J. Korean Math. Soc. **49** (2012), No. 1, pp. 165–185
http://dx.doi.org/10.4134/JKMS.2012.49.1.165

ENTROPY OF NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. In this paper, the topological entropy and measure-theoretic entropy for nonautonomous dynamical systems are studied. Some properties of these entropies are given and the relation between them is discussed. Moreover, the bounds of them for several particular nonautonomous systems, such as affine transformations on metrizable groups (especially on the torus) and smooth maps on Riemannian manifolds, are obtained.

1. Introduction

In the study of the autonomous (i.e., deterministic) dynamical systems which are induced by the iterations of a single transformation, entropies are important invariants. It is well known that many deep results about entropies of the deterministic dynamical systems have been obtained, see [2], for example. In contrast with the autonomous case, the properties of the entropies for the nonautonomous dynamical systems, which are induced by the compositions of a sequence of transformations, have not been fully investigated.

Now we introduce some basic notations for nonautonomous dynamical systems. Let (X,d) be a metric space and $\{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps on X. The identity map on X will be denoted by Id. Let $\mathbb{N}$ be the set of all positive integers. For any $i\in\mathbb{N}$, let $f_i^0 = Id$ and for any $i,\ n\in\mathbb{N}$, let

$$f_i^n = f_{i+(n-1)}\circ\cdots\circ f_{i+1}\circ f_i,\quad f_i^{-n} = (f_i^n)^{-1} = f_i^{-1}\circ f_{i+1}^{-1}\circ\cdots\circ f_{i+(n-1)}^{-1}$$

(f_i^{-1} will be applied to sets, we don't assume that the maps f_i are invertible). Denote by $f_{1,\infty}$ the sequence $\{f_i\}_{i=1}^{\infty}$ and the induced nonautonomous dynamical system $(X;\{f_i\}_{i=1}^{\infty})$.

For any $n\in\mathbb{N}$, define a new metric d_n on X by

$$d_n(x,y) := \max_{0\le i\le n-1} d(f_1^i(x), f_1^i(y)).$$

Received September 27, 2010.

2010 *Mathematics Subject Classification.* 37A35, 37B40, 37B55.

Key words and phrases. nonautonomous dynamical system, random dynamical system, topological entropy, measure-theoretic entropy.

Research Supported by the National Natural Science Foundation of China(No:11071054), the Key Project of Chinese Ministry of Education(No:211020) and the SRF for ROCS, SEM.

Let K be a compact subset of X. For any $\varepsilon > 0$, a subset $E \subset X$ is said to be an $(f_{1,\infty}, n, \varepsilon)$-spanning set of K, if for any $x \in K$, there exists $y \in E$ such that $d_n(x, y) \le \varepsilon$. Let $r(f_{1,\infty}, n, \varepsilon, K)$ denote the smallest cardinality of any $(f_{1,\infty}, n, \varepsilon)$-spanning set of K. A subset $F \subset K$ is said to be an $(f_{1,\infty}, n, \varepsilon)$-separated set of K, if $x, y \in F, x \neq y$, implies $d_n(x, y) > \varepsilon$. Let $s(f_{1,\infty}, n, \varepsilon, K)$ denote the largest cardinality of any $(f_{1,\infty}, n, \varepsilon)$-separated set of K. It's easy to prove that

$$r(f_{1,\infty}, n, \varepsilon, K) \le s(f_{1,\infty}, n, \varepsilon, K) \le r(f_{1,\infty}, n, \frac{\varepsilon}{2}, K).$$

Therefore, we have the following definition.

Definition 1.1. Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps of a metric space (X, d) and K a compact subset of X. Define

$$h(f_{1,\infty}, K) = \lim_{\varepsilon \to 0} r(f_{1,\infty}, \varepsilon, K) = \lim_{\varepsilon \to 0} s(f_{1,\infty}, \varepsilon, K),$$

where

$$r(f_{1,\infty}, \varepsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log r(f_{1,\infty}, n, \varepsilon, K)$$

and

$$s(f_{1,\infty}, \varepsilon, K) = \limsup_{n \to \infty} \frac{1}{n} \log s(f_{1,\infty}, n, \varepsilon, K).$$

The *topological entropy of* $f_{1,\infty}$ is defined by

$$h(f_{1,\infty}) = \sup\{h(f_{1,\infty}, K) : K \subset X \text{ is compact}\}.$$

We sometimes write $h_d(f_{1,\infty})$ to emphasis the dependence on the metric d.

The above definition of topological entropy for nonautonomous dynamical systems was introduced by Kolyada and Snoha in their paper [5]. As in the autonomous cases, calculating the topological entropy for nonautonomous dynamical systems is not a easy task. However, one can give the estimation of the topological entropy for some special nonautonomous systems. For example, S. Kolyada, M. Misiurewicz and L. Snoha [4] showed that if $f_{1,\infty}$ is a finite piecewise monotone, or a bounded totally long-lapped, or a Markov interval nonautonomous dynamical system, then

$$h(f_{1,\infty}) = \limsup_{n \to \infty} \frac{1}{n} \log c_{1,n},$$

where $c_{1,n}$ is the number of the laps of f_1^n. Zhu, Zhang and He [17] proved that for a sequence of equi-continuous monotone maps on circles,

$$h(f_{1,\infty}) = \limsup_{n \to \infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|,$$

where $\deg f_i$ is the degree of f_i. They also showed in another paper [11] that if $\{f_i\}_{i=1}^{\infty}$ is a family of homeomorphisms on a finite graph X, then $h(f_{1,\infty}) = 0$. In their proof they used another entropy-like invariant "preimage entropy", which is based on the preimage structure of the system. Recently, Zhang and

Chen [11] gave the lower bounds of the topological entropy for nonautonomous dynamical systems via the growths of topological complexity in fundamental group and in degree. In particular, if $\{f_i\}_{i=1}^{\infty}$ is a family of C^1 maps on M and Df_i, $i \in \mathbb{N}$, are equi-continuous, then

$$h(f_{1,\infty}) \geq \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|.$$

In the following, we introduce the measure-theoretic entropy for nonautonomous dynamical systems. Let $(X, \mathcal{B}, m)$ be a probability space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of transformations. If all $f_i, i \in \mathbb{N}$, preserve the same probability measure m, then we call that $f_{1,\infty}$ *preserves* m, or m is an $f_{1,\infty}$-*invariant* measure. Now we can define the measure-theoretic entropy of $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ as follows.

Definition 1.2. Let $(X, \mathcal{B}, m)$ be a probability space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ preserves m. If ξ is a finite partition of X, then

$$h_m(f_{1,\infty}, \xi) = \limsup_{n\to\infty} \frac{1}{n} H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi),$$

where $H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi) = -\sum_{A\in\bigvee_{i=0}^{n-1} f_1^{-i}\xi} m(A)\log m(A)$, is called the *entropy of* $f_{1,\infty}$ *with respect to* ξ.

The *measure-theoretic entropy* of $f_{1,\infty}$ is defined by

$$h_m(f_{1,\infty}) = \sup h_m(f_{1,\infty}, \xi),$$

where the supremum is taken over all finite partitions of X.

