

Entanglement Entropy and Conformal Field Theory

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Les Houches Winter School 2010

Mainly joint work with [John Cardy](#)
but also V. Alba, M. Campostrini, F. Essler, M. Fagotti, A.
Lefevre, B. Nienhuis, L. Tagliacozzo, E. Tonni

Review: PC & JC JPA 42, 504005 (2009)

If you want to know more:

Extensive reviews by Amico et al., Eisert et al. [RMP]

⊕ Special issue of JPA

IOP electronic journals

Journal of Physics A:
Mathematical and Theoretical

Volume 42, Number 50, 18 December 2009

SPECIAL ISSUE: ENTANGLEMENT ENTRPY IN EXTENDED QUANTUM SYSTEMS

INTRODUCTION

500301 FREE Entanglement entropy in extended quantum systems

Pasquale Calabrese, John Cardy and Benjamin Doyon (Guest Editors)

Full text [Acceso PDF](#) (214 KB)

REVIEWS

504001 FREE Entanglement and magnetic order

Luisa de la Cruz and Rosario Fazio

Abstract | References Full text [Acceso PDF](#) (930 KB)

504002 FREE A short review on entanglement in quantum spin systems

J. I. Latorre and A. Riera

Abstract | References | Citas | Citas

Full text [Acceso PDF](#) (518 KB)

504003 FREE Reduced density matrices and entanglement entropy in free lattice models

Iago Peschel and Volker Eisler

Abstract | References Full text [Acceso PDF](#) (846 KB)

504004 FREE Renormalization and inner product states in spin chains and lattices

J. Ignacio Cirac and Frank Verstraete

Abstract | References Full text [Acceso PDF](#) (15.5 MB)

504005 FREE Entanglement entropy and conformal field theory

Pasquale Calabrese and John Cardy

Abstract | References | Citas | Citas Full text [Acceso PDF](#) (725 KB)

504006 FREE Bi-partite entanglement entropy in massive (1+1)-dimensional quantum field theories

Óscar A. Castro-Alvaredo and Benjamin Doyon

Abstract | References Full text [Acceso PDF](#) (747 KB)

504007 FREE Entanglement entropy in free quantum field theory

J. J. Castell and M. Huerta

Abstract | References Full text [Acceso PDF](#) (737 KB)

504008 FREE Geometric entanglement entropy: an overview

Tomasz Kania, Steven Ryu and Tadashi Takayanagi

Abstract | References Full text [Acceso PDF](#) (755 KB)

504009 FREE Entanglement entropy in quantum impurity systems and systems with boundaries

Ian Affleck, Nicolas Leporićevic and Erik S. Sørensen

Abstract | References Full text [Acceso PDF](#) (1.29 MB)

504010 FREE Criticality and entanglement in random quantum systems

G. Refael and J. E. Moore

Abstract | References Full text [Acceso PDF](#) (433 KB)

504011 FREE Scaling of entanglement entropy at 2D quantum Lifshitz fixed points and topological fluids

Edwards Fradkin

Abstract | References Full text [Acceso PDF](#) (607 KB)

504012 FREE Entanglement between particle partitions in itinerant many-particle states

Moshef Huse, O. S. Zotov and K. Schoutens

Abstract | References Full text [Acceso PDF](#) (474 KB)

ISSN 1751-8113

Journal of Physics A

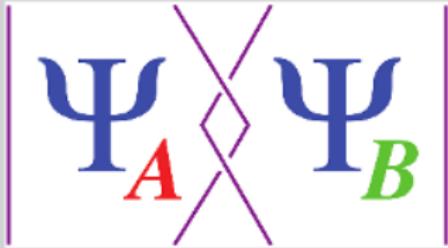
Mathematical and Theoretical

Volume 42 Number 50 18 December 2009

Special Issue

Entanglement entropy in extended quantum systems

Guest Editors: Pasquale Calabrese, John Cardy and Benjamin Doyon



What is new?

$$S_A = \frac{c}{3} \ln \frac{\ell}{a} + c'_1$$



What is new?

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}(n-1/n)}$$

$$S_{\textcolor{red}{A}} = \frac{c}{3} \ln \frac{\ell}{a} + c'_1$$



What is new?

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}(n-1/n)} + \dots ?$$

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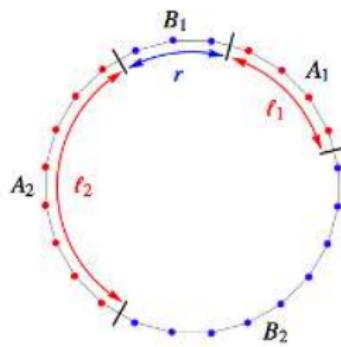


What is new?

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}(n-1/n)} + \dots ?$$

$$S_A = \frac{c}{3} \ln \frac{\ell}{a} + c'_1$$

And what about more intervals?



$$S_A = ? \quad \mathrm{Tr}\rho_A^n = ?$$



Lattice QFT in 1+1 dimensions: $\{\hat{\phi}(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x |\{\phi(x)\}\rangle$

The density matrix at temperature β^{-1} is ($Z = \text{Tr } e^{-\beta \hat{H}}$)

$$\rho(\{\phi_1(x)\} | \{\phi_2(x)\}) = Z^{-1} \langle \{\phi_2(x)\} | e^{-\beta \hat{H}} | \{\phi_1(x)\} \rangle$$

Euclidean path integral:

$$\rho = \frac{\int [d\phi(x, \tau)]}{Z} \prod_x \delta(\phi(x, 0) - \phi_2(x)) \prod_x \delta(\phi(x, \beta) - \phi_1(x)) e^{-S_E}$$

$S_E = \int_0^\beta L_E d\tau$, with L_E the Euclidean Lagrangian

The trace sews together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β .

