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Entire solutions of certain type of difference equations

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Abstract
In this paper, we shall study the conditions regarding the existence of transcendental entire solutions of certain type of difference equations. Our results are either supplements to some results obtained recently, or are relating to the conjecture raised in Yang and Laine (Proc. Jpn. Acad., Ser. A, Math. Sci. 86:10-14, 2010). Finally, two relevant conjectures are posed for further studies.
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1 Introduction, notations and main results

Let f denote a transcendental entire or meromorphic function. Assuming the reader is familiar with the basics of Nevanlinna's value distribution theory, we shall adopt the standard notations associated with the theory, such as the characteristic function $T(r,f)$, the counting function of the poles $N(r,f)$, and the proximity function $m(r,f)$ (see, *e.g.*, [1, 2]).

Among many interesting applications of the Nevanlinna theory, there are studies on the growth and existence of entire or meromorphic solutions of various types of non-linear differential equations, and one can find prototypes for such equations, *e.g.*, in [3–5] and [6]. Recently, the Nevanlinna theory has been applied to study types of non-linear difference equations, see, *e.g.*, [7, 8]. Now, we shall utilize Clunie type of theorems for difference-differential polynomials to study some non-linear difference equations of more general forms and to obtain some improvements of or supplements to [7, 9], and [10].

Notations Given a meromorphic function f, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to f, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to +\infty$, possibly outside a set of r of finite linear measure. For a constant $c \in \mathbb{C}$, $f(z+c)$ is called a shift of f. As for a difference product, we mean a difference monomial of type $\prod_{j=1}^{k} f(z+c_j)^{n_j}$, where $c_1, \ldots, c_k$ are complex constants and $n_1, \ldots, n_k$ are natural numbers.

Definition 1.1 A difference polynomial, respectively, a difference-differential polynomial, in f is a finite sum of difference products of f and its shifts, respectively, of products of f, derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f.

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Consider a transcendental meromorphic function $f(z)$ and let

$$P(z,f) = \sum_{l=1}^{n} a_l(z) \prod_{j=0}^{k} \{f^{(j)}(z + c_{lj})\}^{n_{lj}},$$

where $a_l(z)$ ($l = 1,2,\ldots,n$) are small functions of f, and c_{lj} ($l = 1,2,\ldots,n$; $j = 0,1,\ldots,k$) are complex constants, and n_{lj} ($l = 1,2,\ldots,n$; $j = 0,1,\ldots,k$) are natural numbers.

Two terms $a_l(z) \prod_{j=0}^{k} \{f^{(j)}(z + c_{lj})\}^{n_{lj}}$ and $a_{l'}(z) \prod_{j=0}^{k} \{f^{(j)}(z + c_{lj})\}^{n_{l'j}}$ are called similar, if

$$n_{lj} = n_{l'j} \quad (j = 0,1,\ldots,k).$$

Group together similar terms of $P(z,f)$, if necessary. In the following, we assume that no two terms of $P(z,f)$ are similar and that $a_l(z) \not\equiv 0$ ($l = 1,2,\ldots,n$).

Definition 1.2 We define the total degree d of $P(z,f)$

$$d := d\bigl(P(z,f)\bigr) = \max_{1\le l\le n} \left\{ \sum_{j=0}^{k} n_{lj} \right\}.$$

For the sake of simplicity, we let $\Delta f(z) = f(z+1) - f(z)$, $\Delta^n f(z) = \Delta(\Delta^{n-1} f(z))$ ($n \ge 2$).

Yang and Laine [10] considered the following difference equation and proved it.

Theorem A *A non-linear difference equation*

$$f^3(z) + q(z)f(z+1) = c \sin bz,$$

where $q(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then the above equation possesses three distinct entire solutions of finite order, provided that $b = 3n\pi$ and $q^3 = (-1)^{n+1} c^2 27/4$ for a nonzero integer n.

Now, we shall substitute $f(z+1)$ by $\Delta f(z)$ in Theorem A and prove the following results.

Theorem 1.1 *Let $n \ge 4$ be an integer, $q(z)$ be a polynomial, and p_1, p_2, α_1, α_2 be nonzero constants such that $\alpha_1 \ne \alpha_2$. If there exists some entire solution f of finite order to* (1.1) *below*

$$f^n(z) + q(z)\Delta f(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.1}$$

then $q(z)$ is a constant, and one of the following relations holds:

(1) $f(z) = c_1 e^{\frac{\alpha_1}{n} z}$, *and* $c_1(e^{\alpha_1/n} - 1)q = p_2$, $\alpha_1 = n\alpha_2$,

(2) $f(z) = c_2 e^{\frac{\alpha_2}{n} z}$, *and* $c_2(e^{\alpha_2/n} - 1)q = p_1$, $\alpha_2 = n\alpha_1$,

where c_1, c_2 are constants satisfying $c_1^n = p_1$, $c_2^n = p_2$.

