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ENTROPY, LINEARITY AND CHAIN-RECURRENCE (1)

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We dedicate this paper to Rufus Bowen. Rufus suggested that we write it, and we have done so largely out of respect for him. We held Rufus dear as a friend and as a mathematician. We miss him sorely on both counts.

We will prove two new cases of the entropy conjecture, using volume estimates to bound the growth rate of forms as in [13]. The C^r entropy conjecture $(r \geq 1)$ holds that the topological entropy $h(f)$ of a C^r diffeomorphism $f: M \to M$ of a compact manifold M is bounded below by the logarithm of the spectral radius of $f_*: H_*(M; \mathbf{R}) \to H_*(M; \mathbf{R})$. Several cases of the entropy conjecture are already known ([3], [16]), but it fails for C^0 and PL homeomorphisms [12]. The new cases we deal with are affine mappings and diffeomorphisms with finite chain-recurrence.

In section 1 we will describe the volume estimates we will use in later sections. The C^r entropy conjecture is shown to hold whenever the sets $B_x(\varepsilon, n)$ $(x \in M,\ \varepsilon$ small) lose volume at the asymptotic rate predicted by the differentials $Df^n(x)$.

We demonstrate in section 2 that this volume decay does occur when M is a compact affine manifold and f is an affine map. This establishes the entropy conjecture for many endomorphisms of infrasolvmanifolds ([2], [8]). We note that most known Anosov diffeomorphisms and expanding maps are affine [5]. Our result shows that the difficulties in the C^r entropy conjecture lie in the nonlinearity of smooth mappings.

In section 3 we reprove the entropy conjecture for Axiom A-No Cycle diffeomorphisms [16]. The volume estimates we need are contained in the Bowen-Ruelle Volume Lemma [4] (actually we use a C^1 version).

In section 4 we show that the C^1 entropy conjecture holds whenever the chain-recurrence set $R(f)$ is finite. Here the necessary volume estimates arise from a local,

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nonhyperbolic version of the Volume Lemma. This generalizes the known results for Morse-Smale f by omitting the hyperbolicity conditions at the periodic points ([15], [16]).

We conclude by showing that the necessary homological conditions for a homotopy class to contain a map f with finite $R(f)$ are sufficient on nilmanifolds. If $f: M \to M$ is a nilmanifold automorphism and $f_*: H_*(M, \mathbf{R}) \to H_*(M; \mathbf{R})$ has spectral radius 1 then f is C^∞-approximable by Morse-Smale diffeomorphisms. This extends recent independent work of B. Halpern [6]. In an appendix, we prove the two variations of the Bowen-Ruelle Volume Lemma that we use in sections 3 and 4.

We emphasize that section 4 supplies a computable topological criterion for a diffeomorphism to have infinite chain-recurrence, namely to verify that some homology eigenvalue is not of length 1.

We note that the counterexample to the PL entropy conjecture in [12] has finite chain-recurrent set. This contrast in the behavior of PL and C^1 maps with finite chain-recurrence adds to the evidence for the C^1 entropy conjecture.

Section 1. — Entropy and Volume.

When $f: M \to M$ is a diffeomorphism of a compact Riemannian manifold, let $B_x(\varepsilon, n) = \{y \mid d(f^k x, f^k y) \leq \varepsilon \text{ for } k = 0, 1, \ldots, n\}$. We will relate the entropy conjecture for f to the asymptotic change in the volume of $B_x(\varepsilon, n)$ as $n \to \infty$.

To state the asymptotic behavior we might expect to find, it is necessary to consider the *unstable expansion* $J^u(A)$ of a linear map $A: V_0 \to V_1$, where the V_i are inner product spaces of equal (finite) dimension. One may define $J^u(A)$ as the norm of the induced map $\wedge^*(A): \wedge^*(V_1) \to \wedge^*(V_0)$, where the exterior algebras $\wedge^*(V_i)$ are given the usual inner product. Clearly J^u is submultiplicative, that is $J^u(A \circ B) \leq J^u(A) . J^u(B)$. It is well-known that $J^u(A)$ is the product of the unstable eigenvalues of $(A^t A)^{1/2}$. For a smooth map $f: M \to M$ on a Riemannian manifold, we denote $J^u_x(f) = J^u(Df(x))$ ($= \max_p \theta^p_1(x)$ in the notation of [13]). If M is compact, it is easy to show (using submultiplicativity) that $J^u_x(f)$ changes by a bounded amount if one considers a different Riemannian metric. The unstable expansion of a linear map is related to volume in the following way:

Lemma 1. — Suppose $A: \mathbf{R}^n \to \mathbf{R}^n$ *is a linear map. For the usual inner product on* $\mathbf{R}^n$, $J^u(A) . \mathrm{vol}(B_0(r) \cap A^{-1} B_0(r)) \leq (2r)^n$.

