Research Article

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Entire solutions for several general quadratic trinomial differential difference equations

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Abstract: This paper is devoted to exploring the existence and the forms of entire solutions of several quadratic trinomial differential difference equations with more general forms. Some results about the forms of entire solutions for these equations are some extensions and generalizations of the previous theorems given by Liu, Yang and Cao. We also give a series of examples to explain the existence of the finite order transcendental entire solutions of such equations.

Keywords: Nevanlinna theory, entire solution, differential difference equation

MSC 2020: 39A10, 30D35, 30D20, 30D05

1 Introduction

The main aim of this paper is to investigate the transcendental entire solutions with finite order of the quadratic trinomial difference equation

$$f(z+c)^2 + 2\alpha f(z)f(z+c) + f(z)^2 = e^{g(z)}, \tag{1}$$

and the quadratic trinomial differential difference equation

$$f(z+c)^2 + 2\alpha f(z+c)f'(z) + f'(z)^2 = e^{g(z)}, \tag{2}$$

where $\alpha^2(\neq 0, 1)$, c are constants and $g(z)$ is a polynomial. When $\alpha = 0$ and $g(z) = 0$, the above equations become the Fermat-type difference equation $f(z+c)^2 + f(z)^2 = 1$ and differential difference equations $f(z+c)^2 + f'(z)^2 = 1$, which are discussed by Liu and his colleagues (see [1–3]). They pointed out that the transcendental entire solution with finite order of the latter must satisfy $f(z) = \sin(z \pm Bi)$, where B is a constant and $c = 2k\pi$ or $c = (2k+1)\pi$, k is an integer. For the general Fermat-type functional equation

$$f^2 + g^2 = 1, \tag{3}$$

Gross [4] had discussed the existence of solutions of equation (3) and showed that the entire solutions are $f = \cos a(z)$, $g = \sin a(z)$, where $a(z)$ is an entire function. In recent years, with the development of Nevanlinna theory and difference Nevanlinn theory of meromorphic function [5–8], many scholars obtained lots of results about the solutions of Fermat-type functional equations [1–3,9–17].

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In fact, when $\alpha = \pm 1$, it is easy to get the entire solution of equations $f(z) \pm f(z+c) = \pm e^{\frac{1}{2}g(z)}$ and $f'(z) \pm f(z+c) = \pm e^{\frac{1}{2}g(z)}$, for example, $f(z) = e^{az}$ is a finite order entire solution of the first equation, if $(1 \pm e^{ac}) = \pm e^{b}$ and $g(z) = e^{2az+2b}$, and $f(z) = e^{az}$ is a finite order entire solution of the latter, if $(a \pm e^{ac}) = \pm e^{b}$ and $g(z) = e^{2az+2b}$, where $a(\neq 0)$, b are constants.

For $\alpha^2 \neq 0, 1$, Liu and Yang [9] in 2016 studied the existence and the form of solutions of some quadratic trinomial functional equations and obtained the following results in equations (1) and (2).

Theorem A. (see [9, Theorem 1.6]) *If $\alpha \neq \pm 1, 0$, then equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1 \tag{4}$$

has no transcendental meromorphic solutions.

Theorem B. (see [9, Theorem 1.4]) *If $\alpha \neq \pm 1, 0$, then the finite order transcendental entire functions of equation*

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1 \tag{5}$$

must be of order equal to one.

In recent years, Han and Lü [18] gave the description of meromorphic solutions for the functional equation (3) when $g(z) = f'(z)$ and 1 is replaced by $e^{\alpha z+\beta}$, where $\alpha, \beta \in \mathbb{C}$, and obtained the following results.

Theorem C. (see [18, Theorem 1.1]) *The meromorphic solutions f of the following differential equation*

$$f(z)^n + f'(z)^n = e^{\alpha z+\beta}, \tag{6}$$

must be entire functions, and the following assertions hold.