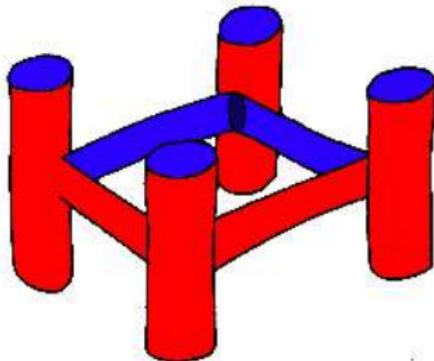
$A = (u_1, v_1), \dots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A , leaving open cuts for (u_j, v_j) along the line $\tau = 0$.

$$\rho_A = \int_{x \in B} [d\phi(x, 0)] \delta(\phi(x, \beta) - \phi(x, 0)) \rho$$



$$S_A = -\text{Tr} \rho_A \log \rho_A = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n$$

$\text{Tr} \rho_A^n$ (for integer n) is the partition function on n of the above cylinders attached to form an n -sheeted Riemann surface



$$= \rho_A^{ij} \rho_A^{jk} \rho_A^{kl} \rho_A^{li}$$

$\text{Tr} \rho_A^n$ has a unique analytic continuation to $\text{Re } n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$



CFT: a remind

- A physical systems at a quantum critical point is scale invariant
 $\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) \rangle = b^{2\Delta_\phi} \langle \phi(b\mathbf{r}_1)\phi(b\mathbf{r}_2) \rangle$ $\langle \phi(\mathbf{r}_1)\phi(\mathbf{r}_2) \rangle = |\mathbf{r}_1 - \mathbf{r}_2|^{-2\Delta_\phi}$
- A Hamiltonian that is invariant under translations, rotations, and scaling transformations has usually the symmetry of the larger *conformal* group defined as the set of transformations that do not change the angles.

• In 2D the consequences are extraordinary:
YOU DON'T NEED THIS!

all the analytic functions $w(z)$ are conformal



$$\langle \phi(z_1)\phi(z_2) \rangle = |w'(z_1)w'(z_2)|^{2\Delta_\phi} \langle \phi(w(z_1))\phi(w(z_2)) \rangle$$

- Under an arbitrary transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu$

$$S \rightarrow S + \delta S, \quad \text{with} \quad \delta S = \int d^2x T^{\mu\nu} \partial_\mu \epsilon_\nu$$

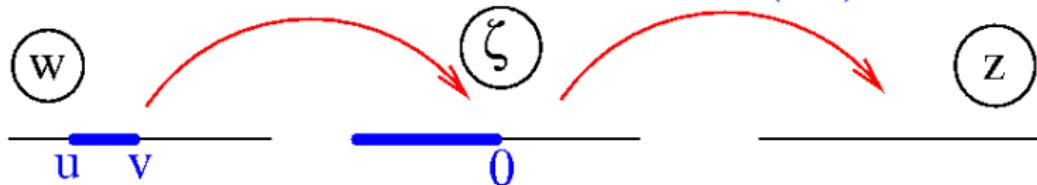
Under a conformal transformation $w \rightarrow z$

$$T(w) = \left(\frac{dz}{dw} \right)^2 T(z) + \frac{c}{12} \frac{z'''z' - 3/2 z''^2}{z'^2}$$



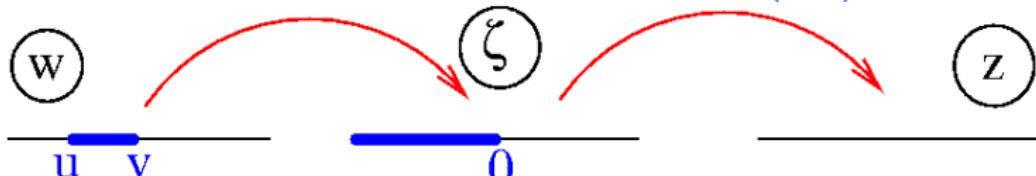
Single interval $A = [u, v]$. We need $Z_n/Z^n = \langle 0|0 \rangle_{\mathcal{R}_n} \Rightarrow$ compute $\langle T(w) \rangle_{\mathcal{R}_n}$

$$w \rightarrow \zeta = \frac{w-u}{w-v}; \quad \zeta \rightarrow z = \zeta^{1/n} \Rightarrow w \rightarrow z = \left(\frac{w-u}{w-v} \right)^{1/n}$$



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$$\langle T(z) \rangle_C = 0 \Rightarrow$$

$$\langle T(w) \rangle_{\mathcal{R}_n} = \frac{c(1 - (1/n)^2)}{24} \frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the
Conformal Ward identities

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v) \rangle_C}{\langle \Phi_n(u)\Phi_{-n}(v) \rangle_C} = \frac{\Delta_\Phi(v-u)^2}{(w-u)^2(w-v)^2}$$

Z_n/Z^n transforms under conformal transformations as n^{th} power of the two point function of a (fake) primary field on the plane with scaling dimension

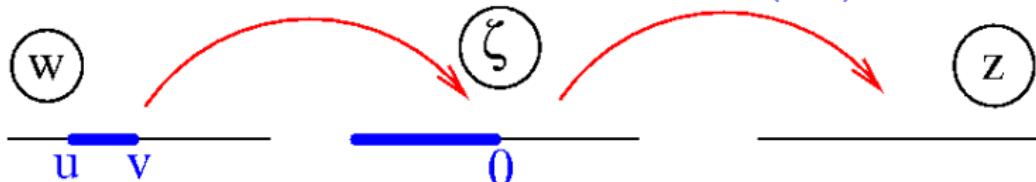
$$\Delta_\Phi = \bar{\Delta}_\Phi = \frac{c}{24} \left(1 - \frac{1}{n^2} \right) \Rightarrow$$

$$\text{Tr } \rho_A^n = \frac{Z_n}{Z^n} = c_n \left(\frac{v-u}{a} \right)^{-(c/6)(n-1/n)}$$