Proof. — Since $A^{-1}(B_0(r)) = (A^t A)^{-1/2}(B_0(r))$ we may assume A is positive definite. Consider $\mathbf{R}^n = E^u \oplus E^{cs}$, where E^u is the span of the unstable eigenvectors and E^{cs} is the span of the eigenvectors whose eigenvalues are of modulus ≤ 1. For $v \in E^{cs} \cap B_0(r)$, $J^u(A) . \mathrm{vol}((v + E^u) \cap B_0(r) \cap A^{-1} B_0(r)) \leq (2r)^u$, where here we use the u-dimensional volume of E^u, $u = \dim E^u$. An easy application of Fubini's Theorem shows that the inequality holds. Q.E.D.

For a diffeomorphism f, we ask whether $\mathrm{vol}(B_x(\varepsilon, n))$ decays asymptotically at the rate predicted by the linearized iterates $Df^n(x)$. Explicitly, we have the

C^r *Volume Question*: Suppose $f: M \to M$ is a C^r diffeomorphism, for some $r \geq 1$, of a compact Riemannian manifold. Given $\delta > 0$, do there exist C, $\varepsilon > 0$ so that, for any $x \in M$ and $n \geq 0$,

$$\mathrm{vol}(B_x(\varepsilon, n)) \leq \frac{C(1+\delta)^n}{J_x^u(f^n)}\ ?$$

To see the importance of this question, note

Proposition 1. — An affirmative answer to the C^r *volume question implies that the* C^r *entropy conjecture holds.*

Proof. — By passing to a double cover, we may assume M is orientable and study the eigenvalues on de Rham cohomology $H^*(M; \mathbf{R}) = (H_*(M; \mathbf{R}))^*$. Let λ be an eigenvalue and c an eigenvector for $f^*: H^*(M; \mathbf{R}) \to H^*(M; \mathbf{R})$. If ω is a form in class c, we may estimate as in [13, section 3]

$$|\lambda|^n \|c\| = \|[f^{*n}\omega]\| \leq K . \int_M \|f^{*n}\omega\| \, d\,\mathrm{vol}$$

for any norm on $H^*(M; \mathbf{R})$ and some constant K.

Let S be a minimal (n, ε)-spanning set, that is $M = \bigcup_{x \in S} B_x(\varepsilon, n)$ and the cardinality of S is minimal. We will estimate $\int_{B_x(\varepsilon, n)} \|f^{*n}\omega\| \, d\,\mathrm{vol}$. Since the Volume Question is assumed to have an affirmative answer, we may choose ε so that

$$\mathrm{vol}\, B_y(2\varepsilon, n) . J_y^u(f^n) \leq C(1+\delta)^n.$$

For $y \in B_x(\varepsilon, n)$, we have $B_x(\varepsilon, n) \subset B_y(2\varepsilon, n)$, so $J_y^u(f^n) \leq \dfrac{C(1+\delta)^n}{\mathrm{vol}\, B_y(2\varepsilon, n)} \leq \dfrac{C(1+\delta)^n}{\mathrm{vol}\, B_x(\varepsilon, n)}$. Thus

$$\int_{B_x(\varepsilon, n)} \|f^{*n}\omega\| \, d\,\mathrm{vol} \leq \|\omega\|_\infty \int_{B_x(\varepsilon, n)} J_y^u(f^n) \, d\,\mathrm{vol} \leq \|\omega\|_\infty . C(1+\delta)^n.$$

Altogether, we have

$$|\lambda|^n \|c\| \leq K . \sum_{x \in S} \int_{B_x(\varepsilon, n)} \|f^{*n}\omega\| \, d\,\mathrm{vol} \leq K . \mathrm{card}\, S . \|\omega\|_\infty . C(1+\delta)^n.$$

Taking logarithms, dividing by n and letting $n \to \infty$ gives

$$\log|\lambda| \leq h_\varepsilon(f) + \log(1+\delta) \leq h(f) + \log(1+\delta).$$

Letting $\delta \to 0$, we obtain $h(f) \geq \log|\lambda|$, as desired. Q.E.D.

Section 2. — The Affine Entropy Conjecture.

We will answer the volume question affirmatively for affine maps. A map $Ax + b : \mathbf{R}^m \to \mathbf{R}^m$, with A linear and $b \in \mathbf{R}^m$, is an *affine transformation.* An *affine structure* on a manifold M is determined by an atlas of coordinate systems where the transition maps are the restrictions of affine transformations. A map $f: M \to M$ is *affine* if it is represented in local coordinates by affine transformations.

Theorem 1 (the affine entropy conjecture). — *The entropy conjecture holds for an affine map f of a compact affine manifold* M. *Moreover, the volume question has an affirmative answer for such f.*

Proof. — We fix a finite atlas of affinely related coordinate systems on M, say $\varphi_i: U_i \to \mathbf{R}^m$. We may assume $\operatorname{diam}(U_i) \leq r < \infty$ and that the Jacobians of the φ_i are bounded away from 0 and infinity.

