Entanglement Rates and Area Laws

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Feb. 04, 2014

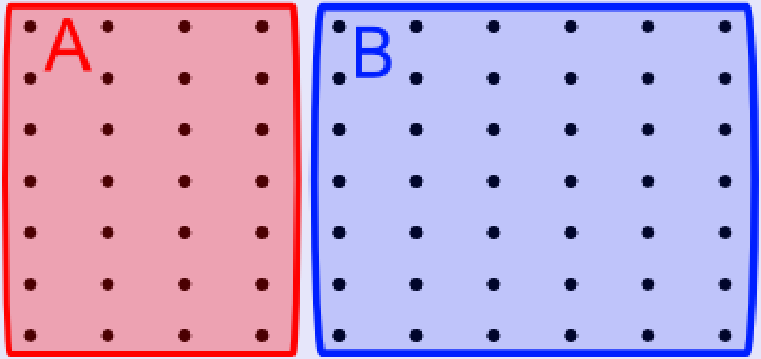
Based on Arxiv: 1304.5931 with Karel Van Acoleyen, Frank Verstraete

Arxiv: 1304.5935 by Koenraad Audenaert

Overview

Part I: Area Law

- Total Hilbert space dimension is d^N
- Dimension of smallest subsystem (A) is D
- Interest is in both A, B big



A random state of a quantum system has entropy

$S\left(\mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)\right) \sim \log D = N \log(d)$ Hayden, Leung, Winter (2004)

For many body systems: volume scaling of entropy

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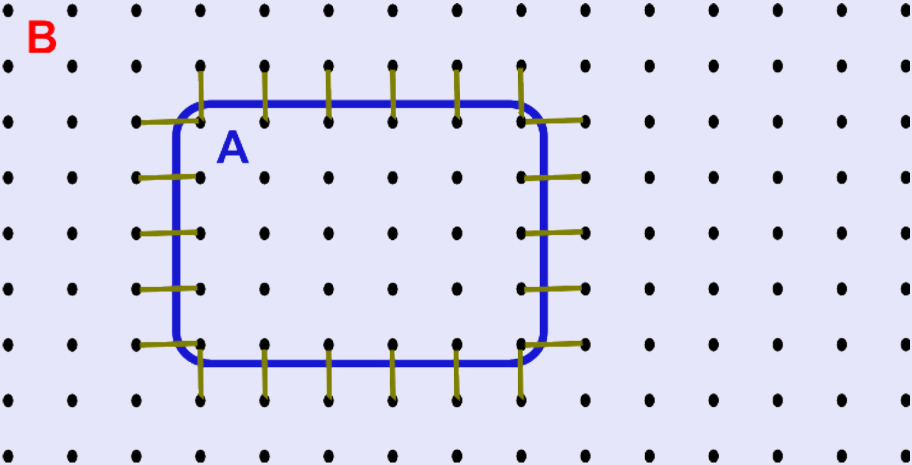
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The area law is the motivation behind variational classes: MPS and PEPS

- Hastings: in 1D, these states have an area law behaviour
- Arad, Kitaev, Landau, Vazirani: improved version

In more than 1 dimension, no rigorous results
Is entanglement a meaningful quantity for many body systems?

$$\begin{aligned} |S(\rho) - S(\sigma)| &\leqslant T \log(D-1) + H(\{T, 1-T\}) \quad \text{(Fannes-Audenaert)} \\ &\lesssim \text{volume scaling} \end{aligned}$$

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Take N qubits and ρ pure and

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\sigma = (1-\varepsilon)\rho + \frac{\varepsilon}{2^N - 1}(\mathbb{1} - \rho) \Rightarrow |S(\rho) - S(\sigma)| \sim \varepsilon N
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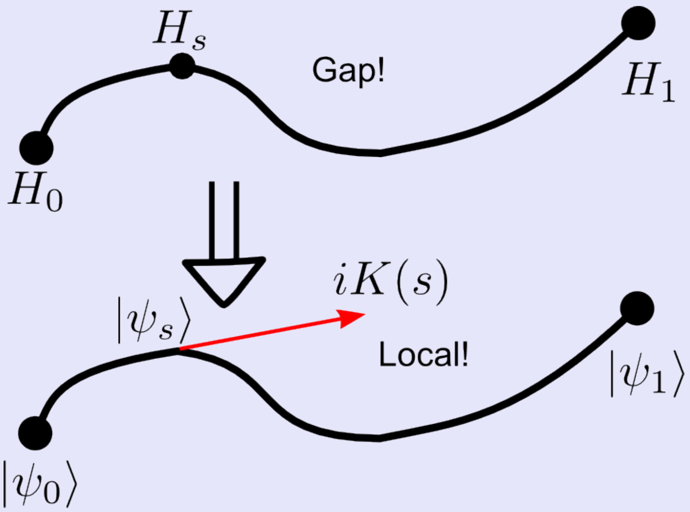
In quantum many body theory, important concept of a phase: states in the same phase have similar properties (not expectation values)

