Entanglement Entropy and Conformal Field Theory

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Les Houches Winter School 2010

Mainly joint work with John Cardy
but also V. Alba, M. Campostrini, F. Essler, M. Fagotti, A. Lefevre, B. Nienhuis, L. Tagliacozzo, E. Tonni

Review: PC & JC JPA 42, 504005 (2009)

If you want to know more:

Extensive reviews by Amico et al., Eisert et al. [RMP]

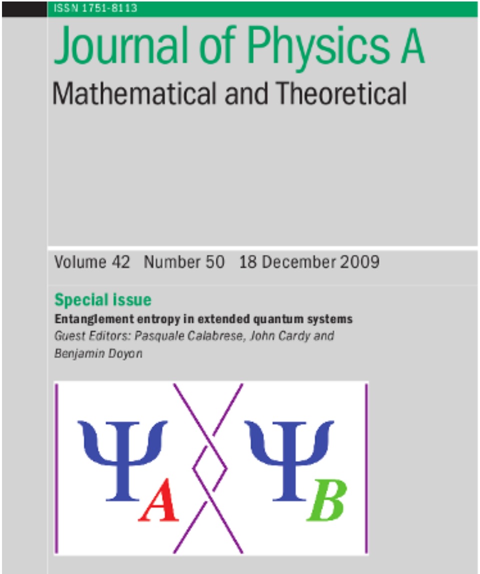
⊕ Special issue of JPA


IOP electronic journals

Journal of Physics A: Mathematical and Theoretical

Volume 42, Number 50, 18 December 2009

SPECIAL ISSUE: ENTANGLEMENT ENTROPY IN EXTENDED QUANTUM SYSTEMS



What is new?

$$S_A = \frac{c}{3} \ln \frac{\ell}{a} + c'_1$$

What is new?

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}(n-1/n)}$$

$$\boxed{S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'}$$

What is new?

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}(n-1/n)} + \ldots ?$$

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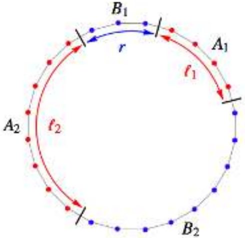
And what about more intervals?



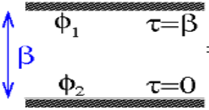
$S_A = ?$ $\mathrm{Tr}\rho_A^n = ?$

Lattice QFT in 1+1 dimensions: $\{\hat\phi(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x |\{\phi(x)\}\rangle$

The density matrix at temperature β^{-1} is ($Z = \mathrm{Tr}\, e^{-\beta \hat H}$)

$$\rho(\{\phi_1(x)\}|\{\phi_2(x)\}) = Z^{-1}\langle\{\phi_2(x)\}|e^{-\beta\hat H}|\{\phi_1(x)\}\rangle$$

Euclidean path integral:

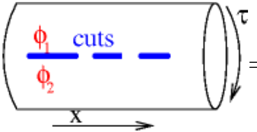
$\rho =$ 

$$= \int \frac{[d\phi(x,\tau)]}{Z} \prod_x \delta(\phi(x,0)-\phi_2(x)) \prod_x \delta(\phi(x,\beta)-\phi_1(x))\, e^{-S_E}$$

$S_E = \int_0^\beta L_E d\tau$, with L_E the Euclidean Lagrangian

The trace sews together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.

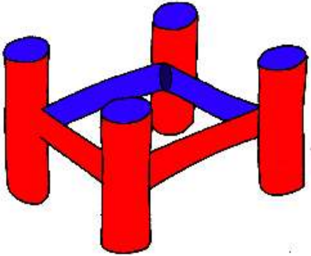
$A = (u_1, v_1), \ldots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A, leaving open cuts for (u_j, v_j) along the the line $\tau = 0$.