Corollary 1.1 *Let $n \geq 4$ be an integer, $q(z) \not\equiv 0$ be a polynomial, and p, α be nonzero constants. Then the non-linear difference equation*

$$f^n(z) + q(z)\Delta f(z) = pe^{\alpha z}$$

has no transcendental entire solutions of finite order provided that $\frac{\alpha}{n} \neq 2k\pi \mathrm{i}$, where k is an integer.

By some further analysis, we can derive the following result.

Theorem 1.2 *A non-linear difference equation*

$$f^3(z) + q(z)\Delta^3 f(z) = c\sin bz, \tag{1.2}$$

where $q(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then (1.2) *possesses solutions of the form $f(z) = c_1 \mathrm{e}^{\frac{b\mathrm{i}}{3}z} + c_2 \mathrm{e}^{-\frac{b\mathrm{i}}{3}z}$, $c_1^3 = -c_2^3 = \frac{c}{2\mathrm{i}}$, provided that $b = 3n\pi$, n is an odd number, $q^3 = \frac{27}{2{,}048}c^2$ or $b = 6n\pi \pm \pi$, $q^3 = -\frac{27}{4}c^2$.*

Examples In the special case of

$$f^3(z) - \frac{3}{8}\Delta^3 f(z) = 2\mathrm{i}\sin 3\pi z,$$

a finite order entire solution is $f(z) = 2\mathrm{i}\sin \pi z = \mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}$.

And $f^3(z) + 3\Delta^3 f(z) = 2\mathrm{i}\sin 7\pi z$ has the entire solution $f(z) = \mathrm{e}^{\frac{7\pi}{3}\mathrm{i}z} - \mathrm{e}^{-\frac{7\pi}{3}\mathrm{i}z}$.

Corollary 1.2 *The non-linear difference equation*

$$f^n + q\Delta f(z) = c\sin bz,$$

where q is a nonzero constant, possesses solutions of the form $f = c_1 \mathrm{e}^{\frac{b\mathrm{i}}{3}z} + c_2 \mathrm{e}^{-\frac{b\mathrm{i}}{3}z}$, if and only if $n = 3$.

Theorem B ([10], Theorem 2.4) *Let p, q be polynomials, then a non-linear difference equation*

$$f^2(z) + q(z)f(z+1) = p(z)$$

has no transcendental entire solutions of finite order.

We shall modify the equation in Theorem B above and derive the following result.

Theorem 1.3 *Let $l \geq 2$, $n \geq 1$ be integers, $a(z)$ and $b(z)$ be meromorphic functions such that $T(r,a) = \lambda_1 T(r,f) + S(r,f)$, $T(r,b) = \lambda_2 T(r,f) + S(r,f)$, where non-negative numbers λ_1, λ_2 satisfy $\lambda_1 + \lambda_2 < 1$. Then*

$$f^l(z) + a(z)\Delta^n f(z) = b(z) \tag{1.3}$$

has no transcendental entire solutions of hyper-order $\sigma_2(f) < 1$.

2 Preliminaries

In order to prove our conclusions, we need some lemmas.

Lemma 2.1 ([7]) *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f), \tag{2.1}$$

where $H(z,f)$, $P(z,f)$, $Q(z,f)$ are difference polynomials in f such that the total degree of $H(z,f)$ in f and its shifts is n, and that the corresponding total degree of $Q(z,f)$ is $\leq n$. If $H(z,f)$ contains just one term of maximal total degree, then for any $\varepsilon > 0$,

$$m\bigl(r, P(z,f)\bigr) = O\bigl(r^{\rho-1+\varepsilon}\bigr) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Remark 2.1 The following result is a Clunie type lemma [11] for the difference-differential polynomials of a meromorphic function f. It can be proved by applying Lemma 2.1 with a similar reasoning as in [10] and stated as follows.

Let $f(z)$ be a meromorphic function of finite order, and let $P(z,f)$, $Q(z,f)$ be two difference-differential polynomials of f. If $f^nP(z,f) = Q(z,f)$ holds and if the total degree of $Q(z,f)$ in f and its derivatives and their shifts is $\leq n$, then $m(r, P(z,f)) = S(r,f)$.