(*A*) *For $n = 1$, the general solutions of (6) are $f(z) = \frac{e^{\alpha z+\beta}}{\alpha+1} + ae^{-z}$ for $\alpha \neq -1$ and $f(z) = ze^{-z+\beta} + ae^{-z}$.*

(*B*) *For $n = 2$, either $\alpha = 0$ and the general solutions of (6) are $f(z) = e^{\frac{\beta}{2}}\sin(z+b)$ or $f(z) = \mathrm{d}e^{\frac{\alpha z+\beta}{2}}$.*

(*C*) *For $n \geq 3$, the general solutions of (6) are $f(z) = \mathrm{d}e^{\frac{\alpha z+\beta}{n}}$.*

Here, $\alpha, \beta, a, b, d \in \mathbb{C}$ with $d^n\left(1 + \left(\frac{\alpha}{n}\right)^n\right) = 1$ for $n \geq 1$.

They also proved that all the trivial meromorphic solutions of $f(z)^n + f(z+c)^n = e^{\alpha z+\beta}$ are the functions $f(z) = \mathrm{d}e^{\frac{\alpha z+\beta}{n}}$ with $d^n(1 + e^{\alpha c}) = 1$ for $n \geq 1$ (see [18, p. 99]).

Theorems A–C suggest the following question as an open problem.

Question 1.1. What will happen when the right side of those equation (1) is replaced by a function e^g in Theorems A and B, where g is a polynomial?

2 Results and some examples

Motivated by the above question, this article is concerned with the entire solutions for the difference equation (1) and the differential difference equation (2). The main tools used in this paper are the Nevanlinna theory and the difference Nevanlinna theory. Our principal results obtained generalize the previous theorems given by Liu, Cao, and Yang [1–3,9]. Here and below, let $\alpha^2 \neq 0, 1$, and

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} - \frac{i}{2\sqrt{1-\alpha}}, \quad A_2 = \frac{1}{2\sqrt{1+\alpha}} + \frac{i}{2\sqrt{1-\alpha}}. \tag{7}$$

The first main theorem is about the existence and the forms of the solutions for the quadratic trinomial difference equation (1).

Theorem 2.1. *Let $\alpha^2 \neq 0, 1$, $c(\neq 0) \in \mathbb{C}$ and $g(z)$ be a polynomial. If the difference equation* (1) *admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = az + b$, where $a, b \in \mathbb{C}$. Furthermore, $f(z)$ must satisfy one of the following cases*:

(*i*)

$$f(z) = \frac{1}{\sqrt{2}}(A_1\eta + A_2\eta^{-1})e^{\frac{1}{2}(az+b)},$$

where $\eta(\neq 0) \in \mathbb{C}$ and a, c, A_1, A_2, η satisfy

$$e^{\frac{1}{2}ac} = \frac{A_2\eta + A_1\eta^{-1}}{A_1\eta + A_2\eta^{-1}};$$

(*ii*)

$$f(z) = \frac{1}{\sqrt{2}}(A_1e^{a_1z+b_1} + A_2e^{a_2z+b_2}),$$

where $a_j, b_j \in \mathbb{C}, (j = 1, 2)$ satisfy

$$a_1 \neq a_2, \quad g(z) = (a_1 + a_2)z + b_1 + b_2 = az + b,$$

and

$$e^{a_1c} = \frac{A_2}{A_1}, \quad e^{a_2c} = \frac{A_1}{A_2}, \quad e^{ac} = 1.$$

The following examples show that the forms of solutions are precise to some extent.

Example 2.1. Let $\alpha = \frac{1}{2}$ and $\eta = 1$. Then it follows that $A_1 = \frac{1}{\sqrt{3}}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$, $A_2 = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$. Let $f(z) = \frac{1}{\sqrt{2}}e^{z+1}$. Thus, $f(z)$ is a solution of (1) with $g(z) = 2z + 2$ and $c = 2\pi i$.

Example 2.2. Let $\alpha = \frac{1}{2}$, $a_1 = \frac{1}{3}$, $a_2 = \frac{2}{3}$ and $b_1 = b_2 = 0$. Then it follows that $A_1 = \frac{1}{\sqrt{3}}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$, $A_2 = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ and

$$f(z) = \frac{1}{\sqrt{3}}\left(e^{\frac{1}{3}z-\frac{1}{3}\pi i} + e^{\frac{2}{3}z+\frac{1}{3}\pi i}\right).$$

Thus, $f(z)$ is a solution of (1) with $g(z) = z$ and $c = 2\pi i$.