$\rho_A =$ 

$$= \int_{x\in B} [d\phi(x,0)]\delta(\phi(x,\beta) - \phi(x,0))\rho$$

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$$

$\mathrm{Tr}\rho_A^n$ (for integer n) is the partition function on n of the above cylinders attached to form an $n-$sheeted Riemann surface



$$= \text{“}\rho_A^{ij}\rho_A^{jk}\rho_A^{kl}\rho_A^{li}\text{”}$$

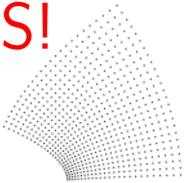
$\mathrm{Tr}\rho_A^n$ has a unique analytic continuation to $\mathrm{Re}\, n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

- A physical systems at a quantum critical point is scale invariant
$\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = b^{2\Delta_\phi}\langle\phi(b\mathbf{r}_1)\phi(b\mathbf{r}_2)\rangle \qquad \langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = |\mathbf{r}_1-\mathbf{r}_2|^{-2\Delta_\phi}$
- A Hamiltonian that is invariant under translations, rotations, and scaling transformations has usually the symmetry of the larger *conformal* group defined as the set of transformations that do not change the angles.
- In 2D the consequences are extraordinary:

YOU DON'T NEED THIS!

all the analytic functions $w(z)$ are conformal



$$\langle\phi(z_1)\phi(z_2)\rangle = |w'(z_1)w'(z_2)|^{2\Delta_\phi}\langle\phi(w(z_1))\phi(w(z_2))\rangle$$

- Under an arbitrary transformation $x^\mu \to x^\mu + \epsilon^\mu$

$$S \to S+\delta S, \qquad \text{with} \quad \delta S = \int d^2x T^{\mu\nu}\partial_\mu\epsilon_\nu$$

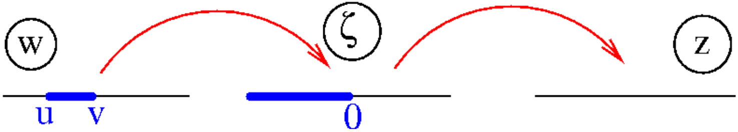
Under a conformal transformation $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\frac{z'''z' - 3/2z''^2}{z'^2}$$

Entropy and CFT PC and John Cardy '04

Single interval $A = [u, v]$. We need $Z_n/Z^n = \langle 0|0\rangle_{\mathcal{R}_n} \Rightarrow$ compute $\langle T(w)\rangle_{\mathcal{R}_n}$

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



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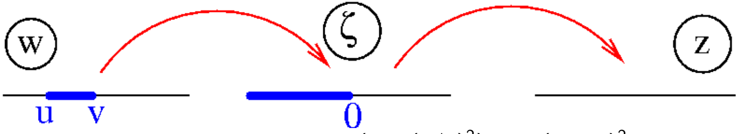

$$\langle T(z)\rangle_{\mathbb{C}} = 0 \Rightarrow \qquad \langle T(w)\rangle_{\mathcal{R}_n} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the Conformal Ward identities

$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbb{C}}}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbb{C}}} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

Z_n/Z^n transforms under conformal transformations as $n^{\rm th}$ power of the two point function of a (fake) primary field on the plane with scaling dimension

$$\Delta_\Phi = \overline{\Delta}_\Phi = \frac{c}{24}\left(1-\frac{1}{n^2}\right) \Rightarrow \qquad \mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n} = c_n\left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)}$$

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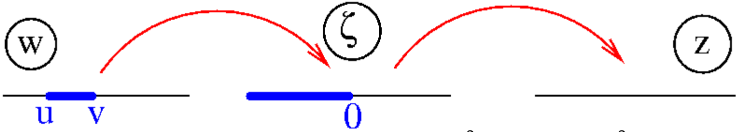

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Finally with the replica trick $(v - u = \ell) \Rightarrow$ $$S_A = \frac{c}{3} \ln \frac{\ell}{a} + c_1'$$

Entanglement and CFT (generalizations) PC and John Cardy '04

Finite temperature

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right) + c_1' \simeq \begin{cases} \dfrac{\pi c}{3}\dfrac{\ell}{\beta}\,, & \ell \gg \beta \qquad \text{classical extensive} \\ \dfrac{c}{3}\log\dfrac{\ell}{a}\,, & \ell \ll \beta \qquad T = 0 \ \text{non} - \text{extensive} \end{cases}$$

Finite size

$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi\ell}{L}\right) + c_1'$ Symmetric $\ell \to L - \ell$. Maximal for $\ell = L/2$