Lemma 2.2 ([12]) *Suppose that m, n are positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there exist no transcendental entire solutions f and g satisfying the equation $af^n + bg^m = 1$, with a, b being small functions of f and g, respectively.*

Lemma 2.3 ([13]) *Assume that $c \in \mathbb{C}$ is a nonzero constant, α is a non-constant meromorphic function. Then the differential equation $f^2 + (cf^{(n)})^2 = \alpha$ has no transcendental meromorphic solutions satisfying $T(r,\alpha) = S(r,f)$.*

Lemma 2.4 ([14, 15]) *Let f be a transcendental meromorphic function of finite order ρ, then for any complex numbers c_1, c_2 and for each $\varepsilon > 0$, $m(r, \frac{f(z+c_1)}{f(z+c_2)}) = O(r^{\rho-1+\varepsilon})$.*

References [16] and [17] further pointed out the following.

Remark 2.2 If f is a non-constant finite order meromorphic function and $c \in \mathbb{C}$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f), \tag{2.2}$$

outside of a possible exceptional set with finite logarithmic measure.

We also know that Remark 2.2 has been improved by Halburd *et al.* [18]. They proved that (2.2) is also true when f is a meromorphic function of hyper-order $\sigma_2(f) < 1$.

3 Proof of Theorem 1.1

Suppose that f is a transcendental entire solution of finite order to (1.1). By differentiating both sides of (1.1), we have

$$nf^{n-1}f' + \big(q(z)\Delta f(z)\big)' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{3.1}$$

From (1.1) and (3.1), we obtain

$$\alpha_2 f^n + \alpha_2 q(z)\Delta f(z) - nf^{n-1}f' - \big(q(z)\Delta f(z)\big)' = (\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}, \tag{3.2}$$

$$\alpha_1 f^n + \alpha_1 q(z)\Delta f(z) - nf^{n-1}f' - \big(q(z)\Delta f(z)\big)' = (\alpha_1 - \alpha_2)p_2 e^{\alpha_2 z}. \tag{3.3}$$

Differentiating (3.2) yields

$$\begin{aligned} &n\alpha_2 f^{n-1}f' + \alpha_2\big(q(z)\Delta f(z)\big)' - n(n-1)f^{n-2}\big(f'\big)^2 - nf^{n-1}f'' - \big(q(z)\Delta f(z)\big)'' \\ &\quad = \alpha_1(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \end{aligned} \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$f^{n-2}\varphi = T(f), \tag{3.5}$$

where

$$\begin{aligned} &\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1 + \alpha_2)f'f + n(n-1)\big(f'\big)^2 + nf''f, \\ &T(f) = -\alpha_1\alpha_2 q(z)\Delta f(z) + (\alpha_1 + \alpha_2)\big(q(z)\Delta f(z)\big)' - \big(q(z)\Delta f(z)\big)''. \end{aligned}$$

Next, we shall prove $\varphi \equiv 0$. In fact, since T is a difference-differential polynomial in f, and its degree at most 1. By (3.5) and Remark 2.1 after Lemma 2.1, we have $m(r,\varphi) = S(r,f)$, and $T(r,\varphi) = S(r,f)$. On the other hand, we can rewrite (3.5) as $f^{n-3}f\varphi = T(f)$, which implies $m(r,f\varphi) = S(r,f)$, and $T(r,f\varphi) = S(r,f)$. If $\varphi \not\equiv 0$, then $T(r,f) = T(r,\frac{f\varphi}{\varphi}) = S(r,f)$, and this is impossible. Hence $\varphi \equiv 0$, and $T \equiv 0$, *i.e.*

$$-\alpha_1\alpha_2 q(z)\Delta f(z) + (\alpha_1 + \alpha_2)\big(q(z)\Delta f(z)\big)' - \big(q(z)\Delta f(z)\big)'' \equiv 0. \tag{3.6}$$

If $q(z)\Delta f(z) \equiv 0$, then (1.1) can be rewritten as

$$\frac{1}{p_1}\big(fe^{-\frac{\alpha_1}{n}z}\big)^n - \frac{p_2}{p_1}\big(e^{\frac{1}{3}(\alpha_2-\alpha_1)z}\big)^3 = 1,$$

which is impossible by Lemma 2.2. Thus, $q(z)\Delta f(z) \not\equiv 0$. It is easily seen from $\alpha_1 \neq \alpha_2$ that

$$\alpha_1 q(z)\Delta f(z) - \big(q(z)\Delta f(z)\big)' \equiv 0$$

and

$$\alpha_2 q(z)\Delta f(z) - \big(q(z)\Delta f(z)\big)' \equiv 0$$

cannot hold simultaneously.

