The entanglement spectrum of chiral fermions on the torus

Pascal Fries[1,*] and Ignacio A. Reyes[2,†]

[1] *Fakultät für Physik und Astronomie, Julius-Maximilians Universität Würzburg, Am Hubland, 97074 Würzburg, Germany*
[2] *Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Potsdam, Germany*

(Dated: May 16, 2019)

We determine the reduced density matrix of chiral fermions on the torus, for an arbitrary set of disjoint intervals and generic torus modulus. We find the resolvent, which yields the modular Hamiltonian in each spin sector. Together with a local term, it involves an infinite series of bi-local couplings, even for a single interval. These accumulate near the endpoints, where they become increasingly redshifted. Remarkably, in the presence of a zero mode, this set of points "condenses" within the interval at low temperatures, yielding continuous non-locality.

INTRODUCTION

Amongst the predictions stemming from the interplay between Quantum Field Theory (QFT) and the causal structure of spacetime, one of the most robust is the celebrated Unruh effect: an accelerated observer in the vacuum measures a thermal bath, with a temperature proportional to its proper acceleration [1–3]. Intimately connected with the thermodynamics of black holes via Hawking radiation, this lies at the heart of our current understanding of the quantum nature of gravity [4]. Therefore, it is natural to explore its generalisations and investigate it further.

In recent years, these phenomena have been extended into the framework of quantum information theory. There, this temperature is understood as arising from the entanglement structure of the vacuum. Starting from a state ρ and some entangling subregion V, one defines the reduced density matrix ρ_V by tracing out the complement of V. Then, just as the entanglement entropy $S_V = -\operatorname{Tr}[\rho_V \log \rho_V]$ generalises the thermal entropy, the usual Hamiltonian is an instance of the more general concept of a *modular* (or entanglement) Hamiltonian $K_V := -\log \rho_V$.

Originally introduced within algebraic QFT [5], the modular Hamiltonian has aroused much interest across a wide community due its close connection to quantum information measures. In the context of many body quantum systems, the spectrum of this operator is known as the "entanglement spectrum" and has been proposed as a fingerprint of topological order [6–8] and investigated in lattice models [9–13], as well as tensor networks [14–16]. In QFT, it is fundamental for the study of relative entropy [17, 18] and its many applications to energy and information inequalities [19–21]. In the context of the AdS/CFT correspondence, it is instrumental in the program of reconstructing a gravitational bulk from the holographic data [22–32].

However, the modular Hamiltonian is known in only a handful of cases. The result is universal and local for the vacuum of any QFT reduced to Rindler space [3, 33] and hence any CFT vacuum on the plane reduced to a ball [22]. For any CFT_2, the same applies for a single interval, for the vacuum on the cylinder or a thermal state on the real line [34, 35]. More generically, modular flows can be non-local, as is the case for multiple intervals in the vacuum of chiral fermions on the plane or the cylinder [36, 37] and scalars on the plane [38]. The exact nature of the transit from locality to non-locality however is not fully understood, and remains an active topic of research.

In this paper we report progress regarding this problem, by providing a new entry to this list. We show that the chiral fermion on the torus (finite temperature on the circle) is a solvable model that undergoes such a transition between locality and non-locality. We compute the reduced density matrix by restating the problem as a singular integral equation, which in turn we solve via complex analysis methods. The resulting modular Hamiltonian exhibits a local flow and, also bi-local couplings between a discrete but infinite set of other points within the subregion. In the low temperature limit, the sector with a zero mode experiences a "condensation" of these points, resulting in a continuously non-local flow.

THE RESOLVENT

We start by introducing the *resolvent*, following [36, 38, 39]. For any spatial region V, the reduced density matrix ρ_V is defined as to reproduce expectation values of local observables supported on V. Now, for free fermions, Wick's theorem implies that it is sufficient that ρ_V reproduces the equal-time Green's function

$$\operatorname{Tr}[\rho_V \psi(x)\psi^\dagger(y)] = \langle \psi(x)\psi^\dagger(y)\rangle =: G(x,y)$$

for $x, y \in V$. This requirement fixes the modular Hamiltonian to be a quadratic operator with kernel $K_V = -\log(G|_V^{-1} - 1)$ [40]. As shown in [36], this can

*Electronic address: pascal.fries@physik.uni-wuerzburg.de
†Electronic address: iareyes@uc.cl

be rewritten as

$$K_V = -\int_{1/2}^{\infty} \mathrm{d}\xi \, [R_V(\xi) + R_V(-\xi)] \tag{1}$$

in terms of the resolvent

$$R_V(\xi) := (G|_V + \xi - 1/2)^{-1}, \tag{2}$$

where the inverse is understood in the sense of distributions.

To obtain the resolvent for a given the propagator G and the subregion V, we do the convenient redefinition

$$R_V(\xi; x, y) = \frac{\delta(x-y)}{\xi - 1/2} - \frac{F_V(\xi; x, y)}{(\xi - 1/2)^2}, \tag{3}$$

which translates (2) into a singular integral equation

$$\begin{aligned} 0 = G(x,y) &- F_V(\xi; x, y) \\ &- \frac{1}{\xi - 1/2} \int_V \mathrm{d}z \, G(x,z) F_V(\xi; z, y). \end{aligned} \tag{4}$$

Provided the global state ρ and the entangling region V, this equation completely determines F_V and thus R_V. Hence, the derivation of the modular Hamiltonian is reduced to finding the function F_V that solves this equation.

All previous considerations hold for free fermions on a generic Riemann surface. The simplest case is the plane where the solution of (4) is a standard result [41], which was used by [36] to derive the corresponding modular Hamiltonian. They found that for multiple intervals, it consists of a local and a bi-local term. The former can be written as

$$\mathcal{K} = \int_V \mathrm{d}x \, \beta(x) T(x) \tag{5}$$

in terms of the stress tensor $T = -\mathrm{i}\psi\partial_x\psi$, where $\beta(x)$ is known as the *entanglement temperature*. On the other hand, the bi-local term couples the field between different intervals and, hence, vanishes in the case of a single interval.