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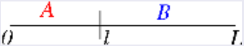
Open systems (generally systems with boundaries)



$$\mathrm{Tr}\,\rho_A{}^n = \tilde{c}_n\left(\frac{2\ell}{a}\right)^{\frac{c}{12}(n-\frac{1}{n})} \Rightarrow S_A = \frac{c}{6}\log\frac{2\ell}{a} + \tilde{c}_1'$$

finite temperature $S_A(\beta) = \frac{c}{6}\log\left(\frac{\beta}{\pi a}\sinh\frac{2\pi\ell}{\beta}\right) + \tilde{c}_1'$

and finite size $S_A(L) = \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \tilde{c}_1'$

$\tilde{c}_1' - c_1'/2 = \ln g$ boundary entropy [Affleck, Ludwig]

Correction to the scaling (Hystory)

$$S_\alpha \equiv \frac{1}{1-\alpha} \ln \mathrm{Tr}\rho_A^\alpha = \frac{c}{6}(1+\alpha^{-1}) \ln \frac{L}{\pi} \sin \frac{\pi \ell}{L} + c'_\alpha$$

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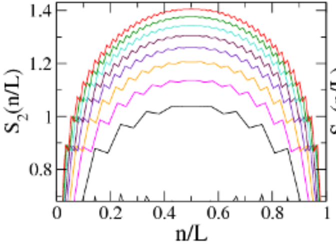

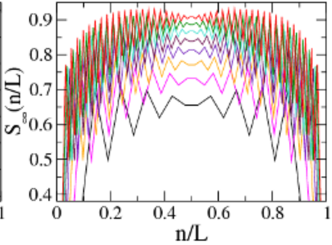

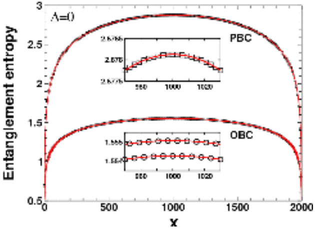

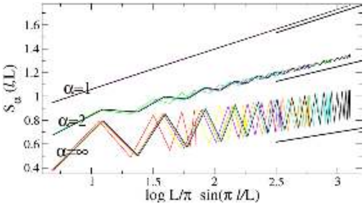

[Laflorencie et al '06]

Corrections to the scaling: Guess?

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$$+ \quad \ell^{-2(x-2)} \qquad \text{with x dim. Irr Op?}$$

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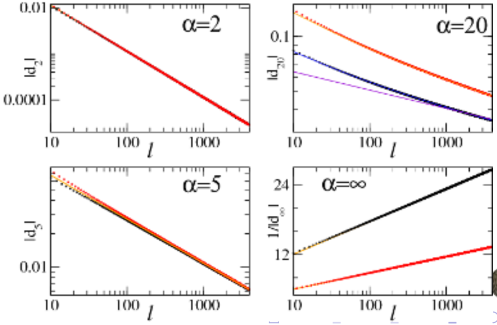
Numerical evidence for XX model (with $x = 1$):

$$d_\alpha(\ell) \equiv S_\alpha(\ell) - S_\alpha^{\rm CFT}(\ell)$$



Exact Solution XX model PC, F Essler

$$S_\alpha(\ell) - S_\alpha^{\rm CFT}(\ell) = (-)^\ell \ell^{-2/\alpha} f_\alpha + \dots$$

$$f_\alpha = \frac{2}{1-\alpha} \frac{\Gamma^2((1+1/\alpha)/2)}{\Gamma^2((1-1/\alpha)/2)} \xrightarrow{\alpha\to 1} 0$$

$$S_\infty(\ell) - S_\infty^{CFT}(\ell) = \begin{cases} \frac{\pi^2}{12} \frac{1}{\ln b\ell} & \ell \text{ odd} \\ -\frac{\pi^2}{24} \frac{1}{\ln b\ell} & \ell \text{ even} \end{cases}$$

In a magnetic field:

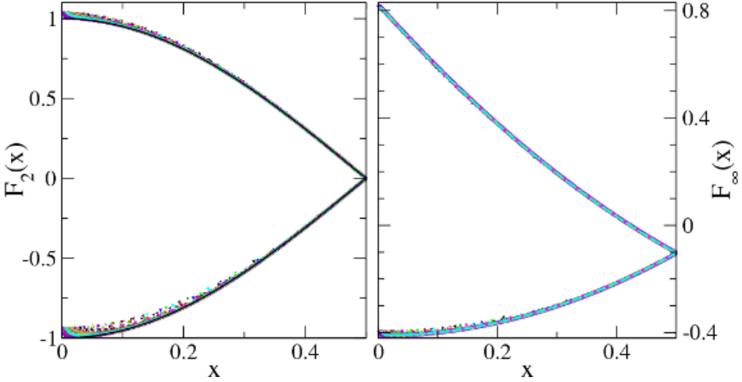
$$S_\alpha(\ell) - S_\alpha^{\rm CFT}(\ell) = f_\alpha \cos(2k_F\ell) |2\ell \sin k_F|^{-2/\alpha} + \dots$$

PS: For the Ising model it is the same $p_\alpha = 2/\alpha$

$$S_\alpha(\ell, L) = S_\alpha^{\rm CFT}(\ell, L) + (-)^\ell f_\alpha \left[\frac{L}{\pi}\sin\frac{\pi\ell}{L}\right]^{-p_\alpha} F_\alpha(\ell/L)$$

XX in Finite Size

All ℓ for several odd L from 17 to 4623 [$\sim 10^5$ points]:

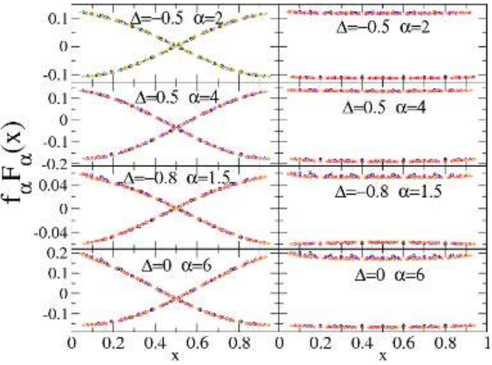


$F_2(x) = \cos \pi x$?

XXZ Heisenberg chain PC, M Campostrini, F Essler, B Nienhuis

$$S_\alpha(\ell, L) = S_\alpha^{\rm CFT}(\ell, L) + (-)^\ell f_\alpha \left[\frac{L}{\pi}\sin\frac{\pi\ell}{L}\right]^{-p_\alpha} F_\alpha(\ell/L)$$

Odd L Even L



- p_α and $F_\alpha(x)$ are universal
- Similar plots for any Δ, odd and even L, any α
- $p_\alpha = 2K/\alpha$
- F_α depends on the parity of L

Add an irrelevant operator Φ of dimension x: $S = S_{CFT} + \lambda \int \Phi(z) d^2 z\,,$

Change in free energy: $-\delta F_n = \sum_{N=1}^{\infty} \frac{(-\lambda)^N}{N!} \int_{\mathcal{R}_n} \cdots \int_{\mathcal{R}_n} \langle \Phi(z_1) \dots \Phi(z_N) \rangle_{\mathcal{R}_n} d^2 z_1 \dots d^2 z_N\,,$

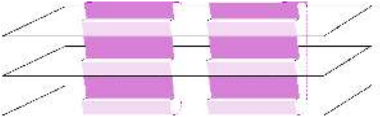
$\zeta = (z/(z-\ell))^{1/n}$ maps $\mathcal{R}_n$ on the plane

$$\delta F_n^{(2)} = -\frac{1}{2} g^2 (n\ell/\epsilon)^{4-2x} \int_{\mathbb{C}'} \int_{\mathbb{C}'} \frac{|\zeta_1 \zeta_2|^{(2-x)(n-1)}}{|\zeta_1^n - 1|^{4-2x} |\zeta_2^n - 1|^{4-2x} |\zeta_1 - \zeta_2|^{2x}} d^2\zeta_1 d^2\zeta_2\,.$$