First of all, we assume that $\alpha_2 q(z)\Delta f(z) - (q(z)\Delta f(z))' \not\equiv 0$, then (3.6) gives

$$\alpha_2 q(z)\Delta f(z) - \big(q(z)\Delta f(z)\big)' = A e^{\alpha_1 z},$$

where A is a nonzero constant. Substituting the above expression into (3.2), we obtain

$$f^{n-1}\big(\alpha_2 f - nf'\big) = \frac{(\alpha_2 - \alpha_1)p_1 - A}{A}\big(\alpha_2 q(z)\Delta f(z) - \big(q(z)\Delta f(z)\big)'\big).$$

Again, Lemma 2.1 shows that $\alpha_2 f - nf' \equiv 0$, $(\alpha_2 - \alpha_1)p_1 = A$. So

$$f^n = B e^{\alpha_2 z}, \tag{3.7}$$

where B is a constant.

Substituting (3.7) into (1.1), we find

$$\left(1 - \frac{p_2}{B}\right) f^n = \frac{p_1(\alpha_2 q(z)\Delta f(z) - (q(z)\Delta f(z))')}{A} - q(z)\Delta f(z).$$

If $B \neq p_2$, by Lemma 2.1, we get $T(r,f) = m(r,f) = S(r,f)$, which is absurd. So $B = p_2$, and $f = c_2 e^{\frac{\alpha_2}{n} z} = c_2 e^{\alpha_1 z}$, $c_2^n = B = p_2$, $c_2(e^{\alpha_2/n} - 1)q = p_1$.

If $\alpha_1 q(z)\Delta f(z) - (q(z)\Delta f(z))' \not\equiv 0$, by (3.3) and using similar arguments as above, we can derive $f = c_1 e^{\frac{\alpha_1}{n} z} = c_1 e^{\alpha_2 z}$, $c_1^n = p_1$, $c_1(e^{\alpha_1/n} - 1)q = p_2$.

This completes the proof of Theorem 1.1.

Remark 3.1 Suggested by the referee, one can also derive the conclusions of Theorem 1.1 when $\varphi \equiv 0$.

In fact, since φ in (3.5) vanishes identically, dividing with f^2, and recalling $f''/f = (f'/f)' + (f'/f)^2$, we get the Riccati equation

$$t' + nt^2 - (\alpha_1 + \alpha_2)t + \alpha_1\alpha_2/n = 0,$$

where $t := f'/f$. This equation has two constant solutions, $t = \alpha_1/n$, $t = \alpha_2/n$. By Corollary 5.2 in the paper by Bank *et al.* [19], all other meromorphic solutions are of infinite order. And from this one can obtain the two entire solutions of exponential type.

4 Proof of Theorem 1.2

Suppose that f is a transcendental entire solution of finite order to (1.2), differentiating (1.2) results in

$$3f^2 f' + q'(z)\Delta^3 f(z) + q(z)\big(f'(z+3) - 3f'(z+2) + 3f'(z+1) - f'(z)\big) = bc\cos bz. \tag{4.1}$$

Combining (1.2) and (4.1), we obtain

$$f^4\big[9\big(f'\big)^2 + b^2 f^2\big] = T_4(f), \tag{4.2}$$

where $T_4(f)$ is a difference-differential polynomial of f, and its total degree at most 4.

If $T_4(f) \equiv 0$, it follows from (4.2) that $9(f')^2 + b^2f^2 \equiv 0$, then $f' = \pm\frac{bi}{3}f$, and there exists a nonzero constant c such that $f = ce^{\pm\frac{bi}{3}z}$. Substituting the expression of f into (1.2), we arrive at a contradiction. Therefore, $T_4(f) \not\equiv 0$. Set $\beta = 9(f')^2 + b^2f^2$, then Lemma 2.1 shows that $m(r,\beta) = S(r,f)$, *i.e.* β is a small function of f. Moreover, from Lemma 2.3, it is easy to see that β is a constant. By differentiating both sides of $\beta = 9(f')^2 + b^2f^2$, we get

$$f'' + \left(\frac{b}{3}\right)^2 f = 0. \tag{4.3}$$

It follows from (4.3) that

$$f = c_1 e^{\frac{bi}{3}z} + c_2 e^{-\frac{bi}{3}z}, \tag{4.4}$$

where c_1, c_2 are constants. Substituting (4.4) into (1.2), we have

$$a_1w^6 + a_2w^4 + a_3w^2 + a_4 = 0,$$

where $w = e^{\frac{bi}{3}z}$, $a_1 = c_1^3 - \frac{c}{2i}$, $a_2 = 3c_1^2c_2 + c_1q(z)e^{bi} - 3q(z)c_1e^{\frac{2bi}{3}} + 3q(z)c_1e^{\frac{bi}{3}} - c_1q(z)$, $a_3 = 3c_1c_2^2 + c_2q(z)e^{-bi} - 3q(z)c_2e^{-\frac{2bi}{3}} + 3q(z)c_2e^{-\frac{bi}{3}} - c_2q(z)$, $a_4 = c_2^3 + \frac{c}{2i}$.