When are two ground states of gapped Hamiltonians in the same phase?

Definition (X.G. Wen, Hastings et al.)

- H_0 and H_1 local gapped Hamiltonians with ground states $|\psi_0\rangle, |\psi_1\rangle$
- The states $|\psi_0\rangle, |\psi_1\rangle$ are in the same phase if there exists a $\gamma > 0$ and a smooth path of gapped, local H_s interpolating between H_0, H_1

Gapped Quantum Phase



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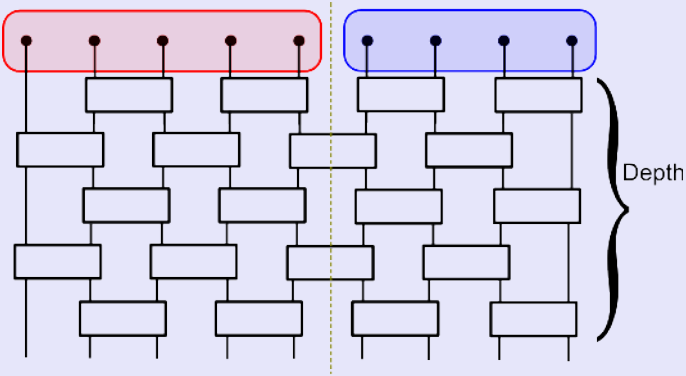
(Almost) equivalent intuitive definition:

- The states $|\psi_0\rangle, |\psi_1\rangle$ are in the same phase if there exists a constant depth local quantum circuit that connects them.

With this intuitive picture in mind:
$|\psi_0\rangle$ obeys an area law iff $|\psi_1\rangle$ does $\Rightarrow$ make this rigorous

Gapped Quantum Phase



$$\Delta S \leq O(\mathrm{Depth.Area})$$

Given a gapped path, how can we go from $|\psi_0\rangle$ to $|\psi_1\rangle$?

Answer $\frac{\partial}{\partial s}|\psi(s)\rangle = iK(s)|\psi(s)\rangle$ with

$$K(s) = -i\int_{\mathbb{R}} F(\gamma t)e^{iH_s t}\left(\partial_s H_s\right)e^{-iH_s t}dt$$

The function F:

- is odd
- decays super polynomially in t
- $\hat{F}(\omega) = -\frac{1}{\omega}, \quad |\omega| \geqslant 1$
- exists, classic result in Fourier analysis

Quasi-Adiabatic Evolution

The existence of K is an exact version of the adiabatic theorem by Kato.

Hastings proved that K itself is a *quasi*-local Hamiltonian!

- Use Lieb-Robinson bounds
- K can be written as $\sum_i \sum_{r \geqslant 0} k_i(r)$ and $\|k(r)\| \leqslant cF(r)$

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Conclusion: $K(s)$ is generator we need

1. Brings $|\psi_0\rangle$ to $|\psi_1\rangle$ in short 'time' $s \in [0, 1]$
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Michalakis (2012):

Extra condition on spectrum of reduced density matrices (decay): use the quasi-adiabatic theorem and techniques from Hasting's proof to find that entanglement changes $\sim A \log A$

