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# Entropy of piecewise monotone mappings

by

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**Abstract.** The topological entropy of a piecewise monotone mapping of an interval is studied. It is proved that the entropy of a piecewise monotone mapping $f$ is equal to $\lim_n \frac{1}{n} \log \operatorname{Var} f^n = \lim_n \frac{1}{n} \log c_n$, where $c_n$ is the smallest number of intervals on which $f^n$ is monotone. If the entropy is positive, then a phenomenon similar to the horseshoe effect is observed. The entropy is also considered as a function of mapping with $C^0$ and $C^2$ topology. There are given some sufficient conditions for $f$ to be a point of semi-continuity and continuity of the entropy. The results are also true for piecewise monotone mappings of the circle.

The aim of the paper is to study the topological entropy of piecewise monotone mappings of intervals and their invariant closed subsets.

In Section 1 there are some formulas connecting the topological entropy of a map $f$ with: (i) the asymptotic behaviour of the numbers $c_n$ of maximal intervals on which $f^n$ is monotone; (ii) the asymptotic behaviour of variation of $f^n$. If $f$ is a piecewise strictly monotone mapping of an interval, then $c_n$ is the number of the points at which $f^n$ has extrema ($\pm 1$, according to whether the end points are taken into account or not). The formulas obtained are as follows:

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log c_n = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n.$$

In Section 2 the action of a piecewise monotone mapping with positive entropy is studied. It turns out that there is a subset on which a phenomenon similar to the horseshoe effect is observed. In the case of a map $f$ of an interval this makes it possible to estimate the asymptotic behaviour of the number of periodic points. Namely, the following inequality holds:

$$h(f) \leqslant \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Card}\{x\colon f^n(x) = x\}.$$

In Section 3 the topological entropy $h(f)$ is regarded as a function of $f$ where $f$ belongs to the set of all $C^0$ or $C^2$ mappings of an interval respectively. There are given some sufficient conditions for $f$ to be a point of semi-continuity and continuity of the function $h(\cdot)$.

Section 4 contains examples showing that the assumption that mappings are piecewise monotone is essential for some theorems in Sections 1 and 3.

In Section 5 it is shown that the results of Sections 1–4 are also true in the case of piecewise monotone mappings of the circle.

One of the examples of Section 4 in the case of the circle indicates some difficulties which may occur if one attempts to prove the entropy conjecture.

Some results of the paper are related to the results of Bowen [4] and Block [2].

**1.** We shall use the following notations. By $X$ we denote the space under consideration. The first capital letters of the alphabet: $A, B, \ldots, G$, denote families of subsets of $X$ (mainly covers or partitions); their elements are denoted by $a, b, \ldots, e$. Some fixed subsets of $X$ are denoted by subsequent capital letters: $J, K, L, \ldots, Y$. The mappings are denoted by $f, g, \varphi, \Phi, \sigma$; the letter $h$ is reserved for entropy only. Numbers are denoted by Greek letters $\alpha, \ldots, \varepsilon$ and by Latin letters $i, \ldots, u$ (also $c_n$).

We assume that the reader is familiar with the common definitions of topological entropy [1], [3]. Let us recall some notions from [6].

Let $X$ be a compact Hausdorff space, $f\colon X \to X$ — a continuous mapping, $Y \subset X$ — an arbitrary subset of $X$, $\mathfrak{A}(X)$ — the set of all finite open covers of $X$, $A$ and $B$ — two finite covers of $X$ (not necessarily open). We set

$$A^n = \bigvee_{i=0}^{n-1} f^{-i}(A).$$

If we consider more than one map $f$, we shall mark $A^n$ by $f$: $A^n = A^n_f$.

$$N(Y, A) = \min\{\operatorname{Card} C\colon C \subset A,\ Y \subset \bigcup_{c\in C} c\} \text{ for } Y \neq \emptyset; \quad N(\emptyset, A) = 1,$$

$$N(A|B) = \max_{b\in B} N(b, A),$$

$$h(f, A|B) = \lim_{n\to\infty} \frac{1}{n} \log N(A^n|B^n),$$

$$h(f|B) = \sup_{A\in\mathfrak{A}(X)} h(f, A|B),$$

$$h^*(f) = \inf_{B\in\mathfrak{A}(X)} h(f|B).$$

The number $h^*(f)$ is called the *topological conditional entropy of $f$*. If $B = \{X\}$, then $h(f, A|B) = h(f, A)$.



We shall use the following results from [6]:

$$h(f, A|B) \leqslant \log N(A|B), \tag{1.1}$$

$$h(f, A) \leqslant h(f, B) + h(f, A|B), \tag{1.2}$$

$$h^*(f^n) = nh^*(f) \quad \text{for every positive integer } n, \tag{1.3}$$

$$h(f) \leqslant h(f, A) + h(f|A). \tag{1.4}$$

From now on we assume $X$ to be a closed subset of the interval $I = \langle 0, 1\rangle$ and $f$ — a continuous mapping: $f\colon X \to X$. Denote by $\mathscr{I}$ the set of all possible subintervals of $I$ (open, closed, half-open, degenerated). For a family of sets $C$ and a set $Y$ we denote by $C|_Y$ the family of sets $\{c \cap Y\colon c \in C\}$. In particular, $\mathscr{I}|_Y$ denotes the family of all subintervals of $\langle 0, 1\rangle$, each restricted to $Y$.

Definition 1. A cover $A$ is called *$f$-mono* if $A$ is finite, $A \subset \mathscr{I}|_X$ and for any $a \in A$ the map $f|_a$ is monotone.

Lemma 1. *If $f, g\colon X \to X$, $A$ is an $f$-mono cover and $B$ is a $g$-mono cover, then $A \vee f^{-1}(B)$ is a $g\circ f$-mono cover.*

Proof. Clearly, $A \vee f^{-1}(B)$ is finite. For $a \in A$, $b \in B$ the map $g\circ f|_{a\cap f^{-1}(b)}$ is monotone as the composition of two monotone maps; furthermore $a \cap f^{-1}(b) \in \mathscr{I}|_X$ because $a \cap f^{-1}(b) = (f|_a)^{-1}(b)$, the map $f|_a$ is monotone and $b \in \mathscr{I}|_X$. ∎

Definition 2. A map $f$ is called *piecewise monotone* (abbreviated to p.m.) if there exists an $f$-mono cover of $X$.

It follows immediately from Lemma 1 that the composition of two p.m. functions is a p.m. function.