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Finally with the replica trick ($v-u=\ell$) \Rightarrow

$$S_A = \frac{c}{3} \ln \frac{\ell}{a} + c'_1$$



Finite temperature

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + c'_1 \simeq \begin{cases} \frac{\pi c}{3} \frac{\ell}{\beta}, & \ell \gg \beta \quad \text{classical extensive} \\ \frac{c}{3} \log \frac{\ell}{a}, & \ell \ll \beta \quad T = 0 \text{ non-extensive} \end{cases}$$

Finite size

$$S_A = \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c'_1 \quad \text{Symmetric } \ell \rightarrow L - \ell. \text{ Maximal for } \ell = L/2$$



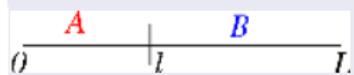
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Open systems (generally systems with boundaries)



$$\text{Tr } \rho_A^n = \tilde{c}_n \left(\frac{2\ell}{a} \right)^{\frac{c}{12}(n-\frac{1}{n})} \Rightarrow S_A = \frac{c}{6} \log \frac{2\ell}{a} + \tilde{c}'_1$$

finite temperature $S_A(\beta) = \frac{c}{6} \log \left(\frac{\beta}{\pi a} \sinh \frac{2\pi\ell}{\beta} \right) + \tilde{c}'_1$

and finite size $S_A(L) = \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi \ell}{L} \right) + \tilde{c}'_1$

$\tilde{c}'_1 - c'_1/2 = \ln g$ boundary entropy [Affleck, Ludwig]



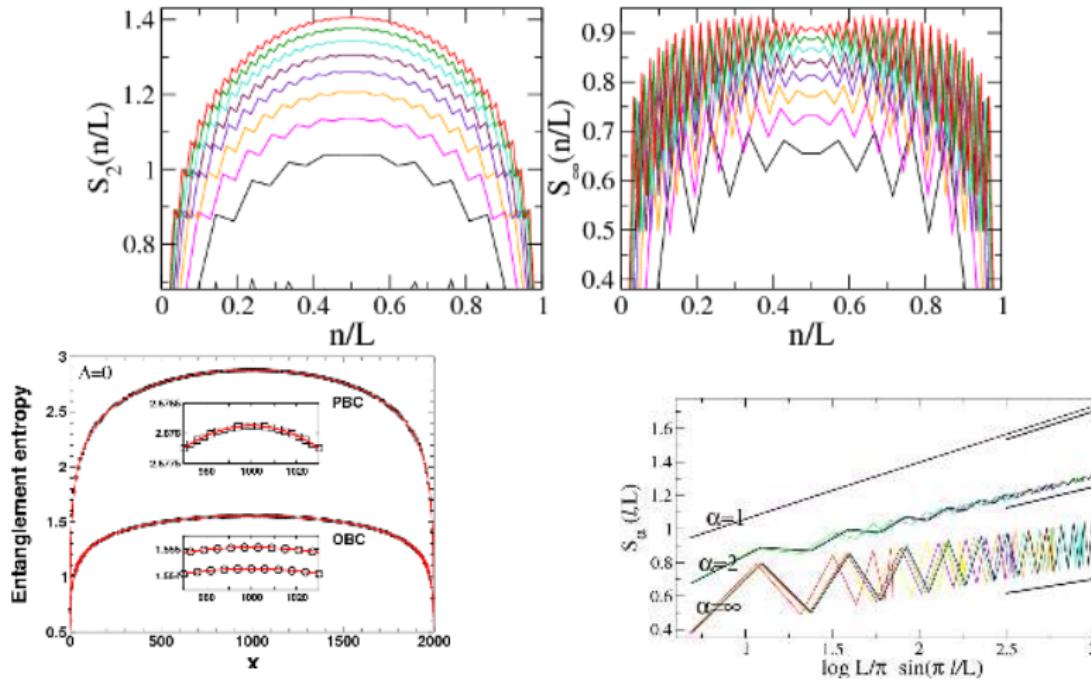
Correction to the scaling (Hystory)

$$S_\alpha \equiv \frac{1}{1-\alpha} \ln \text{Tr} \rho_A^\alpha = \frac{c}{6} (1 + \alpha^{-1}) \ln \frac{L}{\pi} \sin \frac{\pi \ell}{L} + c'_\alpha$$



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[Laflorencie et al '06]

Corrections to the scaling: Guess?

$$S_\alpha = \frac{c}{6}(1 + \alpha^{-1}) \ln \ell + c'_\alpha + ?$$



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+ $\ell^{-2(x-2)}$ with x dim. Irr Op?

You know FSS, but it is WRONG!