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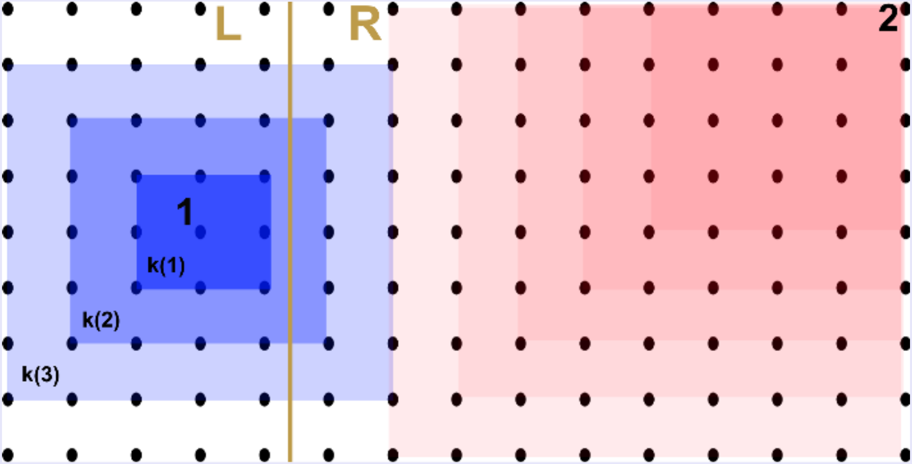
Extra assumption (proof in second part talk):

The maximal rate at which a Hamiltonian H acting on system of dimension D can generate entanglement is $\Gamma(H) \lesssim \|H\| \log D$ independently of ancillas.

Divide a regular 2D lattice in a left and right part with straight cut



Divide a regular 2D lattice in a left and right part with straight cut

$$\begin{aligned}\frac{dS_L(|\psi_s\rangle)}{ds} &= i\,\mathrm{Tr}\left(K(s)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\\ &= i\sum_{r\geqslant 0}\sum_{x}\sum_{y}\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s)|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\\ &= i\sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s)\rangle\langle\psi_s)|\,,\log\rho_L\otimes\mathbb{1}_R]\right).\end{aligned}$$

Hence,

$$\left|\frac{dS_L(|\psi_s\rangle)}{ds}\right| \leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\left|\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\right|$$

Divide a regular 2D lattice in a left and right part with straight cut

$$
\begin{aligned}
\left|\frac{dS_L(|\psi_s\rangle)}{ds}\right| &\leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\left|\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\right| \\
&\leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\Gamma\left(k_{(x,y)}(r)\right) \\
&\leqslant cA_L\sum_{r\geqslant 0} r\|k(r)\|\log\left(d^{P(r)}\right) && \text{log is crucial!} \\
&= cA_L\sum r^3\|k(r)\| && P(r)\sim r^2 \text{ in 2D}
\end{aligned}
$$

Since $k(r)$ decays super polynomially, the sum converges in any dimensions for regular lattices and all partitions.

Part II: Entanglement Rate

How fast can a Hamiltonian generate entanglement between two subsystems?

- Interaction H_{AB} between two subsystems: straightforward (Bravyi)

$$\Gamma(H) \leqslant c\|H\| \log D$$

- What if we allow for ancillas?

Do we really expect ancillas to have an influence on this rate for a local Hamiltonian?

The Swap Operator

Look at unitary gates instead of Hamiltonian evolution

- Can the total change of entanglement change by adding ancillas?

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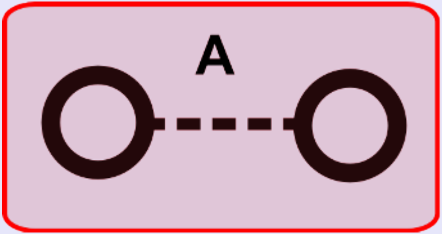

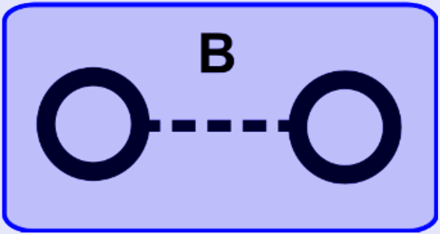

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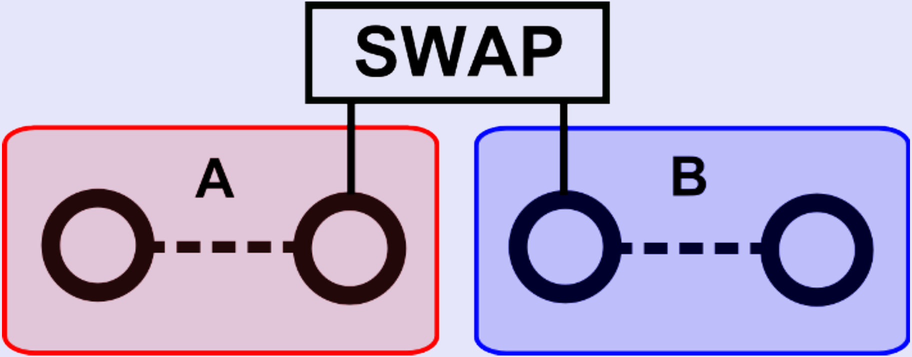

