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ENTIRE FUNCTIONS WITH MAXIMAL DEFICIENCY SUM

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§ **1.** Let $f(z)$ be a transcendental meromorphic function in the finite z-plane. The standard symbols of the Nevanlinna theory

$$m(r, f),\ n(r, f),\ N(r, f),\ T(r, f),\ \delta(a, f), \cdots$$

are used throughout the paper.

Denote by λ_f the order of $f(z)$ and by μ_f its lower order. In addition to the above concepts, we shall consider the total deficiency $\Delta(f)$ of the function $f(z)$:

$$\Delta(f)=\sum_a \delta(a, f),$$

where the summation is to be extended to all the values a, finite or infinite, such that

$$\delta(a, f)>0. \tag{1.1}$$

The number of deficient values of $f(z)$, that is the number of a for which (1.1) holds, will be denoted by $\nu(f)$ $(\leqq\infty)$.

The Nevanlinna second fundamental theorem yields that $\Delta(f)\leqq 2$.

Recently Weitsman [9] proved

(A) *Let $f(z)$ be a meromorphic function of finite lower order μ_f such that $\Delta(f)=2$. Then $\nu(f)\leqq 2\mu_f$.*

The aim in this paper is to prove the following result by the ingenious method developed in Weitsman's paper [9]:

(B) *Let $f(z)$ be a meromorphic function of finite lower order μ_f such that $\Delta(f)=2$, $\delta(\infty, f)=1$. Then $\nu(f)\leqq\mu_f+1$.*

The above result (B) was proved by Pfluger [8] and Edrei and Fuchs [6] in the case of $\lambda_f<\infty$ (Pfluger proved that $\nu(f)\leqq\lambda_f+1$).

§ **2.** An increasing positive sequence

$$r_1, r_2, \cdots, r_m, \cdots$$

is said to be a sequence of Pólya peaks, of order ρ $(0\leqq\rho<\infty)$, of $T(r, f)$, if it is possible to find three sequences

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(2. 1) $$\{r_m'\},\ \{r_m''\},\ \{\varepsilon_m\}$$

such that, as $m\to\infty$,

(2. 2) $$r_m'\to\infty,\ \frac{r_m}{r_m'}\to\infty,\ \frac{r_m''}{r_m}\to\infty,\ \varepsilon_m\to 0,$$

and such that

(2. 3) $$T(r,f)\leqq(1+\varepsilon_m)\left(\frac{r}{r_m}\right)^{\rho}T(r_m,f)\qquad(r_m'\leqq r\leqq r_m'')$$

and

(2. 4) $$T(r,f)\leqq\left(\frac{r}{r_m}\right)^{\rho-1/m}T(r_m,f)\qquad(r_0\leqq r\leqq r_m'),$$

where r_0 is a constant associated with $T(r,f)$.

The main result about Pólya peaks is the following existence theorem:

If $f(z)$ has a finite lower order μ_f, then for each finite number ρ satisfying $\mu_f\leqq\rho\leqq\lambda_f$, $T(r,f)$ has a sequence $\{r_m\}$ of Pólya peaks of order ρ.

A proof of the existence theorem will be found in [2], [3] and [7].

§ **3.** Our basic tool is the following lemma due to Edrei [2]:

LEMMA. *Let $f(z)$ be a meromorphic function and let $f(0)=1$. Denote by $\{a_j\}_{j=1}^{\infty}$ the zeros of $f(z)$ and by $\{b_j\}_{j=1}^{\infty}$ its poles. Put*

$$\gamma_0=0,\ \gamma_m=\frac{1}{\pi\rho^m}\int_0^{2\pi}\log|f(\rho e^{i\theta})|e^{-im\theta}d\theta\qquad(m\geqq 1),$$

where $\rho(>0)$ is so small that the disc $|z|\leqq\rho$ contains neither zeros nor poles of $f(z)$.

