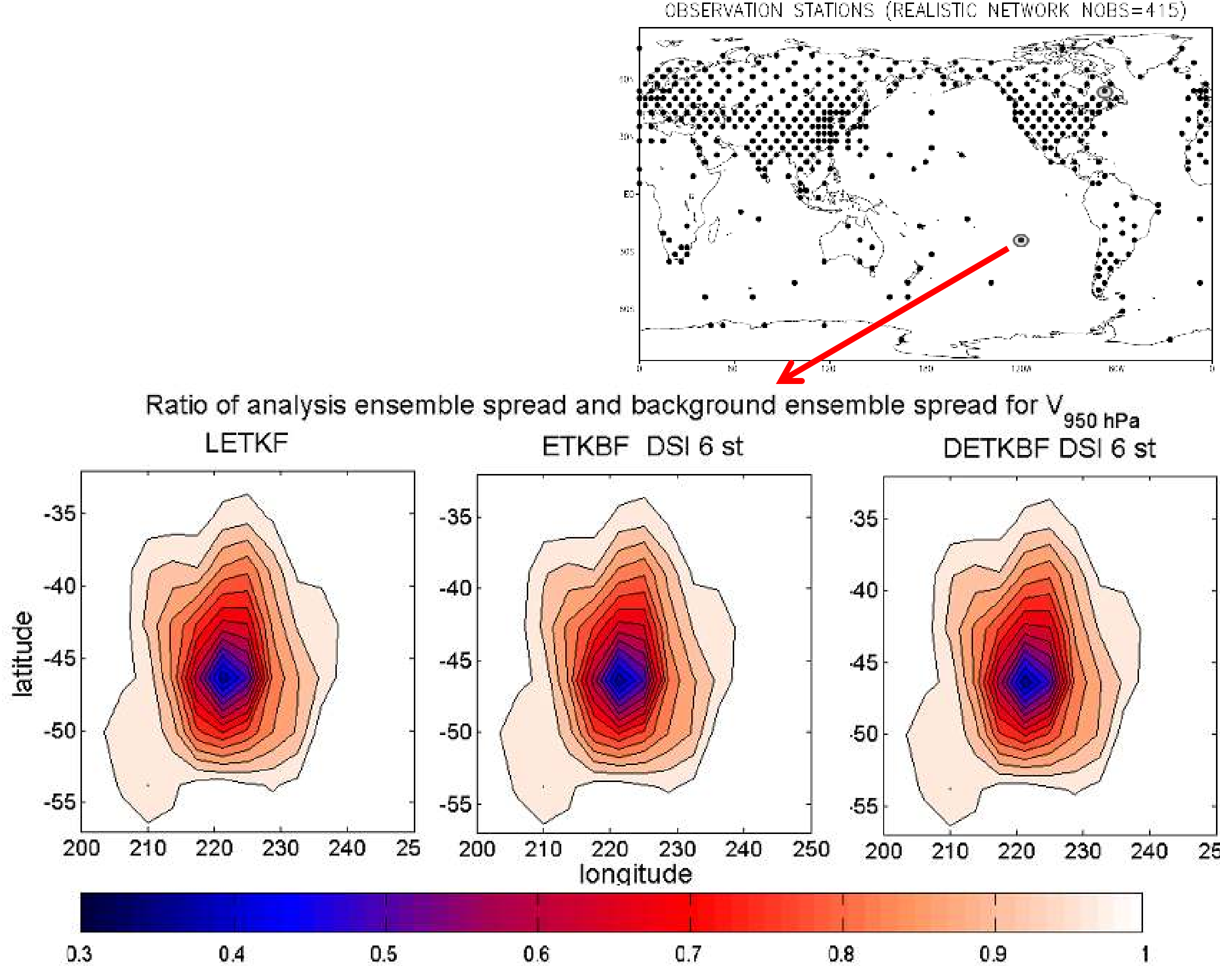


Experiments

SPEEDY

What about poorly-observed regions?

6 steps guarantee good performance for ETKBFs due to the DSI scheme.



The **latitude-weighted analysis RMSE** shows that the **performance** of the **three filters is indistinguishable** for all variables in all regions of the globe.

ETKBFs should be tested in real systems!

References

Amezcua J., Ide K., Kalnay E. and Reich S., 2013. Ensemble transform Kalman-Bucy filters. *Q. J. R. Meteorol. Soc.*, in print.

Bergemann K., Gottwald G. and Reich S., 2009. Ensemble propagation and continuous matrix factorization algorithms. *Q. J. R. Meteorol. Soc.*, **135**, 1560-1572.

Bergemann K. and Reich S., 2010. A localization technique for ensemble Kalman filters. *Q. J. R. Meteorol. Soc.*, **136**, 701-707.

Bergemann K. and Reich S., 2010a. A mollified ensemble Kalman filter. *Q. J. R. Meteorol. Soc.*, **136**, 1636-1643.