Let us now fix a piecewise monotone continuous (p.m.c.) mapping $f\colon X \to X$. Let

$$c_n = \min\{\operatorname{Card} A\colon A \text{ is an } f^n\text{-mono cover}\}.$$

Lemma 2. *Let $A \subset \mathscr{I}|_X$ be a finite cover of $X$. Then there exists a cover $B \in \mathfrak{A}(X)$ such that*

$$h(f, A|B) \leqslant \log 3.$$

Proof. In view of (1.1) it is sufficient to find a $B \in \mathfrak{A}(X)$ such that

$$N(A|B) \leqslant 3. \tag{1.5}$$

It is easy to construct an open cover $B$ satisfying (1.5); as elements of $B$ we take the interiors of elements of $A$ and we add some small intervals in order to get a cover. The number 3 may be attained if an element $b \in B$ contains an $a \in A$ such that $\operatorname{Int} a = \emptyset$. ∎

Lemma 3. *Let $A$ be an $f$-mono cover and let $D \subset \mathscr{I}|_X$ be a finite cover. Then $h(f, D|A) = 0$.*

Proof. By an end point of an element of $\mathscr{I}|_X$ we mean the end point of the minimal interval containing it. Let us fix a positive integer $n$ and an $a \in A^n$. In viev of Lemma 1 the map $f^k|_a$ is monotone for $k = 0, \ldots, n-1$, and therefore for any $d \in D$ the set $a \cap f^{-k}(d) = (f|_a)^{-k}(d)$ belongs to $\mathscr{I}|_X$, and thus it has at most 2 end points. Hence the elements of $D^n|_a$ have at most $2n \operatorname{Card} D$ end points in $a$.

Every element of $D^n|_a$ has at most 2 end points (it may contain them or not — there are 4 possibilities), and so the cardinality of $D^n|_a$ is less than $4(2n \operatorname{Card} D)^2$. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \log N(D^n | A^n) \leqslant \lim_{n\to\infty} \frac{1}{n} \log 16 n^2 (\operatorname{Card} D)^2 = 0,$$

which implies $h(f, D|A) = 0$. ∎

COROLLARY 1. *If a cover $A$ is $f$-mono, then $h(f|A) = 0$.* ∎

THEOREM 1. *If $f\colon X \to X$ is a p.m.c. map, then*

$$\lim_{n\to\infty} \frac{1}{n} \log c_n = h(f)$$

*and* $\dfrac{1}{n} \log c_n \geqslant h(f)$ *for any $n$.*

Proof. Let $A_n$ be an $f^n$-mono cover of minimal cardinality, $n = 1, 2, \ldots$ Let $m$ and $k$ be fixed. By Lemma 1, the cover $f^{-k}(A_m) \vee A_k$ is an $f^{m+k}$-mono cover. Since

$$c_{m+k} \leqslant \operatorname{Card}\big(f^{-k}(A_m) \vee A_k\big) \leqslant c_m \cdot c_k,$$

the sequence $(\log c_n)_{n=1}^{\infty}$ is subadditive and therefore $\lim\limits_{n\to\infty} \dfrac{1}{n} \log c_n$ exists.

By Corollary 1 and (1.4) we have

$$h(f) = \frac{1}{n} h(f^n) \leqslant \frac{1}{n} h(f^n, A_n) \leqslant \frac{1}{n} \log \operatorname{Card} A_n = \frac{1}{n} \log c_n$$

for $n = 1, 2, \ldots$ By Lemma 2 there exists a $B_n \in \mathfrak{A}(X)$ such that $h(f^n, A_n | B_n) \leqslant \log 3$. Hence, by (1.2)

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{k} \log c_k &= \lim_{k\to\infty} \frac{1}{nk} \log c_{nk} \leqslant \lim_{k\to\infty} \frac{1}{nk} \log N\big((A_n)_{f^n}^{k}\big) \\ &= \frac{1}{n} h(f^n, A_n) \leqslant \frac{1}{n} h(f^n, B_n) + \frac{1}{n} h(f^n, A_n | B_n) \\ &\leqslant \frac{1}{n} h(f^n) + \frac{1}{n} \log 3 = h(f) + \frac{1}{n} \log 3.\end{aligned}$$



Since $n$ is arbitrary, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log c_n \leqslant h(f). \ ∎$$

Remark 1. It follows from the proof of Theorem 1 that if $A$ is an $f$-mono cover, then $h(f, A) \geqslant h(f)$. But if we take $A^n$ instead of $A_n$, then we shall obtain also $h(f, A) \leqslant h(f)$ (because $(A^n)_{f^n}^k = A^{nk}$). Therefore, if $A$ is an $f$-mono cover, then $h(f, A) = h(f)$. ∎

THEOREM 2. *If $f\colon X \to X$ is a p.m.c. mapping, then $h^*(f) = 0$.*

Proof. If $A_n$, $B_n$ are as in the proof of Theorem 1, then, in view of (1.3), Lemma 2 and Corollary 1,

$$h^*(f) = \frac{1}{n} h^*(f^n) \leqslant \frac{1}{n} h(f^n | B_n) \leqslant \frac{1}{n} h(f^n, A_n | B_n) + \frac{1}{n} h(f^n | A_n) \leqslant \frac{1}{n} \log 3$$

for $n = 1, 2, \ldots$ Hence $h^*(f) = 0$. ∎

COROLLARY 2. *If $f\colon X \to X$ is a p.m.c. map, then the measure entropy of $f$, regarded as a function of measure, is upper semi-continuous* ([6]). *In particular, there exists a measure with maximal entropy for $f$* ([6]). ∎

Now we shall study the growth of the variation of the iterations of $f$ under the assumption that $f$ has the Darboux property, i.e, for any $J \in \mathscr{I}|_X$, $f(J) \in \mathscr{I}|_X$. Of course, this condition is fulfilled in the case where $X$ is an interval.