Then, if q is a non-negative integer and if

$$0<r=|z|\leqq\frac{R}{2},$$

we have

(3. 1) $$\log|f(z)|=\mathrm{Re}\{\gamma_0+\gamma_1 z+\cdots+\gamma_q z^q\}+\log\left|\prod_{|a_j|\leqq R}E\left(\frac{z}{a_j},q\right)\right|-\log\left|\prod_{|b_j|\leqq R}E\left(\frac{z}{b_j},q\right)\right|+S_q(z,R),$$

where

$$E(u,0)=1-u;\ E(u,q)=(1-u)\exp\left\{u+\frac{u^2}{2}+\cdots+\frac{u^q}{q}\right\}\qquad(q\geqq 1)$$

and

(3. 2) $$|S_q(z,R)|\leqq 14\left(\frac{r}{R}\right)^{q+1}T(2R,f).$$

§ 4. We shall give a proof of the result (B).

Proof of (B). It was proved that $\Delta(f)<2$, if $\mu_f<1$ and $\delta(\infty, f)=1$ [4]. Hence in the following discussion we may assume that $\mu_f\geqq 1$.

It is well known that the following inequality

$$\Delta(f)\leqq 2-\varlimsup_{r\to\infty,\, r\notin\mathcal{E}}\frac{N(r, 1/f')+N(r, f')}{T(r, f')}$$

holds with an exceptional set $\mathcal{E}$ of finite measure, if $\delta(\infty, f)=1$. Hence, if $\Delta(f)=2$, then we have

$$\lim_{r\to\infty,\, r\notin\mathcal{E}}\frac{N(r, 1/f')+N(r, f')}{T(r, f')}=0. \tag{4. 1}$$

Put

$$n_1(r)=n\left(r, \frac{1}{f'}\right)+n(r, f'),$$

$$N_1(r)=N\left(r, \frac{1}{f'}\right)+N(r, f').$$

Let $\{r_m\}$ be a sequence of Pólya peaks, of order μ_f, of $T(r, f)$. Let $\{r_m'\}$, $\{r_m''\}$ and $\{\varepsilon_m\}$ be three sequences satisfying (2. 2), (2. 3) and (2. 4).

By (4. 1), there is a sequence $\{\eta_m\}$ such that

$$\operatorname*{Sup}_{r_m'\leqq t,\, t\notin\mathcal{E}}\frac{N_1(t)}{T(t, f')}<\eta_m, \qquad \lim_{m\to\infty}\eta_m=0. \tag{4. 2}$$

In the following lines we shall study the asymptotic behavior of $f'(z)$ around the sequence $\{r_m\}$.

We use the fact that $\mu_f=\mu_{f'}$, which was proved by Chuang [1]. We set

$$q=[\mu_{f'}]. \tag{4. 3}$$

Put

$$R_m=\frac{1}{4\alpha}\min\{\eta_m{}^{-1/(4\mu_f)}r_m, r_m''\},$$

where $\alpha=\exp(1/(q+1))$.

Denote by $\{a_j\}_{j=1}^{\infty}$ the non-zero zeros of $f'(z)$ and by $\{b_j\}_{j=1}^{\infty}$ its non-zero poles. Set

$$C(r)=\gamma_q+\frac{1}{q}\left\{\sum_{|a_j|\leqq r}\frac{1}{a_j{}^q}-\sum_{|b_j|\leqq r}\frac{1}{b_j{}^q}\right\},$$

where γ_q is defined by

$$\gamma_q=\frac{1}{\pi\rho^q}\int_0^{2\pi}\log|\tilde{f}'(\rho e^{i\theta})|e^{-iq\theta}d\theta$$

with a suitable function $\tilde{f}'(z)$ such that $\tilde{f}'(z)=Az^l f'(z)$, $\tilde{f}'(0)=1$ and a positive number $\rho<\min_j\{|a_j|, |b_j|\}$.