When $f(z + c)$ is replaced by $f'(z)$ in (1), we obtain the second theorem as follows.

Theorem 2.2. *Let $\alpha^2 \neq 0, 1$, $\alpha \in \mathbb{C}$ and $g(z)$ be a polynomial, and if the differential equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = e^{g(z)} \tag{8}$$

admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = az + b$, where $a, b \in \mathbb{C}$.

The following example shows that the forms of solutions are precise to some extent.

Example 2.3. Let $g(z) = -4z + 2$. Then it is easy to get that the function

$$f(z) = \frac{1}{\sqrt{6}}[(1 - \sqrt{3})e^{(-2+\sqrt{3})z} + (1 + \sqrt{3})e^{(-2-\sqrt{3})z}]$$

is a transcendental entire solution of equation (8) with $\alpha = 2$.

From Theorem 2.2, it is easy to get the following corollary.

Corollary 2.1. *Let $\alpha^2 \neq 0, 1$, $\alpha \in \mathbb{C}$ and $g(z)$ be a polynomial with $\deg_z g > 1$. Then the following partial differential difference equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = e^{g(z)} \tag{9}$$

admits no transcendental entire solution with finite order.

For the differential difference counterpart of Theorem 2.2, we have

Theorem 2.3. *Let $\alpha^2 \neq 0, 1$, $c \neq 0$ and $g(z)$ be a nonconstant polynomial. If the differential difference equation* (2) *admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = az + b$, where $a(\neq 0)$, $b \in \mathbb{C}$. Furthermore, $f(z)$ must satisfy one of the following cases*:

(*i*)

$$f(z) = \frac{\sqrt{2}}{a}(A_1\eta^{-1} + A_2\eta)e^{\frac{1}{2}(az+b)},$$

where $\eta(\neq 0) \in \mathbb{C}$ and a, c, A_1, A_2, η satisfy

$$e^{\frac{1}{2}ac} = \frac{a(A_1\eta + A_2\eta^{-1})}{2(A_2\eta + A_1\eta^{-1})};$$

(*ii*)

$$f(z) = \frac{1}{\sqrt{2}}\left(\frac{A_2}{a_1}e^{a_1z+b_1} + \frac{A_1}{a_2}e^{a_2z+b_2}\right),$$

where $a_j(\neq 0)$, $b_j \in \mathbb{C}$, $(j = 1, 2)$ satisfy

$$a_1 \neq a_2, \quad g(z) = (a_1 + a_2)z + b_1 + b_2 = az + b,$$

and

$$e^{a_1c} = \frac{A_2}{A_1}a_1, \quad e^{a_2c} = \frac{A_1}{A_2}a_2, \quad e^{ac} = a_1a_2.$$

The following examples explain the existence of transcendental entire solutions with finite order of (2).

Example 2.4. Let $\alpha = -\frac{1}{2}$ and $\eta = -1$. Then it follows $A_1 = \frac{\sqrt{2}}{\sqrt{3}}e^{-\frac{\pi}{6}i}$ and $A_2 = \frac{\sqrt{2}}{\sqrt{3}}e^{\frac{\pi}{6}i}$. Let $f(z) = e^{-z+b}$, and then $f(z)$ is a transcendental entire solution of equation (2) with $g(z) = -2z + 2b$, $c = \pi i$ and $b \in \mathbb{C}$.

Example 2.5. Let $\alpha = -\frac{1}{2}$, a_1, a_2 satisfy $e^{\pi i\left(a_1 - \frac{1}{3}\right)} = a_1$, $e^{\pi i\left(a_2 + \frac{1}{3}\right)} = a_2$ and $a_1 \neq a_2$. And let

$$f(z) = \frac{1}{\sqrt{3}}\left(\frac{1}{a_1}e^{a_1z+\frac{\pi}{6}i} + \frac{1}{a_2}e^{a_2z-\frac{\pi}{6}i}\right),$$

then $f(z)$ is a transcendental entire solution with finite order of equation (2) with $g(z) = (a_1 + a_2)z_1$ and $c = \pi i$.