LEMMA 4. *If $f\colon X \to X$ is a continuous surjection and $f$ has the Darboux property, then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \geqslant h(f).$$

Proof. Since $f^{n+1} = f^n \circ f$ and $f$ is a surjection having the Darboux property, there exists a monotone function $g\colon X \to X$ such that $f \circ g = \mathrm{id}_X$ and therefore

$$\operatorname{Var} f^{n+1} \geqslant \operatorname{Var} f^n, \quad n = 1, 2, \ldots$$

Let $A \in \mathfrak{A}(X)$ and let $\varepsilon > 0$ be such that $4\varepsilon$ is less than the Lebesgue number of $A$. We take a maximal $(n, \varepsilon)$-separated set $\{x_1, \ldots, x_s\}$, $x_i \in X$, $i = 1, \ldots, s$, $x_1 < x_2 < \ldots < x_s$ (see [3]). For any $i \in \{1, \ldots, s-1\}$ there exists a $k_i \in \{0, \ldots, n-1\}$ such that $|f^{k_i}(x_i) - f^{k_i}(x_{i+1})| > \varepsilon$ and therefore

$$(s-1)\varepsilon \leqslant \sum_{k=0}^{n-1} \operatorname{Var} f^k \leqslant n \cdot \operatorname{Var} f^n$$

(here $s = s(n, \varepsilon)$ is the maximal cardinality of an $(n, \varepsilon)$-separated set). The family of sets

$$\big\{\{x \in X\colon |f^k(x) - f^k(x_i)| < 2\varepsilon \text{ for } k = 0, \ldots, n-1\}\colon i = 1, \ldots, s\big\}$$

is a cover (because of the maximality of $\{x_1, \ldots, x_n\}$) and it is finer than $A^n$. Therefore $N(A^n) \leqslant s$ and hence we have

$$\operatorname{Var} f^n \geqslant \frac{\varepsilon}{n}\big(N(A^n)-1\big).$$

But

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\varepsilon}{n}\big(N(A^n)-1\big)\right) = h(f, A)$$

and therefore

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \geqslant h(f, A).$$

Since $A$ is an arbitrary open cover, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \geqslant h(f). \ \blacksquare$$

THEOREM 3. *Let $f\colon X \to X$ be a p.m.c. map having the Darboux property. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n = h(f).$$

Proof. Let $J = \bigcap_{k=0}^{\infty} f^k(X)$. For every $k$ we have $f^k(X) \in \mathscr{I}|_X$ and therefore also $J \in \mathscr{I}|_X$. The map $f|_J\colon J \to J$ is a surjection and we can apply Lemma 4. Thus

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \geqslant \liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} (f|_J)^n \geqslant h(f|_J) = h(f).$$

Obviously $\operatorname{Var} f^n \leqslant c_n$, and so by Theorem 1 we get

$$\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n \leqslant \lim_{n\to\infty} \frac{1}{n} \log c_n = h(f).$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Var} f^n = h(f). \ \blacksquare$$

**2.** Now we shall investigate the nature of p.m.c. mappings having the Darboux property. In their action one can distinguish a phenomenon which is very similar to Smale's horseshoe effect.

Let $f\colon X \to X$ be a p.m.c. map. Then there exists an $f$-mono cover $A$ which is also a partition. In this case the reader may consider the dynamical system $(X, f)$ in terms of symbolic dynamics. The family of sets $A$ is the alphabet, the elements of $A^n$ are words and $f$ is the shift.



For any $J \in \mathscr{I}|_X$ there are at most two elements of $A$ not contained in $J$ and having with it a non-empty intersection, i.e.

$$\operatorname{Card}\{a \in A\colon a \cap J \neq \emptyset \text{ and } a \setminus J \neq \emptyset\} \leqslant 2. \tag{2.1}$$

Let $E$ be a subfamily of $A$ defined as follows:

$$E = \left\{a \in A\colon \limsup_{n\to\infty} \frac{1}{n} \log N(a, A^n) = h(f)\right\}.$$

It follows from Remark 1 that the family $E$ is non-empty. Let $E^n$ denote, as for covers, the family of sets

$$\left\{\bigcap_{i=0}^{n-1} f^{-i}(e_i)\colon e_i \in E \text{ for } i = 0, \ldots, n-1\right\}.$$

We shall use the following

LEMMA 5. *Let $(\alpha_n)_{n=0}^{\infty}$ and $(\beta_n)_{n=0}^{\infty}$ be two sequences of real non-negative numbers and let*

$$t = \max\left(\limsup_{n\to\infty} \frac{1}{n} \alpha_n, \ \limsup_{n\to\infty} \frac{1}{n} \beta_n\right).$$

*Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{k=0}^{n} e^{\alpha_k + \beta_{n-k}}\right) \leqslant t.$$

Proof. Let us fix an arbitrary $u > t$. Then there exist two numbers $s \geqslant u$ and $p > 0$ such that: $\frac{1}{n}\alpha_n \leqslant u$ and $\frac{1}{n}\beta_n \leqslant u$ for all $n \geqslant p$, $\frac{1}{n}\alpha_n \leqslant s$ and $\frac{1}{n}\beta_n \leqslant s$ for any $n$. If $n \geqslant 2p$ and $k \in \{0, \ldots, n\}$, then either $k \geqslant p$, or $n-k \geqslant p$ and hence $\alpha_k + \beta_{n-k} \leqslant ps + (n-p)u \leqslant ps + nu$. Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{k=0}^{n} e^{\alpha_k + \beta_{n-k}}\right) \leqslant \lim_{n\to\infty}\left(\frac{1}{n}\log(n+1) + \frac{1}{n}(ps + nu)\right) = u.$$

Since $u > t$ is arbitrary, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{k=0}^{n} e^{\alpha_k + \beta_{n-k}}\right) \leqslant t. \ \blacksquare$$

LEMMA 6. *For any $a \in E$ the following equality holds:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Card}(E^n|_a) = h(f).$$

Proof. The inequality

$$\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Card}(E^n|_a) \leqslant h(f)$$

follows from the fact that $\mathrm{Card}(E^n|_a) \leqslant \mathrm{Card}(A^n|_a) = N(a, A^n) \leqslant N(A^n)$ and from Remark 1.

Denote: $\alpha_0 = \beta_0 = 1$, $\alpha_n = \log \mathrm{Card}(E^n|_a)$, $\beta_n = \log\big(\sum_{b\in A\setminus E} \mathrm{Card}(A^n|_b)\big)$ $n = 1, 2, \ldots$ It is easy to see that

$$\mathrm{Card}(A^n|_a) \leqslant \sum_{k=0}^{n} e^{\alpha_k + \beta_{n-k}}$$

($k$ is the smallest number such that the image of a given element of $A^n|_a$ under $f^k$ is contained in an element of $A\setminus E$). For any $b \in A\setminus E$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(A^n|_b) < h(f)$$

and therefore $\limsup_{n\to\infty} \frac{1}{n}\beta_n < h(f)$. By the definition of the set $E$ we have $\limsup_{n\to\infty} \frac{1}{n}\log \mathrm{Card}(A^n|_a) = h(f)$, and so in view of Lemma 5 we must have

$$\limsup_{n\to\infty} \frac{1}{n}\alpha_n \geqslant h(f). \ \blacksquare$$

Now for any $a, b \in E$ we set

$$\gamma(a, b, n) = \mathrm{Card}\{e \in E^n|_a : f^n(e) \supset b\}.$$

LEMMA 7. *Let $f: X\to X$ be a p.m.c. map having the Darboux property and let $h(f) > \log 3$. Then there exists an $a_0 \in E$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \gamma(a_0, a_0, n) = h(f). \tag{2.2}$$