With q defined by (4. 3) we apply the lemma stated in § 3 for $f'(z)$. Then, for $r=|z|\leqq R/2$,

$$\log|f'(z)|=\mathrm{Re}\{\gamma_0+\gamma_1 z+\cdots+\gamma_q z^q\}+\log\left|\prod_{|a_j|\leqq R}E\left(\frac{z}{a_j}, q\right)\right|$$
$$-\log\left|\prod_{|b_j|\leqq R}E\left(\frac{z}{b_j}, q\right)\right|+S_q'(z, R)+O(\log r),$$

where

$$|S_q'(z, R)|\leqq 14\left(\frac{r}{R}\right)^{q+1}T(2R, f').$$

Hence

$$\log|f'(z)|-\mathrm{Re}\{C(r)z^q\}-S_q'(z, R)=\log|g|$$

(4. 4)
$$=\log\left|\prod_{|a_j|\leqq r}E\left(\frac{z}{a_j}, q-1\right)\right|-\log\left|\prod_{|b_j|\leqq r}E\left(\frac{z}{b_j}, q-1\right)\right|$$
$$+\log\left|\prod_{r<|a|\leqq R}E\left(\frac{z}{a_j}, q\right)\right|-\log\left|\prod_{r<|b_j|\leqq R}E\left(\frac{z}{b_j}, q\right)\right|+O(r^{q-1}+\log r).$$

Put

$$\phi(t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|te^{i\theta}-1|}\qquad(t\neq 1).$$

Then we get the following inequality [5]:

(4. 5)
$$\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|E\left(\frac{re^{i\theta}}{a}, p\right)\right|\right|d\theta\leqq r^p\int_{|a|}^{\infty}t^{-p-1}\phi\left(\frac{t}{r}\right)dt.$$

We set

(4. 6)
$$\alpha_m=\left(\frac{r_m}{R_m}\right)^{-(q+1-\mu_f)/(2q+2)},$$

(4. 7)
$$\beta_m=\eta_m{}^{-1/(2\mu_f)},$$

(4. 8)
$$\gamma_m=\delta_m{}^{-1/(2\mu_f-2/m)},$$

where

$$\delta_m=\int_{|d_1|/R_m{}^\alpha}^{r_m'/r_m{}^\alpha}t^{\mu_f-q-1/m}\phi(t)\,dt,\quad |d_1|=\min_j(|a_j|, |b_j|).$$

Further we define

(4. 9)
$$\sigma_m=\min\{\alpha_m, \beta_m, \gamma_m\},$$

(4. 10)
$$r_m{}^*=\frac{1}{4\alpha}\min\{\sigma_m r_m, R_m+r_m\}.$$

By (4. 6), (4. 7) and (4. 8), $\sigma_m\to\infty$, as $m\to\infty$. By (4. 5) we have, as $r\to\infty$,

$$\begin{aligned} & m(r, g)+m\left(r, \frac{1}{g}\right) \\ (4.11)\qquad & \leqq r^{q-1}\int_{|d_1|}^{r} n_1(t)t^{-q}\phi\left(\frac{t}{r}\right)dt+r^{q-1}n_1(r)\int_r^{\infty} t^{-q}\phi\left(\frac{t}{r}\right)dt \\ & +r^q\int_r^{\infty}\{n_1^*(t)-n_1(r)\}\,t^{-q-1}\phi\left(\frac{t}{r}\right)dt+O(r^{q-1}+\log r), \end{aligned}$$

where

$$n_1^*(t)=\begin{cases} n_1(t), & t\leqq R, \\ n_1(R), & t>R. \end{cases}$$

By making use of (4. 2) we have

$$n_1(t)\leqq(q+1)N_1(\alpha t)\leqq(q+1)\eta_m T(\alpha t, f'), \tag{4.12}$$

if $\alpha t\notin\mathcal{E}$, $\alpha t\geqq r_m'$. Since $\delta(\infty, f)=1$, we obtain, as $t\to\infty$,

$$T(t, f')\leqq(1+o(1))T(t, f), \tag{4.13}$$

if $t\notin\mathcal{E}$.