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Sorry, this is the right answer



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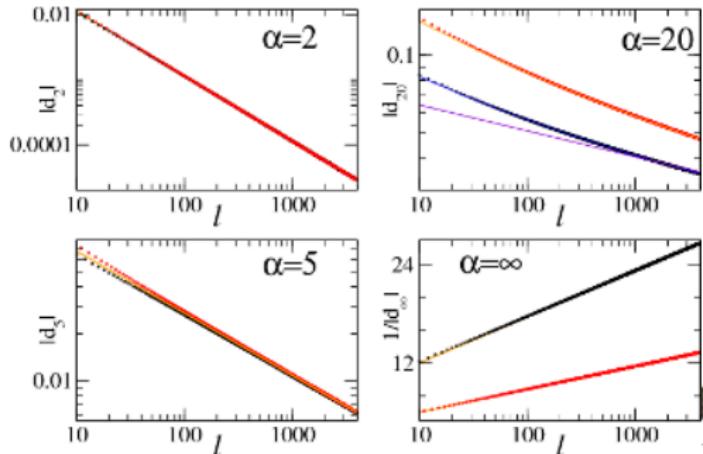
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Numerical evidence for XX
model (with $x = 1$):

$$d_\alpha(\ell) \equiv S_\alpha(\ell) - S_\alpha^{\text{CFT}}(\ell)$$



Exact Solution XX model PC, F Essler

$$S_\alpha(\ell) - S_\alpha^{\text{CFT}}(\ell) = (-)^\ell \ell^{-2/\alpha} f_\alpha + \dots$$

$$f_\alpha = \frac{2}{1-\alpha} \frac{\Gamma^2((1+1/\alpha)/2)}{\Gamma^2((1-1/\alpha)/2)} \xrightarrow{\alpha \rightarrow 1} 0$$

$$S_\infty(\ell) - S_\infty^{\text{CFT}}(\ell) = \begin{cases} \frac{\pi^2}{12} \frac{1}{\ln b\ell} & \ell \text{ odd} \\ -\frac{\pi^2}{24} \frac{1}{\ln b\ell} & \ell \text{ even} \end{cases}$$

In a magnetic field:

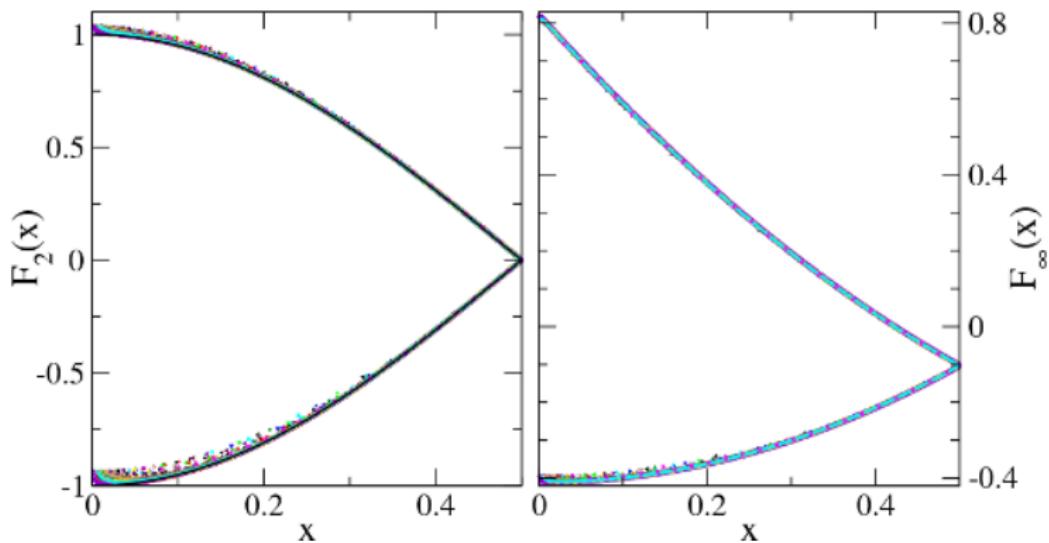
$$S_\alpha(\ell) - S_\alpha^{\text{CFT}}(\ell) = f_\alpha \cos(2k_F \ell) |2\ell \sin k_F|^{-2/\alpha} + \dots$$

PS: For the Ising model it is the same $p_\alpha = 2/\alpha$

$$S_\alpha(\ell, L) = S_\alpha^{\text{CFT}}(\ell, L) + (-)^\ell f_\alpha \left[\frac{L}{\pi} \sin \frac{\pi \ell}{L} \right]^{-\rho_\alpha} F_\alpha(\ell/L)$$

XX in Finite Size

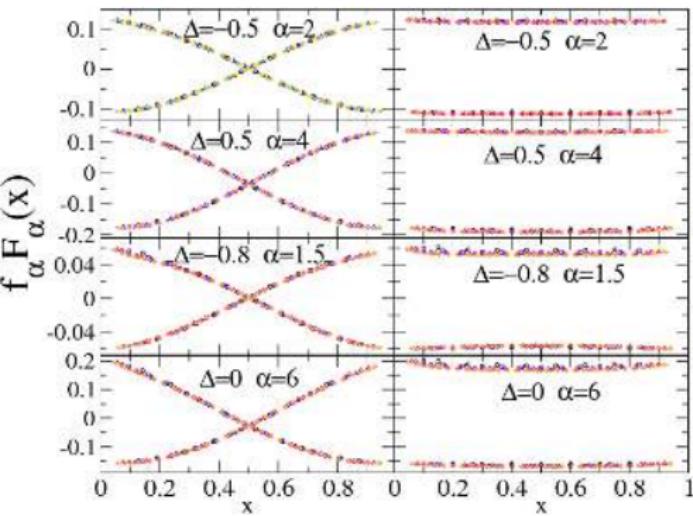
All ℓ for several odd L from 17 to 4623 [$\sim 10^5$ points]:



$F_2(x) = \cos \pi x$?



$$S_\alpha(\ell, L) = S_\alpha^{\text{CFT}}(\ell, L) + (-)^\ell f_\alpha \left[\frac{L}{\pi} \sin \frac{\pi \ell}{L} \right]^{-p_\alpha} F_\alpha(\ell/L)$$

Odd L Even L 

- p_α and $F_\alpha(x)$ are universal
- Similar plots for any Δ , odd and even L , any α
- $p_\alpha = 2K/\alpha$
- F_α depends on the parity of L



Corrections to Scaling in CFT J Cardy, PC

Add an irrelevant operator Φ of dimension x : $S = S_{CFT} + \lambda \int \Phi(z) d^2 z$,