Proof. Let us fix a set $a \in E$ and a real number $u$ such that $\log 3 < u < h(f)$. Suppose that there exists a number $p$ such that for any $n \geqslant p$

$$\frac{1}{n}\log \mathrm{Card}(E^n|_a) > u$$

implies

$$\mathrm{Card}(E^{n+1}|_a) < 3\,\mathrm{Card}(E^n|_a).$$

It is easy to see that then

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathrm{Card}(E^n|_a) \leqslant u,$$



which contradicts Lemma 6. Therefore

(2.3) for every number $p$ there exists an integer $n \geqslant p$ such that

$$\frac{1}{n}\log \mathrm{Card}(E^n|_a) > u$$

and

$$\mathrm{Card}(E^{n+1}|_a) \geqslant 3\,\mathrm{Card}(E^n|_a).$$

Fix a set $e \in E^n|_a$. The set $f^n(e)$ belongs to $\mathscr{F}|_X$ and therefore by (2.1) if it has non-empty intersections with $r$ elements of $E$, then it contains at least $r-2$ of them. But $r = \mathrm{Card}(E^{n+1}|_e)$. Therefore

$$\mathrm{Card}\{b \in E: f^n(e) \supset b\} \geqslant \mathrm{Card}(E^{n+1}|_e) - 2.$$

Summing over $e \in E^n|_a$, we obtain

$$\sum_{b\in E} \gamma(a, b, n) \geqslant \mathrm{Card}(E^{n+1}|_a) - 2\,\mathrm{Card}(E^n|_a).$$

In view of (2.3) we conclude that

$$\limsup_{n\to\infty} \frac{1}{n}\log\Big(\sum_{b\in E}\gamma(a, b, n)\Big) \geqslant u.$$

The number $u$ is an arbitrary number less than $h(f)$; therefore

$$\limsup_{n\to\infty} \frac{1}{n}\log\Big(\sum_{b\in E}\gamma(a, b, n)\Big) \geqslant h(f).$$

Since $E$ is finite, there exists a mapping $\varphi: E\to E$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\log \gamma\big(a, \varphi(a), n\big) \geqslant h(f)$$

for any $a \in E$. The mapping $\varphi$ has to have a periodic point. Denote it by $a_0$, and by $m$ — its period. It is easy to see that

$$\gamma\Big(a_0, a_0, \sum_{i=0}^{m-1} n_i\Big) \geqslant \prod_{i=0}^{m-1} \gamma\big(\varphi^i(a_0), \varphi^{i+1}(a_0), n_i\big)$$

for any $n_i$, $i = 0, 1, \ldots, m-1$. The last inequality implies immediately that

$$\limsup_{n\to\infty} \frac{1}{n}\log \gamma(a_0, a_0, n) \geqslant h(f).$$

On the other hand, it is easy to see that $\gamma(a_0, a_0, n) \leqslant \mathrm{Card}\,A^n$, and so in view of Remark 1 equality (2.2) holds. $\blacksquare$

THEOREM 4. *Let $f\colon X\to X$ be a p.m.c. map having the Darboux property. Then there exist:*

(i) *a set $J\in\mathscr{I}|_X$,*

(ii) *a sequence $(D_n)_{n=1}^\infty$ of partitions of $J$ by elements of $\mathscr{I}|_X$,*

(iii) *a sequence $(k_n)_{n=1}^\infty$ of positive integers such that*

$$\lim_{n\to\infty}\frac{1}{k_n}\log\operatorname{Card}D_n=h(f) \tag{2.4}$$

*and $f^{k_n}(d)\supset J$ for any $d\in D_n$.*

Proof. If $h(f)=0$, then we set $J=\bigcap_{j=0}^{\infty}f^j(X)$, $D_n=\{J\}$, and $k_n=n$, $n=1,2,\ldots$

If $h(f)>0$, then we take a positive integer $r>\dfrac{\log 3}{h(f)}$ and we apply Lemma 7 to $f^r$; we set $J=a_0$, $k_n=rm_n$, where

$$\lim_{n\to\infty}\frac{1}{m_n}\log\gamma(a_0,a_0,m_n)=h(f^r),$$

and $D_n$ is a partition by elements of $\mathscr{I}|_X$ such that for every $d\in D_n$ there exists exactly one element $e\in E_{f^r}^{m_n}|_{a_0}$ for which $e\subset d$ and $f^{rm_n}(e)\supset a_0$. ∎

COROLLARY 3. *If $f\colon I\to I$ is continuous and $X\subset I$ is a closed invariant set such that $h(f|_X)=h(f)$ and $f|_X$ satisfies the hypotheses of Theorem 4, then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{Card}\{x\in I\colon f^n(x)=x\}\geqslant h(f).\ ∎$$

Remark 2. In the case of $X=I$ the result of Theorem 4 can be obtained in another way. Let $A\in\mathscr{I}$ be an $f$-mono partition of minimal cardinality. We say that there is a transition from an $a\in A$ to a $b\in A$ if there exists a positive integer $j$ such that $f^{-j}(a)$ has a component in $\operatorname{Int}b$. Denote by $A'$ the set of all elements of $A$ such that there is no transition from $a$ to any other element of $A$, and by $A''$ the set of all elements of $A$ such that there exists only a finite chain of transitions ending up at $a$. It turns out that $\operatorname{Card}(A^n|_{a''})\leqslant\alpha\cdot 2^n$ for any $a''\in A''$ where $\alpha$ is a constant number. Suppose $h(f)>\log 3$ and let $n$ be an integer such that $c_{n+1}\geqslant 3c_n$. Then for every $a'\in A'$

$$\operatorname{Card}\{a\in A^n\colon a\cap f^{-n}(a')\neq\emptyset\}\leqslant c_n/2.$$

From those two facts we easily obtain Lemma 7 and directly Theorem 4. ∎

**3.** Now we shall study topological entropy as a function of mapping. We shall consider two cases: (i) $f\in C^0(I,I)$ — the space of all continuous mappings of the interval $I$ into itself with $C^0$-topology; (ii) $f\in C^2(I,I)$ — the space of all mappings $f$ of class $C^2$ of the interval $I$ into itself with $C^2$-topology. The results obtained, namely Theorems 1 and 4, enable us to prove some continuity properties of entropy.