From Theorem 2.3, we obtain the following corollary.

Corollary 2.2. *Let $c(\neq 0) \in \mathbb{C}$ and $g(z)$ be not of the form $g(z) = az + b$, where $a, b \in \mathbb{C}$. Then the differential difference equation* (2) *has no transcendental entire solution with finite order.*

2.1 Some lemmas

The following lemmas play the key role in proving our results.

Lemma 2.1. [19] *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*
(*a*) *the internal function h is a polynomial and the external function g is of finite order; or else*
(*b*) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 2.2. [20] *Let $f_j(z)$ $(j = 1, 2, 3)$ be meromorphic functions, $f_1(z)$ be nonconstant. If $\sum_{j=1}^{3} f_j \equiv 1$ and*

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ and $T(r) = \max_{1\le j\le 3}\{T(r, f_j)\}$, then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Remark 2.1. Here, $N_2\left(r, \frac{1}{f}\right)$ is the counting function of the zeros of f in $|z| \le r$, where the simple zero is counted once, and the multiple zero is counted twice.

3 Proof of Theorem 2.1

Suppose that $f(z)$ is a transcendental entire solution with finite order of equation (1). Let

$$f(z) = \frac{1}{\sqrt{2}}(u + v), \quad f(z + c) = \frac{1}{\sqrt{2}}(u - v),$$

where u, v are entire functions. Thus, equation (1) can be written as

$$(1 + \alpha)u^2 + (1 - \alpha)v^2 = e^g. \tag{10}$$

It thus follows from (10) that

$$\left(\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}}\right)^2 + \left(\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}\right)^2 = 1.$$

The above equation leads to

$$\left(\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}}\right) = 1. \tag{11}$$

Since f is a finite order transcendental entire function and g is a polynomial, there thus exists a polynomial $p(z)$ such that

$$\begin{cases} \dfrac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}} + i\dfrac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}} = e^{p(z)}, \\ \dfrac{\sqrt{1+\alpha}\,u}{e^{\frac{g(z)}{2}}} - i\dfrac{\sqrt{1-\alpha}\,v}{e^{\frac{g(z)}{2}}} = e^{-p(z)}. \end{cases} \tag{12}$$

Denote

$$\gamma_1(z) = \frac{g(z)}{2} + p(z), \quad \gamma_2(z) = \frac{g(z)}{2} - p(z). \tag{13}$$

By combining with (12), we have

$$\sqrt{1+\alpha}u=\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2},\qquad \sqrt{1-\alpha}v=\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i}.$$

This leads to

$$f(z)=\frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}}+\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right]=\frac{1}{\sqrt{2}}\left(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}\right), \tag{14}$$

$$f(z+c)=\frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}}-\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right]=\frac{1}{\sqrt{2}}\left(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}\right), \tag{15}$$

where A_1, A_2 are defined in (7). Thus, in view of (14) and (15), it follows that

$$\frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)}-\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)}+e^{\gamma_1(z+c)-\gamma_2(z)}\equiv 1. \tag{16}$$

We will discuss two cases below.

Case 1. Suppose that $e^{\gamma_1(z+c)-\gamma_2(z)}$ is a constant. Then $\gamma_1(z+c)-\gamma_2(z)$ is a constant. Assuming that $\gamma_1(z+c)-\gamma_2(z)=\kappa$, $\kappa\in\mathbb{C}$. Thus, it yields that $\gamma_1(z+c)=\gamma_2(z)=\kappa$. By combining with $\gamma_1(z)-\gamma_2(z)=2p(z)$, it follows from (16) that

$$e^{2p(z)}+(1-\xi)\frac{A_1}{A_2}=e^{-2p(z+c)}\xi, \tag{17}$$

where $\xi=e^{\kappa}$. By using the Nevanlinna second fundamental theorem, we have