Change in free energy: $-\delta F_n = \sum_{N=1}^{\infty} \frac{(-\lambda)^N}{N!} \int_{\mathcal{R}_n} \dots \int_{\mathcal{R}_n} \langle \Phi(z_1) \dots \Phi(z_N) \rangle_{\mathcal{R}_n} d^2 z_1 \dots d^2 z_N$,

$\zeta = (z/(z - \ell))^{1/n}$ maps \mathcal{R}_n on the plane

$$\delta F_n^{(2)} = -\frac{1}{2} g^2 (n\ell/\epsilon)^{4-2x} \int_{\mathbb{C}'} \int_{\mathbb{C}'} \frac{|\zeta_1 \zeta_2|^{(2-x)(n-1)}}{|\zeta_1^n - 1|^{4-2x} |\zeta_2^n - 1|^{4-2x} |\zeta_1 - \zeta_2|^{2x}} d^2 \zeta_1 d^2 \zeta_2.$$

For $n - 1$ small the integral is convergent! \Rightarrow corrections $\ell^{-2(x-2)}$

For larger values of n divergences as $\zeta_j \rightarrow 0$ or ∞ . Regulator ϵ in z plane!

Any of them gives a multiplicative factor $\propto \epsilon^{2-x+(x/n)}$

Summing up: $\ell^{-2(x-2)-(2-x+(x/n))-(2-x+(x/n))} = \ell^{-2x/n}$

But $x > 2$... not only:

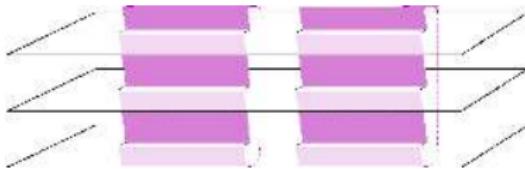
At the conical singularities relevant operators can be generated *locally*

The marginal case $x = 2$ introduces logarithms and has to do with c-theorem



Disjoint intervals: History

$$A = [u_1, v_1] \cup [u_2, v_2]$$



In 2004 we obtained

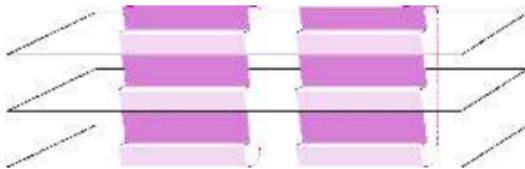
$$\mathrm{Tr} \rho_A^n = c_n^2 \left(\frac{|u_1 - u_2| |v_1 - v_2|}{|u_1 - v_1| |u_2 - v_2| |u_1 - v_2| |u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)}$$

Tested for free fermions in different ways [Casini-Huerta, Florio et al.](#)



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Tested for free fermions in different ways [Casini-Huerta, Florio et al.](#)

For more complicated theories in 2009 [Furukawa-Pasquier-Shiraishi](#) and [Caraglio-Gliozzi](#) showed that it is wrong!

$$\mathrm{Tr} \rho_A^n = c_n^2 \left(\frac{|u_1 - u_2| |v_1 - v_2|}{|u_1 - v_1| |u_2 - v_2| |u_1 - v_2| |u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x)$$

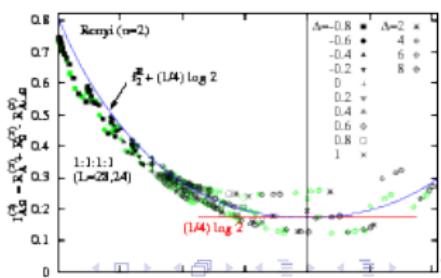
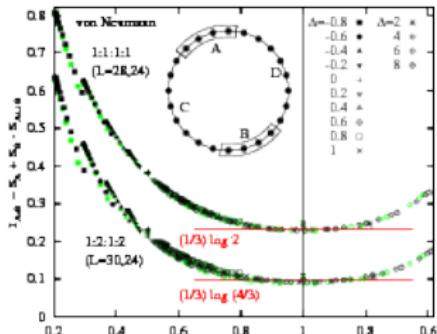
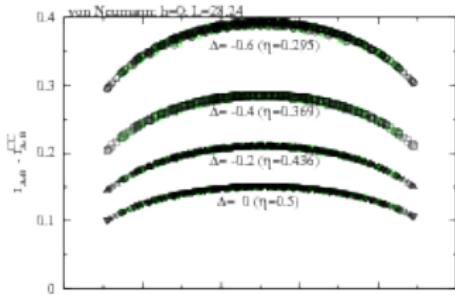
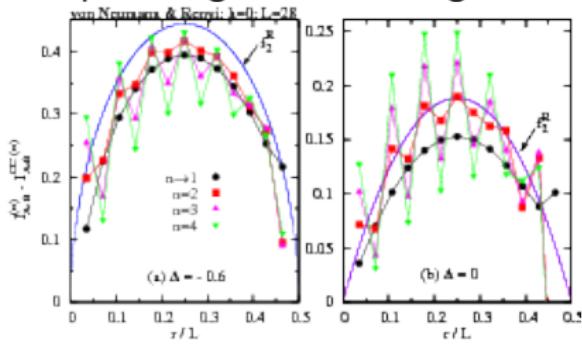
$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)} = 4 - \text{point ratio}$$



Compactified boson (Luttinger) Furukawa Pasquier Shiraishi

$$F_2(x) = \frac{\theta_3(\eta\tau)\theta_3(\tau/\eta)}{[\theta_3(\tau)]^2}, \quad x = \left[\frac{\theta_2(\tau)}{\theta_3(\tau)} \right]^4 \quad \eta = \frac{1}{2K} \propto R^2$$