We shall use the following lemma:

LEMMA 8. *Let $f\colon I\to I$ be a p.m. mapping of class $C^1$ having a local extremum at each critical point. Then for each positive integer $m$ the function $f^m$ also has a local extremum at each critical point.*

Proof. Let $(f^m)'(x)=0$. Then for some $k\in\{0,1,\ldots,m-1\}$ we have $f'(f^k(x))=0$. Thus $f$ has a local extremum at $f^k(x)$. The map $f$ is continuous, and so it maps an open (in $I$) interval containing $f^k(x)$ onto an interval with one of the end points at $f^{k+1}(x)$. Hence $f^m$ maps an open interval containing $x$ onto an interval with one of the end points at $f^m(x)$ and therefore $f^m$ has a local extremum at $x$. ∎

The first continuity property is a slight generalization of a result of Bowen [4].

PROPOSITION 1. *Let $f\colon I\to I$ be a mapping of class $C^2$ such that for any $x\in I$ at least one of the numbers $f'(x)$, $f''(x)$ is non-zero. Then the topological entropy regarded as a function $h\colon C^2(I,I)\to\mathbf{R}$ is upper semi-continuous at $f$.*

Proof. Notice first that if $g\in C^2(I,I)$ is sufficiently close to $f$, then also for any $x\in I$ at least one of the numbers $g'(x)$ and $g''(x)$ is non-zero. Therefore $g$, as well as $f$, is p.m. We can restrict our attention to those mappings $g$. Let $\varepsilon>0$ be fixed. By Theorem 1 there exists a positive integer $m$ such that

$$\frac{1}{m}\log c_m\leqslant h(f)+\frac{\varepsilon}{2}. \tag{3.1}$$

Let $0=x_0,x_1,\ldots,x_{c_m-1},x_{c_m}=1$ be the points at which $f^m$ has local extrema. Let $\{J_i\colon i=0,1,\ldots,c_m\}$ be a family of pairwise disjoint open (in $I$) intervals such that $x_i\in J_i$. It follows from Lemma 8 that if $g$ is sufficiently close to $f$, then $g^m$ has no local extrema in the set $I\setminus\bigcup_{i=0}^{c_m}J_i$. Suppose that for a fixed $i\in\{0,1,\ldots,c_m\}$

$$\operatorname{Card}\{k\in\{0,\ldots,m-1\}\colon f\text{ has a local extremum at }f^k(x_i)\}=r.$$

For $g$ close enough to $f$ the number of maximal intervals contained in $J_i$ on which $g^m$ is monotone is not greater than $2^r$ (for any $k$ for which $g'|_{g^k(J_i)}$ vanishes at one point the number of intervals of monotonicity can grow at most twice). Therefore $g^m$ has at most $2^r+1$ local extrema in the set $J_i$. If there is no $k$ such that $f^k(x_i)$ is periodic and $f$ has a local extremum at $f^k(x_i)$, then $r\leqslant c_1+1$ and $g^m$ has at most $2^{c_1+1}+1$ local extrema in the set $J_i$. Notice that if $f(x)=0$ or $1$ for some $x\in(0,1)$, then $f'(x)=0$. Therefore we need not consider separately the case of a periodic point 0 or 1 (the case of a fixed point 0 or 1 and the case $f(0) = 1$, $f(1) = 0$ are trivial).

Let $Y = \bigcup_{j=0}^{\infty} f^j(\{y \in I: f'(y) = 0 \text{ and } y \text{ is periodic}\})$. If $g$ is close enough to $f$, then for an open (in $I$) set $U \supset Y$ we have $g(U) \subset U$. Moreover, all the points of $U$ except a finite number are $g$-wandering. Denote $X_g = I \setminus \bigcup_{j=0}^{\infty} g^{-j}(U)$. Then $X_g$ is closed, $g$-invariant and $h(g|_{X_g}) = h(g)$.

If $f^k(x_i) \in Y$ for some $k \in \{0, \ldots, m-1\}$, then we may assume that $J_i$ is so small that for any $g$ sufficiently close to $f$ we have $J_i \cap X_g = \emptyset$.

Finally, we see that if $g$ is close enough to $f$, then the minimal number of elements of a $g^m|_{X_g}$-mono cover is not greater than $(c_m+1)(2^{c_1+1}+2)$.

By Theorem 1 and (3.1) we obtain

$$h(g) \leqslant \frac{1}{m}\log[(c_m+1)(2^{c_1+1}+2)] \leqslant \frac{1}{m}\log(c_m\cdot 2^{c_1+3})$$
$$\leqslant h(f) + \frac{\varepsilon}{2} + \frac{c_1+3}{m}\log 2.$$

If $m \geqslant \frac{2(c_1+3)}{\varepsilon}\log 2$, then $h(g) \leqslant h(f)+\varepsilon$. ∎

Now we pass to the second continuity property.

THEOREM 5. *Let $f: I\to I$ be a continuous mapping and let $X$ be a closed invariant subset of $I$ such that $h(f|_X) = h(f)$ and $f|_X: X\to X$ is p.m. and has the Darboux property. Then the topological entropy regarded as a function $h: C^0(I, I)\to \bar{\boldsymbol{R}}$ is lower semi-continuous at $f$.*

Proof. Let $J$, $(k_n)_{n=1}^{\infty}$ and $(D_n)_{n=1}^{\infty}$ be as in Theorem 4 for $f|_X$. If we replace $J$ and the elements of $D_n$ by minimal intervals containing them, then we obtain an interval $K$ and a sequence $(F_n)_{n=1}^{\infty}$ of families of intervals pairwise disjoint (for a fixed $n$) and such that $\operatorname{Card} F_n = \operatorname{Card} D_n$ and $f^{k_n}(d) \supset K$ for all $d \in F_n$, $n = 1, 2, \ldots$

Let $\varepsilon > 0$ be arbitrary and pick $n$ such that

$$\frac{1}{k_n}\log(\operatorname{Card} D_n - 4) \geqslant h(f) - \varepsilon \tag{3.2}$$