For $n-1$ small the integral is convergent! $\Rightarrow$ corrections $\ell^{-2(x-2)}$
For larger values of n divergences as $\zeta_j \to 0$ or ∞. Regulator ϵ in z plane!
Any of them gives a multiplicative factor $\propto \epsilon^{2-x+(x/n)}$
Summing up: $\ell^{-2(x-2)-(2-x+(x/n))-(2-x+(x/n))} = \ell^{-2x/n}$
But $x > 2$ ... not only:
At the conical singularities relevant operators can be generated *locally*
The marginal case $x = 2$ introduces logarithms and has to do with c-theorem

Disjoint intervals: History

$$A = [u_1, v_1] \cup [u_2, v_2]$$



In 2004 we obtained

$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)}$$

Tested for free fermions in different ways Casini-Huerta, Florio et al.

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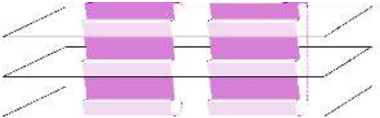

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For more complicated theories in 2009 Furukawa-Pasquier-Shiraishi and Caraglio-Gliozzi showed that it is wrong!

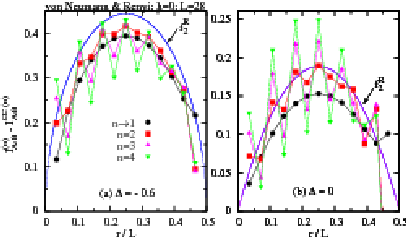
$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x)$$

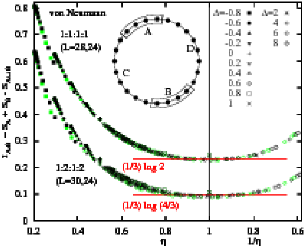
$$x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)} = 4-\text{point ratio}$$

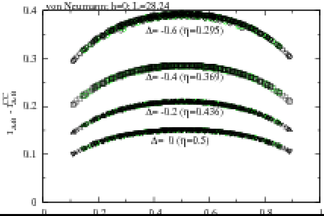
Compactified boson (Luttinger) Furukawa Pasquier Shiraishi

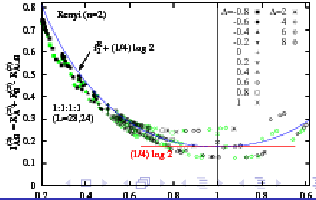
$$F_2(x) = \frac{\theta_3(\eta\tau)\theta_3(\tau/\eta)}{[\theta_3(\tau)]^2}, \qquad x = \left[\frac{\theta_2(\tau)}{\theta_3(\tau)}\right]^4 \quad \eta = \frac{1}{2K} \propto R^2$$

Compared again exact diagonalization in XXZ chain









Compactified boson PC Cardy Tonni

Using old results of CFT on orbifolds Dixon et al 86

$$\mathcal{F}_n(x) = \frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}$$

Γ is an $(n-1)\times(n-1)$ matrix

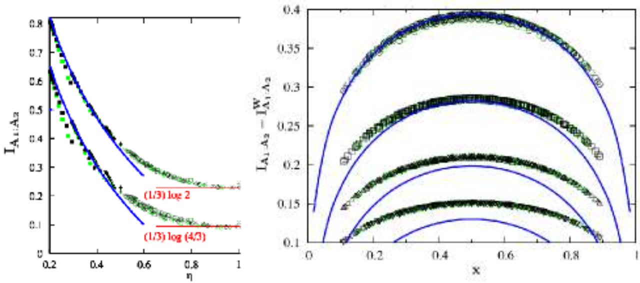
$$\Gamma_{rs} = \frac{2i}{n}\sum_{k=1}^{n-1}\sin\left(\pi\frac{k}{n}\right)\beta_{k/n}\cos\left[2\pi\frac{k}{n}(r-s)\right],\quad \beta_y = \frac{{}_2F_1(y,1-y;1;1-x)}{{}_2F_1(y,1-y;1;x)}$$

Riemann theta function $\Theta(z|\Gamma) \equiv \sum_{m\in \mathbf{Z}^{n-1}} \exp\left[\,i\pi\, m\cdot\Gamma\cdot m + 2\pi i m\cdot z\,\right]$