At first glance, it seems that the condition of $f_i, i \in \mathbb{N}$, preserving the same measure m is in a sense strong. However, when each $f_i, i \in \mathbb{N}$, is taken from a set of conservative systems, especially from a set of volume preserving systems, the induced nonautonomous system satisfies this condition naturally. Furthermore, we will see in Section 2 that many known results about entropy for autonomous systems may not hold anymore for nonautonomous systems even under such strong conditions. We also notice that some other aspects about the measure of the nonautonomous systems were studied. For example, W. Ott, M. Stenlund and L.-S. Young recently discussed the evolution of probability distributions for certain nonautonomous systems in [9]. Exponential loss of memory was proved in their paper for expanding maps and for one-dimensional piecewise expanding maps with slowly varying parameters.

There are some reasons why we are interested in nonautonomous systems in particular in their entropies. For example, when take a computational experiment or study random dynamical systems, we often work with a sequence of maps in place of a single map. We also note that the notion of sequence entropy (with respect to an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers) of an autonomous dynamical system $(X; f)$ is nothing else than the entropy of the nonautonomous dynamical system $(X; f^{n_1}, f^{n_2-n_1}, f^{n_3-n_2}, \ldots)$.

Our goal is to study the properties of the entropies of the nonautonomous dynamical systems. This paper is organized in the following way. In Section 2, we obtain some properties of the topological entropy. And the bounds of the topological entropy for smooth maps on Riemannian manifolds and for nonautonomous endomorphisms on torus are given. In Section 3 we give some properties of the measure-theoretic entropy. The relation between the measure-theoretic entropy and the topological entropy, especially for affine transformations on metrizable groups, are obtained. Moreover, we also try to explain the reasons why we may not obtain the analogues of some known results for deterministic systems in the case of nonautonomous systems.

2. Topological entropy

Through out this section (X, d) is a metric space, not necessarily compact, and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ is a sequence of continuous maps on X. It is obvious that the topological entropy is independent of the metric.

Proposition 2.1. *If d' is another metric on X which is uniformly equivalent to d and $f_{1,\infty}$ is a sequence of (equi-continuous) maps on X, then $h_d(f_{1,\infty}) = h_{d'}(f_{1,\infty})$.*

The following proposition will be useful to calculate the topological entropy of the nonautonomous systems on non-compact metric spaces.

Proposition 2.2. *Let $\delta > 0$. Then*

$$h_d(f_{1,\infty}) = \sup\{h(f_{1,\infty}, K) : K \subset X \text{ and } \operatorname{diam}(K) < \delta\}. \tag{2.1}$$

Proof. It is suffice to prove that for any compact subsets $K \subset \bigcup_{k=1}^{m} K_i$ of X we have

$$h_d(f_{1,\infty}, K) \le \max_{1\le i\le m} h_d(f_{1,\infty}, K_i). \tag{2.2}$$

Indeed, once this is done, any compact subset K of X can be covered by a finite number of balls $B_1, \ldots, B_m$ of diameter $\frac{\delta}{2}$ and hence

$$h_d(f_{1,\infty}, K) \le \max_{1\le i\le m} h_d(f_{1,\infty}, K \cap \overline{B_i}),$$

which gives (2.1).

Now we prove (2.2). It is clear that

$$s(f_{1,\infty}, n, \varepsilon, K) \le \sum_{i=1}^{m} s(f_{1,\infty}, n, \varepsilon, K_i).$$

Fix $\varepsilon > 0$. For each n choose $K_{i(n,\varepsilon)}$ such that

$$s(f_{1,\infty}, n, \varepsilon, K_{i(n,\varepsilon)}) = \max_{1\le i\le m} s(f_{1,\infty}, n, \varepsilon, K_i).$$

Then

$$s(f_{1,\infty}, n, \varepsilon, K) \le m \cdot s(f_{1,\infty}, n, \varepsilon, K_{i(n,\varepsilon)})$$

and so

$$\log s(f_{1,\infty}, n, \varepsilon, K) \leq \log m + \log s(f_{1,\infty}, n, \varepsilon, K_{i(n,\varepsilon)}).$$

Choose $n_j \to \infty$ such that

$$\lim_{j\to\infty} \frac{1}{n_j} \log s(f_{1,\infty}, n_j, \varepsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log s(f_{1,\infty}, n, \varepsilon, K)$$

and $K_{i(n_j,\varepsilon)}$ does not depend on j (i.e., $K_{i(n_j,\varepsilon)} = K_{i(\varepsilon)}$ for any j). Therefore $s(f_{1,\infty}, \varepsilon, K) \leq s(f_{1,\infty}, \varepsilon, K_{i(\varepsilon)})$. Choose $\varepsilon_t \to 0$ such that $K_{i(\varepsilon_t)}$ is constant. Then

$$h_d(f_{1,\infty}, K) \leq h_d(f_{1,\infty}, K_i) \leq \max_{1\leq i\leq m} h_d(f_{1,\infty}, K_i),$$

i.e., (2.2) holds. □

Theorem 2.3. *Let M be a k-dimensional Riemannian manifold and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of C^1 maps on M which satisfy*

$$\sup_{i\in\mathbb{N},\, x\in M} \|d_x f_i\| < \infty.$$

Then

$$h(f_{1,\infty}) \leq \max\{0, \limsup_{n\to\infty} \frac{k}{n} \sum_{i=1}^{n-1} \log \sup_{x\in M} \|d_x f_i\|\}.$$

Proof. For $i \in \mathbb{N}$, let $a_i = \sup_{x\in M} \|d_x f_i\|$. By the mean-value theorem,

$$d(f_i(x), f_i(y)) \leq a_i d(x, y), \quad x, y \in M.$$

Suppose K is a compact subset of M of diameter less than some positive number δ. Assume δ is small enough such that we can select one convenient chart on M that covers K. Let $|||\cdot|||$ denote the norm on $\mathbb{R}^k$ given by

$$|||u||| = \max_{1\leq i\leq k} |u_i|$$

for $u = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$ and let $B(0, r)$ denote the open ball in $\mathbb{R}^k$ with center 0 and radius r in this norm. Choose a differentiable map $\varphi : B(0, 3) \to M$ such that $K \subset \varphi(B(0, 1))$. Let $b > 0$ be so that

$$d(\varphi(u), \varphi(v)) \leq b|||u - v|||, \quad \forall u, v \in B(0, 2).$$

For any $c \in (0, 1)$, let

$$E(c) = \{(t_1 c, \ldots t_k c) \in \mathbb{R}^k \mid t_i \in \mathbb{Z}\} \cap B(0, 2).$$

The cardinality of $E(c)$ is at most $(\frac{4}{c})^k$. Each point of $B(0, 2)$ is within distance c of a point of $E(c)$. The set $\varphi(E(c))$ is clearly an $(f_{1,\infty}, n, (\prod_{i=1}^{n-1} a_i)bc)$-spanning set of K. Given $\varepsilon > 0$ and put $c = \frac{\varepsilon}{(\prod_{i=1}^{n-1} a_i)b}$, then

$$r(f_{1,\infty}, n, \varepsilon, K) \leq \Big(\frac{4\prod_{i=1}^{n-1} a_i b}{\varepsilon}\Big)^k = \big(\prod_{i=1}^{n-1} a_i\big)^k \cdot \big(\frac{4b}{\varepsilon}\big)^k.$$

Therefore

$$r(f_{1,\infty},\varepsilon,K)\le \max\{0,\limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log a_i\}.$$

Letting ε tend to 0 we have

$$h(f_{1,\infty},K)\le \max\{0,\limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log a_i\}.$$

By Proposition 2.2,

$$h(f_{1,\infty})\le \max\{0,\limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log a_i\}.$$

□

In the following, we will give the lower and the upper bounds of the entropy of the nonautonomous expanding endomorphisms on $\mathbb{R}^n$ and $\mathbb{T}^n$.