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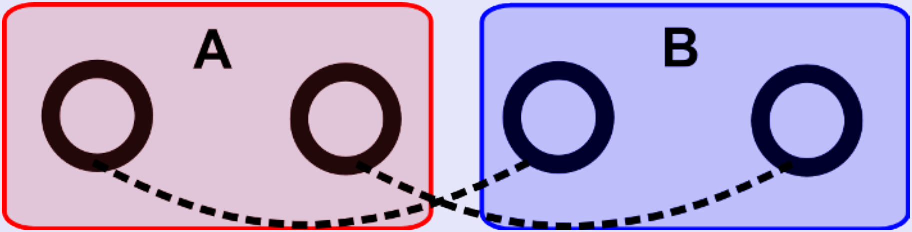

Influence of Ancillas

- The swap operator is the worst case scenario
- In general, the upper bound changes by factor (Bennett et al. 2003),

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- The swap operator is the worst case scenario
- In general, the upper bound changes by factor (Bennett et al. 2003),

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- How about the (infinitesimal) rate at which entanglement can be created?
- Kitaev conjectured the analogous bound

$$\Gamma := \left| \frac{dS(\rho_{Aa})}{dt} \right| \leqslant c \|H\| \log D$$

this conjecture is the *Small Incremental Entangling* (SIE)

- Example were ancillas increase the entanglement rate given by Dür et al. (2001)
- Several authors obtained partial results,
  1. Dür, et al. (2001): qubits without ancillas
  2. Childs, et al. (2002): Ising and anisotropic Heisenberg interaction
  3. Wang, et al. (2002): Self-inverse product Hamiltonians
  4. Childs, et al. (2004): Simulation of product Hamiltonians
- Bennett, Harrow, Leung, Smolin: first general bound independent of ancillas

The last authors found an upper bound of the form

$$\Gamma \leqslant O(\|H\| D^4)$$

History of the Problem

The last bound is a polynomial in the system's dimension, further refinements:

- Bravyi (2007): obtained several results,
  1. $\Gamma \leqslant O(\|H\| D^2)$
  2. general case without ancillas: $\Gamma \lesssim c\|H\| \log D$ (tight, $c \approx 2$)
  3. rewrote the problem to make it tractable (see later)
- Lieb, Vershynina (2013): corollary $\Gamma \leqslant O(\|H\| D) \sim O(\|H\| d^N)$

Numerical evidence suggests that Kitaev was right,

$$\Gamma \leqslant 2\|H\| \log D \sim 2\|H\| N \log d \qquad \text{SIE-Conjecture}$$

Bravyi's Trick

Suppose $D_A \geqslant D_B$, we replace $A \Rightarrow A \otimes a$.

- The entanglement rate reads

$$\Gamma = -i \operatorname{Tr}\left(H_{AB}[\rho_{AB}, \log(\rho_A) \otimes \mathbb{1}_B]\right)$$

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- The entanglement rate reads

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- Find an ensemble $\{(1-p, \rho_0), (p, \rho_{AB})\}$ such that

$$p = \frac{1}{D_B^2} \quad \text{and} \quad (1-p)\rho_0 + p\rho_{AB} = \rho_A \otimes \frac{\mathbb{1}_B}{D_B}$$

Look at *Small Incremental Mixing* (SIM)

$$\Lambda(p) = \frac{dS}{dt}\underbrace{\left((1-p)\rho_0 + pe^{-iHt}\rho_{AB}e^{iHt}\right)}_{\tau(t)}\Big|_{t=0}$$

Bravyi's Trick

We see that for this ensemble

$$\Lambda(p) = p\Gamma$$

If we proof that

$$\Lambda(p) \leqslant c\|H\| p \log(1/p) \qquad \text{SIM-Conjecture}$$

we conclude that

$$\Gamma \leqslant c\|H\| \log(D_B^2) = 2c\|H\| \log D_B$$

Bravyi's Trick

We now bound $\Lambda(p)$ under the restrictions $\|H\| = 1$ and $p < e^{-2}$

It suffices to proof that

$$|\Lambda(p)| \leqslant \max_{X,Y} \|[X, \log(Y)]\|_1 \leqslant -cp \log p$$

with

$$\operatorname{Tr} X = p, \quad \operatorname{Tr} Y = 1, \quad 0 \leqslant X \leqslant Y$$

