# Entire functions and $m$-convex structure in commutative Baire algebras

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**Abstract**

We show that a unitary commutative locally convex algebra, with a continuous product which is a Baire space and in which entire functions operate is actually $m$-convex. Whence, as a consequence, the same result of Mitiagin, Rolewicz and Zelazko, in commutative $B_0$-algebras.

It is known that entire functions operate in complete $m$-convex algebras [1]. In [3] Mitiagin, Rolewicz and Zelazko show that a unitary commutative $B_0$-algebra in which all entire functions operate is necessarily $m$-convex. Their proof is quite long and more or less technical. They use particular properties of $B_0$-algebras, a Baire argument and the polarisation formula. Here we show that any unitary commutative locally convex algebra, with a continuous product which is a Baire space and in which all entire functions operate is actually $m$-convex. The proof is short, direct and selfcontained.

A locally convex algebra $(A, \tau)$, l. c. a. in brief, is an algebra over a field $K$ ($K = R$ or $C$) with a Hausdorff locally-convex topology for which the product is separately continuous. If the product is continuous in two variables, $(A, \tau)$ is said to be with continuous product. A l. c. a. $(A, \tau)$ is said to be $m$-convex (l. m. c. a.) if the origin 0 admits a fundamental system of idempotent neighbourhoods ([2]). An entire function $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in K$, operates in a unitary l. c. a. $(A, \tau)$ if, for every $x$ in $A$, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, converges in $(A, \tau)$.

Lemma 1.5 in [3], given in $B_0$-algebras, is actually valid in any l. c. a. and with the same proof.

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**Lemma.** *Let $(A,\tau)$ be a l. c. a. and $(p_\lambda)_{\lambda\in\Lambda}$ a family of seminorms defining $\tau$. If any entire function operate in $A$, then for every $x$ in $A$, $\underset{n}{Sup}\ [p_\lambda(x^n)]^{\frac{1}{n}} < +\infty$, for every $\lambda \in \Lambda$.*

*Proof:* If not then there is an $\lambda_0$ and $x_0$ such that $p_{\lambda_0}(x_0^{k_n}) \geq n^{k_n}$ for a certain increasing sequence $(k_n)_n$ of integers. This implies that the entire function $\sum_{n=0}^{+\infty} n^{-k_n} z^{k_n}$ diverges at $x_0$. ∎

**Theorem.** *Let $(A,\tau)$ a unitary commutative l. c. a. with a continuous product which is a Baire space. If entire functions operate in $A$, then it is $m$-convex.*

*Proof:* Let $V$ be a closed absolutely convex neighbourhood of zero, in $A$, and $p$ its gauge. The product being continuous, there is another continuous seminorm $q$ such that

$$p(ab) \leq q(a)q(b); \quad a, b \in A.$$

By the lemma, we have $f_q(a) = \underset{n}{Sup}\ [q(a^n)]^{\frac{1}{n}} < +\infty$ for every $a$ in $A$. Since $f_q$ is lower semicontinuous, the set $A_n = \{a \in A\colon f_q(a) \leq n\}$ is closed, for every integer $n$. By Baire's argument, there is an integer $m$ such that $A_m$ is of non void interior. Hence, there is an $a_0$ in $A_m$ and a neighbourhood $W$ of zero such that, for every $a$ in $W$,

$$q[(a_0+a)^n] \leq m^n, \quad n = 1, 2, \ldots.$$

Whence,

$$\begin{aligned} p(a^n) &= p[(a_0 + a - a_0)^n] \\ &\leq \sum_{k=0}^{n} \binom{n}{k} p[(a_0+a)^k(-a_0)^{n-k}] \\ &\leq \sum_{k=0}^{n} \binom{n}{k} q[(a_0+a)^k] q(a_0^{n-k}) \\ &\leq (2m)^n. \end{aligned}$$

So we have

$$\left(\frac{1}{2m}a\right)^n \in V, \text{ for every } a \text{ in } W.$$

Consider the polarisation formula

$$x_1x_2 \ldots x_n = \frac{1}{n!}\sum_I (-1)^{n-c(I)} \left(\sum_{i\in I} x_i\right)^n$$

where $I$ runs over the collection of all finite subsets of $\{1, 2, \ldots, n\}$, $c(I)$ the cardinal of $I$ and $x_1, x_2, \ldots, x_n$ elements of $A$.

For $t > 0$, if $x_i \in \frac{t}{2m}W$, $1 \leq i \leq n$, we have $x_1x_2\ldots x_n \in \frac{(2nt)^n}{n!}V$. Then, for $t$ small enough, $V$ contains an idempotent neighbourhood of zero. ∎

## References

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