By selecting ε less than a Lebesgue number for $\{U_i\}$, we may assume $B_x(\varepsilon) \subset U_{i(x)}$. If $a = \varphi_{i(x)}(x)$, $b = \varphi_{i(f^n x)}(f^n x)$ and $A = \varphi_{i(f^n x)} \circ f^n \circ \varphi_{i(x)}^{-1}$, then Lemma 1 gives

$$\operatorname{vol}(\varphi_{i(x)} B_x(\varepsilon, n)) \leq \operatorname{vol}(D_a(r) \cap A^{-1} D_b(r)) \leq \frac{(2r)^m}{J_a^u(A)}.$$

Passing from this computation in coordinates to one on M, we obtain

$$\operatorname{vol} B_x(\varepsilon, n) \leq \frac{C}{J_x^u(f^n)}. \qquad \text{Q.E.D.}$$

Section 3. — *Entropy and Volume for Axiom* A *Diffeomorphisms.*

We reprove the entropy conjecture for Axiom A-No Cycles diffeomorphisms [16] in a natural way, using relative versions of the volume considerations of section 1. We will need the following relative version of a result in [13].

Lemma 2. — *Let* M_0 *and* M_1 *be oriented compact m-dimensional manifolds with* $M_0 \subset \operatorname{int} M_1$. *Let f be a* C^1 *diffeomorphism with* $f(M_i) \subset \operatorname{int} M_i$, *for* $i = 0, 1$. *For fixed* $B > 0$, *let*

$$V(n) = f^B(M_1) - f^{-(n+B)}(M_0).$$

Then the logarithm of the spectral radius of $f^*: H^*(M_1, M_0; \mathbf{C}) \to H^*(M_1, M_0; \mathbf{C})$ *does not exceed* $\limsup_{n \to \infty} \left(\frac{1}{n} \log \int_{V(n)} J_y^u(f^n)\, d\operatorname{vol}\right)$.

Proof. — For any Riemannian metric on M_1, $\|C\| = \min\left\{\int \|\eta\|\, d\operatorname{vol} \mid [\eta] = C\right\}$ is a norm on $H^*(M_1, M_0; \mathbf{C})$, where η runs over closed relative forms $(d\eta = 0 = \eta \mid M_0)$. (The definiteness property is [13, Prop. (3.1)], using the appropriate relative De Rham theory [17, Prop. (4.2)].)

We choose $C \neq 0$ to be an eigenclass, say $f^* C = \lambda C$, where $C = [\omega]$, $d\omega = 0 = \omega \mid M_0$, $\lambda \neq 0$. Then $\operatorname{support}(f^{*(n+2B)}\omega) \subset M_1 - f^{-(n+2B)} M_0 = f^{-B} V(n)$. Hence

$$\begin{aligned} \|\lambda^{n+2B} C\| &\leq \int_{M_1} \|f^{*(n+2B)}\omega\|\, d\operatorname{vol} = \int_{f^{-B}V(n)} \|f^{*(n+2B)}\omega\|\, d\operatorname{vol} \\ &\leq K \int_{V(n)} \|f^{*(n+B)}\omega\|\, d\operatorname{vol} \leq K \|f^{*B}\omega\|_\infty \int_{V(n)} J_y^u(f^n)\, d\operatorname{vol}, \end{aligned}$$

where K is a constant arising from the change of variables formula. Taking logarithmic growth rates gives $\log|\lambda| = \limsup_{n\to\infty} \frac{1}{n} \log \|\lambda^{n+2B} C\| \leq \limsup_{n\to\infty} \frac{1}{n} \log \int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol}$.

Q.E.D.

We recall that a filtration for $\Omega(f)$, where f is a C^1 diffeomorphism of a compact manifold M, consists of a sequence of compact submanifolds M_i, $i = 0, 1, \ldots, k$, where

1) $M_0 = \emptyset$, $M_k = M$, $\dim M_i = \dim M$;
2) $M_i \subset \operatorname{int} M_{i+1}$;
3) $f(M_i) \subset \operatorname{int} M_i$, and
4) $\Omega(f) = \bigcup_{i=1}^{k} \Lambda_i$, where $\Lambda_i = \bigcap_{n=-\infty}^{+\infty} f^n(M_i - M_{i-1})$.

Proposition. — *If* $\{M_i\}$ *is a filtration for* $\Omega(f)$ *and if for each* $i = 1, \ldots, k$

$$h(f\,|\,\Lambda_i) \geq \liminf_{B\to\infty} \left(\limsup_{n\to\infty} \frac{1}{n} \log \int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol} \right)$$

then the entropy conjecture holds for f. (As in the previous lemma, for given i, $B > 0$, $V(n) = f^B M_i - f^{-(n+B)} M_{i-1}$.)

Proof. — By passing to an oriented cover, we may assume M is orientable.

By an induction on the length of our filtration using the cohomology sequence of a triple, the spectral radius of $f^* : H^*(M; \mathbf{C}) \to H^*(M; \mathbf{C})$ does not exceed the spectral radius of $\bigoplus_{i=1}^{k} f_i^*$, where $f_i^* : H^*(M_i, M_{i-1}; \mathbf{C}) \to H^*(M_i, M_{i-1}; \mathbf{C})$ [16].

On the other hand, $h(f) \geq h(f\,|\,\Lambda_i)$ for all i. Our conclusion follows from the preceding lemma. Q.E.D.

We now reprove [16, Theorem 1].

Theorem 2. — *If* f *is an Axiom A-No Cycles diffeomorphism of a compact manifold* M, *then the entropy conjecture holds for* f.