We use variational characterization of the trace norm

$$\|[X, \log(Y)]\|_1 \leqslant 2 \max_{0 \leqslant P \leqslant \mathbb{1}} |\operatorname{Tr}(P[X, \log(Y)])|$$

Proof

1. Use the eigenbasis of Y,

$$2\left|\sum_{i<j} \log \frac{y_i}{y_j}\left(X_{ij}P_{ji} - X_{ji}P_{ij}\right)\right|$$

2. Order its eigenvalues $y_{i_k} \in [p^k, p^{k-1})$ and the summation

$$\sum_{i<j} = \underline{\left(\sum_{i_1<j_1} + \sum_{i_2,j_2} + \sum_{i_2<j_2}\right)} + \underline{\left(\sum_{i_2<j_2} + \sum_{i_2,j_3} + \sum_{i_3<j_3}\right)} + \dots$$
$$- \underline{\left(\sum_{i_2<j_2}\right)} - \underline{\left(\sum_{i_3<j_3}\right)} - \dots\dots + \underline{\left(\sum_{i_1,i_{k>2}} + \sum_{i_2,i_{k>3}} + \dots\right)}$$

Reordering the Summations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix}$$

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Reordering the Summations

$$a_{ij} = \log \frac{y_i}{y_j} \left(X_{ij} P_{ji} - X_{ji} P_{ij} \right)$$

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\end{pmatrix}$$

$\frac{y_j}{y_i} \le p$

$\frac{y_j}{y_i} \ge p^2$

Cauchy-Schwarz

Last braces has summations over pairs of eigenvalues far from each other:

$$y_j < py_i \Rightarrow \sqrt{y_j/y_i}\log\left(\frac{y_i}{y_j}\right) \leqslant -\sqrt{p}\log(p)$$

We use Cauchy-Schwarz and $X = Y^{1/2}ZY^{1/2}$ with $0 \leqslant Z \leqslant \mathbb{1}$,

$$\begin{aligned}
\text{Summations} &= 2\left|\tilde{\sum}_{i<j}\log\frac{y_i}{y_j}\left(X_{ij}P_{ji} - X_{ji}P_{ij}\right)\right| \\
&\leqslant \left(\tilde{\sum}\log\frac{y_i}{y_j}\sqrt{y_iy_j}Z_{ij}Z_{ji}\right)^{1/2}\left(\tilde{\sum}\log\frac{y_i}{y_j}\sqrt{y_iy_j}P_{ij}P_{ji}\right)^{1/2} \\
&\leqslant 4\sqrt{p}\log(1/p)\left(\sum y_iZ_{ij}Z_{ji}\right)^{1/2}\left(\sum y_iP_{ij}P_{ji}\right)^{1/2} \\
&\leqslant 4p\log(1/p)
\end{aligned}$$

First braces: matrices restricted to small subspaces spanned by eigenvectors with close eigenvalues

$$\begin{aligned}\text{First term} &= 2\left|\sum_{i}^{n_2}\sum_{j>i}^{n_2}\log\frac{y_i}{y_j}\left(X_{ij}P_{ji}-X_{ji}P_{ij}\right)\right| \\ &\leqslant \left\|[\tilde{X},\log\tilde{Y}]\right\|_1 \\ &\leqslant \left\|[\tilde{X},\log\tilde{Y}/\tilde{y}_{\min}]\right\|_1 \\ &\leqslant \left\|\log\left(\tilde{Y}/\tilde{y}_{\min}\right)\right\|\,\|X\|_1\end{aligned}$$

We continue:

$$\text{First term} = \log \frac{\tilde{y}_{\max}}{\tilde{y}_{\min}} \operatorname{Tr} \tilde{X}$$
$$\leqslant 2 \log(1/p) \sum_{i}^{n_2} X_{ii}$$

The first line in the decomposition is bounded by $4p \log(1/p)$, the last contribution is bounded by $p \log(1/p)$

We obtain the final bound

$$\Lambda(p) \leqslant 9p \log(1/p) \quad \Rightarrow \quad \Gamma \leqslant 18 \|H\| \log D$$

The quantum relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

has the well known problem of divergence if $\mathrm{supp}(\rho) \subsetneq \mathrm{supp}(\sigma)$

Quantum Skew Divergence

The quantum relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

has the well known problem of divergence if $\mathrm{supp}(\rho) \subsetneq \mathrm{supp}(\sigma)$

One solution is:

$$SD_\alpha(\rho||\sigma) = \frac{1}{-\log\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma)$$

Is the Quantum Skew Divergence SD_α useful?