Proposition 2.4. *Let $(\tilde{X},\tilde{d})$ and (X,d) be metric spaces and $\pi:\tilde{X}\to X$ a continuous surjection such that there exists $\delta>0$ with*

$$\pi|_{B(\tilde{x},\delta)}:B(\tilde{x},\delta)\to B(\pi(\tilde{x}),\delta) \tag{2.3}$$

an isometric surjection for all $\tilde{x}\in\tilde{X}$. If $\tilde{f}_{1,\infty}=\{\tilde{f}_i\}_{i=1}^{\infty}$ and $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ are two sequences of equi-continuous maps on $\tilde{X}$ and X, respectively, and satisfy $\pi\tilde{f}_i=f_i\pi$ for any $i\in\mathbb{N}$, then

$$h(\tilde{f}_{1,\infty})=h(f_{1,\infty}).$$

Proof. If $\tilde{K}$ is compact in $\tilde{X}$ and $\mathrm{diam}(\tilde{K})<\delta$, then $\pi(\tilde{K})$ is compact in X and $\mathrm{diam}(\pi(\tilde{K}))<\delta$. Every compact subset of X of diameter less than δ is of this form, that is, if K is compact in X and $\mathrm{diam}(K)<\delta$, then there exists at least one compact set $\tilde{K}$ in $\tilde{X}$ such that $\mathrm{diam}(\tilde{K})<\delta$ and $\pi(\tilde{K})=K$.

Since $\tilde{f}_i$, $i\in\mathbb{N}$, are equi-continuous, we can take $\varepsilon\in(0,\delta)$ such that

$$\tilde{d}(\tilde{x},\tilde{y})<\varepsilon\Rightarrow\tilde{d}(\tilde{f}(\tilde{x}),\tilde{f}(\tilde{y}))<\delta,\ \ \tilde{x},\tilde{y}\in\tilde{X},i\in\mathbb{N}. \tag{2.4}$$

Let $\tilde{K}$ be compact in $\tilde{X}$ with $\mathrm{diam}(\tilde{K})<\delta$ and $\tilde{E}\subset\tilde{K}$ an $(\tilde{f}_{1,\infty},n,\varepsilon)$-separated set. By (2.3) and (2.4), it is obviously that $\pi(\tilde{E})$ is an $(f_{1,\infty},n,\varepsilon)$-separated subset of $\pi(\tilde{K})$. Therefore

$$s(\tilde{f}_{1,\infty},n,\varepsilon,\tilde{K})\le s(f_{1,\infty},n,\varepsilon,\pi(\tilde{K})). \tag{2.5}$$

To prove the converse inequality, suppose E is an $(f_{1,\infty},n,\varepsilon)$-separated subset of $\pi(\tilde{K})$. By (2.3) and (2.4), $\tilde{E}=\pi^{-1}(E)\cap\tilde{K}$ is an $(\tilde{f}_{1,\infty},n,\varepsilon)$-separated subset of $\tilde{K}$. Hence

$$s(f_{1,\infty},n,\varepsilon,\pi(\tilde{K}))\le s(\tilde{f}_{1,\infty},n,\varepsilon,\tilde{K}). \tag{2.6}$$

From (2.5) and (2.6),

$$s(f_{1,\infty},n,\varepsilon,\tilde{K})=s(f_{1,\infty},n,\varepsilon,\pi(\tilde{K})),$$

and hence

$$h(\tilde{f}_{1,\infty}, \tilde{K}) = h(f_{1,\infty}, \pi(\tilde{K})).$$

By Proposition 2.2,

$$h(\tilde{f}_{1,\infty}) = h(f_{1,\infty}). \qquad \square$$

Corollary 2.5. *Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous maps of $\mathbb{T}^k$. For $i \in \mathbb{N}, \tilde{f}_i$ is a lift of f_i. Then*

$$h(f_{1,\infty}) = h(\tilde{f}_{1,\infty}),$$

where $\tilde{f}_{1,\infty} = \{\tilde{f}_i\}_{i=1}^{\infty}$.

Lemma 2.6. *Let $A_{1,\infty} = \{A_i\}_{i=1}^{\infty}$ be a sequence of linear operators on $\mathbb{R}^k$, m the Lebesgue measure on $\mathbb{R}^k$ and ρ a metric on $\mathbb{R}^k$ determined by a norm. Then*

$$h_\rho(A_{1,\infty}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(0, \varepsilon, A_{1,\infty})),$$

where $D_n(0, \varepsilon, A_{1,\infty}) = \bigcap_{i=0}^{n-1} A_1^{-i} B_\rho(0, \varepsilon)$. Also, $h_\rho(A_{1,\infty})$ does not depend on the norm chosen.

Proof. Since all norms on $\mathbb{R}^k$ are equivalent, they induce uniformly equivalent metrics on $\mathbb{R}^k$, so by Proposition 2.1,

$$h_\rho(A_{1,\infty}) = h_d(A_{1,\infty}),$$

where d is the Euclidean distance. Hence we may as well suppose ρ is the Euclidean distance.

First note the fact that for any $\varepsilon > 0$ and $x \in \mathbb{R}^k$,

$$D_n(x, \varepsilon, A_{1,\infty}) = x + D_n(0, \varepsilon, A_{1,\infty}),$$

where $D_n(x, \varepsilon, A_{1,\infty}) = \bigcap_{i=0}^{n-1} A_1^{-i} B_\rho(A_1^i(x), \varepsilon)$.

Let K be a compact subset of $\mathbb{R}^k$ with $m(K) > 0$. If F is an $(A_{1,\infty}, n, \varepsilon)$-spanning set of K, then

$$K \subset \bigcup_{x\in F} D_n(x, 2\varepsilon, A_{1,\infty}) = \bigcup_{x\in F} x + D_n(0, 2\varepsilon, A_{1,\infty}).$$

Therefore $m(K) \le r(A_{1,\infty}, n, \varepsilon, K) \cdot m(D_n(0, 2\varepsilon, A_{1,\infty}))$. This gives

$$r(A_{1,\infty}, n, \varepsilon, K) \ge \frac{m(K)}{m(D_n(0, 2\varepsilon, A_{1,\infty}))}$$

and hence

$$h_\rho(A_{1,\infty}) \ge r(A_{1,\infty}, \varepsilon, K) \ge \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(0, 2\varepsilon, A_{1,\infty})).$$

Therefore

$$h_\rho(A_{1,\infty}) \ge \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(0, \varepsilon, A_{1,\infty})). \tag{2.7}$$