(this is possible because of (2.4) and the fact that $h(f|_X) = h(f)$). It is easy to see that there exists an interval $K_n$ such that $\bar{K}_n \subset \operatorname{Int} K$, no element of $F_n$ has non-empty intersections with both $K_n$ and $K\setminus K_n$ and $\operatorname{Card} F_n|_{K_n} \geqslant \operatorname{Card} F_n - 4$. Then $\bar{K}_n \subset \operatorname{Int}(f^{k_n}(d))$ for all $d \in F_n|_{K_n}$, and hence, if $g$ is sufficiently close to $f$ (in $C^0$ topology), then $\bar{K}_n \subset \operatorname{Int}(g^{k_n}(d))$ for all $d \in F_n|_{K_n}$. Now we slightly modify $F_n|_{K_n}$ to obtain a partition $G_n$ of $K_n$ by intervals such that the images of the end points of elements of $G_n$ (perhaps except the end points of $K_n$) under $g^{k_n}$ do not belong to $\bar{K}_n$, $\operatorname{Card} G_n = \operatorname{Card} F_n|_{K_n}$ and $\bar{K}_n \subset \operatorname{Int}(g^{k_n}(d))$ for all $d \in G_n$. Denote $Y = \bigcap_{i=0}^{\infty} (g^{k_n})^{-i}(\bar{K}_n)$. Then $G_n|_Y$ is an open partition and for any $d \in G_n|_Y$ we have $g^{k_n}(d) = Y$. Therefore $h(g^{k_n}|_Y) \geqslant \log \operatorname{Card} G_n$. Thus we have

$$h(g) = \frac{1}{k_n}h(g^{k_n}) \geqslant \frac{1}{k_n}h(g^{k_n}|_Y) \geqslant \frac{1}{k_n}\log\operatorname{Card} G_n \geqslant \frac{1}{k_n}\log(\operatorname{Card} D_n - 4).$$

The last inequality and (3.2) give $h(g) \geqslant h(f) - \varepsilon$. ∎

THEOREM 6. *If $f: I\to I$ satisfies the hypotheses of Proposition 1, then $f$ is a point of continuity of topological entropy $h: C^2(I, I)\to \boldsymbol{R}$.* ∎

**4.** Now we shall present two examples showing that some assumptions of Theorem 3 and Proposition 1 (and therefore of Theorem 6) cannot be omitted.

We define two auxiliary functions.

1. $\varphi: \boldsymbol{R}\to\boldsymbol{R}$ is given by the formula:

$$\varphi(x) = \tfrac{1}{4}(x-\tfrac{1}{4})^{2r+1}\cdot\sin\frac{1}{x-\frac{1}{4}} + \tfrac{3}{4} \quad \text{for} \quad x \neq \tfrac{1}{4} \quad \text{and} \quad \varphi(\tfrac{1}{4}) = \tfrac{3}{4}.$$

The function $\varphi$ is of class $C^r$ and $\varphi^{(i)}(\frac{1}{4}) = 0$ for $i = 1, \ldots, r$.

2. $\Phi: \boldsymbol{R}\to\langle 0, 1\rangle$ is a function of class $C^\infty$ such that $\Phi(x) = 0$ for $x \in (-\infty, 1\rangle \cup \langle 5, +\infty)$ and $\Phi(x) = 1$ for $x \in \langle 2, 4\rangle$.

THEOREM 7. *For every non-negative integer $r$ the topological entropy regarded as a function of $C^r$ p.m. mapping of the interval $I$ into itself is not upper semi-continuous in $C^r$-topology.*

Proof. Let $f: I\to I$ be a mapping of class $C^r$ such that:

(i) $f(0) = 0, f(1) = 1, f(\frac{1}{4}) = \frac{3}{4}$, $f'(\frac{1}{4}) = f'(\frac{1}{2}) = 0$ and $f'(x) \neq 0$ for $x \neq \frac{1}{4}, \frac{1}{2}$; moreover $f^{(n)}(\frac{1}{4}) = 0$ for $n = 1, 2, \ldots, r$.

(ii) $f(x) = \frac{3}{4} + a(x-\frac{3}{4})$ for $x \in \langle \frac{3}{4} - \frac{1}{2a}, \frac{3}{4}\rangle$

where $a$ is a fixed number such that

(iii) $a > 3^{2r+1}$

(see Fig. 1). There exists an $f$-mono cover of $I$ of cardinality 3, and so $h(f) \leqslant \log 3$. For $t > 0$ we set

$$f_t(x) = f(x) + (\varphi(x) - f(x))\,\Phi(t(x-\tfrac{1}{4})).$$

If $t$ is large enough, then $f_t(I) \subset I$. Obviously, $f_t$ is of class $C^r$.

We claim that the perturbation $f_t - f$ tends to 0 in $C^r$-topology as $t\to+\infty$. To show that, we have to estimate its $k$th derivative, $k = 0, 1, \ldots, r$. The support of the perturbation is contained in the interval

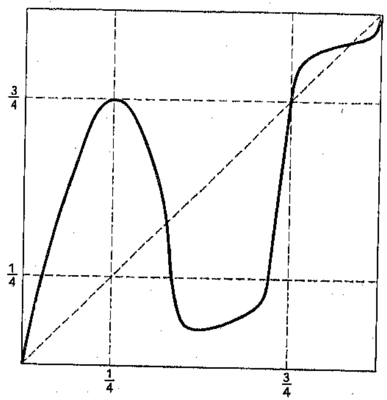

Fig. 1

$\langle \frac{1}{4}-\frac{5}{t}, \frac{1}{4}+\frac{5}{t}\rangle$. The following equality holds:

(4.1) $\quad \big(\varphi(x)-f(x)\big)\Phi\big(t(x-\frac{1}{4})\big)^{(k)}$

$$= \sum_{i=0}^{k}\binom{k}{i}\big(\varphi^{(i)}(x)-f^{(i)}(x)\big)t^{k-i}\Phi^{(k-i)}\big(t(x-\tfrac{1}{4})\big).$$

The derivatives from 0th to $(k-i)$th of the function $\varphi^{(i)}-f^{(i)}$ are equal to 0 at the point $\frac{1}{4}$; therefore

$$\sup_{x\in\langle\frac{1}{4}-\frac{5}{t},\frac{1}{4}+\frac{5}{t}\rangle}\big(\varphi^{(i)}(x)-f^{(i)}(x)\big)t^{k-i}\to 0 \quad \text{as} \quad t\to+\infty.$$

But $\big|\Phi^{(k-i)}\big(t(x-\frac{1}{4})\big)\big|$ is bounded by the $C^r$-norm of $\Phi$, and thus, in view of (4.1), the $C^r$-norm of $f_t-f$ tends to 0 as $t\to+\infty$.

