Entropy and Channel Capacity in the Regenerative Setup with Applications to Markov Channels.

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Abstract — **We obtain new entropy and mutual information formulae for regenerative stochastic processes. We use them on Markov channels to generalize the results in Goldsmith and Varaiya [3]. Also we obtain tighter bounds on capacity and better algorithms than in [3].**

I. Introduction

We obtain new formulae for the entropy and mutual information for regenerative processes including many Harris recurrent Markov and long range dependent processes. Using our results we improve upon the lower bound for the capacity of the Markov channels without feedback considered in [3]. We generalize all the results in [3] while significantly simplifying the proofs, we also modify the algorithm in [3] to obtain the mutual information and obtain a substantially simpler algorithm.

II. Entropy and Mutual Information

Let $\{X_k, k \geq 0\}$ be a discrete valued regenerative stochastic process (see [1] chapter V). Let $1 \leq T_1 < T_2 < T_3 < \ldots$ be the regeneration epochs for $\{X_k\}$ with $\tau = T_2 - T_1$ a regeneration length. Denote $X_m^n = (X_m, \ldots, X_n)$. Entropy per sample of X is given by $H(X) = \lim_{n\to\infty} \frac{1}{n} H(X_1^n)$, if the limit exists.

The next theorem and lemma provides existence and rates of convergence of $H(X)$. For X starting in any initial distribution, define $Y_0 = -\log P(X_1^{T_1})$, $Y_n = -\log P(X_{T_n+1}^{T_{n+1}})$, $n \geq 1$, where $P(\cdots)$ denotes the distribution of X. Let $S_n = -\log P(X_1^n)$, $Y_0(n) = -\log P(X_1^{T_1 \wedge n})$. The following theorem can be proved using regenerative theory (see e.g. [1] Chapter V).

Theorem 1 *Let $E[\tau] < \infty$, $E[Y_1] < \infty$ and $P[Y_0 < \infty] = 1$. Then $S_n/n \to EY_1/E\tau$ a.s.. Furthermore, if $E[Y_0] < \infty$ then the limit $H(X)$ also exists and equals $E[Y_1]/E\tau$. If in addition, $0 < var(Y_1) < \infty$ then $[S_n - nEY_1/E\tau]n^{-\frac{1}{2}}$ converges in distribution to normal r.v. $\mathcal{N}(0, \sigma^2)$ (CLT), where $\sigma^2 = var(Y_1) + (\frac{EY_1}{E\tau})^2 Var(\tau) - \frac{2E[Y_1]}{E[\tau]} cov(Y_1, \tau)$. Under the same conditions the law of iterated logarithm also holds.* □

Lemma 1 *For an irreducible countable Markov chain $\{X_k\}$ with transition matrix $P(j \mid i)$, $E[Y_1^\alpha] < \infty$ for any $\alpha \geq 1$ if for some $\epsilon > 0$, $P(j|i) \geq \epsilon$ whenever $P(j \mid i) > 0$ and $E[\tau^{2\alpha}] < \infty$.* □

For an irreducible, ergodic Markov chain, the expression $E[Y_1]/E[\tau]$ reduces to the well known expression. Theorem 1, along with its proof is valid for continuous time regenerative processes with values in a general Polish space and m-dependence in cycles. Theorem 1 can be used on regenerative processes X and Y with common regeneration epochs to obtain limits for $I(X_1^n, Y_1^n)/n \triangleq I(X, Y)$.

III. Application to Markov Channels

Consider one user transmitting to another via a Markov fading channel. Let (Z_n, S_n) be a countable, irreducible, aperiodic, ergodic Markov chain. Intervisit time to a particular state will represent a regeneration epoch. S_n is the channel state. Given Z_n, X_n is independent of everything else. Unlike in [3], our model allows inter-symbol interference.

Following [3], define $\rho_n(s, z) = P(S_n = s, Z_n = z \mid Y^{n-1})$ and $\pi_n(s, z) = P(S_n = s, Z_n = z \mid Y_1^{n-1}, X_1^{n-1})$. The proofs in [3] do not hold for Markov inputs in general. Our key observation is that $\{\pi_n\}$ and $\{\rho_n\}$ are regenerative sequences. The regeneration epochs of $\{\rho_n\}$ and $\{\pi_n\}$ are same as that of $\{(Z_n, S_n)\}$. Therefore, $\pi_n \to \pi$ and $\rho_n \to \rho$ converge in total variation to their unique distributions.

Next we provide recursive formulae for $I(X_1^n, Y_1^n)$. We have shown that

$$I(X_1^n, Y_1^n) = \sum_{i=1}^n [-E[\log(\sum_{s,z} \sum_x P(Y_i \mid S_i = s, X_i = x) P(X_i = x \mid Z_i = z)\rho_i(s, z))] + E[\log(\sum_{s,z} P(Y_i \mid X_i, S_i = s) \frac{P(X_i|S_i=s, Z_i=z)\pi_i(s,z)}{\sum_{s',z'} P(X_i|S_i=s', Z_i=z')\pi_i(s',z')})]].$$

We have obtained recursive formulae for ρ_n and π_n as in [3].

Now as in [3], we can obtain the limit of $I(X, Y)/n$. However unlike for the *iid* inputs, computing these distributions for Markov inputs is extremely complicated. Therefore, we consider another algorithm. Observe that, from our results $[\log P(Y_1^n \mid X_1^n)/P(X_1^n)/n] \to I(X, Y)$ *a.s.* and hence we can obtain recursive algorithms to compute an aproximation for the limit from formulae for ρ_n and π_n.

We obtain a lower bound on channel capacity using the above algorithm, by calculating $\sup_{P(X)} I(X, Y)$ by restricting the supremum to the set of (hidden) Markov chain inputs. This lower bound is obviously tighter than I_{iid} obtained in [3]. As in [3], to compute this bound via optimization algorithms, it helps to know if $\lim_{n\to\infty} I(X_1^n, Y_1^n)/n$ is a continuous function of the distribution of $\{Z_n\}$. We have proved it, using the regenerative setup for finite state spaces.

The decision feedback decoder designed in the section VI of [3] can be directly extended to our setup. We have demonstrated the utility of our algorithm by applying on some Markov channels.

References

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