Let K_q be the closed q-cube with center $0 \in \mathbb{R}^k$ and side length $2q$. If E is an $(A_{1,\infty}, n, \varepsilon)$-separated subset of K_q, then $\bigcup_{x\in E} D_n(x, \frac{\varepsilon}{2}, A_{1,\infty})$ is a disjoint union and

$$\bigcup_{x\in E} D_n(x, \frac{\varepsilon}{2}, A_{1,\infty}) = \bigcup_{x\in E}(x + D_n(0, \frac{\varepsilon}{2}, A_{1,\infty})) \subset K_{q+\varepsilon}.$$

Therefore

$$s(A_{1,\infty}, n, \varepsilon, K_q)\cdot m(D_n(0, \frac{\varepsilon}{2}, A_{1,\infty})) \leq 2^k(q+\varepsilon)^k$$

and so

$$s(A_{1,\infty}, \varepsilon, K_q) \leq \limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(0, \frac{\varepsilon}{2}, A_{1,\infty})).$$

If K is any compact subset of $\mathbb{R}^n$, then $K \subset K_q$ for some q so

$$s(A_{1,\infty}, \varepsilon, K) \leq s(A_{1,\infty}, \varepsilon, K_q) \leq \limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(0, \frac{\varepsilon}{2}, A_{1,\infty})).$$

Therefore

$$h_\rho(A_{1,\infty}) = \sup_K \lim_{\varepsilon\to 0} s(A_{1,\infty}, \varepsilon, K) \leq \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(0, \varepsilon, A_{1,\infty})). \tag{2.8}$$

From (2.7) and (2.8), the desired equality

$$h_\rho(A_{1,\infty}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(0, \varepsilon, A_{1,\infty}))$$

holds. □

Now we can give the main result of this section.

Theorem 2.7. *Let $A_{1,\infty} = \{A_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous linear automorphisms of $\mathbb{R}^k$. If for each $i \in \mathbb{N}$, all eigenvalues of A_i are of modulus greater than or equal to 1, then*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{k}\log|\lambda_i^{(j)}| \leq h(f_{1,\infty}) \leq \limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log\Lambda_i^{(1)}, \tag{2.9}$$

where $\lambda_i^{(1)}, \ldots, \lambda_i^{(k)}$ are the eigenvalues of $A_i, i \in \mathbb{N}$, counted with their multiplicities, and $\Lambda_i^{(1)}$ is the biggest eigenvalue of $\sqrt{A_i A_i^T}, i \in \mathbb{N}$.

In particular, in the case $k = 1$, we have

$$h(A_{1,\infty}) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\log|\lambda_i|,$$

where λ_i is the proportionality constant of $A_i : \mathbb{R} \to \mathbb{R}, x \mapsto \lambda_i x, i \in \mathbb{N}$.

Proof. Let d be the Euclidean distance on $\mathbb{R}^k$. By Lemma 2.6, we have

$$h_d(A_{1,\infty}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(0,\varepsilon, A_{1,\infty})).$$

Note that for any linear operator $A : \mathbb{R}^k \to \mathbb{R}^k$ and any Borel set $B \subset \mathbb{R}^k$, we have $m(A(B)) = |\det A| m(B)$. Since

$$D_n(0,\varepsilon, A_{1,\infty}) = \bigcap_{i=1}^{n-1} A_1^{-i}(B_d(0,\varepsilon)) \subset A_1^{-(n-1)}(B_d(0,\varepsilon))$$

then

$$\begin{aligned} m(D_n(0,\varepsilon, A_{1,\infty})) &\le m(A_1^{-(n-1)}(B_d(0,\varepsilon))) \\ &\le |\det A_1^{-(n-1)}| \cdot m(B_d(0,\varepsilon)) \\ &= \frac{1}{\prod_{i=1}^{n-1} |\det A_i|} \cdot m(B_d(0,\varepsilon)) \\ &= \frac{1}{\prod_{i=1}^{n-1} \prod_{j=1}^{k} |\lambda_i^{(j)}|} \cdot m(B_d(0,\varepsilon)). \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{n} \log m(D_n(0,\varepsilon, A_{1,\infty})) &\ge \frac{1}{n} \log \prod_{i=1}^{n-1} \prod_{j=1}^{k} |\lambda_i^{(j)}| - \frac{1}{n} \log m(B_d(0,\varepsilon)) \\ &= \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=1}^{k} \log |\lambda_i^{(j)}| - \frac{1}{n} \log m(B_d(0,\varepsilon)). \end{aligned}$$

Hence

$$h(A_{1,\infty}) \ge \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=1}^{k} \log |\lambda_i^{(j)}|. \tag{2.10}$$

Note that for any linear operator $A : \mathbb{R}^k \to \mathbb{R}^k$ we have $\|A\| = \Lambda^{(1)}$, where $\Lambda^{(1)}$ is the biggest eigenvalue of $\sqrt{AA^T}$. So for any $x \in \mathbb{R}^k$, $\|Ax\| \le \Lambda^{(1)} \|x\|$ and hence

$$A^{-1}(B_d(0,\varepsilon)) \supset B_d\big(0, \frac{1}{\Lambda^{(1)}}\varepsilon\big).$$

Therefore,

$$D_n(0,\varepsilon, A_{1,\infty}) \supset B_d\Big(0, \frac{1}{\prod_{i=1}^{n-1} \Lambda_i^{(1)}}\varepsilon\Big),$$

and so

$$-\frac{1}{n} \log m(D_n(0,\varepsilon, A_{1,\infty})) \le \frac{1}{n} \log \prod_{i=1}^{n-1} (\Lambda_i^{(1)})^k + \frac{1}{n} \log m(B_d(0,\varepsilon))$$

$$= \frac{k}{n}\sum_{i=1}^{n-1}\log\Lambda_i^{(1)} + \frac{1}{n}\log m(B_d(0,\varepsilon)).$$

Hence

$$h(A_{1,\infty}) \le \lim_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log\Lambda_i^{(1)}. \tag{2.11}$$

Form (2.10) and (2.11), the desired inequalities (2.9) hold. □

From Corollary 2.5 and Theorem 2.7, we can immediately deduce the result we set out to prove.

Theorem 2.8. *Let $A_{1,\infty} = \{A_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous surjective endomorphisms of $\mathbb{T}^k$ and $\tilde{A}_{1,\infty} = \{\tilde{A}_i\}_{i=1}^{\infty}$ the corresponding linear automorphisms of $\mathbb{R}^k$. If for each $i \in \mathbb{N}$, all eigenvalues of $\tilde{A}_i$ are of modulus greater than or equal to 1, then*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{k}\log|\lambda_i^{(j)}| \le h(A_{1,\infty}) \le \limsup_{n\to\infty}\frac{k}{n}\sum_{i=1}^{n-1}\log\Lambda_i^{(1)}, \tag{2.12}$$

where $\lambda_i^{(1)},\dots,\lambda_i^{(k)}$ are the eigenvalues of $\tilde{A}_i, i \in \mathbb{N}$, counted with their multiplicities, and $\Lambda_i^{(1)}$ is the biggest eigenvalue of $\sqrt{\tilde{A}_i\tilde{A}_i^T}, i \in \mathbb{N}$.