Let us now proceed to the case of a chiral fermion on the torus. As is customary, we take the periods to be $1, \tau$ with $\Im(\tau) > 0$, such that the nome $q := \mathrm{e}^{\mathrm{i}\pi\tau}$ is inside the unit disk. We move to radial coordinates $w = \mathrm{e}^{\mathrm{i}\pi z}$, and work with the Dedekind eta and Jacobi theta functions given by

$$\begin{aligned} \eta(q^2) &:= q^{1/12} \prod_{k \geq 1} (1 - q^{2k}), \\ \vartheta_3(w|q) &:= \sum_{k \in \mathbb{Z}} q^{k^2} w^{2k}, \\ \vartheta_4(w|q) &:= \vartheta_3(\mathrm{i}w|q), \\ \vartheta_2(w|q) &:= q^{1/4} w \vartheta_3(\sqrt{q} w|q), \quad \text{and} \\ \vartheta_1(w|q) &:= -\mathrm{i} q^{1/4} w \vartheta_3(\mathrm{i}\sqrt{q} w|q). \end{aligned}$$

Since we are dealing with fermions, the correlator $G(u,v)$ with $u = \mathrm{e}^{\mathrm{i}\pi x}$ and $v = \mathrm{e}^{\mathrm{i}\pi y}$ is either periodic (Ramond; R) or anti-periodic (Neveu-Schwarz; NS) with respect to either of the two periods of the torus. We shall restrict to the "thermal" case, with NS periodicity with respect to τ. Combining this with the requirement to reproduce the UV correlator $G^{\mathrm{UV}}(x,y) = [2\pi\mathrm{i}(x-y)]^{-1}$ on small scales, this fully determines the standard Green's functions [42]

$$G^\nu(u,v) = \frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}\mathrm{e}^\epsilon|q)} \frac{\vartheta_\nu(uv^{-1}|q)}{\vartheta_\nu(1|q)}. \tag{6}$$

Here, the superscript

$$\nu = 2, 3 = (\mathrm{R}, \mathrm{NS}), (\mathrm{NS}, \mathrm{NS})$$

labels the different spin-structures, and we introduced a regulator ϵ in order to treat the distribution G^ν as a function. The sign of ϵ depends on the chirality—without loss of generality, we choose $\epsilon > 0$.

In radial coordinates, the integral equation (4) reads

$$\begin{aligned} 0 = G^\nu(u,v) &- F_V^\nu(\xi; u, v) \\ &- \frac{1}{\xi - 1/2} \frac{1}{\mathrm{i}\pi} \int_A \frac{\mathrm{d}w}{w} G^\nu(u,w) F_V^\nu(\xi; w, v) \end{aligned} \tag{7}$$

with $A := \mathrm{e}^{\mathrm{i}\pi V}$ being the entangling region.

The key strategy of this paper to solving (7) is to reformulate it as a statement about contour integrals. To this end, we start by listing a set of sufficient properties that F_V^ν must possess in order to solve this equation. First, it must have the same periodicities in the w argument as G^ν, such that $G^\nu F_V^\nu$ is doubly periodic in w, i.e. well defined on the torus. Since this yields an elliptic function, its residue along the boundary γ of a fundamental region (see fig. 1) must vanish:

$$0 = \frac{1}{\mathrm{i}\pi} \oint_\gamma \frac{\mathrm{d}w}{w} G^\nu(u,w) F_V^\nu(\xi; w, v). \tag{8}$$

Our aim is now to rewrite this in the form of (7).

The next property we demand is that F_V^ν have a simple pole $F_V^\nu(u,v) \sim 1/2(uv^{-1} - 1)$ at $u \to v$, together with a branch cut along the entangling region A, which we specify below. Everywhere else it must be analytic. Note that, similarly to G^ν, we need to introduce a regulator $\epsilon' > 0$ for the pole of F_V^ν—the choice of sign has to be the same as that for ϵ to make the argument work.

If these conditions are met, a simple residue analysis shows that (8) reduces to

$$\begin{aligned} 0 = G^\nu(u, v\mathrm{e}^{-\epsilon'}) &- F_V^\nu(\xi; u\mathrm{e}^\epsilon, v\mathrm{e}^{-\epsilon'}) \\ &- \frac{1}{\xi - 1/2} \frac{1}{\mathrm{i}\pi} \int_{A^\circlearrowleft} \frac{\mathrm{d}w}{w} G^\nu(u\mathrm{e}^\epsilon, w) F_V^\nu(\xi; w, v\mathrm{e}^{-\epsilon'}), \end{aligned} \tag{9}$$

where we made the regulators explicit and $A^\circlearrowleft$ denotes a snug path around the cut on A. Note the distinction

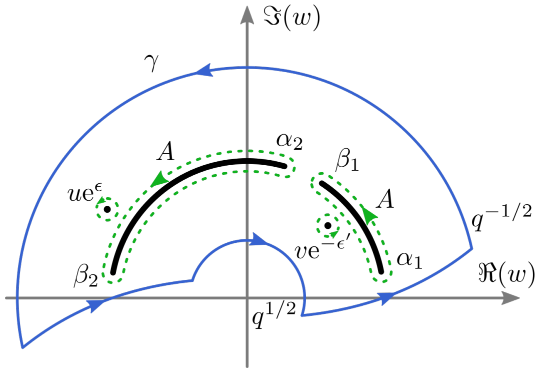

FIG. 1: The coordinate system used in the argument. The black solid line is the entangling region—here for simplicity the union of two intervals. The blue line represents the contour of integration γ in (8), which leads to the residues evaluated along the green dotted curves.

between (7) and this last equation: whereas the integral in the former is an ordinary one, the latter involves a contour integration.

This last integral decomposes into three contributions: one along A just inside the unit circle, one along A just outside the unit circle, and contributions from the boundary points $\alpha_n = e^{i\pi a_n}, \beta_n = e^{i\pi b_n}$ of A as can be seen from fig. 1. Our final requirements on F_V^ν are that the residues at ∂A vanish, while F_V^ν has to have a multiplicative branch cut along A. This means that, at every point along the cut, the ratio of the function just above and below the cut is a fixed number:

$$\frac{F_V^\nu(u\mathrm{e}^{-\epsilon''}, v)}{F_V^\nu(u\mathrm{e}^{+\epsilon''}, v)} = \frac{\xi + 1/2}{\xi - 1/2} =: \mathrm{e}^{2\pi h}. \tag{10}$$

Then, it is easy to show that such an $F_V^\nu(u\mathrm{e}^{\epsilon'}, v)$ indeed solves the problem: Eq. (9) becomes exactly (7). The requirement that the residues on ∂A vanish is equivalent to demanding that the modular flow behaves like Rindler space in the vicinity of ∂A. This is analogous to the derivation of the black hole temperature by the smoothness condition at the event horizon.