In the following discussion we assume that $r\in[r_m, r_m^*]$. Hence $T(r_m, f)\leqq T(r, f)$. Put

$$\begin{aligned} r^{q-1}\int_{|d_1|}^{r} n_1(t)t^{-q}\phi\left(\frac{t}{r}\right)dt &= r^{q-1}\left\{\int_{|d_1|}^{(r_0-1)/\alpha}+\int_{(r_0-1)/\alpha}^{(r_m'-1)/\alpha}+\int_{(r_m'-1)/\alpha}^{r}\right\} \\ &\equiv I_m^1+I_m^2+I_m^3, \end{aligned}$$

where r_0 is a sufficiently large value such that (2. 4), (4. 13) and $N_1(t)\leqq 2T(t, f')$ hold for all $t\geqq(r_0-1)/\alpha$, $t\notin\mathcal{E}$. Then

$$I_m^1=O(r^{q-1}). \tag{4.14}$$

Since $\mathcal{E}$ is a set of finite measure, we can find a point u such that $u\notin\mathcal{E}$, $u\in[t, t+1]$, if t is sufficiently large. Hence $T(\alpha t, f')\leqq(1+o(1))\,T(\alpha t+1, f)$, if t is sufficiently large. Thus, by (2. 4), (4. 8), (4. 10), (4. 12) and (4. 13), we get

$$\begin{aligned} I_m^2 & \leqq(q+1)r^{q-1}\int_{(r_0-1)/\alpha}^{(r_m'-1)/\alpha} N_1(\alpha t)t^{-q}\phi\left(\frac{t}{r}\right)dt \\ & \leqq 2(q+1)r^{q-1}\int_{(r_0-1)/\alpha}^{(r_m'-1)/\alpha} T(\alpha t+1, f')\,t^{-q}\phi\left(\frac{t}{r}\right)dt \\ (4.15)\qquad & \leqq 4(q+1)r^{q-1}\int_{(r_0-1)/\alpha}^{(r_m'-1)/\alpha} T(\alpha t+2, f)\,t^{-q}\phi\left(\frac{t}{r}\right)dt \\ & \leqq 4(q+1)T(r_m, f)r^{q-1}\int_{(r_0-1)/\alpha}^{(r_m'-1)/\alpha}\left(\frac{\alpha t+2}{r_m}\right)^{\mu_f-1/m} t^{-q}\phi\left(\frac{t}{r}\right)dt \end{aligned}$$

$$\leqq 4(q+1)(2\alpha)^{\mu_f-1/m}\left(\frac{r}{r_m}\right)^{\mu_f-1/m}\delta_m T(r, f)=o(T(r, f)).$$

Similary we have

$$(4.16)\quad \begin{aligned} I_m{}^3 &\leqq (q+1)r^{q-1}\int_{(r_m'-1)/\alpha}^{r} N_1(\alpha t)t^{-q}\phi\left(\frac{t}{r}\right)dt \\ &\leqq (q+1)\eta_m r^{q-1}\int_{(r_m'-1)/\alpha}^{r} T(\alpha t+1, f')t^{-q}\phi\left(\frac{t}{r}\right)dt \\ &\leqq 2(q+1)\eta_m(1+\varepsilon_m)T(r_m, f)r^{q-1}\int_{(r_m'-1)/\alpha}^{r}\left(\frac{\alpha t+2}{r_m}\right)^{\mu_f}t^{-q}\phi\left(\frac{t}{r}\right)dt \\ &\leqq 2(q+1)(2\alpha)^{\mu_f}\eta_m(1+\varepsilon_m)\sigma_m{}^{\mu_f}T(r, f)\int_{r_m'/r}^{1} t^{\mu_f-q}\phi(t)\,dt=o(T(r, f)), \end{aligned}$$

since

$$\int_0^1 t^{\mu_f-q}\phi(t)\,dt \leqq \int_0^1 t^{-1/2}\phi(t)\,dt<\infty.$$

Since $q\geqq 1$, as above, we have

$$(4.17)\quad \begin{aligned} & r^{q-1}n_1(r)\int_r^{\infty} t^{-q}\phi\left(\frac{t}{r}\right)dt \leqq 2(q+1)\eta_m T(\alpha r+2, f)r^{q-1}\int_r^{\infty} t^{-q}\phi\left(\frac{t}{r}\right)dt \\ &\leqq 2(q+1)(2\alpha)^{\mu_f}(1+\varepsilon_m)\eta_m\sigma_m{}^{\mu_f}T(r, f)\int_1^{\infty} t^{-1/2}\phi(t)\,dt=o(T(r, f)). \end{aligned}$$