Compared again exact diagonalization in XXZ chain



Using old results of CFT
on orbifolds Dixon et al 86

Γ is an $(n - 1) \times (n - 1)$ matrix

$$\boxed{\mathcal{F}_n(x) = \frac{\Theta(0|\eta\Gamma) \Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}}$$

$$\Gamma_{rs} = \frac{2i}{n} \sum_{k=1}^{n-1} \sin\left(\pi \frac{k}{n}\right) \beta_{k/n} \cos\left[2\pi \frac{k}{n}(r-s)\right], \quad \beta_y = \frac{{}_2F_1(y, 1-y; 1; x)}{{}_2F_1(y, 1-y; 1; x)}$$

Riemann theta function $\Theta(z|\Gamma) \equiv \sum_{m \in \mathbb{Z}^{n-1}} \exp [i\pi m \cdot \Gamma \cdot m + 2\pi i m \cdot z]$

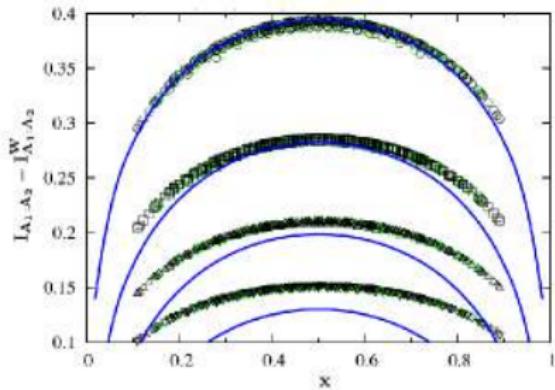
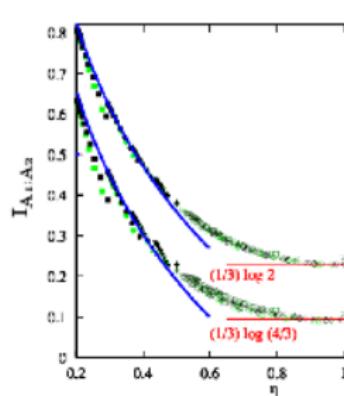
- $\mathcal{F}_n(x)$ invariant under $x \rightarrow 1 - x$ and $\eta \rightarrow 1/\eta$
- We are unable to analytic continue to real n for general x and η
- Only for $\eta \ll 1$ and for $x \ll 1$



Compactified boson II PC Cardy Tonni

$\eta \ll 1$

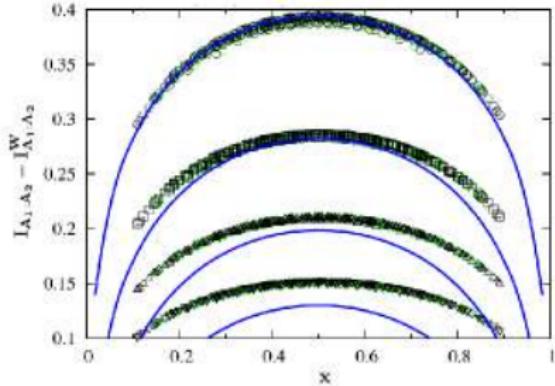
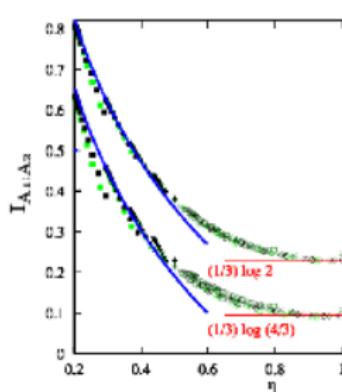
$$-F_1'(x) = \frac{1}{2} \ln \eta - \frac{D_1'(x) + D_1'(1-x)}{2} \quad \text{with } D_1'(x) = - \int_{-i\infty}^{i\infty} \frac{dz}{i} \frac{\pi z}{\sin^2 \pi z} \log H_z(x)$$



Compactified boson II PC Cardy Tonni

$\eta \ll 1$

$$-F'_1(x) = \frac{1}{2} \ln \eta - \frac{D'_1(x) + D'_1(1-x)}{2} \quad \text{with } D'_1(x) = - \int_{-i\infty}^{i\infty} \frac{dz}{i} \frac{\pi z}{\sin^2 \pi z} \log H_z(x)$$



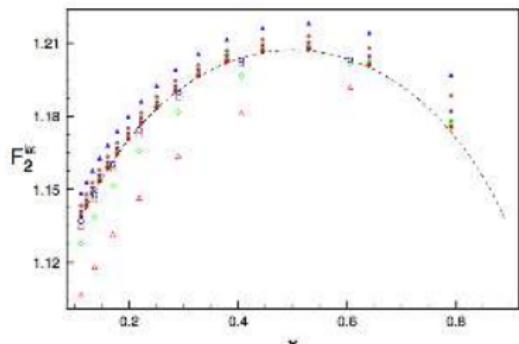
$x \ll 1$

$$F_n(x) = 1 + 2n \left(\frac{x}{4n^2} \right)^\alpha P_n + \dots \quad \alpha = \min(\eta, 1/\eta) \quad P_n = \sum_{l=1}^{n-1} \frac{l/n}{[\sin(\pi l/n)]^{2\alpha}}$$

$$-F'_1(x) = 2^{1-2\alpha} x^\alpha P'_1 + \dots$$



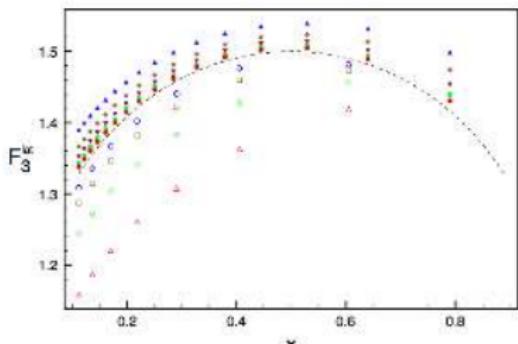
The RDM of two intervals is not trivial because of JW string



