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Entangled coherent-state qubits in an ion trap

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We show how entangled qubits can be encoded as entangled coherent states of two-dimensional center-of-mass vibrational motion for two ions in an ion trap. The entangled qubit state is equivalent to the canonical Bell state, and we introduce a proposal for entanglement transfer from the two vibrational modes to the electronic states of the two ions in order for the Bell state to be detected by resonance fluorescence shelving methods.

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The qubit, or quantum bit, is the fundamental component of information in quantum computation. In the case of a spin-1/2 system, for some axis of orientation, one can identify the *up* state with an *on* state, generally written as $|1\rangle_L \equiv |\mathbf{1}\rangle$, for **1** indicating *on* and the L subscript indicating that this is a logical state. The $|0\rangle_L \equiv |\mathbf{0}\rangle$, or *off* state, then corresponds to *down* for this orientation. Qubits can thus be realized, in principle, in any spin-1/2 system, such as the electronic state of a two-level atom, the polarization of a single photon, or the vibrational state of an ion that is restricted to either zero- or one-phonon excitations. The concept of qubits is useful for quantum information considerations, but the qubit is also a useful construct for Bell inequality tests [1,2] and for considering the maximally entangled canonical Bell states.

It is not necessary to restrict a qubit encoding to systems with a two-dimensional Hilbert space. For example, a more exotic form of qubit can be constructed from superpositions of coherent states [3] and, as we show here, by employing entangled coherent states [4]. Despite both the nonorthogonality of coherent states and the unbounded Hilbert space, Bell inequality violations are possible in both limits $\alpha\rightarrow 0$ [4] and $\alpha\rightarrow\infty$ [5], for α the dimensionless amplitude of the coherent state. The $\alpha\rightarrow\infty$ limit is achieved by representing the entangled coherent states in a subspace corresponding to two-coupled spin-1/2 systems, and ideal canonical Bell states are realized in the $\alpha\rightarrow\infty$ limit. Entangled coherent states can include the entanglement of even and odd coherent states [6], which can also be treated as coupled spin-1/2 systems. The advantage of entangled even and odd coherent states, as we show, is that the states are distinguishable by parity, so that heating that changes the vibrational quanta corresponds to bit-flip errors, which can be detected and corrected via the appropriate circuit [7].

Here we show how the desired entangled coherent states can be created for the two-dimensional center-of-mass vibrational mode state of two trapped ions. This proposal involves the generalization of experimental techniques for generating even coherent states for the motional state of one ion in one dimension [8,9]. The advantage of distinguishing the logical states by phonon number parity has been shown for the case of one-dimensional motion [3,10]. We demonstrate that these entangled coherent states can be represented as entangled qubit states, and, moreover, such a state is equivalent to a canonical Bell state up to unitary transformation with respect to one of the two vibrational modes, which is up to a local unitary transformation. In order to make measurements on the entangled coherent states we give a procedure for swapping entanglement from the vibrational to the internal electronic states of the ions, which can then be read by resonance shelving methods.

The two-mode coherent state

$$|\alpha,\beta\rangle \equiv |\alpha\rangle_a \otimes |\beta\rangle_b \tag{1}$$

can be prepared in an entangled coherent state via the mutual phase-shift interaction $H_I=\hbar\chi a^\dagger a b^\dagger b$; this interaction has been studied in detail in the context of quantum nondemolition measurements [11] and for implementing phase gates for photon qubits [12]. In the ion trap, the two-mode coherent state corresponds to a two-dimensional Gaussian wave packet for the center-of-mass motion of the two trapped ions. The mutual phase-shift interaction between these two vibrational modes of freedom for the ion can be achieved by an appropriate Raman laser excitation [13].

After an interaction time $t=\pi/\chi$, the output state is [14]

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}(|\alpha\rangle_a\otimes|+\rangle_b + |-\alpha\rangle_a\otimes|-\rangle_b)\\ &= \frac{1}{\sqrt{2}}(|+\rangle_a\otimes|\beta\rangle_b + |-\rangle_a\otimes|-\beta\rangle_b),\end{aligned} \tag{2}$$

where the even and odd coherent states are defined by

$$\begin{aligned}|\pm\rangle_a &\equiv \mathcal{N}_\pm(\alpha)(|\alpha\rangle_a \pm |-\alpha\rangle_a),\\ |\pm\rangle_b &\equiv \mathcal{N}_\pm(\beta)(|\beta\rangle_b \pm |-\beta\rangle_b),\end{aligned} \tag{3}$$

with $\mathcal{N}_\pm$ being the appropriate normalization coefficients given by

$$\mathcal{N}_\pm(\xi) = 1/\sqrt{2\pm 2e^{-2|\xi|^2}}. \tag{4}$$

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We will generally ignore these normalization coefficients unless otherwise stated.