In the supplementary material, we explicitly derive F_V^ν satisfying all of the above assumptions. The general procedure is as follows:

1. Start with the standard solution for the requirement of a multiplicative branch cut (10) on the cylinder [37].

2. Average over all fundamental domains in the direction of τ. This yields a quasiperiodic function.

3. Multiply with a slightly modified form of the Green's function (6) to turn the quasiperiodicity into a periodicity and introduce the correct pole.

We are now in position to state one of the main results of this paper: the resolvent for a finite union of disjoint intervals on the torus, $V = \cup_{n=1}^N (a_n, b_n)$. The exact expression lives in the complex plane, but is vastly simplified along A. Introducing the shorthand notation

$$\lambda := \left[\prod_{n=1}^N \frac{\alpha_n}{\beta_n}\right]^{\mathrm{i}h} = \mathrm{e}^{\pi h L}, \tag{11}$$

where L is the total length of V, our result is

$$F_V^\nu(\xi; u, v) = \frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}\mathrm{e}^{\epsilon'}|q)} \frac{\vartheta_\nu(\lambda u v^{-1}|q)}{\vartheta_\nu(\lambda|q)} \times \mathrm{e}^{-2\pi h}\left[\frac{\Omega_V(u)}{\Omega_V(v)}\right]^{\mathrm{i}h} \tag{12}$$

with h defined in (10), and

$$\Omega_V(w) := -\prod_{n=1}^N \frac{\vartheta_1(w\alpha_n^{-1}|q)}{\vartheta_1(w\beta_n^{-1}|q)}. \tag{13}$$

Some comments are in order. Equation (12) is essentially the product of two factors. The second one is the complex power of a quotient, which introduces the required branch cut along A. This function is actually quasi-periodic, acquiring a factor of λ^2 when translated into the next fundamental domain. The first factor resembles the propagator (6) and introduces the desired pole, as described above. Additionally, the extra factor of λ in the argument of ϑ_ν is there to precisely cancel the quasi-periodicity of the second term. This allows the product $G^\nu F_V^\nu$ to be exactly doubly periodic, making the total residue vanish and, hence, the argument work.

MODULAR HAMILTONIAN

Now that we have found the resolvent R_V^ν, we can go back to (1) to obtain the modular Hamiltonian K_V^ν. First, note that the leading divergence of $F_V^\nu(u, v) \sim 1/2(uv^{-1}\mathrm{e}^{\epsilon'} - 1)$ at $u \to v$ can be rewritten as a Cauchy principle value

$$\frac{1}{2}\frac{1}{uv^{-1}\mathrm{e}^{\epsilon'} - 1} = \frac{\delta(x-y)}{2} + \mathcal{P}\frac{1}{2}\frac{1}{uv^{-1} - 1}. \tag{14}$$

For the sake of readability, we shall keep $\mathcal{P}$ implicit for the rest of this paper. Equation (14) implies that the δ-terms from (3) drop out in (1), yielding

$$K_V^\nu = \int_{1/2}^\infty \frac{\mathrm{d}\xi}{(\xi - 1/2)^2}\Big[F_V^\nu(\xi) + F_V^\nu(-\xi)\Big]. \tag{15}$$

In the following, we restrict to purely imaginary modulus $\tau = i\beta$, where β is the inverse temperature – the general case can be recovered by analytic continuation. Since the integrand is oscillatory, the expression (15) has to be understood in the sense of distributions.

In the supplementary material, we evaluate (15) analytically for an arbitrary set of intervals. The main steps are the following:

1. Regularize (15) in a way compatible with its symmetries and evaluate it by contour integration.
2. Use the quasiperiodicity of ϑ_ν to isolate the highly oscillatory contribution of the regulator.
3. Remove the regulator, leading to standard Dirichlet kernel representations of multiple Dirac deltas.

The final expression for the modular Hamiltonian depends on the spin sector. Let us focus on the results for a single interval. Both sectors $\nu = 2, 3$ have a local and a bi-local term. The local term is identical in both cases and takes the form

$$K_{\mathrm{loc}}(x,y) = \beta(x)[\mathrm{i}\partial_x + f(x)]\delta(x-y), \tag{16}$$

with the entanglement temperature

$$\beta(x) = \frac{2\pi\beta}{2\pi + \beta\partial_x \log\Omega_V(\mathrm{e}^{\mathrm{i}\pi x})}, \tag{17}$$

where Ω_V is as defined in (13) and the function $f(x)$ is fixed by requiring that K_{loc} is hermitian. Note that the expression (16) is equivalent to the more familiar representation (5).

The bi-local term represents the central result of this paper and shows a novel feature: In both sectors, it involves a coupling between an infinite but discrete set of points, and is given by

$$K^{\pm}_{\text{bi-loc}}(x,y) = \frac{\mathrm{i}\pi}{L\sinh\pi\mu(x,y)} \times \sum_{k\in\mathbb{Z}\setminus\{0\}} (\pm 1)^k \delta(x-y+\beta\mu(x,y)-k), \tag{18}$$

where the sign $\pm$ corresponds to $\nu = \frac{2}{3}$. Here, we used the function

$$\mu(x,y) = \frac{1}{2\pi L}\log\frac{\Omega_V(\mathrm{e}^{\mathrm{i}\pi x})}{\Omega_V(\mathrm{e}^{\mathrm{i}\pi y})}, \tag{19}$$

which will play an important role in the analysis below.

Note that $K^{\pm}_{\text{bi-loc}}$ couples pairs (x,y) which are solutions of

$$x - y + \beta\mu(x,y) - k = 0, \quad k \in \mathbb{Z}\setminus\{0\}. \tag{20}$$

Since $\mu(x,y)$ is increasing in x and diverges at the endpoints, eq. (20) possesses a unique solution for every k, as shown in fig. 2. Solutions accumulate near the endpoints. In the next section, we analyse the above expressions and discuss their physical meaning. A summary of the results is presented in table I.