$$T(r,e^{2p})\le N\left(r,\frac{1}{e^{2p}}\right)+N\left(r,\frac{1}{e^{2p}-\delta}\right)+S(r,e^{2p})\le N\left(r,\frac{1}{e^{-2p(z+c)}\xi}\right)+S(r,e^{2p})=S(r,e^{2p}),$$

where $\delta=-(1-\xi)\frac{A_1}{A_2}$. This is a contradiction, which implies that $p(z)$ is a constant. Let $\eta=e^{p}$. Substituting this into (14) and (15), we have

$$f(z)=\frac{1}{\sqrt{2}}(A_1\eta+A_2\eta^{-1})e^{\frac{1}{2}g(z)}, \tag{18}$$

$$f(z+c)=\frac{1}{\sqrt{2}}(A_2\eta+A_1\eta^{-1})e^{\frac{1}{2}g(z)}. \tag{19}$$

From (18) and (19), it follows that

$$(A_1\eta+A_2\eta^{-1})e^{\frac{g(z+c)-g(z)}{2}}=A_2\eta+A_1\eta^{-1}. \tag{20}$$

In view of $\alpha^2\ne1$, it follows that $A_2\eta+A_1\eta^{-1}=0$ and $(A_1\eta+A_2\eta^{-1})=0$ cannot hold at the same time. Hence, we have $A_2\eta+A_1\eta^{-1}\ne0$ and $(A_1\eta+A_2\eta^{-1})\ne0$. Since $g(z)$ is a polynomial, then (20) implies that $g(z+c)-g(z)$ is a constant in $\mathbb{C}$. Otherwise, we obtain a contradiction from the fact that the left of the above equation is not transcendental but the right is transcendental. Thus, it follows that $g(z)=az+b$, where a, b are constants satisfying

$$e^{\frac{1}{2}ac}=\frac{A_2\eta+A_1\eta^{-1}}{(A_1\eta+A_2\eta^{-1})}.$$

This completes the proof of Theorem 2.1(i).

Case 2. Suppose that $e^{\gamma_1(z+c)-\gamma_2(z)}$ is not a constant. Since $\gamma_1(z), \gamma_2(z)$ are polynomials and $e^{\gamma_1(z+c)-\gamma_2(z)}$ is not a constant, and by applying Lemma 2.2 for (16), it follows that

$$-\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)}\equiv1\quad\text{or}\quad\frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)}\equiv1.$$

If $-\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_2(z)} \equiv 1$, it follows from (16) that $-\frac{A_1}{A_2}e^{\gamma_1(z+c)-\gamma_2(z+c)} \equiv 1$. Thus, in view of (13), we have

$$-\frac{A_2}{A_1}e^{2p(z)} \equiv 1, \quad -\frac{A_1}{A_2}e^{2p(z+c)} \equiv 1, \tag{21}$$

which imply that $p(z)$ is a constant and $\frac{A_2}{A_1} = \frac{A_1}{A_2}$. This leads to $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 \neq 0, 1$.

If $\frac{A_2}{A_1}e^{\gamma_2(z+c)-\gamma_2(z)} \equiv 1$, then it follows that $\gamma_2(z)$ is of the form $\gamma_2(z) = a_2z + b_2$, where a_2, b_2 are constants satisfying $e^{a_2c} = \frac{A_1}{A_2}$. Moreover, it follows from (16) that $\frac{A_2}{A_1}e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. This means that $\gamma_1(z)$ is of the form $\gamma_1(z) = a_1z + b_1$, where a_1, b_1 are constants satisfying $e^{a_1c} = \frac{A_2}{A_1}$. Since $e^{\gamma_1(z+c)-\gamma_2(z)}$ is not a constant, it follows that $a_1 \neq a_2$. In view of the definitions of γ_1, γ_2, we have

$$e^{\gamma_1(z+c)+\gamma_2(z+c)-(\gamma_1(z)+\gamma_2(z))} \equiv e^{g(z+c)-g(z)} \equiv 1, \tag{22}$$

which means that $g(z)$ is of the form $g(z) = az + b$ and $ac = 2k\pi i$, $k \in \mathbb{Z}$. Substituting these into (14), we have