Closed formula, linear and operator monotonous, jointly convex, contractivity, ...

- $0 \leqslant SD_\alpha \leqslant 1$ and $SD_\alpha = 1$ iff $\rho \perp \sigma$, $SD_\alpha = 0$ iff $\rho = \sigma$
- $SD_\alpha(\rho||\sigma) \leqslant \|\rho - \sigma\|_1/2$
- Continuity in first and second argument

Special case $\sigma_2 = \sigma$ and $\sigma_1 = e^{itH}\sigma e^{-itH}$:

$$|SD_\alpha(\rho||\sigma_1) - SD_\alpha(\rho||\sigma_2)| \leqslant \frac{1-\alpha}{-\alpha \log \alpha} t \|H\|$$

Holevo Chi

Consider an ensemble of states $\mathcal{E} = \{(p, \rho), (1-p, \sigma)\}$.

The Holevo-Chi quantity is given by

$$\begin{aligned}\chi &= S(p\rho + (1-p)\sigma) - pS(\rho) - (1-p)S(\sigma) \\ &= -p \log pSD_p(\rho||\sigma) - (1-p)\log(1-p)SD_{1-p}(\sigma||\rho) \\ &\leqslant h(\{p, 1-p\})\|\rho - \sigma\|_1/2\end{aligned}$$

Improvement of both $\chi \leqslant h(\{p, 1-p\})$ and $\chi \leqslant \|\rho - \sigma\|_1/2$.

Small Incremental Mixing

Remember the small incremental mixing from Bravyi's trick:

$$\Lambda(p) = \frac{dS}{dt}\underbrace{\left((1-p)\rho_1 + pe^{-iHt}\rho_2 e^{iHt}\right)}_{\tau_t} = \frac{d\chi(\mathcal{E})}{dt}$$

We obtain

$$\begin{aligned} S(\tau(t)) - S(\tau(0)) &= \chi(\mathcal{E}(t)) - \chi(\mathcal{E}(0)) \\ &\leqslant t\|H\| \end{aligned}$$

by rewriting χ and using continuity of SD_α

The factor $h(\{p, 1-p\})$ is missing.

Differential Skew Divergence

Improve the continuity inequality to give us the correct bound

We need to look at *Differential Skew Divergence*

$$
\begin{aligned}
DSD_\alpha(\rho||\sigma) &= \frac{d}{d(-\log(\alpha))} S(\rho||\alpha\rho + (1-\alpha)\sigma) \\
&= -\alpha \frac{d}{d\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma)
\end{aligned}
$$

Differential Skew Divergence

Improve the continuity inequality to give us the correct bound

We need to look at *Differential Skew Divergence*

$$\begin{aligned} DSD_\alpha(\rho||\sigma) &= \frac{d}{d(-\log(\alpha))} S(\rho||\alpha\rho + (1-\alpha)\sigma) \\ &= -\alpha \frac{d}{d\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma) \end{aligned}$$

Same nice properties as Skew Divergence itself, similar proofs, stronger bounds.

Relation is given by averaging procedure:

$$SD_\alpha(\rho||\sigma) = \frac{1}{-\log\alpha} \int_0^{-\log\alpha} DSD_\alpha(\rho||\sigma) d(-\log\alpha)$$

Conclusion

- We considered the rate at which entanglement can be generated by a Hamiltonian H_{AB} in the most general case with ancillas.
- We used Bravyi's trick to rewrite the problem
- Two different methods to proof the upper bound: direct calculation and quantum skew divergence

$$\left| \frac{dS(\rho_{Aa})}{dt} \right| \leqslant c \|H\| \log D$$

- The log and quasi-adiabatic evolution gives the stability of the area law
- Area law is property of phase: suffices to find one state in each phase (fixed point, string net models, ...)
- Details in Arxiv:1304.5931 and Arxiv:1304.5935

THANK YOU