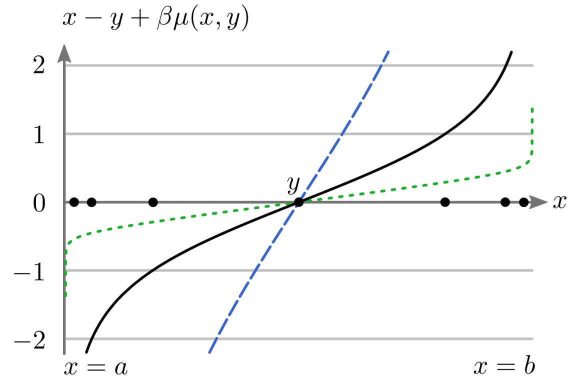

FIG. 2: The function $x - y + \beta\mu(x,y)$ for a single interval $V = (a,b)$ and fixed y. For finite values of β (black solid line), equation (20) has an infinite number of solutions (black dots) in the interval. For large β (blue dashed line), the solutions distribute densely, whereas for small β (green dotted line), they all accumulate at the endpoints.

DISCUSSION

As we saw above, the main result of this paper is that, for arbitrary torus modulus, the modular Hamiltonian contains a local term, as well as an infinite number of bi-local contributions, even for a single interval. Let us now analyse the bi-local terms in more detail: These couple solutions of (20), which are depicted in fig. 2. The behaviour of the function $\mu(x,y)$ from (19) near the endpoints of the interval is particularly interesting. Since μ diverges, there is an infinite number of solutions near these points. Looking at (18), one sees that solutions close to the boundaries are exponentially damped, resembling a redshift factor associated to a Rindler horizon located at the endpoints.

As a next step, let us see how to recover the known results at very high [35] and low [37] temperatures. We start with the high temperature limit $\beta \to 0$. One easily sees from (17) that the local term goes as the inverse temperature, $\beta(x) \sim \beta$, as expected. On the other hand, as depicted in fig. 2, the bi-local contributions (18) all approach the endpoints, where they vanish exponentially.

Moving now to the low temperature limit $\beta \to \infty$, the entanglement temperature (17) approaches the well known result for the cylinder [37]

$$\lim_{\beta\to\infty}\beta(x) = \frac{2\pi}{\partial_x \log\frac{\sin(x-a)}{\sin(b-x)}}. \tag{21}$$

The bi-local contributions however behave remarkably. As can be understood from fig. 2, as we lower the temperature, the curve gets increasingly steep. Thus, the solutions to (20) form a partition of the interval which becomes denser and denser in the limit $\beta \to \infty$. Now, recall that the modular Hamiltonian must always be thought of as a distribution, i.e. as integrated against regular test functions. In this limiting procedure, the solutions to (20) "condense" in the interval, and it can be shown that the sequence of Dirac deltas in (18) reproduce precisely the definition of a Riemann integral. Indeed, one can show that in this sense (18) becomes continuously non-local

$$\lim_{\beta\to\infty} K^{+}_{\text{bi-loc}}(x,y) = \frac{\mathrm{i}\pi}{L\sinh \pi\mu(x,y)}, \qquad (22)$$

in agreement with [37], whereas $\lim_{\beta\to\infty} K^{-}_{\text{bi-loc}} = 0$ due to the oscillating $(-1)^k$.

The previous analysis provides a new insight into the structure of fermionic entanglement: At any finite temperature, non-locality couples a given point only to an infinite but discrete set of other points. The characteristic scale needed to resolve this discreteness goes as $1/\beta$. Hence, continuous non-locality emerges strictly in the limit of zero temperature. We summarize the structure of the modular Hamiltonian in table I.

TABLE I: Summary of our results for the modular Hamiltonian in different spin sectors. The definitions for K_{loc} and $K^{\pm}_{\text{bi-loc}}$ are in (16)–(18). The local and non-local terms at low temperature ($\beta\to\infty$) are given in (21) and (22).

ν	$\beta\to\infty$	β finite	$\beta\to 0$
2	local + cont. non-local	$K_{\text{loc}} + K^{+}_{\text{bi-loc}}$	$\beta\mathrm{i}\partial_x\delta(x-y)$
3	local	$K_{\text{loc}} + K^{-}_{\text{bi-loc}}$	$\beta\mathrm{i}\partial_x\delta(x-y)$

For multiple intervals, the analysis is very similar, with the only difference that (20) now possesses one solution *per interval* for a given k. In particular, we must also consider the non-trivial ($x\neq y$) solutions for $k=0$. In the low temperature limit, these extra terms yield precisely the well known bi-local terms of [36, 37].

CONCLUSIONS

In this paper we computed the modular Hamiltonian of chiral fermions in a thermal state on the circle, reduced to an arbitrary set of disjoint intervals. The key strategy was to reformulate the problem as a singular integral equation, which in turn we solved by complex analysis methods. We were then able to obtain closed form expressions because analytic functions on the torus are highly constrained by the double periodicity.

In summary, our results provide new insights into the structure of entanglement in QFT. We hope they will aid the study of non-local correlations in many body quantums systems, black holes and the emergence of spacetime in gauge/gravity duality.

Even though we restricted to NS boundary conditions in direction of the modulus, an entirely analogous calculation also works in the R sector, as we shall discuss in another paper [43].

Interestingly, an alternative approach to the problem has been developed by Blanco and Pérez-Nadal using method of images, in a paper to appear soon [44].

During the final stage of this project, related results were independently reported in [45]. Since their approach differs significantly from the one presented here, it will be interesting to explore the connection amongst both methods further.

ACKNOWLEDGMENTS

We are very grateful to D. Blanco and G. Pérez-Nadal for collaboration in the initial stages of this project.

We thank J. Camps, B. Czech, M. Heller, H. Hinrichsen, C. Northe and G. Wong for helpful discussions on the subject. The Gravity, Quantum Fields and Information group at AEI is generously supported by the Alexander von Humboldt Foundation and the Federal Ministry for Education and Research through the Sofja Kovalevskaja Award. IR also acknowledges the hospitality of Perimeter Institute, where part of this work was done.