Applying the method from the proof of Theorem 5, we estimate $h(f_t)$ from below. For any $t$

$$f_t(x) = \varphi(x) \quad \text{for} \quad x\in\left\langle\frac{1}{4}+\frac{2}{t},\frac{1}{4}+\frac{4}{t}\right\rangle.$$

This interval can be divided into $\mathrm{E}\left(\frac{t/2-t/4}{\pi}\right)-2 = \mathrm{E}\left(\frac{t}{4\pi}\right)-2$ intervals such that the image of any of them under $f_t$ contains the interval $\left\langle\frac{3}{4}-\frac{1}{4}\left(\frac{2}{t}\right)^{2r+1},\frac{3}{4}\right\rangle$. For $x\in\left\langle\frac{3}{4}-\frac{1}{2a},\frac{3}{4}\right\rangle$ and $t$ large enough we have $f_t(x)=f(x)$ and therefore it follows from (ii) that if $\frac{1}{4}\left(\frac{2}{t}\right)^{2r+1}a^n \geqslant \frac{1}{2}$,



then

$$f_t^n\left(\left\langle\frac{3}{4}-\frac{1}{4}\left(\frac{2}{t}\right)^{2r+1},\frac{3}{4}\right\rangle\right)\supset\left\langle\frac{1}{4},\frac{3}{4}\right\rangle.$$

Thus, the interval $\left\langle\frac{1}{4}+\frac{2}{t},\frac{1}{4}+\frac{4}{t}\right\rangle$ can be divided into $\mathrm{E}\left(\frac{t}{4\pi}\right)-2$ intervals such that the image of any of them under $f_t^n$ (where $n = \mathrm{E}\left(\log\left(2\left(\frac{t}{2}\right)^{2r+1}\right)/\log a\right)+2$) contains $\left\langle\frac{1}{4}+\frac{2}{t},\frac{1}{4}+\frac{4}{t}\right\rangle$ in its interior. Therefore the arguments used in the proof of Theorem 5 show that for any $\tilde{f}_t$ sufficiently $C^0$-close to $f_t$ we have

$$h(\tilde{f}_t)\geqslant\frac{\log\left(\mathrm{E}\left(\frac{t}{4\pi}\right)-2\right)}{\mathrm{E}\left(\frac{\log\left(2\left(\frac{t}{2}\right)^{2r+1}\right)}{\log a}\right)+2} \tag{4.2}$$

(we consider $\tilde{f}_t$ because we are not sure whether $f_t$ is p.m.).

The right-hand side of (4.2) tends to $\frac{\log a}{2r+1}$ as $t\to+\infty$. For any $t$ we pick a polynomial function $\tilde{f}_t\colon I\to I$ such that inequality (4.2) holds and $\tilde{f}_t\to f$ in $C^r$-topology. In view of (iii) we have

$$\limsup_{t\to+\infty} h(\tilde{f}_t)\geqslant\frac{\log a}{2r+1}>\log 3\geqslant h(f). \ ■$$

THEOREM 8. *For any non-negative integer there exists a $C^r$-mapping $g\colon I\to I$ such that*

$$\liminf_{n\to\infty}\frac{1}{n}\log\operatorname{Var}g^n > h(g).$$

Proof. There exists a $C^r$-mapping $g\colon I\to I$ such that

(i) $\quad g(0)=0, \quad g(1)=1, \quad g(\frac{1}{2})=\frac{1}{2};$

(ii) $\quad$ for a certain $c>0$ we have $g(x)=\varphi(x)$ for $x\in\langle\frac{1}{4}-c,\frac{1}{4}+c\rangle$;

(iii) $\quad g(x)>x \quad$ for $\quad x\in(0,\frac{1}{2})$;

(iv) $\quad g$ is increasing on the interval $\langle\frac{1}{2},1\rangle$;

(v) $\quad g(x)=\frac{3}{4}+a(x-\frac{3}{4}) \quad$ for $\quad x\in\langle\frac{3}{4}-c,\frac{3}{4}+c\rangle$

($a$ is as in the proof of Theorem 7) (see Fig. 2).

By (i) and (iv) we have $g(\langle\frac{1}{2},1\rangle)=\langle\frac{1}{2},1\rangle$ and in view of (iii) all the points of $(0,\frac{1}{2})$ are wandering. Hence, by (iv), we have $h(g)=0$.

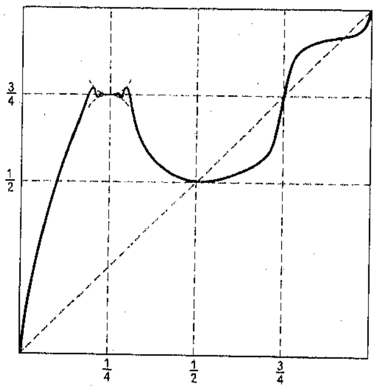


Fig. 2

But the same estimations as in the proof of Theorem 7 show that

$$\liminf_{n\to\infty}\frac{1}{n}\log\operatorname{Var}g^n \geqslant \frac{\log\alpha}{2r+1} > 0. \ \blacksquare$$

Remark 3. Slightly modifying $g$ in the neighbourhood of the point $\frac{1}{4}$, one can easily obtain a $C^r$-mapping $\tilde{g}\colon I\to I$ for which $\lim\limits_n \frac{1}{n}\log\operatorname{Var}\tilde{g}^n$ does not exist. ∎

**5.** After some small modifications the results of Sections 1–4 are valid also in the case of mappings of the circle into itself.

We shall consider the circle as the set $S^1 = \{z\in C\colon |z| = 1\}$. Let $\sigma\colon \langle 0, 1)\to S^1$ be given by the formula $\sigma(x) = e^{2\pi i x}$ and let $f\colon S^1\to S^1$ be a continuous mapping.

DEFINITION 1′. A cover $A$ of $S^1$ is called *f-mono* if $\sigma^{-1}(A)$ is $(\sigma^{-1}\circ f\circ\sigma)$-mono.

The map $\sigma^{-1}\circ f\circ\sigma$ is not necessarily continuous on the interval $\langle 0, 1)$. On Fig. 3 we present $\sigma^{-1}\circ f\circ\sigma$ for $f(z) = z^2$.

DEFINITION 2′. A map $f\colon S^1\to S^1$ is called *piecewise monotone* if there exists an $f$-mono cover of $S^1$.

Denote

$$c_n = \min\{\operatorname{Card}A\colon\ A \text{ is an } f^n\text{-mono cover}\}.$$

COROLLARY 1′. *If $f\colon S^1\to S^1$ is a p.m.c. map and $A$ is an f-mono cover, then $h(f|A) = 0$.*



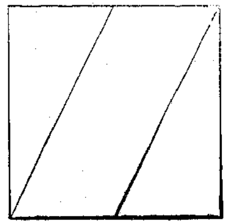

Fig. 3

THEOREM 1′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then*

$$\lim_n \frac{1}{n}\log c_n = h(f)$$

*and for any $n$*

$$\frac{1}{n}\log c_n \geqslant h(f).$$

Remark 1′. If $f\colon S^1\to S^1$ is a p.m.c. map and $A$ is an $f$-mono cover, then $h(f, A) = h(f)$.

