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Article

The entropy of the entangled Hawking radiation

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Abstract – Entropic information theory, as a unified informational theory, presents a new informational theoretical framework capable of fully describing the evaporation of the black holes phenomenon while resolving the information paradox, reconciling quantum formalism and relativistic formalism in a single approach. With a set of five new equivalent equations expressing entropy, and by introducing the Hawking temperature into one of them, it is possible to solve the black holes information paradox by being able to calculate the entropy of entangled Hawking radiation, entangled with the fields inside black holes, allowing us to extract information from inside black holes. The proposed model solves the information paradox of black holes by calculating a new entropy formula for the entropy of black holes as equal to the entropy of the pure state of entangled Hawking radiation, itself equal to the fine-grained entropy or von Neumann entropy, itself according to the work of Casini and Bousso equal to the Bekenstein bound which is itself equal, being saturated by Bekenstein-Hawking entropy, at this same entropy. Moreover, since the law of the entropy horizon of black holes turns out to be a special case of the Ryu-Takayanagi conjecture, this general formula for the fine-grained entropy of quantum systems coupled to gravity, equalizes the entropy of entangled Hawking radiation with the gravitational fine-grained entropy of black holes, and makes it possible to relate this resolution of the information paradox of black holes based on the concept of mass of the information bit to quantum gravity explaining the emergence of the quantum gravity process through the fundamentality of entangled quantum information.

Keywords – Information paradox; Quantum gravity; Mass-energy-information principle; Entropy; Black hole; Black hole thermodynamics; Quantum Information.

1. Introduction

After giving a brief introduction to the black hole information paradox and explaining its crucial importance in the problem of quantum gravity theory, we review the evaporation of black holes, performing the analysis of Hawking radiation by making its presentation and description, leading to a more accurate representation, and to a better understanding of the need for a theory of quantum gravity to fully describe this phenomenon. A set of five new equivalent formulations expressing the notion of entropy based on the principle of mass-energy-information equivalence [1] is presented [2]. The relationship with the thermodynamics of black holes comes from the injection of the Hawking temperature [3] into one of these equations which makes it possible to calculate the Bekenstein Hawking entropy based on the mass of the information bit, as a new formula for black hole entropy, and to perfectly describe black holes system up to the quantum level. Since the Bekenstein Hawking entropy saturates exactly the Bekenstein bound, it is equal to it. With the help of Casini [4] and Bousso's work on the Bekenstein bound [5-13] which is equal to von Neumann entropy, this fine-grained entropy is itself equal to the entropy of Hawking radiation as “In other words, if the black hole degrees of freedom together with the radiation are producing a pure state, then the fine-grained entropy of the black hole should be equal to that of the radiation [14]. ” Moreover, the Hawking radiation is entangled with the interior of black holes. “In fact, this is precisely what happens with Hawking radiation. The radiation is entangled with the fields living in the black hole interior [14].” “The Hawking radiation, are allowing us to extract information that resides (from the semiclassical viewpoint) in a distant spacelike-separated region: the black hole interior [15].” This approach explains the black hole information paradox by calculating the entropy of entangled Hawking radiation as equal to the fine-grained entropy of the black hole itself equal to the Bekenstein

Hawking entropy. But the Ryu–Takayanagi conjecture [16,17] as a general formula for the fine-grained entropy of quantum systems coupled to gravity [14] as being equivalent to the Bekenstein Hawking entropy, the entropy of entangled Hawking radiation is equal to the gravitational fine-grained entropy of black holes.

2. Information paradox

The goal of quantum gravity is to provide a coherent framework that unifies quantum mechanics and general relativity, allowing us to understand the nature of gravity at the macroscopic and microscopic scales. A crucial question in the study of quantum gravity is understanding the process by which information escapes from the interior of the black hole. According to classical general relativity, when matter collapses under its own gravity to form a black hole, a region called the event horizon forms. The event horizon is essentially the limit beyond which nothing, including information, can escape. Any information that falls into the black hole seems to be lost forever to an outside observer. On the other hand, quantum mechanics, which is very successful in describing the behavior of elementary particles and their interactions, assumes that information is always preserved. In quantum mechanics, the evolution of a physical system is described by a unitary principle that preserves information, which means that if you know the state of a system at one time, you can in principle determine its state at any other time. This conflict between general relativity and quantum mechanics gives rise to the "information paradox" or the "black hole information paradox". It essentially wonders what happens to the information contained in matter that falls into a black hole. According to quantum mechanics, this information cannot be destroyed, but according to classical general relativity, it is apparently lost. According to quantum mechanics, information is always preserved, but classical black holes seem to destroy information, leading to a violation of the unitarity principle. That is the paradox.