In particular, in the case $k = 1$, we have

$$h(A_{1,\infty}) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}\log|\lambda_i|,$$

where λ_i is the degree of the automorphism A_i of S^1, $i \in \mathbb{N}$.

It is well known that for the autonomous linear system, i.e., the system generated by the iteration of a single linear map A of $\mathbb{R}^k$, we have

$$h(A) = \sum_{|\lambda^{(j)}|>1}\log|\lambda^{(j)}|, \tag{2.13}$$

where $\lambda^{(1)},\dots,\lambda^{(k)}$ are the eigenvalues of A, counted with their multiplicities. The proof of the formula (2.13) relies on the invariance of the Jordan decomposition of A.

Let M be a closed Riemannian manifold and f a differentiable map on M. For any $x \in M$, the sequence of differentials of f along the orbit of x naturally generates a nonautonomous linear system. By the variational principle and Ruelle's inequality, we have

$$h(f) \le \sup\left\{\int_M \sum_i \lambda_i^+(x)m_i(x)d\mu(x) \mid \mu \text{ is an invariant measure of } f\right\},$$

where $\lambda_i^+(x)$ are the positive Lyapunov exponents and $m_i(x)$ are their multiplicities.

However, with respect to the nonautonomous linear system in the setting of this paper, we can only obtain the lower bound and the upper bound (which seems a little bit coarse) of the topological entropy for the expanding case. The main reason that we can't get the estimation of the topological entropy for the general nonautonomous linear systems without the "expanding" assumption is that there is neither invariant decomposition nor the theory of Lyapunov exponents available any more. Therefore, it is not a easy task to calculate the entropy of nonautonomous systems even under strong conditions.

Example 2.9. Let A and B be the hyperbolic automorphisms on $\mathbb{T}^2$ induced by the matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

respectively. For the nonautonomous system $A_{1,\infty} = \{A_i\}_{i=1}^{\infty}$, where $A_i = A$ or B, its entropy relays on the frequency p_A of A appearing in this sequence. In fact, it is easy to prove that

$$h(A_{1,\infty}) = (1 - 2\min\{p_A, 1 - p_A\}) \log \frac{3+\sqrt{5}}{2}.$$

3. Measure-theoretic entropy

We first give some basic properties of the measure-theoretic entropy of the nonautonomous dynamical systems.

Proposition 3.1. *Let $(X, \mathcal{B}, m)$ be a probability space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ preserves m. If ξ, η are finite partitions of X, then*

(1) $h_m(f_{1,\infty}, \xi) \le H_m(\xi)$.

(2) $h_m(f_{1,\infty}, \xi \vee \eta) \le h_m(f_{1,\infty}, \xi) + h_m(f_{1,\infty}, \eta)$.

(3) $\xi \le \eta \Rightarrow h_m(f_{1,\infty}, \xi) \le h_m(f_{1,\infty}, \eta)$.

(4) $h_m(f_{1,\infty}, \xi) \le h_m(f_{1,\infty}, \eta) + H_m(\xi|\eta)$.

Proof. We can get the desired results easily from the following facts respectively.

(1)

$$\frac{1}{n} H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi) \le \frac{1}{n}\sum_{i=0}^{n-1} H_m(f_1^{-i}\xi) = \frac{1}{n}\sum_{i=0}^{n-1} H_m(\xi) = H_m(\xi).$$

(2)

$$\begin{aligned} H_m(\bigvee_{i=0}^{n-1} f_1^{-i}(\xi \vee \eta)) &= H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi \vee \bigvee_{i=0}^{n-1} f_1^{-i}\eta) \\ &\le H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi) + H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\eta). \end{aligned}$$

(3) If $\xi \le \eta$, then for any $n \ge 1$ we have

$$\bigvee_{i=0}^{n-1} f_1^{-i}\xi \le \bigvee_{i=0}^{n-1} f_1^{-i}\eta.$$

(4)

$$\begin{aligned} H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi) &\le H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi \vee \bigvee_{i=0}^{n-1} f_1^{-i}\eta) \\ &= H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) + H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\xi | \bigvee_{i=0}^{n-1} f_1^{-i}\eta) \\ &\le H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) + \sum_{i=0}^{n-1} H_m(f_1^{-i}\xi | f_1^{-i}\eta) \\ &= H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) + nH_m(\xi|\eta). \end{aligned}$$

□

Proposition 3.2. *Let $(X, \mathcal{B}, m)$ be a probability space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ preserves m. If ξ is a finite partition of X, then for any $i, n \in \mathbb{N}$, we have*

$$H_m(\bigvee_{j=0}^{n-1} f_i^{-j}\xi) = H_m(\xi) + \sum_{j=1}^{n-1} H_m(\xi | \bigvee_{k=1}^{n-j} f_{i+j-1}^{-k}\xi). \tag{3.1}$$

Proof. We show (3.1) by induction. For any $i \in \mathbb{N}$ and $n = 1$, it is clear. Assume that it is true for any $i \in \mathbb{N}$ and $n = p$, then

$$\begin{aligned} H_m(\bigvee_{j=0}^{p} f_i^{-j}\xi) &= H_m(\bigvee_{j=1}^{p} f_i^{-j}\xi \vee \xi) \\ &= H_m(\bigvee_{j=1}^{p} f_i^{-j}\xi) + H_m(\xi | \bigvee_{j=1}^{p} f_i^{-j}\xi) \\ &= H_m(f_i^{-1} \bigvee_{j=0}^{p-1} f_{i+1}^{-j}\xi) + H_m(\xi | \bigvee_{j=1}^{p} f_i^{-j}\xi) \\ &= H_m(\bigvee_{j=0}^{p-1} f_{i+1}^{-j}\xi) + H_m(\xi | \bigvee_{j=1}^{p} f_i^{-j}\xi). \end{aligned} \tag{3.2}$$

For the first part of the right-hand side of the equation (3.2), we have by the induction assumption,

$$H_m(\bigvee_{j=0}^{p-1} f_{i+1}^{-j}\xi) = H_m(\xi) + \sum_{j=1}^{p-1} H_m(\xi | \bigvee_{k=1}^{p-j} f_{(i+1)+j-1}^{-k}\xi)$$

$$= \quad H_m(\xi) + \sum_{j=2}^{p} H_m(\xi | \bigvee_{k=1}^{(p+1)-j} f_{i+j-1}^{-k}\xi).$$

For the second part of the right-hand side of the equation (3.2), we have

$$H_m(\xi | \bigvee_{j=1}^{p} f_i^{-j}\xi) = H_m(\xi | \bigvee_{k=1}^{(p+1)-1} f_{(i+1)-1}^{-k}\xi).$$

Therefore,

$$H_m(\bigvee_{j=0}^{p} f_i^{-j}\xi) = H_m(\xi) + \sum_{j=1}^{p} H_m(\xi | \bigvee_{k=1}^{p+1-j} f_{i+j-1}^{-k}\xi).$$

Hence (3.1) holds for all $i, n \in \mathbb{N}$. □

Remark 3.3. Take $i = 1$ in the formula (3.1), we get

$$H_m(\bigvee_{j=0}^{n-1} f_1^{-j}\xi) = H_m(\xi) + \sum_{j=1}^{n-1} H_m(\xi | \bigvee_{k=1}^{n-j} f_j^{-k}\xi).$$

Moreover, if we take $f_i = f$ for any $i \in \mathbb{N}$, then we have

$$H_m(\bigvee_{j=0}^{n-1} f^{-j}\xi) = H_m(\xi) + \sum_{j=1}^{n-1} H_m(\xi | \bigvee_{k=1}^{j} f^{-k}\xi),$$

which is exactly the formula in the proof of Theorem 4.14 in [10] for the autonomous dynamical system.