XX chain

$L \rightarrow \infty$

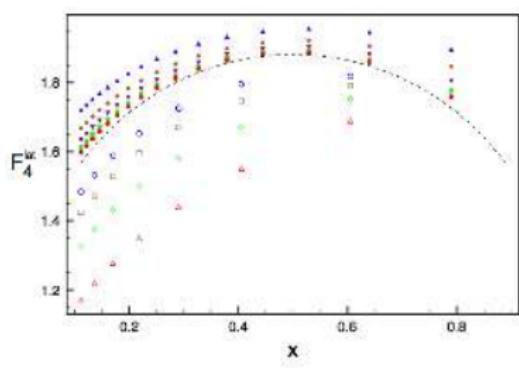
- $\delta = 16$
- $\delta = 32$
- $\delta = 64$
- $\delta = 128$
- $\delta = 256$
- $\delta = 7$
- $\delta = 21$
- $\delta = 63$
- $\delta = 189$
- CFT



XX chain

$L \rightarrow \infty$

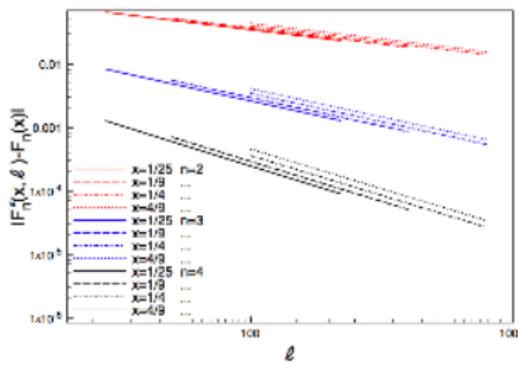
- $\delta = 16$
- $\delta = 32$
- $\delta = 64$
- $\delta = 128$
- $\delta = 256$
- $\delta = 7$
- $\delta = 21$
- $\delta = 63$
- $\delta = 189$
- CFT



XX chain

$L \rightarrow \infty$

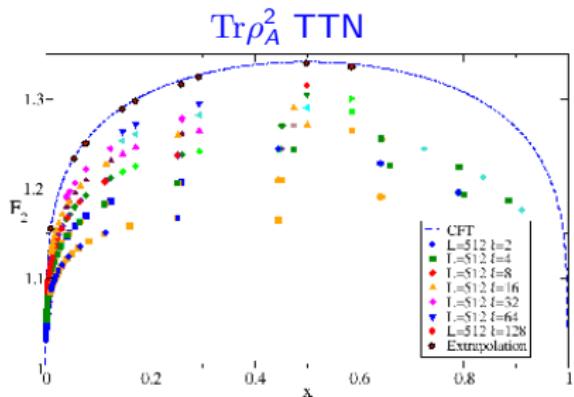
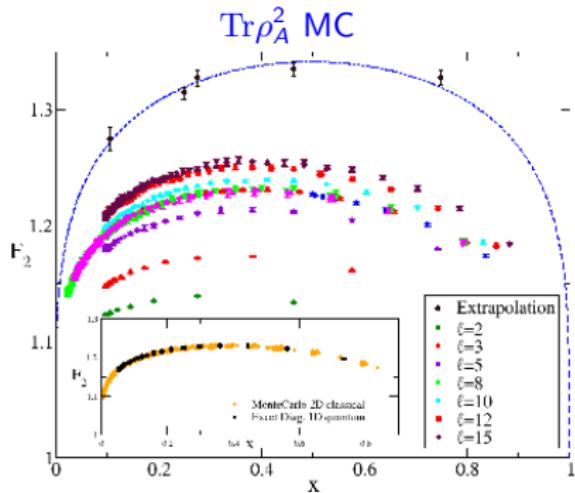
- $\delta = 16$
- $\delta = 32$
- $\delta = 64$
- $\delta = 128$
- $\delta = 256$
- $\delta = 7$
- $\delta = 21$
- $\delta = 63$
- $\delta = 189$
- CFT