3. Hawking Radiation

3.1. Presentation

A proposed solution to this paradox was discovered in 1974, by physicist Stephen Hawking, proposing that black holes are not completely black, but rather emit faint radiation due to quantum effects near the event horizon. This radiation is now known as Hawking radiation. The derivation of Hawking radiation by Hawking in 1974 was done within the framework of a free quantum field in a fixed classical curved spacetime. Now, this framework used by Hawking in 1974 is called semiclassical quantum gravity or quantum field theory in curved spacetime (QFTCS). This approach studies the interaction of quantum fields in a fixed classical space-time and predicts, among other things, the creation of particles by time-varying space-time. QFTCS provides a theoretical framework for understanding the behavior of quantum fields in the presence of gravitational fields, bridging the gap between quantum mechanics and general relativity. According to Hawking's theory, black holes can slowly lose mass and energy over time through the emission of this radiation. Hawking radiation reduces the mass and energy of the black hole and is therefore also known as black hole evaporation. The evaporation of a black hole, which results in Hawking radiation [18,19,20] is the phenomenon that an observer looking at a black hole can detect tiny blackbody radiation, black hole evaporation, emitted from the area near its event horizon.

3.2. Description

Hawking radiation is a form of black body radiation, where the black hole is considered the black body. The connection between Hawking radiation and black body radiation lies in the mathematical form of Hawking radiation. When studying the quantum effects near the event horizon, Hawking used the principles of quantum field theory and treated the event horizon as an effective temperature. The radiation that escapes from the event horizon of a black hole has a spectrum that resembles the spectrum of black body radiation at a certain temperature. This means that the temperature of the black hole, known as the Hawking temperature, determines the characteristics of the emitted radiation, just as the temperature of a black body determines the characteristics of its emitted radiation according to Planck's law. In this sense, Hawking radiation can be seen as a form of black body radiation, where the black hole event horizon acts as the equivalent of a black body emitting radiation. However, while black body radiation is a natural consequence of the thermal properties of a black body, Hawking radiation is a result of quantum mechanical effects near the event horizon of a black hole. Additionally, black body radiation is emitted by all parts of a black body, while Hawking radiation is specific to the event horizon of a black hole. This Hawking radiation gradually causes the black hole to lose mass and energy over time, eventually leading to its evaporation. The energy emitted by Hawking radiation ultimately comes from the black hole itself. As a result, the black hole loses energy, and its mass decreases over time. This energy loss through Hawking radiation is a fundamental aspect of the phenomenon and leads to the eventual evaporation of black holes over an extremely long timescale. Hawking radiation describes the process of particle emission from black holes due to quantum effects near the event horizon, while the holographic principle suggests that the information within a black hole can be represented as a hologram on its boundary. According to the conjectured gauge-gravity duality (also known as the AdS/CFT correspondence), black holes in certain cases (and perhaps in general) are equivalent to solutions of quantum field theory at a non-zero temperature. This means that no information loss is expected in black holes (since no such loss exists in the quantum field theory), and the radiation emitted by a black hole is probably the usual thermal radiation. The most rigorous realization of the holographic principle is the AdS/CFT correspondence by Juan Maldacena. By 2015, Maldacena's article had over 10,000 citations, becoming the most highly cited article in the field of high energy physics [21]. The temperature of the radiation depends on the mass of the black hole, with smaller black holes having higher temperatures. The temperature of the black hole is inversely proportional to its mass, meaning smaller black holes have higher temperatures and emit more intense radiation. Hawking radiation follows the Boltzmann distribution, which is a statistical distribution that describes the behavior of particles in a thermal equilibrium, which means that the emission of particles is consistent with the principles of thermodynamics and statistical mechanics. According to Hawking, when black holes absorb some photons in a pure state described by a wave function, they re-emit new photons in a thermal mixed state described by a density matrix. In fact, if we had a very complex quantum system which starts in a pure state, it will appear to thermalize and will emit radiation that is very close to thermal [14]. But Hawking radiation is in a pure state, is that this is in apparent contradiction to the fact that Hawking radiation is also said to be thermal. How can it be both pure and thermal? An observer located far away from the black hole, at infinity, will see the radiation from the black hole as a thermal bath of particles. This is because the observer's reference frame is influenced by the curvature of spacetime caused by the black hole. From their perspective, the radiation appears to be in a mixed state, following the laws of thermodynamics. On the other hand, an observer located very close to the event horizon, in the local region near the black hole, would describe the quantum fields as being in a vacuum state. This is because their reference frame is influenced by the strong gravitational field near the black hole. From their perspective, there are no particles present, and the fields are in a pure state. The notion of a vacuum state depends on the observer's reference frame and the geometry of the spacetime they are in. The apparent contradiction between the pure state and thermal nature of Hawking radiation is resolved by recognizing that different observers in different regions of spacetime will have different definitions of what constitutes the vacuum state and therefore the whole Hilbert space. An observer at infinity sees Hawking radiation as a thermal bath of particles in a mixed state, while an observer near the black hole sees the quantum fields in a vacuum state, in a pure state. The different perspectives arise due to the different reference frames and the curvature of spacetime near the black hole. The apparent contradiction is solved when one realizes that in a general curved spacetime there is no unique definition of the vacuum state and therefore the whole Hilbert. Consequently, an observer at infinity will see a thermal bath of particles (i.e., in a mixed state) coming from the horizon, even though the quantum fields are in the local vacuum state near the horizon. But if the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems. “In fact, this is precisely what happens with Hawking radiation. The radiation is entangled with the fields living in the black hole interior [14].”