Similar to the autonomous case, we can show that the entropy map of the nonautonomous systems is affine.

Theorem 3.4. *Let $(X, \mathcal{B})$ be a measurable space and $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ a sequence of measurable transformations of X. Then for any $f_{1,\infty}$-invariant probability measure m, μ and $p \in [0, 1]$, we have*

$$h_{p\mu+(1-p)m}(f_{1,\infty}) = p h_\mu(f_{1,\infty}) + (1-p) h_m(f_{1,\infty}).$$

Proof. As in the proof of Theorem 8.1 of [10], we have

$$0 \le H_{p\mu+(1-p)m}(\xi) - pH_\mu(\xi) - (1-p)H_m(\xi) \le \log 2 \tag{3.3}$$

for any finite partition ξ of X.

If η is any finite partition of X, then by putting $\xi = \bigvee_{i=0}^{n-1} f_1^{-i}(\eta)$ in (3.3), we have

$$h_{p\mu+(1-p)m}(f_{1,\infty}, \eta) = p h_\mu(f_{1,\infty}, \eta) + (1-p) h_m(f_{1,\infty}, \eta). \tag{3.4}$$

Clearly

$$h_{p\mu+(1-p)m}(f_{1,\infty}) \le p h_\mu(f_{1,\infty}) + (1-p) h_m(f_{1,\infty}). \tag{3.5}$$

We now show the opposite inequality. Let $\varepsilon > 0$, choose $\eta_1, \eta_2 > 0$ such that

$$h_\mu(f_{1,\infty}, \eta_1) > \begin{cases} h_\mu(f_{1,\infty}) - \varepsilon & \text{if } h_\mu(f_{1,\infty}) < \infty \\ \dfrac{1}{\varepsilon} & \text{if } h_\mu(f_{1,\infty}) = \infty. \end{cases}$$

$$h_m(f_{1,\infty}, \eta_2) > \begin{cases} h_m(f_{1,\infty}) - \varepsilon & \text{if } h_m(f_{1,\infty}) < \infty \\ \dfrac{1}{\varepsilon} & \text{if } h_m(f_{1,\infty}) = \infty. \end{cases}$$

Putting $\eta = \eta_1 \vee \eta_2$ in (3.4) gives

$$h_{p\mu+(1-p)m}(f_{1,\infty}, \eta)$$

$$> \begin{cases} ph_\mu(f_{1,\infty}) + (1-p)h_m(f_{1,\infty}) - \varepsilon, & \text{if } h_\mu(f_{1,\infty}), h_m(f_{1,\infty}) < \infty \\ \dfrac{1}{\varepsilon}, \quad \text{if either } h_m(f_{1,\infty}) = \infty & \text{or } h_\mu(f_{1,\infty}) = \infty. \end{cases}$$

Therefore

$$h_{p\mu+(1-p)m}(f_{1,\infty}) \geq ph_\mu(f_{1,\infty}) + (1-p)h_m(f_{1,\infty}). \tag{3.6}$$

From (3.5) and (3.6), the desired equality holds. □

It is well known that there is a power rule for the measure-theoretic entropy of the autonomous system, that is, for any transformation f which preserves m we have

$$h_m(f^k) = kh_m(f),$$

where $k \in \mathbb{N}$. And we can apply the power rule to obtain the variational principle which relates the topological and measure-theoretic entropies. However, we do not know whether the power rule holds or not for the measure-theoretic entropy of the nonautonomous system. So far, we can only get the following inequality and have no idea to prove the reverse inequality.

Proposition 3.5. *Let $(X, \mathcal{B}, m)$ be a probability space and $f_{1,\infty} = \{f_i\}_{i=1}^\infty$ preserves m. Then for any $k \in \mathbb{N}$, we have*

$$h_m(f_{1,\infty}^{[k]}) \leq kh_m(f_{1,\infty}),$$

where $f_{1,\infty}^{[k]}$ is the nonautonomous system induced by the sequence of maps $\{f_i^{[k]} = f_{(i-1)k+1}^k\}_{i=1}^\infty$.

Proof. Note that for any finite partition ξ of X, we have

$$\begin{aligned} h_m(f_{1,\infty}^{[k]}, \xi) &= \limsup_{n\to\infty} \frac{1}{n} \log H_m(\xi \vee \bigvee_{i=0}^{n-1} f_1^{-k}(f_{k+1}^{-k}(\cdots(f_{ik+1}^{-k}\xi)\cdots)) \\ &\leq k \cdot \limsup_{n\to\infty} \frac{1}{nk} \log H_m(\bigvee_{i=0}^{nk-1} f_1^{-i}\xi) \\ &\leq k \cdot h_m(f_{1,\infty}, \xi) \\ &\leq k \cdot h_m(f_{1,\infty}). \end{aligned}$$

Therefore,

$$h_m(f_{1,\infty}^{[k]}) = \sup_{\xi} h_m(f_{1,\infty}^{[k]}, \xi) \le k \cdot h_m(f_{1,\infty}). \qquad \square$$

Now we consider the relation between the topological entropy and the measure-theoretic entropies of the nonautonomous systems.

Theorem 3.6. *Let X be a compact metric space and $f_{1,\infty}$ a sequence of continuous maps of X. Then for any $f_{1,\infty}$-invariant Borel probability measure m, we have*

$$h_m(f_{1,\infty}) \le h(f_{1,\infty}) + \log 2.$$

Proof. Let $\xi = \{A_1, \ldots, A_k\}$ be a finite partition of M. For any $a > 0$, choose $\varepsilon > 0$ so that $\varepsilon < \frac{a}{k \log k}$. Since m is regular, there exist compact sets $B_j \subset A_j, 1 \le j \le k$ with $m(A_j \backslash B_j) < \varepsilon$. Let η be the partition $\eta = \{B_0, B_1, \ldots, B_k\}$ where $B_0 = X - \bigcup_{j=1}^k B_j$. We have $m(B_0) < k\varepsilon$ and

$$\begin{aligned} H_m(\xi|\eta) &= -\sum_{i=0}^{k}\sum_{j=1}^{k} m(B_i)\frac{m(B_i \cap A_j)}{m(B_i)} \log \frac{m(B_i \cap A_j)}{m(B_i)} \\ &= -m(B_0)\sum_{j=1}^{k} \frac{m(B_0 \cap A_j)}{m(B_0)} \log \frac{m(B_0 \cap A_j)}{m(B_0)} \\ &\le m(B_0) \log k \\ &< k\varepsilon \log k < a. \end{aligned}$$