The state in Eq. (2) is equivalent, up to a local (single-oscillator) unitary transformation, to a Bell state for a particular encoding. Following Ref [3], the logical states are encoded in terms of even and odd coherent states, viz.,

$$|\mathbf{0}\rangle \leftrightarrow |+\rangle, \quad |\mathbf{1}\rangle \leftrightarrow |-\rangle, \tag{5}$$

and the Discrete Fourier transform states are represented by

$$\overline{|\mathbf{0}\rangle} \leftrightarrow |\mathbf{0}\rangle + |\mathbf{1}\rangle, \quad \overline{|\mathbf{1}\rangle} \leftrightarrow |\mathbf{0}\rangle - |\mathbf{1}\rangle. \tag{6}$$

We can ignore normalization coefficients and write the state (2) as

$$|\psi\rangle = \overline{|\mathbf{0}\rangle}_a \otimes |\mathbf{0}\rangle_b + \overline{|\mathbf{1}\rangle}_a \otimes |\mathbf{1}\rangle_b = |\mathbf{0}\rangle_a \otimes \overline{|\mathbf{0}\rangle}_b + |\mathbf{1}\rangle_a \otimes \overline{|\mathbf{1}\rangle}_b . \tag{7}$$

A single-qubit rotation on either oscillator a or b of the form

$$\overline{|\mathbf{0}\rangle} \rightarrow |\mathbf{0}\rangle, \quad \overline{|\mathbf{1}\rangle} \rightarrow |\mathbf{1}\rangle \tag{8}$$

leads to $|\psi\rangle$ in Eq. (7) being in the maximally entangled Bell state

$$|\phi^+\rangle \equiv |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle. \tag{9}$$

The Bell state (9) is entangled with respect to phonon number parity. That is, the two-dimensional oscillations are either both in even coherent states or in odd coherent states. A bit-flip error would destroy this parity entanglement. We now show how the prepared state $|\psi\rangle$ in Eq. (2) can be transformed into the Bell state (9).

We must be able to implement the qubit rotation in the logical basis of the mode, namely,

$$\begin{pmatrix} |\psi_0(\theta)\rangle \\ |\psi_1(\theta)\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\mathbf{0}\rangle \\ |\mathbf{1}\rangle \end{pmatrix}. \tag{10}$$

We present one simple, but approximate, scheme to achieve this rotation. For $D(\beta) \equiv \exp(\beta a^\dagger - \beta^* a)$ as the displacement operator, we can express the displaced coherent state as

$$D(\beta)|\alpha\rangle = e^{i\,\mathrm{Im}(\alpha\beta^*)}|\alpha+\beta\rangle, \tag{11}$$

which acquires a phase shift $\mathrm{Im}(\alpha\beta^*)$. Displacements can be effected in ion traps via the Raman laser scheme [8,9]. We assume that bosonic coding employs coherent states with real amplitudes, and we assume that $\epsilon \equiv -i\beta$ is real to obtain

$$D(i\epsilon)|\alpha\rangle \approx e^{i\alpha\epsilon}|\alpha+i\epsilon\rangle. \tag{12}$$

If we let $\theta = \alpha\epsilon$ be fixed, with $\epsilon \to 0$ and $\alpha \to \infty$, then we obtain the rotation (10) for

$$|\psi_0(\theta)\rangle \sim D(i\epsilon)|\mathbf{0}\rangle, \quad |\psi_1(\theta)\rangle \sim D(i\epsilon)|\mathbf{1}\rangle. \tag{13}$$

The displacement-effected rotation is approximate but adequate for sufficiently small ϵ. In order to quantify the effectiveness of this approach to rotation, we consider the fidelity of the operation

$$F = |\langle \psi_0(\theta)|D(i\epsilon)|\mathbf{0}\rangle|^2 = |\langle \psi_1(\theta)|D(i\epsilon)|\mathbf{1}\rangle|^2 = \exp(-\epsilon^2). \tag{14}$$

Here we have explicitly taken the normalization in Eq. (3) into account. The fidelity approaches unity exponentially with respect to ϵ^2 and hence is a good approximation for small ϵ.