$$f(z) = \frac{1}{\sqrt{2}}\left(A_1e^{a_1z+b_1} + A_2e^{a_2z+b_2}\right).$$

Therefore, this completes the proof of Theorem 2.1. □

4 Proof of Theorem 2.2

Suppose that $f(z)$ is a transcendental entire solution with finite order of equation (8). By using the same argument as in the proof of Theorem 2.1, we have (14) and

$$f'(z) = \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} - \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2\sqrt{1-\alpha}i}\right] = \frac{1}{\sqrt{2}}\left(A_2e^{\gamma_1(z)} + A_1e^{\gamma_2(z)}\right). \tag{23}$$

Thus, it follows from (14) and (23) that

$$f'(z) = \frac{1}{\sqrt{2}}\left(A_1\gamma_1'(z)e^{\gamma_1(z)} + A_2\gamma_2'(z)e^{\gamma_2(z)}\right) = \frac{1}{\sqrt{2}}\left(A_2e^{\gamma_1(z)} + A_1e^{\gamma_2(z)}\right),$$

which leads to

$$e^{\gamma_1(z)}(A_1\gamma_1'(z) - A_2) = e^{\gamma_2(z)}(A_1 - A_2\gamma_2'(z)). \tag{24}$$

By combining with (13) and (24), we have

$$e^{2p(z)}(A_1\gamma_1'(z) - A_2) = A_1 - A_2\gamma_2'(z). \tag{25}$$

If $p(z)$ is not a constant, then it follows from (25) that

$$A_1\gamma_1'(z) - A_2 = 0, \quad A_1 - A_2\gamma_2'(z) = 0.$$

Otherwise, we have

$$e^{2p(z)} = \frac{A_1 - A_2\gamma_2'(z)}{A_1\gamma_1'(z) - A_2}. \tag{26}$$

Since $p(z)$, $g(z)$ are polynomials, the left of equation (26) is transcendental, but the right of equation (26) is a polynomial. Thus, a contradiction can be obtained from (26). Hence, it follows that

$$\gamma_1(z) = \frac{A_2}{A_1}z + b_1, \quad \gamma_2(z) = \frac{A_1}{A_2}z + b_2,$$

where b_1, b_2 are constants. Thus, we have $g(z) = \gamma_1(z) + \gamma_2(z) = \left(\frac{A_2}{A_1} + \frac{A_1}{A_2}\right)z + b = -2\alpha z + b$, where $b = b_1 + b_2$.

If $p(z)$ is a constant, then $\gamma_1'(z) = \gamma_2'(z) = \frac{1}{2}g'(z)$. Let $\xi = e^{2p}$, in view of (25), it follows that

$$\left(\frac{1}{2}A_1 g' - A_2\right)\xi = \left(A_1 - \frac{1}{2}A_2 g'\right),$$

which leads to

$$g' = \frac{2(A_1 + A_2\xi)}{A_1\xi + A_2}.$$

Thus, we have $g(z) = \frac{2(A_1 + A_2\xi)}{A_1\xi + A_2}z + b$. Hence, $g(z)$ must be of the form $g(z) = az + b$.

Therefore, this completes the proof of Theorem 2.2. □

5 Proof of Theorem 2.3

Suppose that $f(z)$ is a transcendental entire solution with finite order of equation (2). By using the same argument as in the proof of Theorem 2.1, we have (23) and

$$f(z+c) = \frac{1}{\sqrt{2}}\left[\frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} + \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2\sqrt{1-\alpha}\,i}\right] = \frac{1}{\sqrt{2}}\left(A_1 e^{\gamma_1(z)} + A_2 e^{\gamma_2(z)}\right), \tag{27}$$

where $p(z)$ is a polynomial and $\gamma_1(z), \gamma_2(z)$ are stated as in (13). In view of (23) and (27), it follows that

$$f'(z+c) = \frac{1}{\sqrt{2}}\left(A_1\gamma_1'(z)e^{\gamma_1(z)} + A_2\gamma_2'(z)e^{\gamma_2(z)}\right) = \frac{1}{\sqrt{2}}\left(A_2 e^{\gamma_1(z+c)} + A_1 e^{\gamma_2(z+c)}\right).$$

Thus, we have

$$\frac{A_1}{A_2}\gamma_1'(z)e^{\gamma_1(z)-\gamma_1(z+c)} + \gamma_2'(z)e^{\gamma_2(z)-\gamma_1(z+c)} - \frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1. \tag{28}$$

Now, we will discuss two cases below.