SUPPLEMENTARY MATERIAL

Deriving the resolvent

In this section, we derive the solution (12) F^{ν}_{V} to the singular integral equation (7). Let us start with the functions [37]

$$\omega_n(w) := \frac{\sin(\pi(a_n - z))}{\sin(\pi(b_n - z))} = \frac{\beta_n}{\alpha_n}\frac{\alpha_n^2 - w^2}{\beta_n^2 - w^2}, \qquad (23)$$

which provide the correct branch-cut on the cylinder. Choosing the branch cut of the logarithm along the negative real line, we see that

$$\prod_{n=1}^{N} \frac{\omega_n^{\mathrm{i}h}(w\mathrm{e}^{-\epsilon''})}{\omega_n^{\mathrm{i}h}(w\mathrm{e}^{\epsilon''})} = \mathrm{e}^{2\pi h} \qquad (24)$$

for $w\in A$. Note that (23) is not well defined on the torus since it transforms non-trivially under $w\to qw$. We shall remedy this by defining

$$\log\Omega_n(w) := \sum_{k\in\mathbb{Z}} \left[\log(\omega_n(q^k w)) - \log(\omega_n(q^k))\right], \qquad (25)$$

where the second term in the brackets is to ensure absolute convergence. We made the logarithms explicit in order not to break the behaviour (24) at the branch cut.

At first sight, $\Omega(w)$ seems doubly-periodic: by construction, ω is periodic with respect to the spatial circle, and now we sum over all translations along imaginary time. However, $\Omega(w)$ has a non-vanishing residue within each fundamental region due to the branch-cut, and thus cannot be elliptic. Instead, it turns out to be quasi-periodic, as is seen by putting a cutoff in the sum (25), and then computing $\Omega(qw)$. Then, the series acquires a prefactor originating in

$$\lim_{k\to\pm\infty}\omega_n(q^k w) = \left[\frac{\beta_n}{\alpha_n}\right]^{\mp 1}.$$

This yields the quasi-periodicity

$$\prod_{n=1}^{N}\Omega_n^{\mathrm{i}h}(qw) = \lambda^2\prod_{n=1}^{N}\Omega_n^{\mathrm{i}h}(w),$$

where λ is defined in (11). To cancel off the quasi-periodicity and to introduce the desired pole, we multiply with a combination of theta functions. We find

$$F_V^\nu(\xi;u,v) = \frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}\mathrm{e}^{\epsilon'}|q)}\frac{\vartheta_\nu(\lambda uv^{-1}|q)}{\vartheta_\nu(\lambda|q)} \times\prod_{n=1}^{N}\frac{\Omega_n^{\mathrm{i}h}(u\mathrm{e}^{\epsilon''})}{\Omega_n^{\mathrm{i}h}(v\mathrm{e}^{-\epsilon''})}, \qquad (26)$$

where we made our choice of branches in $\Omega_n^{\mathrm{i}h}$ explicit (our choice is such that the residue evaluation in the main body of the paper does not cross the branch cut). Eq. (26) now solves (7).

Finally, let us rewrite this in terms of more familiar elliptic functions. Note that (24) implies

$$\prod_{n=1}^{N}\frac{\Omega_n^{\mathrm{i}h}(u\mathrm{e}^{\epsilon''})}{\Omega_n^{\mathrm{i}h}(v\mathrm{e}^{-\epsilon''})} = \mathrm{e}^{-2\pi h}\prod_{n=1}^{N}\frac{\Omega_n^{\mathrm{i}h}(u\mathrm{e}^{\epsilon''})}{\Omega_n^{\mathrm{i}h}(v\mathrm{e}^{\epsilon''})}$$

and, now that the numerator and denominator are on the same side of the branch cut, we can move the product into the complex power to find

$$\prod_{n=1}^{N}\frac{\Omega_n^{\mathrm{i}h}(u\mathrm{e}^{\epsilon''})}{\Omega_n^{\mathrm{i}h}(v\mathrm{e}^{-\epsilon''})} = \mathrm{e}^{-2\pi h}\left[\prod_{n=1}^{N}\prod_{k\in\mathbb{Z}}\frac{\omega_n(q^k u\mathrm{e}^{\epsilon''})}{\omega_n(q^k v\mathrm{e}^{\epsilon''})}\right]^{\mathrm{i}h}$$

for $u,v\in A$. After some algebra and an application of the Jacobi triple product [46], this simplifies the solution (26) to (12) with Ω_V from (13).

Deriving the modular Hamiltonian

In this section, we provide the mains steps to evaluate the integral expression (15) for the modular Hamiltonians. We restrict to purely imaginary $\tau = \mathrm{i}\beta$—the general case can be restored by analytic continuation. Let us first change the variable of integration from ξ to

$$\Lambda := \lambda^2 = \mathrm{e}^{2\pi L h} = \left[\frac{\xi+1/2}{\xi-1/2}\right]^L,$$

such that (15) turns into

$$K_V^\nu = \frac{1}{L}\int_0^\infty\frac{\mathrm{d}\Lambda}{\Lambda}\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)}\frac{\vartheta_\nu(\sqrt{\Lambda}uv^{-1}|q)}{\vartheta_\nu(\sqrt{\Lambda}|q)}\Lambda^{\mathrm{i}\mu},$$

where we use the shorthand notation

$$\mu := \frac{1}{2\pi L}\log\frac{\Omega_V(u)}{\Omega_V(v)}.$$

To evaluate the above integral, note the following two facts:

- The integrand is oscillatory for $\Lambda\to 0,\infty$.
- Since we merged the two occurences of $F_V^\nu(\pm\xi)$ in (15) into a single integral, integration has to be done symmetrically with respect to $\xi\to-\xi$, i.e., $\Lambda\to\Lambda^{-1}$.