Proof. — Smale has shown that there exists a filtration for $\Omega(f)$ [14]. By our proposition, we need to estimate $\int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol}$, for fixed i and suitable $B > 0$.

Given $\delta > 0$, choose ε as in the C^1 Volume Lemma for Basic Sets (see Appendix). By Bowen's Shadow Lemma [1] we may pick a neighborhood U of $\Lambda = \Lambda_i$ such that, if $y \in \bigcap_{j=0}^{n} f^{-j} U$, then $y \in B_x(\varepsilon/2, n)$ for some $x \in \Lambda$. We select B large enough so that $V(0) = f^B(M_i) - f^{-B}(M_{i-1}) \subset U$.

If $y \in B_x(n, \varepsilon)$, with $x \in \Lambda$, we estimate $J^u_y(f^n)$ as follows. Introduce a continuous inner product on TM agreeing with our Riemannian metric on E^u_Λ but in which $E^u_x \perp E^s_x$

for all $x \in \Lambda$. For this inner product, we may find $\eta > 0$ so that for all $\ell \in \Lambda$, $m \in M$ with $d(\ell, m) \leq \varepsilon$ we have $J^u_m(f) \leq (1+\eta) J^u_\ell(f)$, where $\eta \to 0$ as $\varepsilon \to 0$. We compute

$$J^u_y(f^n) \leq J^u_y(f) \cdot \ldots \cdot J^u_{f^{n-1}y}(f) \leq (1+\eta)^n J^u_x(f) \cdot \ldots \cdot J^u_{f^{n-1}x}(f) = (1+\eta)^n \operatorname{Jac}(Df^n : E^u_x \to E^u_{f^n x}).$$

Passing back to our Riemannian metric, we find that for $x \in \Lambda$ and $y \in B_x(n, \varepsilon)$

$$(*) \qquad J^u_y(f^n) \leq C_0 (1+\eta)^n . \operatorname{Jac}(Df^n : \ E^u_x \to E^u_{f^n x}),$$

where C_0 is some positive constant and where $\eta \to 0$ with ε.

Now let S be a minimal $(n, \varepsilon/2)$-spanning set for $f \mid \Lambda$. By our choice of U and B,

$$\begin{aligned} V(n) &= f^B M_i - f^{-(n+B)} M_{i-1} \\ &\subset \bigcap_{j=0}^{n} f^{-j} V(0) \subset \bigcap_{j=0}^{n} f^{-j} U \subset \bigcup_{x \in S} B_x(\varepsilon, n). \end{aligned}$$

Thus

$$\begin{aligned} \int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol} &\leq \sum_{x \in S} \int_{B_x(\varepsilon, n)} J^u_y(f^n)\, d\,\mathrm{vol} \\ &\leq \sum_{x \in S} C_0 (1+\eta)^n . \operatorname{Jac}(Df^n : \ E^u_x \to E^u_{f^n x}) . \mathrm{vol}\, B_x(\varepsilon, n) \\ &\leq \mathrm{Card}\, S . C_0 (1+\eta)^n . C(1+\delta)^n \end{aligned}$$

where we have used $(*)$ and the C^1 Volume Lemma for Basic Sets (see Appendix). Taking logarithmic growth rates,

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \log \int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol} &\leq h_{\varepsilon/2}(f \mid \Lambda) + \log(1+\eta)(1+\delta) \\ &\leq h(f \mid \Lambda) + \log(1+\eta)(1+\delta). \end{aligned}$$

Letting $\delta \to 0$, we have $\varepsilon \to 0$, $\eta \to 0$ and $B \to +\infty$, so the proposition applies. Q.E.D.

Section 4. — Quasi-unipotence and Finite Chain-recurrence

We now prove the entropy conjecture for diffeomorphisms f whose chain-recurrence set $R(f)$ (see [14]) is finite. As $h(f) = h(f \mid R(f)) = 0$, this amounts to showing that all eigenvalues of $f_* : H_*(M; \mathbf{R}) \to (H_*(M; \mathbf{R}))$ are of modulus 1 (that is, f_* is quasi-unipotent). We show, as a partial converse, that any nilmanifold diffeomorphism which is quasi-unipotent on homology is homotopic to a diffeomorphism with finite R (indeed, to a Morse-Smale diffeomorphism).

Theorem 3. — The entropy conjecture is valid for diffeomorphisms $f : M \to M$ *when* M *is compact and* $R(f)$ *is finite.*

Proof. — We will show f_* is quasi-unipotent. By passing to a power, we may assume $R(f)$ consists of fixed points. Then [14] shows that there is a filtration $\{M_i\}$ for $\Omega(f) = R(f)$.

By Proposition 2, we need only show $\limsup \frac{1}{n} \log \int_{V(n)} J^u_y(f^n)\, d\,\mathrm{vol} \to 0$ as $B \to \infty$, where i is fixed and $V(n)=f^{B}M_i^u - f^{-(n+B)}M_{i-1}$. Given $\delta>0$, choose ε as in the Local C^1 Volume Lemma (see Appendix). Choose B large enough that $V(0)\subset B_x(\varepsilon)$, where $\Lambda_i=\{x\}$.