CFT OK and $\delta_\alpha = 2/\alpha$

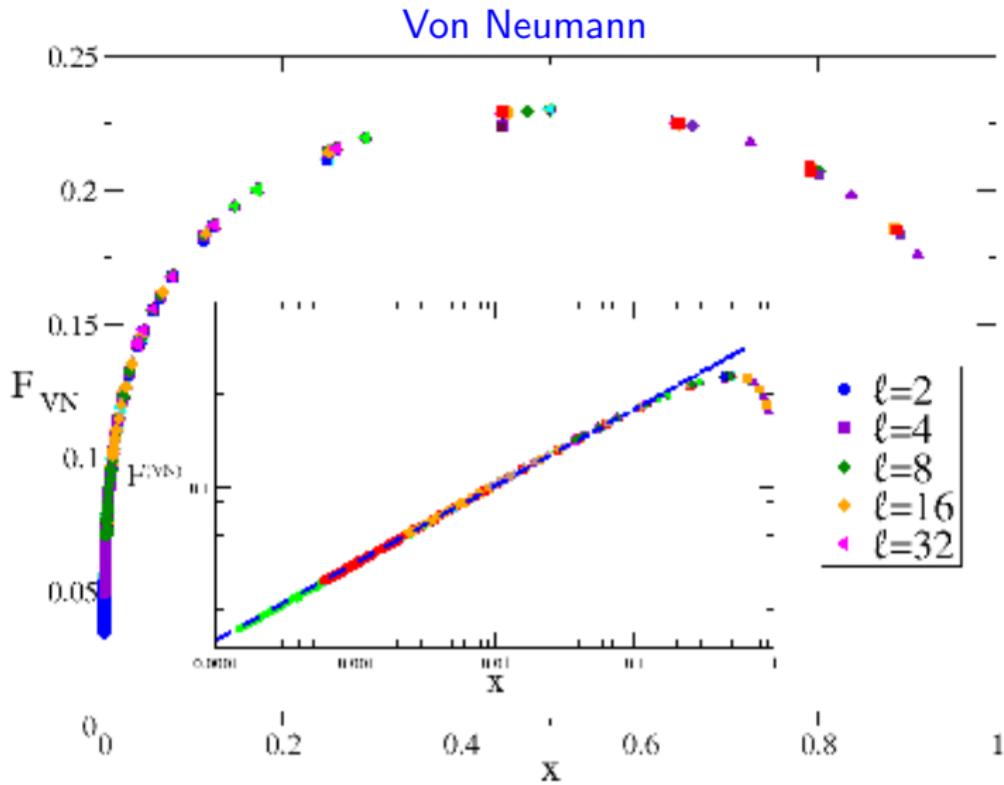


Monte Carlo for 2D and TTN for 1D



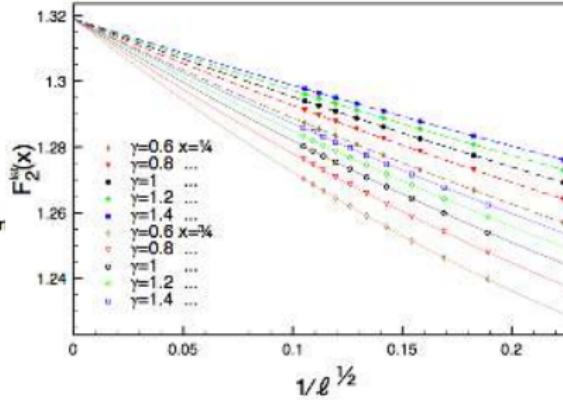
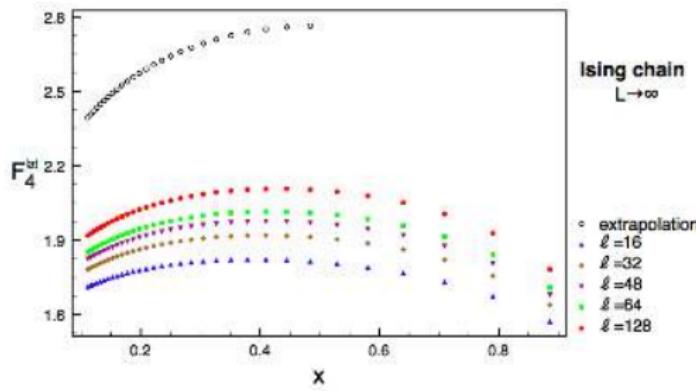
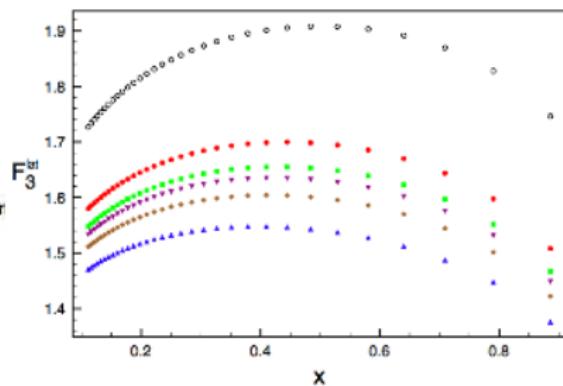
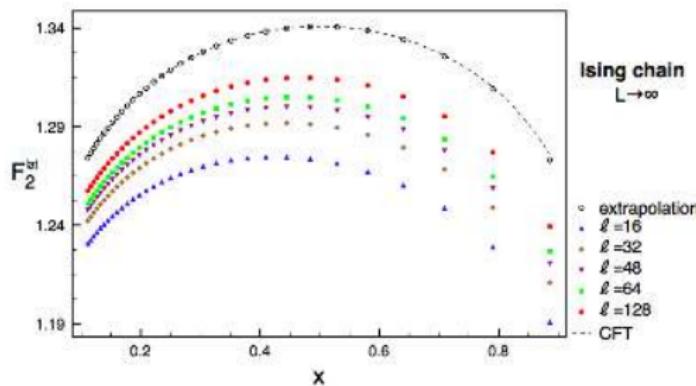
Large monotonic correction to the scaling! FSS analysis confirms:

$$F_2(x) = \left[\left(\frac{(1 + \sqrt{x})(1 + \sqrt{1 - x})}{2} \right)^{1/2} + \frac{(x^{1/4} + ((1 - x)x)^{1/4} + (1 - x)^{1/4})}{2} \right]^{1/2}$$

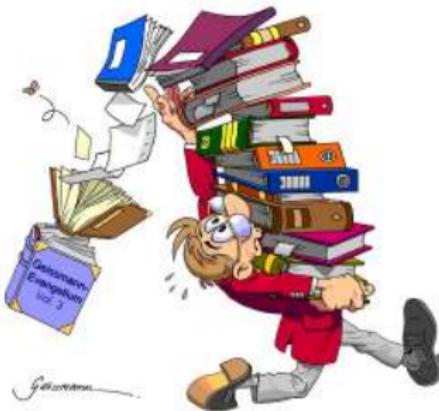


The Ising model Fagotti PC

$\delta_\alpha = 1/\alpha$ because of Ising fermion!



Take home message



Entanglement entropy naturally encodes
universal features of quantum systems