- $\mathcal{F}_n(x)$ invariant under $x\to 1-x$ and $\eta\to 1/\eta$
- We are unable to analytic continue to real n for general x and η
- Only for $\eta \ll 1$ and for $x \ll 1$

$\eta \ll 1$

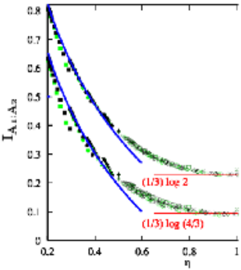
$$-F_1'(x) = \frac{1}{2}\ln\eta - \frac{D_1'(x)+D_1'(1-x)}{2} \qquad \text{with } D_1'(x) = -\int_{-i\infty}^{i\infty}\frac{dz}{i}\frac{\pi z}{\sin^2\pi z}\log H_z(x)$$

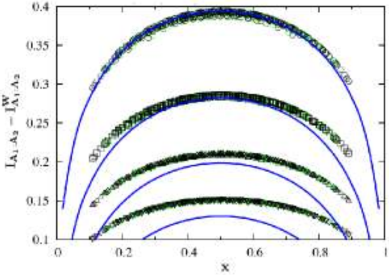


Compactified boson II PC Cardy Tonni

$\eta \ll 1$

$$-F_1'(x) = \frac{1}{2}\ln\eta - \frac{D_1'(x) + D_1'(1-x)}{2} \qquad \text{with } D_1'(x) = -\int_{-i\infty}^{i\infty} \frac{dz}{i} \frac{\pi z}{\sin^2 \pi z} \log H_z(x)$$





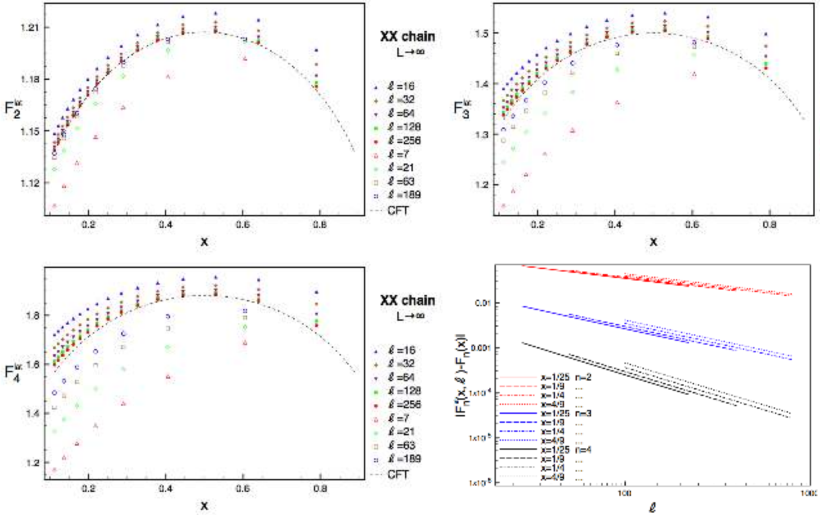
$x \ll 1$

$$F_n(x) = 1 + 2n\left(\frac{x}{4n^2}\right)^{\alpha} P_n + \dots \qquad \alpha = \min(\eta, 1/\eta) \qquad P_n = \sum_{l=1}^{n-1} \frac{l/n}{[\sin(\pi l/n)]^{2\alpha}}$$