This requires that we introduce a symmetric regulator $r_\epsilon(\Lambda) = r_\epsilon(\Lambda^{-1})$ to tame the integral, allowing us to evaluate it via standard complex analysis methods. We choose

$$r_\epsilon(\Lambda) := \frac{(1+\epsilon)^2}{(\Lambda+\epsilon)(\Lambda^{-1}+\epsilon)} \qquad (27)$$

to otain

$$K_V^\nu = \lim_{\epsilon\searrow 0}\frac{1}{L}\int_0^\infty\mathrm{d}\Lambda\,\frac{(1+\epsilon)(1+\epsilon^{-1})}{(\Lambda+\epsilon)(\Lambda+\epsilon^{-1})} \times\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)}\frac{\vartheta_\nu(\sqrt{\Lambda}uv^{-1}|q)}{\vartheta_\nu(\sqrt{\Lambda}|q)}\Lambda^{\mathrm{i}\mu}. \qquad (28)$$

The integral (28) can now be evaluated using contour integration. To this end, consider the integral

$$I_\epsilon^\nu := \frac{1}{L}\oint_\gamma\mathrm{d}\Lambda\,\frac{(1+\epsilon)(1+\epsilon^{-1})}{(\Lambda+\epsilon)(\Lambda+\epsilon^{-1})} \times\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)}\frac{\vartheta_\nu(\sqrt{\Lambda}uv^{-1}|q)}{\vartheta_\nu(\sqrt{\Lambda}|q)}\Lambda^{\mathrm{i}\mu}, \qquad (29)$$

where the contour γ is as depicted in fig. 3.

The circular contributions vanish due to the falloff of the regulator. Choosing the branch cut of $\Lambda^{\mathrm{i}\mu}$ along the positive real axis, we see that two remaining horizontal contributions yields two almost identical terms, differing differ only by a global prefactor of $-\mathrm{e}^{-2\pi\mu}$. We thus find

$$\lim_{\epsilon\searrow 0}I_\epsilon^\nu = (1-\mathrm{e}^{-2\pi\mu})K_V^\nu. \qquad (30)$$

By Cauchy's theorem, I_ϵ^ν can also be expressed as a sum over residues, yielding a series expression for K_V^ν. We shall do this explicitly for $\nu=3$ and briefly mention the differences for $\nu=2,4$ at the end.

The poles of the integrand are of two types (see fig. 3): Two come from the regulator (27), located at $\Lambda\to-\epsilon$ and $\Lambda\to-\epsilon^{-1}$. The other (infinitely many) poles come

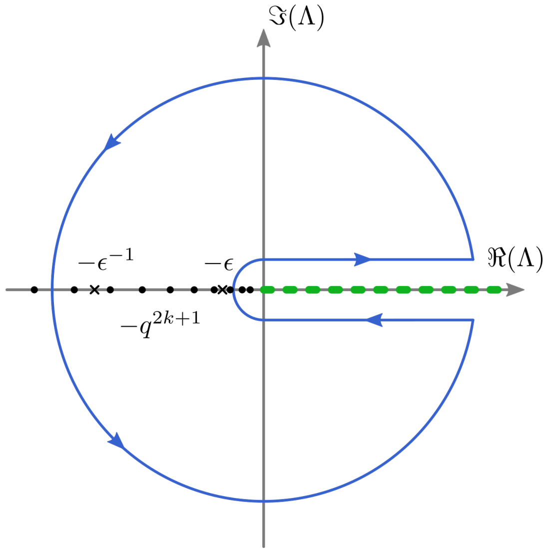

FIG. 3: Contour for the integral (29). The integral along the blue solid line is equal to the sum of all residues at $\Lambda \to -q^{2k+1}$ (black dots) and at $\Lambda \to -\epsilon^{\pm 1}$ (black crosses). The contour avoids the branch cut along the positive real axis (green dashed line).

from the poles of the 'propagator-like' term. As can be seen from either the Laurent expansion of this term (see section below) or directly from the Jacobi triple product, we have the leading divergences

$$\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)}\frac{\vartheta_3(\sqrt{\Lambda}uv^{-1}|q)}{\vartheta_3(\sqrt{\Lambda}|q)} \sim -\frac{(q^{-1}uv^{-1})^{-2k-1}}{\Lambda + q^{2k+1}}$$

at $\Lambda \to -q^{2k+1}$ for $k \in \mathbb{Z}$. Keeping in mind that the negative sign of poles always has to be written as $\mathrm{e}^{+\mathrm{i}\pi}$ due to our choice of branch cut, this yields

$$K_V^3 = \frac{2\pi\mathrm{i}}{L}\frac{1}{\mathrm{e}^{\pi\mu} - \mathrm{e}^{-\pi\mu}} \lim_{\epsilon\searrow 0}\Bigg[\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)} \times \left(\frac{\vartheta_4(\sqrt{\epsilon}uv^{-1}|q)}{\vartheta_4(\sqrt{\epsilon}|q)}\epsilon^{\mathrm{i}\mu} - (\epsilon \to \epsilon^{-1})\right) + \sum_{k\in\mathbb{Z}} \frac{(uv^{-1}q^{-\mathrm{i}\mu})^{-2k-1}}{(q^{2k+1}-\epsilon)(q^{-2k-1}-\epsilon)}\Bigg]. \quad (31)$$

Let us have a look at the series in the last line: Using the Laurent expansions below, this can be rewritten as

$$\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}q^{-i\mu}|q)}\frac{\vartheta_4(\sqrt{\epsilon}uv^{-1}q^{-i\mu}|q)}{\vartheta_4(\sqrt{\epsilon}|q)} - (\epsilon \to \epsilon^{-1}).$$

We choose the cutoff to be $\epsilon = q^{2m}$ with very large $m \in \mathbb{Z}$ to avoid the poles at q^{2k+1}, so that we only deal with simple poles. Then, putting everything together into (31) and using the quasiperiodicities of ϑ_4, one finds

$$K_V^3(x,y) = \lim_{m\to\infty} P(x,y)\sin\big(2m\pi(x-y+\beta\mu)\big)$$

with

$$P(x,y) = \frac{2\pi}{L\sinh\pi\mu(x,y)}\Bigg[\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}|q)}\frac{\vartheta_4(uv^{-1}|q)}{\vartheta_4(1|q)} - \frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(uv^{-1}q^{-i\mu}|q)}\frac{\vartheta_4(uv^{-1}q^{-i\mu}|q)}{\vartheta_4(1|q)}\Bigg].$$

As already stated above, this limit must be understood in the sense of distributions.

We see that K_V^3 contains essentially two factors: the term involving the sine function is highly oscillatory for $m \to \infty$, except at solutions of

$$x - y + \beta\mu(x,y) = k \in \mathbb{Z}. \quad (32)$$

As a distribution, it vanishes when integrated against any regular test function. However, the remaining factor $P(x,y)$ is not regular since it has poles, and thus we must examine its behaviour in their vicinity, which will lead to finite contributions. These poles coincide precisely with the solutions to (32), which are of two kinds: the trivial solution $x = y$ will lead to a local term, while the other solutions $x \neq y$ will give bi-local contributions. Let us start with the latter.