Let $\delta = \min_{1 \le i,j \le k,;i \neq j} d(B_i, B_j)$. Choose a maximal $(f_{1,\infty}, n, \frac{\delta}{2})$-spanning set E_n of X with the cardinality $r(f_{1,\infty}, n, \frac{\delta}{2})$. It is obvious that

$$X = \bigcup_{x \in E_n} \bigcap_{i=0}^{n-1} f_1^{-i} \bar{B}_d(f_1^i x, \frac{\delta}{2}).$$

Since each $\frac{\delta}{2}$-ball in X intersects at most two elements of η, then we have

$$\operatorname{card}(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) \le r(f_{1,\infty}, n, \frac{\delta}{2}) \cdot 2^n.$$

Hence

$$\begin{aligned} H_m(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) &\le \log \operatorname{card}(\bigvee_{i=0}^{n-1} f_1^{-i}\eta) \\ &\le \log r(f_{1,\infty}, n, \frac{\delta}{2}) + n \log 2. \end{aligned}$$

Therefore

$$h_m(f_{1,\infty}, \eta) \le h(f_{1,\infty}) + \log 2.$$

So by (4) of Proposition 3.1,

$$h_m(f_{1,\infty}, \xi) \le h_m(f_{1,\infty}, \eta) + H_m(\xi|\eta)$$

$$\leq \quad h(f_{1,\infty}) + \log 2 + a.$$

This gives

$$h_m(f_{1,\infty}) \leq h(f_{1,\infty}) + \log 2 + a.$$

Since a is chosen arbitrarily, we get the desired inequality

$$h_m(f_{1,\infty}) \leq h(f_{1,\infty}) + \log 2$$

immediately. □

We ever tried to borrow some idea from [8] to prove the following inequality

$$h_m(f_{1,\infty}) \leq h(f_{1,\infty}). \tag{3.7}$$

However, it seems that the definition of entropies for the $\mathbb{Z}_+^n$ action in this paper are not reasonable and the proof of the first part of the corresponding variational principle is not acceptable.

Question 3.7. Can we get a "nonautonomous version" of the variational principle or the inequality (3.7)?

Now we study the Harr measure entropy and the topological entropy for the nonautonomous affine transformations.

Theorem 3.8. *Let G be a compact metrizable group and $\{f_i = a_i \cdot A_i\}$ a sequence of affine transformations of G. Let m denote normalized Haar measure on G and d a left-invariant metric on G. Then $h_m(f_{1,\infty}) = h_m(A_{1,\infty})$ and $h(f_{1,\infty}) = h(A_{1,\infty})$, and*

$$h_m(A_{1,\infty}) \geq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(\bigcap_{i=0}^{n-1} A_1^{-i} B(e,\varepsilon)) = h(A_{1,\infty}), \tag{3.8}$$

where e denotes the identity element of G and $B(e,\varepsilon)$ is the open ball center e and radius ε with respect to the metric d (This limit clearly exists or is ∞).

Proof. Suppose d is a left-invariant metric on G. Put

$$D_n(x,\varepsilon,f_{1,\infty}) = \bigcap_{k=0}^{n-1} f_1^{-i} B(f_1^i x, \varepsilon).$$

By induction we shall show that

$$f_1^{-k}(B(f_1^k x, \varepsilon)) = x \cdot A_1^{-k} B(e,\varepsilon).$$

It is true for $k=0$ by the invariance of the metric d. Assuming it holds for k, we prove it for $k+1$:

$$\begin{aligned} f_1^{-(k+1)} B(f_1^{k+1} x, \varepsilon) &= f_1^{-1}(f_2^{-k} B(f_2^k(f_1 x), \varepsilon)) \\ &= f_1^{-1}(f_1 x \cdot A_2^{-k} B(e,\varepsilon)) \\ &= x \cdot (A_1^{-k} B(e,\varepsilon)). \end{aligned}$$

Hence

$$D_n(x,\varepsilon,f_{1,\infty}) = x\cdot\bigcap_{k=0}^{n-1} A_1^{-k}B(e,\varepsilon) = x\cdot D_n(e,\varepsilon,A_{1,\infty})$$

and

$$m(D_n(x,\varepsilon,f_{1,\infty})) = m(D_n(e,\varepsilon,A_{1,\infty})).$$

Let $\varepsilon>0$ and $\xi=\{C_1,\ldots,C_k\}$ be a partition of G into Borel sets of diameter less than ε. If $x\in\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j})$, then $\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j})\subset x\cdot D_n(e,\varepsilon,A_{1,\infty})$. Thus $m(\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j}))\le m(D_n(e,\varepsilon,A_{1,\infty}))$ and taking logarithms we see that

$$\begin{aligned}
&\sum_{i_0,\ldots,i_{n-1}=1}^{k} m(\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j}))\log m(\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j}))\\
\le\ &\sum_{i_0,\ldots,i_{n-1}=1}^{k} m(\bigcap_{j=0}^{n-1} f_1^{-j}(C_{i_j}))\log m(D_n(e,\varepsilon,A_{1,\infty}))\\
=\ &\log m(D_n(e,\varepsilon,A_{1,\infty})).
\end{aligned}$$

Therefore

$$\begin{aligned}
h_m(f_{1,\infty})\ge h_m(f_{1,\infty},\xi) &= \limsup_{n\to\infty}\frac{1}{n}H_m(\bigvee_{j=0}^{n-1} f_1^{-j}\xi)\\
&\ge \limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(e,\varepsilon,A_{1,\infty})).
\end{aligned}$$

Since ε was arbitrary, we have

$$h_m(f_{1,\infty}) = h_m(A_{1,\infty}) \ge \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(e,\varepsilon,A_{1,\infty})). \tag{3.9}$$

Consider now an $(f_{1,\infty},n,\varepsilon)$-separated set E of G with cardinality $s(f_{1,\infty},n,\varepsilon,G)$. Then

$$\bigcup_{x\in E} D_n(x,\frac{\varepsilon}{2},f_{1,\infty}) = \bigcup_{x\in E} x\cdot D_n(e,\frac{\varepsilon}{2},A_{1,\infty})$$

is a disjoint union because of the choice of E. Therefore

$$s(f_{1,\infty},n,\varepsilon,G)\cdot m(D_n(e,\frac{\varepsilon}{2},A_{1,\infty}))\le 1$$

and so

$$s(f_{1,\infty},n,\varepsilon,G)\le\frac{1}{m(D_n(e,\frac{\varepsilon}{2},A_{1,\infty}))}.$$

Therefore

$$s(f_{1,\infty},\varepsilon,G)\le\limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(e,\frac{\varepsilon}{2},A_{1,\infty})),$$

and letting $\varepsilon\to 0$ we see that

$$h(f_{1,\infty}) = h(A_{1,\infty}) \le \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log m(D_n(e,\frac{\varepsilon}{2},A_{1,\infty})). \tag{3.10}$$

If F is an $(f_{1,\infty}, n, \varepsilon)$-spanning set with cardinality $r(f_{1,\infty}, n, \varepsilon, G)$, then

$$G = \bigcup_{x\in F} D_n(x, 2\varepsilon, f_{1,\infty}) = \bigcup_{x\in F} x \cdot D_n(e, 2\varepsilon, A_{1,\infty}).$$