Choose an inner product on TM that agrees with our Riemannian metric on E_x^u and satisfies $\|Df\,|\,E_x^s\|<1$, $\|Df\,|\,E_x^c\|\leq 1+\varepsilon$. Then for $y\in B_x(\varepsilon, n)$, we have

$$\begin{aligned} J_y^u(f^n) &\leq J_y^u(f)\cdot \ldots \cdot J_{f^{n-1}y}^u(f) \\ &\leq (1+\eta_0)^n J_x^u(f)\cdot \ldots \cdot J_x^u(f) \\ &\leq (1+\eta_0)^n (1+\varepsilon)^{nc} (J^u(Df\,|\,E_x^u))^n \\ &= (1+\eta_0)^n (1+\varepsilon)^{nc} J^u(Df^n\,|\,E_x^u) \\ &\leq (1+\eta_0)^n (1+\varepsilon)^{nc} J_x^u(f^n) \end{aligned}$$

where $\eta_0\to 0$ as $\varepsilon\to 0$. Setting $(1+\eta)=(1+\eta_0)(1+\varepsilon)^c$ and passing back to our original Riemannian metric, we obtain

$$(*)\qquad J_y^u(f^n)\leq C_0(1+\eta)^n J_x^u(f^n),$$

for $y\in B_x(\varepsilon, n)$, where C_0 is independent of y and n and where $\eta\to 0$ as $\varepsilon\to 0$.

Since $V(n)=\bigcap_{i=0}^{n} V(0)\subset B_x(\varepsilon, n)$, we have

$$\begin{aligned} \int_{V(n)} J_y^u(f^n)\,d\,\mathrm{vol} &\leq C_0(1+\eta)^n J_x^u(f^n)\,.\,\mathrm{vol}\; B_x(\varepsilon, n) \\ &\leq C_0(1+\eta)^n\,.\,C(1+\delta)^n \end{aligned}$$

where we have used $(*)$ and the Local C^1 Volume Lemma (see Appendix). Taking logarithmic growth rates, we obtain

$$\limsup \int_{V(n)} J_y^u(f^n)\,d\,\mathrm{vol}\leq \log(1+\eta)(1+\delta).$$

As $\delta\to 0$, we have $\varepsilon\to 0$, $\eta\to 0$ and $B\to+\infty$.

Thus proposition 2 does apply. Q.E.D.

We finally consider the case where $M=N/\Gamma$ is a nilmanifold. Then any diffeomorphism of M is homotopic to one induced by a Lie group automorphism of N which preserves Γ. The product of the latter with a translation by an element of N will here be called an *automorphism* of N/Γ [11]. We now show:

Theorem 4. — If α is an automorphism of a compact nilmanifold N/Γ *and if α induces a quasi-unipotent map on* $H_1(N/\Gamma;\mathbf{R})$, *then α is* C^∞ *close to a Morse-Smale diffeomorphism.*

Proof. — We first perturb the translational part of α to be rational (relative to the lattice Γ).

Using the standard form of [9, Theorem III.12] for the map

$$\alpha^*:\ H^1(N/\Gamma;\mathbf{Z})\to H^1(N/\Gamma;\mathbf{Z}),$$

we find a fibration $\pi : N/\Gamma \to \mathbf{T}^r$ which induces a factor map $\bar{\alpha} : \mathbf{T}^r \to \mathbf{T}^r$ with $\bar{\alpha}^* : H^1(\mathbf{T}^r; \mathbf{Z}) \to H^1(\mathbf{T}^r; \mathbf{Z})$ irreducible. Since α^* was quasi-unipotent, $\bar{\alpha}^*$ is also. But by [14, p. 149], the eigenvalues of $\bar{\alpha}^*$ are roots of unity. Thus $\bar{\alpha}^*$ is periodic. Since the translational part of α is rational, $\bar{\alpha}$ is itself periodic and so admits an equivariant Morse function $f : M \to \mathbf{R}$ with finite singularity set S.

The nilmanifolds $\pi^{-1}(S)$ are mapped amongst themselves by α so that each return map is an automorphism with rational translation part. By [11, Lemma 2], these lower dimensional automorphisms are also quasi-unipotent on H_1. So, inducting on dimension (N/Γ), we assume there are flows $\varphi^1, \ldots, \varphi^k$ on $\pi^{-1}(S)$ so that, for any $t > 0$, $\Omega(\varphi^k_t \circ \ldots \circ \varphi^1_t \circ (\alpha \mid \pi^{-1}S))$ is finite, hyperbolic and has no cycles.