Close to these solutions, a straightforward calculation shows that

$$P(x,y) \sim \frac{\mathrm{i}\pi}{L\sinh\pi\mu(x,y)}\frac{1}{\sin(\pi[x-y+\beta\mu(x,y)])}.$$

Combined with the oscillatory term, we recognize the Dirichlet kernel [47] representation of the anti-periodic Dirac delta

$$\lim_{m\to\infty}\frac{\sin 2m\pi z}{\sin\pi z} = \sum_{k\in\mathbb{Z}}(-1)^k\delta(z-k), \quad (33)$$

yielding the final expression for the modular Hamiltonian for $x \neq y$,

$$\frac{\mathrm{i}\pi}{L\sinh\pi\mu}\sum_{k\in\mathbb{Z}}(-1)^k\delta(x-y+\beta\mu(x,y)-k). \quad (34)$$

Now we turn to the solution $x = y$, which is special as it leads to a second order pole in P. Similarly to before, in the vicinity of that solution, $P(x,y)$ takes the form

$$-\frac{\mathrm{i}\beta}{L}\frac{1}{x-y}\frac{1}{\sin(\pi[x-y+\beta\mu(x,y)])},$$

which together with the oscillatory term leads to

$$-\frac{\mathrm{i}\beta}{L}\frac{\delta(x-y+\beta\mu(x,y))}{x-y}. \quad (35)$$

Note that we did not need to consider the terms with $k \neq 0$ as in (33) since we only deal with the solution $x = y$. As a last step, we use the methods from [36] to rewrite the singular fraction (35) as

$$\frac{\beta}{L}\frac{[\mathrm{i}\partial_x + f(x)]\delta(x-y)}{1+\beta(\partial_x\mu)(y,y)}, \quad (36)$$

where $f(x)$ is fixed by hermiticity.

Now we focus on the case of a single interval. Again we begin by considering on the bi-local terms. Since $\mu(x, y)$ is monotonically increasing with respect to x in the interval, eq. (32) has a unique solution for each $k \in \mathbb{Z}$. In particular, note that for $k = 0$, the solution is $x = y$. Since we already consider this contribution separately in (36), we can explicitly exclude it from the series (34). The final expression for the modular Hamiltonian for a single interval is then given by the sum of (16) and (18).

An analogous calculation holds for $\nu = 2$, with one small adjustment: Since the poles of the Laurent expansion are instead located at $-q^{2k}$, we obtain the periodic version of the Dirichlet kernel in (33). The rest of the calculation is identical.

LAURENT EXPANSION

To better understand the location and behaviour of the poles of the "propagator-like" terms in (12), we derived their Laurent expansions. The coefficients may be computed as contour integrals which vastly simplify due to the quasi-periodicities of the theta functions. In the fundamental domain $|q|^{1/2} < |w| < |q|^{-1/2}$, the result then takes the form of Lambert series

$$\begin{aligned}
\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(w|q)} \frac{\vartheta_3(\lambda w|q)}{\vartheta_3(\lambda|q)} &= \frac{1}{w - w^{-1}} \\
&\quad + \sum_{\substack{k \geq 1 \\ k \text{ odd}}} \left[\frac{w^k q^k}{\lambda^{-2} + q^k} - \frac{w^{-k} q^k}{\lambda^2 + q^k} \right], \\
\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(w|q)} \frac{\vartheta_4(\lambda w|q)}{\vartheta_4(\lambda|q)} &= \frac{1}{w - w^{-1}} \\
&\quad - \sum_{\substack{k \geq 1 \\ k \text{ odd}}} \left[\frac{w^k q^k}{\lambda^{-2} - q^k} - \frac{w^{-k} q^k}{\lambda^2 - q^k} \right], \\
\frac{\eta^3(q^2)}{\mathrm{i}\vartheta_1(w|q)} \frac{\vartheta_2(\lambda w|q)}{\vartheta_2(\lambda|q)} &= \frac{1}{2} \frac{w + w^{-1}}{w - w^{-1}} + \frac{1}{2} \frac{\lambda^2 - 1}{\lambda^2 + 1} \\
&\quad + \sum_{\substack{k \geq 2 \\ k \text{ even}}} \left[\frac{w^k q^k}{\lambda^{-2} + q^k} - \frac{w^{-k} q^k}{\lambda^2 + q^k} \right].
\end{aligned}$$

[1] S. A. Fulling, "Nonuniqueness of canonical field quantization in Riemannian space-time," *Phys. Rev.*, vol. D7, pp. 2850–2862, 1973.

[2] P. C. W. Davies, "Scalar particle production in Schwarzschild and Rindler metrics," *J. Phys.*, vol. A8, pp. 609–616, 1975.

[3] W. G. Unruh, "Notes on black hole evaporation," *Phys. Rev. D*, vol. 14, p. 870, 1976.

[4] S. W. Hawking, "EUCLIDEAN QUANTUM GRAVITY," *NATO Sci. Ser. B*, vol. 44, p. 145, 1979.

[5] R. Haag, *Local quantum physics: Fields, particles, algebras*. Texts and monographs in physics, Berlin, Germany: Springer, 1992.

[6] H. Li and F. D. M. Haldane, "Entanglement Spectrum as a Generalization of Entanglement Entropy: Identification of Topological Order in Non-Abelian Fractional Quantum Hall Effect States," *Phys. Rev. Lett.*, vol. 101, p. 010504, Jul 2008.

[7] A. Chandran, M. Hermanns, N. Regnault, and B. A. Bernevig, "Bulk-edge correspondence in entanglement spectra," *Phys. Rev. B*, vol. 84, p. 205136, Nov 2011.

[8] M. Dalmonte, B. Vermersch, and P. Zoller, "Quantum Simulation and Spectroscopy of Entanglement Hamiltonians," *Nature Phys.*, vol. 14, no. 8, pp. 827–831, 2018.

[9] I. Peschel and V. Eisler, "Reduced density matrices and entanglement entropy in free lattice models," *J. Phys. A: Math. Gen.*, vol. 42, p. 504003, Dec 2009.

[10] V. Eisler and I. Peschel, "Properties of the entanglement hamiltonian for finite free-fermion chains," *arXiv e-print*, Apr 2018.