Case 1. Suppose that $\gamma_1(z+c) - \gamma_2(z+c)$ is a constant. In view of $\gamma_1(z+c) - \gamma_2(z+c) = 2p(z+c)$, it follows that $p(z)$ is a constant. Let $\xi = e^p$. In view of (13) and (27), it follows that

$$f(z+c) = \frac{1}{\sqrt{2}}(A_1\xi + A_2\xi^{-1})e^{\frac{1}{2}g(z)}, \quad f'(z) = \frac{1}{\sqrt{2}}(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}g(z)}. \tag{29}$$

Thus, we can deduce from (29) that

$$(A_2\xi + A_1\xi^{-1})e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}(A_1\xi + A_2\xi^{-1})g'(z). \tag{30}$$

If $\deg_z g \geq 2$, it thus follows that $g'(z) \not\equiv 0$ and $g(z+c) - g(z)$ is not a constant. Equation (30) implies that $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi + A_2\xi^{-1} = 0$. Otherwise, we have

$$e^{\frac{1}{2}(g(z+c)-g(z))} = \frac{1}{2}g'(z)\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}. \tag{31}$$

The left side of equation (31) is transcendental, but the right of equation (31) is a polynomial. Thus, a contradiction can be obtained from (31). If $A_2\xi + A_1\xi^{-1} = 0$ and $A_1\xi + A_2\xi^{-1} = 0$, we can deduce that $A_1^2 = A_2^2$, which is a contradiction with $\alpha^2 = 1$.

If $\deg_z g = 1$, that is, $g(z) = az + b$, $a(\neq 0)$, b are constants, it follows from (31) that

$$e^{\frac{1}{2}ac} = \frac{1}{2}\frac{A_1\xi + A_2\xi^{-1}}{A_2\xi + A_1\xi^{-1}}a. \tag{32}$$

By combining with (29) and (32), we have

$$f(z)=\frac{1}{\sqrt{2}}(A_1\xi+A_2\xi^{-1})e^{\frac{1}{2}(az+b)-\frac{1}{2}ac}=\frac{\sqrt{2}}{a}(A_2\xi+A_1\xi^{-1})e^{\frac{1}{2}(az+b)}. \tag{33}$$

Thus, in view of (32) and (33), this completes the proof of Theorem 2.3(i).

Case 2. Suppose that $\gamma_1(z+c)-\gamma_2(z+c)$ is not a constant, it follows from (13) that $p(z)$ is not a constant. Then we have that γ_1' and γ_2' cannot be equal to 0 at the same time. Otherwise, it yields that $\gamma_1(z+c)-\gamma_2(z+c)$ is a constant, this is a contradiction. If $\gamma_1'\equiv 0$ and $\gamma_2'\not\equiv 0$, it thus follows from (28) that

$$\gamma_2'(z)e^{\gamma_2(z)-\gamma_1(z+c)}-\frac{A_1}{A_2}e^{\gamma_2(z+c)-\gamma_1(z+c)}\equiv 1. \tag{34}$$

Obviously, $\gamma_2(z)-\gamma_1(z+c)$ is not a constant. Otherwise, $\gamma_2(z+c)-\gamma_1(z+c)$ is a constant because γ_1,γ_2 are polynomials. By applying the Nevanlinna second fundamental theorem for $e^{\gamma_2(z+c)-\gamma_1(z+c)}$, we have from (34) that

$$\begin{aligned}T\left(r,e^{\gamma_2(z+c)-\gamma_1(z+c)}\right)&\le N\left(r,\frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}}\right)+N\left(r,\frac{1}{e^{\gamma_2(z+c)-\gamma_1(z+c)}+\frac{A_2}{A_1}}\right)+S\left(r,e^{\gamma_2(z+c)-\gamma_1(z+c)}\right)\\&\le N\left(r,\frac{1}{\gamma_2'(z)e^{\gamma_2(z)-\gamma_1(z+c)}}\right)+S\left(r,e^{\gamma_2(z+c)-\gamma_1(z+c)}\right)=S\left(r,e^{\gamma_2(z+c)-\gamma_1(z+c)}\right),\end{aligned}$$

which is a contradiction.