Extend the φ^i to C^∞ flows on N/Γ tangent to the fibration π. Introduce right invariant Riemannian metrics on N/Γ and $\mathbf{T}^r$, chosen so that $T\pi : T(N/\Gamma) \to T(\mathbf{T}^r)$ and $T^*\pi : T^*(\mathbf{T}^r) \to T^*(N/\Gamma)$ are adjoint, and let φ^{k+1} and ψ be the gradient flows for $f \circ \pi$ and f respectively. Then $\varphi^{k+1}_t \circ \ldots \circ \varphi^1_t \circ \alpha : N/\Gamma \to N/\Gamma$ factors over $\psi_t \circ \bar{\alpha} : \mathbf{T}^r \to \mathbf{T}^r$ but agrees with $\varphi^k_t \circ \ldots \circ \varphi^1_t \circ \alpha$ on $\pi^{-1}S = \pi^{-1}\Omega(\psi_t \circ \bar{\alpha})$. Thus

$$\Omega(\varphi^{k+1}_t \circ \ldots \circ \varphi^1_t \circ \alpha) = \Omega(\varphi^k_t \circ \ldots \circ \varphi^1_t \circ \alpha) \cap \pi^{-1}S$$

is finite, hyperbolic and has no cycles.

By the Ω-Stability Theorem, any Kupka-Smale C^∞ approximation to $\varphi^{k+1}_t \circ \ldots \circ \varphi^1_t \circ \alpha$ will be Morse-Smale [10]. Choosing t small gives the desired C^∞ Morse-Smale approximation to α. Q.E.D.

We obtain from Theorems 3 and 4 (and the discussion in between):

Corollary. — A diffeomorphism $g : N/\Gamma \to N/\Gamma$ *of a compact nilmanifold is homotopic to a diffeomorphism with finite* R *iff it is homotopic to a Morse-Smale diffeomorphism.*

It would be interesting to understand the differences between isotopy classes of diffeomorphisms with Morse-Smale and finite R representatives. Morse-Smale diffeomorphisms are virtual permutations on homology, whereas finite R diffeomorphisms are quasi-unipotent [15]. It is known that the "virtual permutation" condition is sufficient for isotopy to Morse-Smale, at least for M simply connected, $\dim M \geq 6$ [15].

We do not know in general (even for simply connected manifolds of dimension bigger than or equal to six) whether a diffeomorphism f with $R(f)$ finite can be approximated by a Morse-Smale diffeomorphism or even if f is isotopic or homotopic to a Morse-Smale diffeomorphism.

M. Misiurewicz has constructed a two-dimensional Axiom A-No Cycles diffeomorphism with finite R for which the growth rate on 1-forms is positive. Thus the C^r Volume Question ($r < \infty$) can have a negative answer even when Theorems 2 and 3 apply.

To conclude, we observe that by filtering $R(f)$ as in [14] we may easily combine Theorems 2 and 3 into our most general result:

Theorem 5. — If f is a diffeomorphism of a compact manifold and $R(f)$ is the union of a closed hyperbolic set and a finite set, then the entropy conjecture holds for f.

APPENDIX

In this appendix we modify the Bowen-Ruelle C^2 Volume Lemma to produce the volume estimates used in the text.

We first assume that the diffeomorphism is only C^1 and aim at a weaker estimate (Bowen-Ruelle did not allow our $(1+\delta)^n$ factor). This lets us shorten their argument, as follows.

The C^1 *Volume Lemma for Basic Sets. — Let* Λ *be a basic set for an Axiom A diffeomorphism* f. *Then, given* $\delta>0$, *one may find* $\varepsilon>0$ *and* $C>1$ *so that, for all* $x\in\Lambda$ *and* $n\geq 0$,

$$\mathrm{Vol}\, B_x(\varepsilon, n)\,.\,\mathrm{Jac}(Df^n:\ E^u_x\to E^u_{f^n x})\leq C(1+\delta)^n.$$

Proof. — There is a continuous splitting $T_xM=E^u_x\oplus E^s_x$, for $x\in\Lambda$, of the tangent bundle over Λ into unstable and stable bundles. By a result of Mather, a Riemannian metric $(\ ,\ )$ may be chosen on M so that for $x\in\Lambda$, $v=(v_u, v_s)\in T_xM$

$$|||D_xf(V_s)|||\leq\lambda|||V_s||| \quad \text{and} \quad |||D_xf(V_u)|||\geq\lambda^{-1}|||V_u|||$$

where $0<\lambda<1$ is independent of x and $|||w|||=(w, w)^{1/2}$.

We shall use the modified inner product $\langle\ ,\ \rangle$ on T_xM, $x\in\Lambda$, where

$$\langle v, w\rangle=(v_s, w_s)+(v_u, w_u).$$

We will denote the stable and unstable seminorms $(v_s, v_s)^{1/2}$ and $(v_u, v_u)^{1/2}$ by $||v||_s$ and $||v||_u$ so that our modified inner product corresponds to the norm $(||v||_s^2+||v||_u^2)^{1/2}=||v||$. The letter x will denote an element of Λ.

For a subspace $E'\subset T_xM$, $E'(\varepsilon)=\{v\in E'(\varepsilon)\mid ||v||\leq\varepsilon\}$.

For ε_0 small, we fix a continuous family of C^1 charts near Λ.

For $x\in\Lambda$, $\varphi_x: T_xM(\varepsilon_0)\to M$ can be chosen [4] to satisfy:

1) $\varphi_x(0)=x$ and $D_0\varphi_x: T_xM\to T_xM$ is the identity;
2) $\varphi_x(E^s_x(\varepsilon_0))\subset W^s_x$, and
3) $\varphi_x(E^u_x(\varepsilon_0))\subset W^u_x$,

where W^s_x and W^u_x are the stable and unstable manifolds at x. The map $F=\varphi_{fx}^{-1}\circ f\circ\varphi_x$ represents f in these coordinates. Clearly $F: TM(\varepsilon_0)\to TM$ preserves E^u and E^s.