THEOREM 2′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then $h^*(f) = 0$.*

COROLLARY 2′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then the measure entropy of $f$, regarded as a function of measure, is upper semi-continuous. In particular, there exists a measure with maximal entropy for $f$.*

THEOREM 3′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then*

$$\lim_{n\to\infty}\frac{1}{n}\log\operatorname{Var}f^n = h(f).$$

THEOREM 4′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then there exist: an arc $J\subset S^1$, a sequence $(D_n)_{n=1}^{\infty}$ of partitions of $J$ by arcs and a sequence $(k_n)_{n=1}^{\infty}$ of positive integers such that*

$$\lim_{n\to\infty}\frac{1}{k_n}\log\operatorname{Card}D_n = h(f)$$

*and $f^{k_n}(d)\supset J$ for any $d\in D_n$.*

COROLLARY 3′. *If $f\colon S^1\to S^1$ is a p.m.c. map, then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{Card}\{x\in S^1\colon f^n(x) = x\}\geqslant h(f).$$

As usual $C^r(S^1, S^1)$ denotes the space of all $C^r$ mappings of the circle $S^1$ into itself with $C^r$ topology, $r\geqslant 0$.

PROPOSITION 1′. *If $f \in C^2(S^1, S^1)$ and for any $x \in S^1$ either $f'(x) \neq 0$ or $f''(x) \neq 0$, then the topological entropy regarded as a function $h\colon C^2(S^1, S^1) \to \mathbf{R}$ is upper semi-continuous at $f$.*

THEOREM 5′. *If $f \in C^0(S^1, S^1)$ is p.m., then the topological entropy regarded as a function $h\colon C^0(S^1, S^1) \to \overline{\mathbf{R}}$ is lower semi-continuous at $f$.*

THEOREM 6′. *If a map $f$ satisfies the hypotheses of Proposition 1′, then $f$ is a point of continuity of topological entropy in $C^2$-topology.*

THEOREM 7′. *For any non-negative integer $r$ the topological entropy, regarded as a function of p.m. $C^r$-mapping of $S^1$ into itself, is not upper semi-continuous (in $C^r$-topology).*

THEOREM 8′. *For any non-negative integer $r$ there exists a $C^r$ mapping $g\colon S^1 \to S^1$ such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log \operatorname{Var} g^n > h(g).$$

Remark 3′. For any non-negative integer $r$ there exists a $C^r$ mapping $\tilde{g}\colon S^1 \to S^1$ such that $\lim_n \frac{1}{n} \log \operatorname{Var} \tilde{g}^n$ does not exist.

We shall also make two other remarks.

Remark 4. Every proper subset of $S^1$ is homeomorphic to a subset of $I$, and so the results obtained for subsets of $I$ are valid for subsets of $S^1$. ■

Remark 5. Sacksteder and Shub define [7] for a smooth mapping $f\colon M \to M$ ($M$ is a manifold) some numbers $h_1, h_2, h_3$ related to the topological entropy $h = h(f)$. They prove the inequalities $\log\lambda \leqslant h_1 \leqslant h_2 \geqslant h_3 \leqslant h$, where $\lambda$ is the spectral radius of the transformation induced in the homology groups. The number $h_1$ is defined as $\limsup_{n\to\infty} \frac{1}{n} \log \Lambda_n$, where $\Lambda_n$ is the integral of the maximal "multidimensional expansion" of $f^n$ over the whole manifold. In the case $M = S^1$ we have

$$\Lambda_n = \int_{S^1} |(f^n)'| = \operatorname{Var} f^n.$$

Hence it follows from Theorem 8′ that for some $g$ we have $h_1(g) > h(g)$ and therefore $h_2(g) > h_3(g)$. Thus one cannot hope to prove the entropy conjecture in this way. ■

Nearly all of the proofs in Sections 1–4 may be repeated almost word by word in the case of $S^1$; only some small modifications have to be made. We shall now list them.

1. We must put $S^1$ instead of $X$, $\sigma(\mathscr{I}|_{\langle 0,1)})$ or the set of all arcs instead of $\mathscr{I}|_X$ and sometimes consider $\sigma^{-1} \circ f \circ \sigma$ instead of $f$, and the points and subsets of either $S^1$ or $\langle 0, 1)$.



2. In Lemma 3′ we must replace "$D \subset \mathscr{I}|_X$ be a finite cover" by "$D$ be a finite cover of $S^1$ consisting of arcs of length $< \pi$". Furthermore, since we consider the whole circle, there is no need to make any assumptions of the Darboux property.

3. In the proof of Theorem 3′, if $J \neq S^1$, then we use Lemma 4; if $J = S^1$, then we use the corresponding Lemma 4′.

4. In (2.1) for $S^1$ the set $J$ is an arc.

5. In the proof of Proposition 1′ we must consider various cases. If $f$ is a local homeomorphism, then $h(f) = \log|\deg f|$ (see [5]) and the same holds for all mappings $C^1$-close to $f$. If $f$ is not a local homeomorphism, then we may consider, instead of $f$, a mapping $f_0$ given by the formula $f_0(z) = \frac{1}{z_0} f(z \cdot z_0)$, where $f$ has a local extremum at $z_0$. Then $f_0$ is smoothly conjugate to $f$ and $f_0$ has a local extremum at 1. Then if we add to the set $\{x_i\}_{i=0}^{\tilde{c}_m}$ (from the proof of Proposition 1) the inverse images of 1 under $f_0^m$, we obtain the end points of the elements of the minimal $f_0^m$-mono cover. The cardinality of the inverse image is not greater than $\tilde{c}_m$ (the number of all points in which $(f_0^m)'$ is equal to 0) and therefore

$$\lim_{m\to\infty} \frac{1}{m} \log \tilde{c}_m = h(f_0) = h(f).$$

Furthermore, if there exists a periodic point of $f_0$ at which $f_0'$ is equal to 0, then we apply Theorem 1 for $g$; otherwise we apply Theorem 1′.

6. In the proof of Theorem 7′ we must use trigonometric polynomials multiplied by the identity instead of ordinary polynomials.

Theorems 1′–8′ generalize some results of Block [2] and answer some questions formulated there.

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