3.3. Von Neumann Entropy

In particular, in the early stages, if we computed the von Neumann entropy of the emitted radiation it would be almost exactly thermal because the radiation is entangled with the quantum system [14].
“In other words, if the black hole degrees of freedom together with the radiation are producing a pure state, then the fine-grained entropy of the black hole should be equal to that of the radiation $S_{Black\ Hole} = S_{rad}$ [14]."
“Where fine grained entropy is the entropy of the density matrix calculated by the standard methods of quantum field theory in curved space time. In the literature, this is often simply called the von Neumann entropy”. It is Shannon’s entropy with distribution replaced by density matrix.

$$S = -\mathrm{tr}[\rho \log(\rho)] \tag{1}$$

Entanglement entropy is a measure of “quantumness” that vanishes for classical states, and it is large when quantum correlations are important. It is also a measure of complexity [22].

The von Neumann entropy is a measure of the information contained in a quantum system.
For a pure state, the von Neumann entropy is zero, indicating that the state is perfectly defined and contains no

uncertainty. In contrast, for a mixed state, which is a statistical ensemble of different pure states, the von Neumann entropy is non-zero, indicating the presence of uncertainty.

3.4. Pure states

A pure quantum state refers to a state of a quantum system that can be described by a single, unique wavefunction. A pure state is one where the wavefunction provides a complete description of the system, meaning that there is no uncertainty or lack of knowledge about the system's properties. In the context of quantum entanglement, a pure quantum state can be maximally entangled. Entanglement is a phenomenon in which the quantum states of two or more particles become correlated in such a way that the state of one particle cannot be described independently of the others. When two particles are maximally entangled, their quantum states are completely intertwined, and any measurement or change in the state of one particle instantaneously affects the state of the other, regardless of the distance between them.

3.5. GHZ states

A maximally entangled state, also known as a Bell state, is a specific type of entangled state that exhibits the highest degree of entanglement possible for a given system.
Maximally entangled states are often referred to as "maximally entangled quantum states".
The GHZ state is an example of a pure state in quantum mechanics, specifically in the context of multi-qubit systems. GHZ states and Bell states represent different levels of entanglement. GHZ states involve entanglement among three or more particles, while Bell states are specific instances of entanglement between two particles.
GHZ states can be seen as generalizations of Bell states, encompassing the same kind of correlation but extended to multiple particles. The GHZ state is characterized by the fact that all the particles are entangled such that their outcomes are perfectly correlated. If one particle is measured in the $|0\rangle$ state, the other two particles will also be found in the $|0\rangle$ state with certainty, and the same applies to the $|1\rangle$ state. This correlation holds regardless of the spatial separation between the particles. The GHZ state are used to demonstrate the non-local correlations between particles.

Nota-Bene: a semi-classical perspective can explain the notion of entanglement; indeed, the entanglement of photons can be explained in terms of relativistic properties of space-time as defined by Einstein as well as by quantum mechanics. Regarding photons and the theory of special relativity, since all photons move at the speed of light, the separation between these two photons would be zero from the point of view of these photons. Entangled photons that share one or more degrees of freedom at the quantum level between each photon cannot be described independently of the quantum state of the others because they share degrees of freedom. We can therefore say that quantum theory is local in the strict sense defined by special relativity and, as such, the term "quantum non-locality" is sometimes considered a misnomer [23].