Since w is transcendental, we must have $a_1 = a_2 = a_3 = a_4 = 0$. It follows from $a_2 = a_3 = 0$ that

$$\left(e^{\frac{bi}{3}} - 1\right)^3 = \left(e^{-\frac{bi}{3}} - 1\right)^3. \tag{4.5}$$

Let $v = e^{\frac{bi}{3}}$, then from (4.5) we get

$$v - 1 = u_k\left(\frac{1}{v} - 1\right), \quad k = 1, 2, 3, \tag{4.6}$$

where $u_1 = 1$, $u_2 = \frac{-1+\sqrt{3}i}{2}$, $u_3 = \frac{-1-\sqrt{3}i}{2}$.

Now we distinguish three cases to discuss.

Case 1. Let $u_k = 1$, we conclude $e^{\frac{2bi}{3}} = 1 = e^{2n\pi i}$, so $b = 3n\pi$, which and $a_2 = 0$ imply that

$$3c_1c_2 + q(z)\left(v^3 - 3v^2 + 3v - 1\right) = 0,$$

and $3c_1c_2 = 4q(z)(1 - (-1)^n)$, where $c_1 \neq 0$, $c_2 \neq 0$ are constants. Therefore n is odd and $q^3 = \frac{27}{2{,}048}c^2$.

Case 2. Assume that $u_k = \frac{-1+\sqrt{3}i}{2}$; by a similar method as Case 1, we get $b = 6n\pi$ or $b = 6n\pi - \pi$.

If $b = 6n\pi$, combining with $a_2 = 0$, we find $c_1c_2 = 0$, which is a contradiction.

If $b = 6n\pi - \pi$, substituting this expression of b into $a_2 = 0$, we have $q^3 = -\frac{27}{4}c^2$.

Case 3. While $u_k = \frac{-1-\sqrt{3}i}{2}$, using a similar way as above, we get $b = 6n\pi$ or $b = 6n\pi + \pi$. If $b = 6n\pi$, we can get a contradiction, hence $b = 6n\pi + \pi$, and $q^3 = -\frac{27}{4}c^2$.

This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Suppose that f is a transcendental entire solutions of hyper-order $\sigma_2(f) < 1$ to (1.3). Without loss of generality, we assume that $a \not\equiv 0$. Remark 2.2 after Lemma 2.4 and (1.3) will give

$$\begin{aligned} lm(r,f) &= m\bigl(r, b - a\Delta^n f(z)\bigr) \\ &\leq m(r,f) + O\bigl(r^{\rho-1+\varepsilon}\bigr) + (\lambda_1 + \lambda_2)T(r,f) + S(r,f). \end{aligned}$$

So

$$(l-1)T(r,f) = (l-1)m(r,f) \leq O\bigl(r^{\rho-1+\varepsilon}\bigr) + (\lambda_1 + \lambda_2)T(r,f) + S(r,f).$$

For $l \geq 2$, $\lambda_1 + \lambda_2 < 1$, we get a contradiction, thus (1.3) has no transcendental entire solutions of hyper-order $\sigma_2(f) < 1$.

This also completes the proof of Theorem 1.3.

6 Final remark and conjectures

The current Clunie types of theorems regarding difference-differential polynomials are mainly useful or effective to deal with problems relating to entire or meromorphic solutions of finite order for certain types of difference-differential equations. Thus, it is very natural for us to pose the following two conjectures, for further studies.

Conjecture 6.1 There are no entire solutions of infinite order for any equations (1.1), (1.2), and (1.3).

Conjecture 6.2 Equations of the forms

$$f^n(z) + P\bigl(z, f(z), f(z+c_1), \ldots, f(z+c_k)\bigr) = q_1(z)e^{\alpha_1 z} + q_2(z)e^{\alpha_2 z}$$

have no entire solutions of infinite order, where $P(z, f(z), f(z+c_1), \ldots, f(z+c_k))$ is a difference polynomial, q_1, q_2 are polynomials, $n \geq 2$ and $k \geq 1$ are integers, and α_1, α_2, c_j $(j = 1, 2, \ldots, k)$ are constants.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read and approved the final manuscript.

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