If $\gamma_1'\not\equiv 0$ and $\gamma_2'\equiv 0$, using the same argument as in the above, we can get a contradiction. Hence, we have $\gamma_1'\not\equiv 0$ and $\gamma_2'\not\equiv 0$. By Lemma 2.2, it follows that

$$\frac{A_1}{A_2}\gamma_1'(z)e^{\gamma_1(z)-\gamma_1(z+c)}\equiv 1,\quad\text{or}\quad \gamma_2'(z)e^{\gamma_2(z)-\gamma_1(z+c)}\equiv 1.$$

Subcase 2.1. If $\frac{A_1}{A_2}\gamma_1'(z)e^{\gamma_1(z)-\gamma_1(z+c)}\equiv 1$, it yields that $\gamma_1(z)$ is a linear form of $\gamma_1(z)=a_1z+b_1$, and $e^{a_1c}=a_1\frac{A_1}{A_2}$, where $a_2(\neq 0)$, b_2 are constants. In view of (28), it follows

$$\frac{A_2}{A_1}\gamma_2'(z)e^{\gamma_2(z)-\gamma_2(z+c)}\equiv 1,$$

which implies that $\gamma_2(z)$ is a linear form of $\gamma_2(z)=a_2z+b_2$, and $e^{a_2c}=a_2\frac{A_2}{A_1}$, where $a_2(\neq 0)$, b_2 are constants. Since $\gamma_1(z+c)-\gamma_2(z+c)$ is not a constant, it follows that $a_1\neq a_2$. In view of (13) and (27), it follows that $g(z)=\gamma_1(z)+\gamma_2(z)=(a_1+a_2)z+b_1+b_2=az+b$ and

$$\begin{aligned}f(z)&=\frac{1}{\sqrt{2}}\left(A_1e^{a_1z+b_1-a_1c}+A_2e^{a_2z+b_2-a_2c}\right)\\&=\frac{1}{\sqrt{2}}\left(A_1\frac{A_2}{a_1A_1}e^{a_1z+b_1}+A_2\frac{A_1}{a_2A_2}e^{a_2z+b_2}\right)\\&=\frac{1}{\sqrt{2}}\left(\frac{A_2}{a_1}e^{a_1z+b_1}+\frac{A_1}{a_2}e^{a_2z+b_2}\right).\end{aligned} \tag{35}$$

Subcase 2.2. If $\gamma_2'(z)e^{\gamma_2(z)-\gamma_1(z+c)}\equiv 1$, this means

$$\gamma_2(z)-\gamma_1(z+c)=\varepsilon_1, \tag{36}$$

where ε_1 is a constant. In view of (28), it thus follows that $\gamma_1'(z)e^{\gamma_1(z)-\gamma_2(z+c)}\equiv 1$, this means

$$\gamma_1(z)-\gamma_2(z+c)\equiv\varepsilon_2, \tag{37}$$

where ε_2 is a constant. In view of (36) and (37), it yields that

$$\gamma_1(z)-\gamma_2(z)+\gamma_1(z+c)-\gamma_2(z+c)=\varepsilon_2-\varepsilon_1.$$

By combining with (13), we have

$$p(z) + p(z + c) = \frac{1}{2}(\varepsilon_2 - \varepsilon_1),$$

this is a contradiction with the assumption that $\gamma_1(z + c) - \gamma_2(z + c) = 2p(z + c)$ is not a constant. Thus, we get the conclusions of Theorem 2.3(ii) from Case 2.

Therefore, this completes the proof of Theorem 2.3. □

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