We apply (4.11) with $R=R_m$. Put

$$r^q\int_r^{\infty}\{n_1{}^*(t)-n_1(r)\}t^{-q-1}\phi\left(\frac{t}{r}\right)dt=r^q\left\{\int_r^{R_m}+\int_{R_m}^{\infty}\right\}\equiv I_m{}^4+I_m{}^5.$$

Then, as above, we get

$$(4.18)\quad \begin{aligned} I_m{}^5 &\leqq r^q n_1(R_m)\int_{R_m}^{\infty} t^{-q-1}\phi\left(\frac{t}{r}\right)dt \\ &\leqq 2(q+1)(2\alpha)^{\mu_f}(1+\varepsilon_m)\eta_m{}^{3/4}T(r, f)\int_{R_m/r}^{\infty} t^{-1/2}\phi(t)\,dt=o(T(r, f)), \end{aligned}$$

$$(4.19)\quad \begin{aligned} I_m{}^4 &\leqq (q+1)r^q\int_r^{R_m} N_1(\alpha t)t^{-q-1}\phi\left(\frac{t}{r}\right)dt \\ &\leqq 2(q+1)(2\alpha)^{\mu_f}(1+\varepsilon_m)\eta_m{}^{1/2}T(r, f)\int_1^{R_m/r} t^{-q-1}\phi(t)\,dt=o(T(r, f)), \end{aligned}$$

$$14\left(\frac{r}{R_m}\right)^{q+1}T(2R_m, f')\leqq 28\left(\frac{r}{R_m}\right)^{q+1}T(2R_m+1, f)$$

$$\leqq 28\cdot 2^{\mu_f}\left(\frac{r}{R_m}\right)^{q+1}(1+\varepsilon_m)\left(\frac{R_m}{r_m}\right)^{\mu_f} T(r_m, f) \tag{4.20}$$

$$\leqq 28\cdot 2^{\mu_f}(1+\varepsilon_m)\left(\frac{r_m}{R_m}\right)^{(q+1-\mu_f)/2} T(r, f)=o(T(r, f)).$$

Consequently, by (4.14), (4.15), (4.16), (4.17), (4.18), (4.19) and (4.20), we have in $[r_m, r_m^*]$

$$m(r, g)+m\left(r, \frac{1}{g}\right)=o(T(r, f)). \tag{4.21}$$

Let $\Gamma(r)$ be the set of θ satisfying

$$\frac{1}{2\pi}\int_{\Gamma(r)} \log|f'(re^{i\theta})|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f'(re^{i\theta})|d\theta\equiv m(r, f').$$

Then, by (4.4) and (4.21), as $r\to\infty$ in $[r_m, r_m^*]$, meas $\Gamma(r)\to\pi$.

On the other hand, by (4.1) and a lemma in [9], we get

$$m\left(r, \frac{1}{f'}\right)\sim T(r, f')\sim T(r, f),$$

as $r\to\infty$, $r\notin\mathcal{E}$.

Therefore, in $[r_m, r_m^*]-\mathcal{E}$, the measure of the set $J(r)$ of θ satisfying

$$\frac{1}{2\pi}\int_{J(r)}\log^+|f'(re^{i\theta})|d\theta\sim T(r, f)$$

tends to π, as $r\to\infty$.

By carefully tracing the procedure in [9], especially pp. 137–138, we can see that the number of finite deficient value is at most μ_f. Hence $\nu(f)\leqq\mu_f+1$, since $\delta(\infty, f)=1$.

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