Choose $\varepsilon_1\leq\varepsilon_0$ and $\omega\in(0, 1)$ so that, if $L\in\mathrm{Hom}(E^u_x, E^s_x)$ with $v\in T_xM(\varepsilon_1)$ and $||L||<\omega$, one has

$$(*)\qquad \mathrm{Jac}\, D_v(F\mid v+\mathrm{graph}\, L)\geq\frac{1}{1+\delta}\mathrm{Jac}\, D_0(F\mid E^u_x).$$

Here δ is the fixed constant mentioned in our hypothesis.

For $\varepsilon_2\leq\varepsilon_1$ sufficiently small, there is a constant $\gamma\in(0, 1)$ such that

$$v\in T_xM(\varepsilon_2)\Rightarrow ||F_v||_u\geq\gamma^{-1}||v||_u \quad \text{and} \quad ||Fv||_s\leq\gamma||v||_s.$$

We may choose $\varepsilon_3 \leq \varepsilon_2$ so that if $v, w \in T_x M(\varepsilon_3)$ satisfy $||v-w||_s \leq \omega ||v-w||_u$, one has

$$||F_v - F_w||_s \leq \omega ||F_v - F_w||_u.$$

Hence if $h : E_x^u(\varepsilon) \to E_x^s(\varepsilon)$ is C^1, $\varepsilon \leq \varepsilon_3$, and $||Dh|| \leq \omega$ then

$$\{v \in F(\text{graph } h) \mid ||v||_u \leq \varepsilon\}$$

is the graph of a C^1 function $\Gamma_F(h) : E_{fx}^u(\varepsilon) \to E_{fx}^s(\varepsilon)$, called the graph transform of h by F [7]. Clearly $||D\Gamma_F(h)|| \leq \omega$.

For $\varepsilon \leq \varepsilon_3$ and $w \in E_x^s(\varepsilon)$, let

$$N_w(\varepsilon, n) = \{v \in T_x M \mid v_s = w \text{ and } ||F^k v||_u \leq \varepsilon \text{ for } k = 0, \ldots, n\}.$$

By the properties of γ, $N_w(\varepsilon, n) = \{v \in T_x M \mid v_s = w \text{ and } ||F^n v||_u \leq \varepsilon\}$.

Thus $F^n N_w(\varepsilon, n) = \text{graph } g$, where $g : E_{f^n x}^u(\varepsilon) \to E_{f^n x}^s(\varepsilon)$ is the graph transform of the constant function $w : E_x^u(\varepsilon) \to E_x^s(\varepsilon)$ by F^n. By the preceding paragraph, g is C^1 with $||Dg|| \leq \omega$. Computing areas relative to the Riemannian metric induced by $\langle\ ,\ \rangle$, we obtain $\text{area}(\text{graph } g) \leq K$, where K depends only on ω and ε.

We may apply $(*)$ n times to obtain

$$\text{Jac } D_v(F^n \mid v + E_x^u) \geq \frac{1}{(1+\delta)^n} \text{Jac } D_0(F^n \mid E_x^u), \quad \text{for} \quad v \in N_w(\varepsilon, n).$$

Thus

$$\begin{aligned} \text{area}(\text{graph } g) &= \int_{N_w(\varepsilon, n)} \text{Jac } D_v(F^n \mid v + E_x^u)\, d \text{ area} \\ &\geq \frac{1}{(1+\delta)^n} \text{Jac } D_0(F^n \mid E_x^u) . (\text{area } N_w(\varepsilon, n)). \end{aligned}$$

So $(\text{area } N_w(\varepsilon, n)) \leq \dfrac{K(1+\delta)^n}{\text{Jac } D_0(F^n \mid E_x^u)}$.

Clearly $D_x(\varepsilon, n) = \{v \in T_x M \mid ||F^k v|| \leq \varepsilon,\ k = 0, \ldots, n\} \subset \bigcup_{w \in E_\varepsilon^s(x)} N_w(\varepsilon, n)$.

Thus

$$\text{vol } D_x(\varepsilon, n) \leq \int_{E_\varepsilon^s(x)} (\text{area } N_w(\varepsilon, n)), \quad \text{so}$$

$$\text{vol } D_x(\varepsilon, n) \leq \frac{\sup_\varepsilon(\text{area } E_\varepsilon^s(x)) K(1+\delta)^n}{\text{Jac } D_0(F^n \mid E_x^u)}.$$

By passing back to the Riemannian metric on M, we change our distances, volumes and Jacobians only by positive factors bounded away from 0 and ∞, and obtain the desired estimate. Q.E.D.

One can amend the preceding proof to obtain a lower bound $(C(1+\delta)^n)^{-1}$ for $\text{vol } B_x(\varepsilon, n) . \text{Jac}(Df^n : E_x^u \to E_{f^n x}^u)$. Since we don't use this lower bound, we leave the details to the reader.