The Bell state can thus be created for the state of the two-dimensional vibrational mode. However, direct detection of the Bell state is not possible with current technology. An entanglement transfer from the vibrational mode to the internal electronic states of the ions would allow detection of the entanglement due to the existence of the Bell state. The electronic state of an ion can be ''rotated'' and read with current technology.

In order to transfer entanglement from vibrational to electronic degrees of freedom, we need to be able to effect the transfer

$$(c_0|\mathbf{0}\rangle + c_1|\mathbf{1}\rangle)|0\rangle_e \rightarrow |\mathbf{0}\rangle(c_0|0\rangle_e + c_1|1\rangle_e), \tag{15}$$

for $\{|0\rangle_e, |1\rangle_e\}$ the two electronic states of the ion. The transfer (15) is achieved via the swap operation

$$|\mathbf{0}\rangle \otimes |0\rangle_e \rightarrow |\mathbf{0}\rangle \otimes |0\rangle_e, \tag{16a}$$

$$|\mathbf{0}\rangle \otimes |1\rangle_e \rightarrow |\mathbf{1}\rangle \otimes |0\rangle_e, \tag{16b}$$

$$|\mathbf{1}\rangle \otimes |0\rangle_e \rightarrow |\mathbf{0}\rangle \otimes |1\rangle_e, \tag{16c}$$

$$|\mathbf{1}\rangle \otimes |1\rangle_e \rightarrow |\mathbf{1}\rangle \otimes |1\rangle_e. \tag{16d}$$

The swap operation can be realized via a sequence of three controlled-NOT gates. The vibrational qubit is the control and the electronic qubit is the target for the first and third gates, and the reverse holds for the second qubit. We now discuss how to realize these two types of controlled-NOT gates.

In the first case, where the vibrational qubit is the control, it is necessary for the electronic qubit to be prepared in the ground state and to become excited if and only if the vibrational qubit contains an odd number of phonons. This transformation is achieved via the unitary operator

$$U_{ve} = \exp[-i\pi a^\dagger a \sigma_y]\exp[i\pi a^\dagger a|1\rangle_e\langle 1|], \tag{17}$$

which can be achieved by employing several Raman pulses at the carrier frequency. Schneider *et al.* [16] explicitly considered a unitary operator of the form $\exp[-i\pi a^\dagger a\sigma_z]$. Noting that $\sigma_y = U\sigma_z U^\dagger$, where U is a single-qubit rotation, the $\exp[-i\pi a^\dagger a\sigma_y]$ operator can be achieved by first applying a single-qubit rotation to the electronic state and then by performing the $\exp[-i\pi a^\dagger a\sigma_z]$ operation via Raman pulses.

The $\exp[-i\pi a^\dagger a\sigma_y]$ part of the unitary operator (17) transforms the input states as follows:

$$\exp[-i\pi a^\dagger a\sigma_y]|\mathbf{0}\rangle \otimes |0\rangle_e = |\mathbf{0}\rangle \otimes |0\rangle_e, \tag{18a}$$

$$\exp[-i\pi a^\dagger a\sigma_y]|\mathbf{0}\rangle \otimes |1\rangle_e = |\mathbf{0}\rangle \otimes |1\rangle_e, \tag{18b}$$

$$\exp[-i\pi a^\dagger a\sigma_y]|\mathbf{1}\rangle \otimes |0\rangle_e = -|\mathbf{1}\rangle \otimes |1\rangle_e, \tag{18c}$$

$$\exp[-i\pi a^{\dagger}a\sigma_y]|\mathbf{1}\rangle\otimes|1\rangle_e=|\mathbf{1}\rangle\otimes|0\rangle_e, \tag{18d}$$

whereas the operator $\exp[i\pi a^{\dagger}a|1\rangle_e\langle 1|]$ flips the sign of the $|\mathbf{1}\rangle\otimes|1\rangle_e$ term

$$\exp[i\pi a^{\dagger}a|1\rangle_e\langle 1|]|\mathbf{1}\rangle\otimes|1\rangle_e=-|\mathbf{1}\rangle\otimes|1\rangle_e, \tag{19}$$

while leaving the other states $|\mathbf{0}\rangle\otimes|0\rangle_e$, $|\mathbf{0}\rangle\otimes|1\rangle_e$ and $|\mathbf{1}\rangle\otimes|0\rangle_e$ unchanged. Hence the unitary transformation (17) is a CNOT with the vibrational modes being the control bit and the electronic mode the target.