Therefore

$$r(f_{1,\infty}, n, 2\varepsilon) \cdot m(D_n(e, 2\varepsilon, A_{1,\infty})) \geq 1$$

and so

$$r(f_{1,\infty}, n, 2\varepsilon, G) \geq \frac{1}{m(D_n(e, 2\varepsilon.A_{1,\infty}))}.$$

Hence,

$$r(f_{1,\infty}, \varepsilon, G) \geq \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(e, 2\varepsilon, A_{1,\infty})),$$

and letting $\varepsilon \to 0$ we see that

$$h(f_{1,\infty}) = h(A_{1,\infty}) \geq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log m(D_n(e, 2\varepsilon, A_{1,\infty})). \tag{3.11}$$

From (3.9), (3.10) and (3.11), the desired equation (3.8) holds. □

Corollary 3.9. *Let $\mathbb{T}^k$ be the k-dimensional torus, m the Lebesgue probability measure on $\mathbb{T}^k$ and $A_{1,\infty} = \{A_i\}_{i=1}^{\infty}$ a sequence of endomorphisms on $\mathbb{T}^k$. Then*

$$h_m(A_{1,\infty}) \geq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log m(D_n(0, \varepsilon, A_{1,\infty})) = h(A_{1,\infty}), \tag{3.12}$$

where $D_n(0, \varepsilon, A_{1,\infty}) = \bigcap_{i=0}^{n-1} A_1^{-i} B(0, \varepsilon)$.

Remark 3.10. It is obvious that if we have

$$h_m(f_{1,\infty}) \leq h(f_{1,\infty}),$$

then we can obtain the equality

$$h_m(f_{1,\infty}) = h(f_{1,\infty})$$

for both the nonautonomous affine transformations on a compact metrizable group in Theorem 3.8 and the nonautonomous endomorphisms on torus in Corollary 3.9.

From Proposition 3.5, Theorem 3.6 and Theorem 3.8 we can see that many classical results on entropy for autonomous systems may not hold any more for nonautonomous cases even under strong conditions such as the maps preserve the same measure. However, we shall see that many things takes on a new look if we consider the nonautonomous systems generated by applying at each time a transformation chosen randomly from a given family according to some probability distribution.

Let (X, d) be a compact metric space, $\mathcal{B}(X)$ the Borel σ-algebra and m a probability measure on X. Denote by $C^0(X, X)$ the space of continuous maps on X equipped with the C^0-topology. Now consider the subspace $\mathcal{U} \subset$

$C^0(X, X)$ of the maps preserving m. Let μ be a probability measure on $(\mathcal{U}, \mathcal{B}(\mathcal{U}))$. Denote

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{U}^{\mathbb{N}}, \mathcal{B}(\mathcal{U})^{\mathbb{N}}, \mu^{\mathbb{N}}) = \prod_{0}^{\infty}(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$$

the infinite product of copies of the measure space $(\mathcal{U}, \mathcal{B}(\mathcal{U}), \mu)$. For each $\omega = \{\omega_0, \omega_1, \ldots\} \in \Omega$ and $n \geq 0$, define

$$f_\omega^0 = \mathrm{id}, \; f_\omega^n = \omega_{n-1} \circ \omega_{n-2} \circ \cdots \circ \omega_0.$$

It is very useful to study the dynamical behavior of these composed maps as n tends to infinity for $\mathbb{P}$-a.e. ω, and the random dynamical systems generated by $\{f_\omega^n : n \geq 0, \omega \in \Omega\}$ will be referred to as F. The measure-theoretic entropy of F with respect to the measure m is defined by

$$h_m(F) = \int_\Omega h_m(\omega) d\mathbb{P}(\omega),$$

where $h_m(\omega)$ is the measure-theoretic entropy of the nonautonomous systems generated by ω. The topological entropy of F is defined by

$$h(F) = \int_\Omega h(\omega) d\mathbb{P}(\omega),$$

where $h(\omega)$ is the topological entropy of ω.

In the setting of random dynamical system, we can improve some of the results for the nonautonomous systems. Let τ be the left shift operator on Ω, namely,

$$(\tau\omega)_{i+1} = \omega_i$$

for all $\omega = \{\omega_0, \omega_1, \ldots\} \in \Omega^{\mathbb{N}}$, and Θ the induced skew product transformation on $\Omega \times X$ which is defined by

$$\Theta(\omega, x) = (\vartheta\omega, \phi(\omega)x).$$

Proposition 3.11. *Let F be the RDS as above. Then for any $k \in \mathbb{N}$, we have*

$$h_m(F^k) = k h_m(F),$$

where F^k is the RDS generated by $\{f_{\tau^{nk}\omega}^k\}, n \in \mathbb{N}, \omega \in \Omega$.

Proof. By the Abramov-Rokhlin formula in [1], we have

$$h_{\mathbb{P}\times m}(\Theta) = h_{\mathbb{P}}(\tau) + h_m(F).$$

Therefore, by the power rules of $h_{\mathbb{P}\times m}(\Theta)$ and $h_{\mathbb{P}}(\tau)$, we have

$$h_{\mathbb{P}\times m}(\Theta^k) = k h_{\mathbb{P}\times m}(\Theta) = k h_{\mathbb{P}}(\tau) + k h_m(F)$$

and

$$h_{\mathbb{P}\times m}(\Theta^k) = h_{\mathbb{P}}(\tau^k) + h_m(F^k) = k h_{\mathbb{P}}(\tau) + h_m(F^k).$$

Then the desired equality

$$h_m(F^k) = k h_m(F)$$

holds. □

Proposition 3.12. *Let $\mathcal{U} \subset C^0(X, X)$ be the family of the equi-continuous maps preserving m and F the RDS as above. Then*

$$h_m(F) \le h(F).$$

Proof. By Theorem 3.6,

$$h_m(F) \le h(F) + \log 2.$$

Since $\mathcal{U} \subset C^0(X, X)$ is the family of the equi-continuous maps, we can easily show that the power rule for the topological entropy $h(F)$ holds, i.e., for any $k \in \mathbb{N}$ we have

$$h(F^k) = kh(F).$$

Together with the power rule for $h_m(F)$(Proposition 3.11), we have for any $k \in \mathbb{N}$,

$$h_m(F) = \frac{1}{k} h_m(F^k) \le \frac{1}{k} h(F^k) + \frac{1}{k}\log 2 = h(F) + \frac{1}{k}\log 2.$$

Letting $k \to \infty$ gives

$$h_m(F) \le h(F).$$ □

By Theorem 3.8, Corollary 3.9 and Proposition 3.12, we have the following result immediately.

Theorem 3.13. *Let F be a random equi-continuous endomorphism on $\mathbb{T}^k$. Then*

$$h_m(F) = h(F) = \int \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log m(D_n(0, \varepsilon, \omega)) d\mathbb{P}(\omega).$$

We emphasis that in our settings we only consider the random dynamical systems generated by the particular collection of the transformations which all preserve the same measure m. In fact, the notions invariant measures for random dynamical systems can be given more generally. For more general theory of RDS, we refer to [3], [6, 7]. And for some recent results about the entropy, especially the preimage entropy which relays on the preimage structure, of RDS, we can see [13, 14, 15, 16].

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