Of course the Volume Lemma just proved applies to the trivial basic set composed of one hyperbolic fixed point. We now adapt this argument to non-hyperbolic fixed points.

The Local C^1 *Volume Lemma.* — *If* $f: (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *is a* C^1 *diffeomorphism, then for any* $\delta > 0$ *there are* $\varepsilon > 0$ *and* $C > 1$ *such that for any* $n \geq 0$

$$\operatorname{vol} B_0(\varepsilon, n) \leq \frac{C(1+\delta)^n}{J_0^u(f^n)}.$$

Proof. — The idea, as in the preceding argument, is to use the graph transform to estimate the area of slices of $B_0(\varepsilon, n)$ parallel to the unstable manifold.

The tangent space at 0 splits into the direct sum $\mathbf{R}^n = E^u \oplus E^{cs}$, where E^u is the generalized eigenspace of eigenvalues of modulus greater than one and E^{cs} is the generalized eigenspace of eigenvalues of modulus less than or equal to one for $D_0 f$. We may fix coordinates so that $E^{cs} \supset W_0^{cs}(\varepsilon_0) =$ the local center-stable manifold at 0, and $E^u \supset W_0^u(\varepsilon_0) =$ the local unstable manifold at 0 (see [7] for details about these manifolds).

We choose an inner product on $\mathbf{R}^n$ so that E^u and E^{cs} are perpendicular and so that $\|D_0(f^{-1} | E^u)\| = \lambda > 1$ and $\|D_0(f | E^{cs})\| = \mu < \lambda$.

Choose $\omega \in (0, 1)$ and $\varepsilon_1 \leq \varepsilon_0$ so that if $L \in \operatorname{Hom}(E^u, E^{cs})$ with $\|L\| \leq \omega$ and if $\|v\| \leq \varepsilon_1$, then

$$(*) \qquad \operatorname{Jac} D_v(f | v + \operatorname{graph} L) \geq \frac{1}{1+\delta} \operatorname{Jac} D_0(f | E_x^u).$$

For $\|v\| \leq \varepsilon_2 \leq \varepsilon_1$, we have $\|F_v\|_u \geq \lambda_1 \|v\|_u$ and $\|F_v\|_{cs} \leq \mu_1 \|v\|_{cs}$, where $1 < \mu_1 < \lambda_1$.

We choose $\varepsilon_3 \leq \varepsilon_2$ so that when $\|v\|, \|w\| \leq \varepsilon_3$ satisfy $\|v - w\|_{cs} \leq \omega \|v - w\|_u$ then $\|Fv - Fw\|_{cs} \leq \omega \|Fv - Fw\|_u$. This is possible because $\mu < \lambda$.

For $\varepsilon < \varepsilon_3$, we may apply graph transform to a C^1 function $h: E^u(\varepsilon) \to E^{cs}(\varepsilon)$ with $\|Dh\| \leq \omega$. Unfortunately $\Gamma_f(h): E^u(\varepsilon) \to E^{cs}(\mu . \varepsilon)$ may no longer take values in $E^{cs}(\varepsilon)$. We therefore consider the truncated graph transform

$$\{v \in f(\operatorname{graph} h) \mid \|v\|_u \leq \varepsilon \text{ and } \|v\|_{cs} \leq \varepsilon\},$$

which is the graph of a function $\Gamma_f'(h)$ from a closed subset of $E^u(\varepsilon)$ into $E^s(\varepsilon)$. If h is defined on a closed subset of $E^u(\varepsilon)$ and extends to a C^1 function h^* with $\|\operatorname{Lip}(h^*)\| \leq \omega$ on $\operatorname{domain}(h) \subset \operatorname{int}(\operatorname{domain}(h^*))$, then $\Gamma_f'(h)$ will satisfy the same properties. We may say that h and $\Gamma_f'(h)$ are C^1 with $\|\operatorname{Lip}(h)\| \leq \omega$ and $\|\operatorname{Lip}(\Gamma_f'(h))\| \leq \omega$.

By applying this truncated graph transform n times to a constant function $w: E^u(\varepsilon) \to E^{cs}(\varepsilon)$ we obtain graph g, where $\operatorname{domain}(g) \subset E^u(\varepsilon)$, g is C^1 and $\|\operatorname{Lip}(g)\| \leq \omega$. Clearly

$$\begin{aligned} f^{-n}(\operatorname{graph} g) &= N_w(\varepsilon, n) \\ &= \{v \mid v_{cs} = w, \|F^k v\|_u \leq \varepsilon \text{ and } \|F^k v\|_{cs} \leq \varepsilon \text{ for } k = 0, \ldots, n\}. \end{aligned}$$

Also $B_0(\varepsilon, n) \subset \bigcup_{w \in E^{cs}(\varepsilon)} N_w(\varepsilon, n)$.

Just as in the conclusion of the preceding proof, we may estimate the area of $N_w(\varepsilon, n)$ from above (using the change of variables formula, n applications of $(*)$, and a bound on area(graph g)) and Fubini's Theorem yields the desired estimate of $\operatorname{vol} B_0(\varepsilon, n)$.

Q.E.D.

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