The second controlled-NOT gate reverses the roles of the vibrational and electronic qubits. Therefore, phonon number parity must be changed if the ion is in the excited state. The required unitary transformation is the conditional displacement of the vibrational mode if and only if the ion is in the excited state, and such conditional displacements have been achieved experimentally [8,9]. The corresponding unitary operator is

$$U_{\mathrm{ev}}=\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]\exp[-i\pi|1\rangle_e\langle 1|/2], \tag{20}$$

with $\theta=\alpha\epsilon=\pi/2$. The $\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]$ part in Eq. (20) gives

$$\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]|\mathbf{0}\rangle\otimes|0\rangle_e=|\mathbf{0}\rangle\otimes|0\rangle_e, \tag{21a}$$

$$\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]|\mathbf{0}\rangle\otimes|1\rangle_e=-|\mathbf{1}\rangle\otimes|1\rangle_e, \tag{21b}$$

$$\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]|\mathbf{1}\rangle\otimes|0\rangle_e=|\mathbf{1}\rangle\otimes|0\rangle_e, \tag{21c}$$

$$\exp[i\epsilon(a+a^{\dagger})|1\rangle_e\langle 1|]|\mathbf{1}\rangle\otimes|1\rangle_e=-|\mathbf{0}\rangle\otimes|1\rangle_e, \tag{21d}$$

while the second term $\exp[-i\pi|1\rangle_e\langle 1|/2]$ flips the sign of the $|\mathbf{0}\rangle\otimes|1\rangle_e$ and $|\mathbf{1}\rangle\otimes|1\rangle_e$ states. Hence the unitary operator (20) performs the required controlled-NOT operation with the electronic mode as the control and the vibrational mode as the target.

It is straightforward to then show that the sequence

$$U_{\mathrm{swap}}=U_{\mathrm{ve}}U_{\mathrm{ev}}U_{\mathrm{ve}} \tag{22}$$

produces the desired entanglement swap. This sequence should be achievable with current experimentally technology.

In current ion trap experiments heating of the vibrational mode, though small, cannot be neglected. A simple model of heating for a vibrational mode with annihilation operator a is described by the master equation [15]

$$\frac{d\rho}{dt}=\frac{\gamma}{2}[2a^{\dagger}\rho a+2a\rho a^{\dagger} \tag{23}$$

$$-(a^{\dagger}a+aa^{\dagger})\rho-\rho(a^{\dagger}a+a^{\dagger}a)]. \tag{24}$$

This master equation describes two conditional point processes; one corresponds to an upward transition in phonon number at rate $\gamma\langle aa^{\dagger}\rangle$, and the other, to a downward transition at the rate $\gamma\langle a^{\dagger}a\rangle$. For the states discussed in this paper these two rates are approximately the same, at least initially. The mean value of the amplitude does not decay, but the average energy increases at the constant rate γ. We can thus model the heating by two independent jump processes. We will assume that over each run of the experiment, taking time τ, the heating rate is low enough that we only need to consider at most a single jump, either up or down, with probability $\delta=\gamma|\alpha|^2T$. If only a single jump occurs, no matter which way upward or downward, it flips the parity of the state. In other words, heating leads to bit-flip errors.