[11] W. Zhu, Z. Huang, and Y.-c. He, "Reconstructing Entanglement Hamiltonian via Entanglement Eigenstates," *arXiv e-print*, Jun 2018.

[12] F. Parisen Toldin and F. F. Assaad, "Entanglement Hamiltonian of Interacting Fermionic Models," *Phys. Rev. Lett.*, vol. 121, p. 200602, Nov 2018.

[13] D. J. Luitz, N. Laflorencie, and F. Alet, "Participation spectroscopy and entanglement Hamiltonian of quantum spin models," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2014, p. 08007, Aug 2014.

[14] J. I. Cirac, D. Poilblanc, N. Schuch, and F. Verstraete, "Entanglement spectrum and boundary theories with projected entangled-pair states," *Phys. Rev. B*, vol. 83, p. 245134, Jun 2011.

[15] T. H. Hsieh, L. Fu, and X.-L. Qi, "Tensor Network Implementation of Bulk Entanglement Spectrum," *Phys. Rev.*, vol. B90, no. 8, p. 085137, 2014.

[16] L. Vanderstraeten, M. Mariën, J. Haegeman, N. Schuch, J. Vidal, and F. Verstraete, "Bridging perturbative expansions with tensor networks," *Phys. Rev. Lett.*, vol. 119, p. 070401, Aug 2017.

[17] G. Sarosi and T. Ugajin, "Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields," *JHEP*, vol. 01, p. 012, 2018.

[18] H. Casini, E. Teste, and G. Torroba, "Modular Hamiltonians on the null plane and the Markov property of the vacuum state," *J. Phys. A: Math. Gen.*, vol. 50, no. 36, p. 364001, 2017.

[19] H. Casini, "Relative entropy and the Bekenstein bound," *Class. Quant. Grav.*, vol. 25, p. 205021, 2008.

[20] D. D. Blanco and H. Casini, "Localization of Negative Energy and the Bekenstein Bound," *Phys. Rev. Lett.*, vol. 111, no. 22, p. 221601, 2013.

[21] T. Faulkner, R. G. Leigh, O. Parrikar, and H. Wang, "Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition," *JHEP*, vol. 09, p. 038, 2016.

[22] H. Casini, M. Huerta, and R. C. Myers, "Towards a derivation of holographic entanglement entropy," *JHEP*, vol. 05, p. 036, 2011.

[23] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, "Relative Entropy and Holography," *JHEP*, vol. 08, p. 060, 2013.

[24] D. L. Jafferis and S. J. Suh, "The Gravity Duals of Mod-

ular Hamiltonians," *JHEP*, vol. 09, p. 068, 2016.

[25] D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, "Relative entropy equals bulk relative entropy," *JHEP*, vol. 06, p. 004, 2016.

[26] N. Lashkari, M. B. McDermott, and M. Van Raamsdonk, "Gravitational dynamics from entanglement 'thermodynamics'," *JHEP*, vol. 04, p. 195, 2014.

[27] J. Koeller, S. Leichenauer, A. Levine, and A. Shahbazi-Moghaddam, "Local Modular Hamiltonians from the Quantum Null Energy Condition," *Phys. Rev. D*, vol. 97, no. 6, p. 065011, 2018.

[28] Y. Chen, X. Dong, A. Lewkowycz, and X.-L. Qi, "Modular Flow as a Disentangler," *JHEP*, vol. 12, p. 083, 2018.

[29] A. Belin, N. Iqbal, and S. F. Lokhande, "Bulk entanglement entropy in perturbative excited states," *SciPost Phys.*, vol. 5, no. 3, p. 024, 2018.

[30] R. Abt and J. Erdmenger, "Properties of Modular Hamiltonians on Entanglement Plateaux," *JHEP*, vol. 11, p. 2, 2018.

[31] R. Jefferson, "Comments on black hole interiors and modular inclusions," *SciPost Phys.*, vol. 6, p. 042, 2019.

[32] D. Blanco, M. Leston, and G. Prez-Nadal, "Gravity from entanglement for boundary subregions," *JHEP*, vol. 18, p. 130, 2018.

[33] J. J. Bisognano and E. H. Wichmann, "On the Duality Condition for Quantum Fields," *J. Math. Phys.*, vol. 17, pp. 303–321, 1976.

[34] T. Hartman and N. Afkhami-Jeddi, "Speed Limits for Entanglement," *arXiv e-print*, Dec 2015.

[35] J. Cardy and E. Tonni, "Entanglement hamiltonians in two-dimensional conformal field theory," *J. Stat. Mech.*, vol. 1612, no. 12, p. 123103, 2016.

[36] H. Casini and M. Huerta, "Reduced density matrix and internal dynamics for multicomponent regions," *Class. Quant. Grav.*, vol. 26, p. 185005, 2009.

[37] I. Klich, D. Vaman, and G. Wong, "Entanglement Hamiltonians for chiral fermions with zero modes," *Phys. Rev. Lett.*, vol. 119, no. 12, p. 120401, 2017.

[38] R. E. Arias, H. Casini, M. Huerta, and D. Pontello, "Entropy and modular Hamiltonian for a free chiral scalar in two intervals," *Phys. Rev. D*, vol. 98, no. 12, p. 125008, 2018.

[39] I. Klich, D. Vaman, and G. Wong, "Entanglement Hamiltonians and entropy in 1+1D chiral fermion systems," *Phys. Rev.*, vol. B98, p. 035134, 2018.

[40] I. Peschel, "Calculation of reduced density matrices from correlation functions," *J. Phys. A: Math. Gen.*, vol. 36, pp. 205–208, Apr 2003.

[41] N. I. Muskhelishvili, *Singular Integral Equations*. Springer Netherlands, 1 ed., 1958.

[42] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics, New York: Springer-Verlag, 1997.

[43] P. Fries and I. A. Reyes , to appear.

[44] D. Blanco and G. Perez-Nadal , to appear.

[45] S. Hollands, "On the modular operator of mutli-component regions in chiral cft," *arXiv e-print*, Apr 2019.

[46] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge University Press, 4 ed., 2009.

[47] W. Rudin, *Principles of Mathematical Analysis*. International Series in Pure & Applied Mathematics, New York: McGraw Hill, 3 ed., 1976.