$$-F_1'(x) = 2^{1-2\alpha} x^{\alpha} P_1' + \dots$$

The XX model Fagotti PC

The RDM of two intervals is not trivial because of JW string

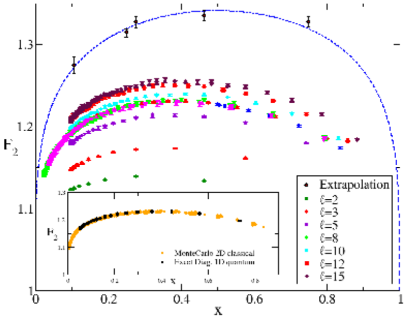


CFT OK and $\delta_\alpha = 2/\alpha$

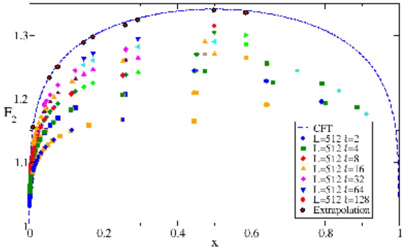
The Ising model Alba Tagliacozzo PC

Monte Carlo for 2D and TTN for 1D

$\mathrm{Tr}\rho_A^2$ MC

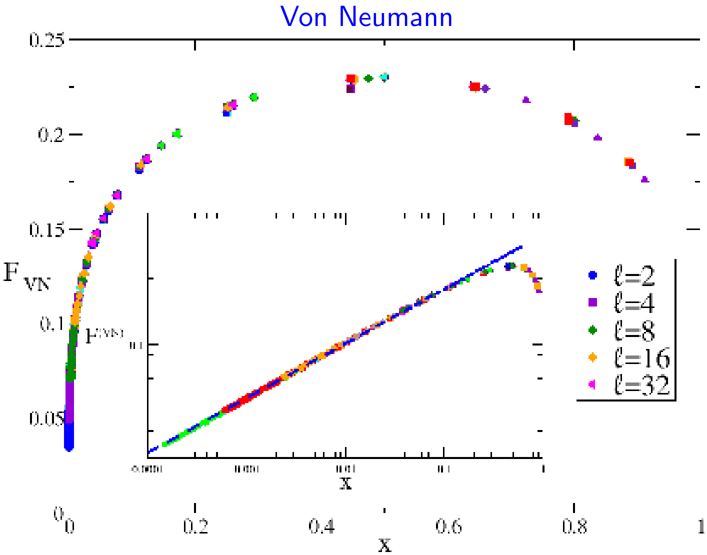


$\mathrm{Tr}\rho_A^2$ TTN



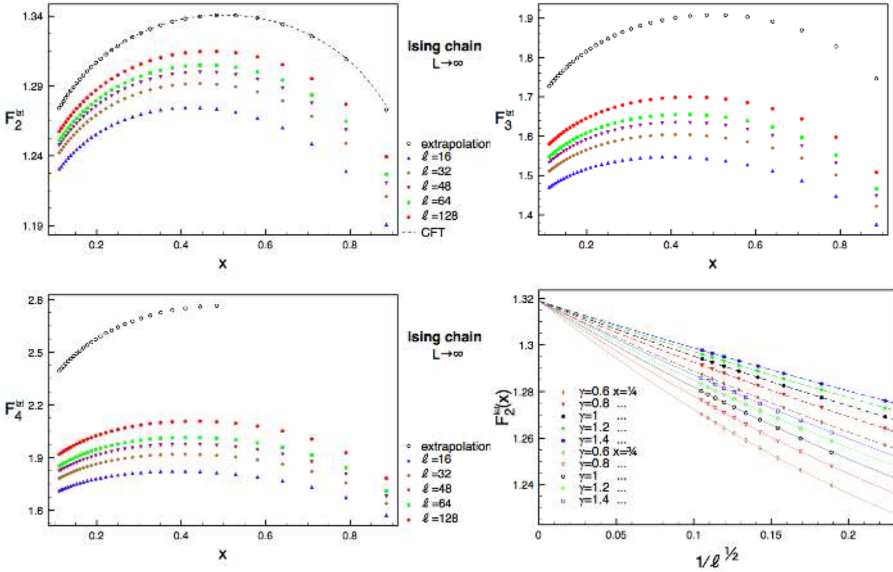
Large monotonic correction to the scaling! FSS analysis confirms:

$$F_2(x) = \left[\left(\frac{(1+\sqrt{x})(1+\sqrt{1-x})}{2} \right)^{1/2} + \frac{(x^{1/4} + ((1-x)x)^{1/4} + (1-x)^{1/4})}{2} \right]^{1/2}$$


Von Neumann
0.25
0.2
0.15
0.1
0.05
0
F VN
0
0.2
0.4
0.6
0.8
1
x
ℓ=2
ℓ=4
ℓ=8
ℓ=16
ℓ=32
x

The Ising model Fagotti PC

$\delta_\alpha = 1/\alpha$ because of Ising fermion!



Take home message

Entanglement entropy naturally encodes universal features of quantum systems