Up to a fidelity of $\exp(-\epsilon^2)$, the pure Bell state $\rho=\frac{1}{2}|\phi^+\rangle\langle\phi^+|$ is obtained. In order to test a Bell inequality with the entangled coherent states, a large number of runs of the experiment would need to be performed, and the state may vary from one run to the next if bit-flip errors occur. Thus the test of the Bell inequality is actually performed on a mixed state. Provided that a time interval τ is chosen such that the probability of more than one bit-flip error due to heating is negligible, the density matrix for the state can be expressed as

$$\rho=\frac{1}{2}(1-\delta)|\phi^+\rangle\left\langle\phi^+\right|+\frac{1}{2}\delta\left|\psi^+\right\rangle\langle\psi^+|, \tag{25}$$

with

$$|\psi^+\rangle\equiv|\mathbf{0}\rangle\otimes|\mathbf{1}\rangle+|\mathbf{1}\rangle\otimes|\mathbf{0}\rangle \tag{26}$$

being another of the four maximally entangled Bell states. The Bell state $|\psi^+\rangle$ is orthogonal to the desired state $|\phi^+\rangle$.

The state given by Eq. (25) for δ sufficiently small must violate the spin Bell inequality [1,2]

$$B=|E(\theta_1,\theta_2)+E(\theta_1,\theta_2')+E(\theta_1',\theta_2)-E(\theta_1',\theta_2')|\le 2, \tag{27}$$

where the correlation function $E(\theta_1,\theta_2)$ is given by the expectation value

$$E(\theta_1,\theta_2)=\langle\sigma_{\theta_1}^{(1)}\sigma_{\theta_2}^{(2)}\rangle. \tag{28}$$

Here the operator $\sigma_{\theta_i}^{(i)}$ may be defined as

$$\sigma_{\theta_i}^{(i)}=\cos\theta_i\sigma_x^{(i)}+\sin\theta_i\sigma_y^{(i)}, \tag{29}$$

where the operators $\sigma_a^{(i)}$ (with $a=x$, y or z) are the a Pauli spin operators for the two−level system of atom i. The tunable parameters θ_i ($i=1,2$) control the proportion of $\sigma_x^{(i)}$ to $\sigma_y^{(i)}$ in $\sigma_{\theta_i}^{(i)}$ and function like variable polarizers in the single−photon experiments.

In an ion trap this correlation function (28) is achieved by first applying single-qubit rotations to both ions and then by performing a simultaneous measurement of $\hat{\sigma}_z$ on both ions. The $\hat{\sigma}_z$ measurement is achieved with high precision via the shelving fluorescence technique [18]. The experiment is repeated over many runs and the average gives the desired correlation function $E(\theta_1,\theta_2)$. Mathematically, this correlation function can be expressed in the form

$$E(\theta_1,\theta_2)=\mathrm{Tr}[\rho\hat{V}_1^{1/2}(\theta_1)\sigma_z^{(1)}(\hat{V}_1^{1/2}(\theta_1))^{\dagger}\hat{V}_2^{1/2}(\theta_2)\sigma_z^{(2)}\times(\hat{V}_2^{1/2}(\theta_2))^{\dagger}] \tag{30}$$

where the Cirac and Zoller single-qubit rotations [17] $\hat{V}_i^k(\phi)$ on the ith ion are given by

$$\hat{V}_i^k(\phi)=\exp\left[-ik\frac{\pi}{2}(|1\rangle_i\langle 0|e^{-i\theta_i}+|0\rangle_i\langle 1|e^{i\theta_i})\right]. \tag{31}$$

This single-qubit rotation is achieved by applying a carrier pulse of length $k\pi$ with a phase θ_i.

Returning to the density matrix given by Eq. (25), it is easily shown that the correlation function (28) has the simplified form

$$E(\theta_1,\theta_2)=(1-\delta)\cos(\theta_1+\theta_2)+\delta\cos(\theta_1-\theta_2); \tag{32}$$

hence, the spin Bell inequality (27) for optimal-angles choices [19] reduces to

$$B=2\sqrt{2}(1-\delta). \tag{33}$$

A violation of this inequality is possible for $B>2$, when $\delta<1-1/\sqrt{2}$. Whereas the Bell inequality is technically violated, this does not present a loophole-free test due to the limited temporal separation of the ions. It does, however, completely close the detection loophole.

To summarize, we have described how entangled qubits can be encoded as entangled coherent states of two-dimensional center-of-mass vibrational motion for two ions in an ion trap. The entangled qubit state is equivalent to the canonical Bell state, and by transferring the entanglement from the two vibrational modes to the electronic states of the two ions, the Bell state can be detected by resonance fluorescence shelving methods.

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