3.6. Representation

In apparent contradiction to the fact that Hawking radiation is also said to be thermal, this radiation is in its pure state. As the radiation is entangled with the fields living inside the black hole, the black hole radiation can lead to a multipartite entanglement of GHZ states. "The Hawking radiation, are allowing us to extract information that resides from the semiclassical viewpoint in a distant spacelike-separated region: the black hole interior [15]." Moreover, since the combined degrees of freedom of a black hole and the radiation it produces form a pure state, then the fine-grained entropy of the black hole should be equal to the entropy of the radiation. The relationship between Hawking radiation and quantum gravity is close, since Hawking radiation involves both, gravity, due to the presence of black holes, and quantum mechanics, due to its quantum origin, it is often considered an indication of the need for a theory of quantum gravity to fully describe this phenomenon. It is nevertheless possible to calculate the entropy of entangled Hawking radiation based on informational theory, entropic information theory capable of fully describing the evaporation of black hole phenomena and solving the information paradox.

4. Entropic Information Approach

4.1. New formulations of entropy

In the approach of entropic information, the choice of the terms "entropic information" must be related to the term "entropic" used in the perspective of entropic gravity, where entropic gravity is a theory in physics that describes gravity as an entropic force, a consequence of a system's tendency to increase its entropy, not as a fundamental interaction. As according to the entropic information approach, quantum information is considered an emergent form of the degree of freedom as: "Information is a quantum state change due to the modification of a degree of freedom with respect to the quantum system considered." Entropic information approach is based on the mass of information bit formula from mass-energy-information equivalence principle from Melvin Vopson, principle testable by a given experimental protocol [24], leading to the fundamentality of quantum information considered as physical with a finite and quantifiable mass.

We start this entropic information approach by introducing the mass of information into the hidden thermodynamics of Louis De Broglie.

About the hidden thermodynamics of isolated particles, it is an attempt to bring together the three furthest principles of physics: the principles of Fermat, Maupertuis, and Carnot, that De Broglie has had his final idea. Entropy becomes a sort of opposite to action with an equation that relates the only two universal dimensions of the form:

$$\frac{\text{action}}{\text{h}} = -\frac{\text{entropy}}{\text{k}} \tag{2}$$

Where:

k: Boltzmann constant.
h: Planck constant.
With $action = Energy * Time$
and $Energy = mc^2$

with mass of information bit:

$$\text{mass}_{\text{bit}} = \frac{\text{k T ln(2)}}{\text{c}^2} \tag{3}$$

Where:

k: Boltzmann's constant.
T: the temperature at which the bit of information is stored.
c: speed of light.

$$\frac{\text{action}}{\text{h}} = \frac{\text{mc}^2\text{t}}{\text{h}} = -\frac{\text{entropy}}{\text{k}} = -\frac{\text{k ln(w)}}{\text{k}} \tag{4}$$

$$\frac{\text{action}}{\text{h}} = \frac{\text{mc}^2\text{t}}{\text{h}} = \frac{\frac{\text{k T ln(2)}}{\text{c}^2}\text{c}^2\text{t}}{\text{h}} = -\frac{\text{entropy}}{\text{k}} = -\frac{\text{k ln(w)}}{\text{k}} \tag{5}$$

$$\frac{\text{action}}{\text{h}} = \frac{\text{mc}^2\text{t}}{\text{h}} = \frac{\text{k T ln(2)t}}{\text{h}} = -\frac{\text{entropy}}{\text{k}} = -\ln(\text{W}) \tag{6}$$

$$\ln(\text{W}) = -\frac{\text{kT ln(2)t}}{\text{h}} \tag{7}$$

Moreover, as Entropy:

$$S = k \ln(W) \tag{8}$$

We obtain a new value for the general entropy S formula based on the hidden thermodynamics of de Broglie with the introduction of mass of bit of information:

$$k \ln(W) = -k \frac{k\,T \ln(2) t}{h} \tag{9}$$

$$\boxed{S = -k^2 \frac{T \ln(2) t}{h}} \tag{10}$$

With S *entropy* expressed in the number of bits of information. The negative sign refers to a state in which the disorder or randomness of a system decreases, or the uncertainty or information content decreases, implicating a movement towards a more organized, structured, or predictable state.

Where:

k: Boltzmann's constant.
h: Planck constant.
T: the temperature at which the bit of information is stored.
t: time required to change the physical state of the information bit.

We obtain the validity proof by the Landauer limit as:

$$\frac{k\,T \ln(2) t}{h} = \frac{mc^2 t}{h} \tag{11}$$

$$kT \ln(2) = mc^2 \tag{12}$$

Indeed, the Landauer limit is the minimum possible amount of energy required to erase one bit of information, known as the Landauer limit:

As $E = mc^2$

$$E = kT \ln(2) \tag{13}$$

This limit is the minimum amount of energy possible needed to erase a bit of information, known as the Landauer's limit. Landauer's principle is fully compatible with the laws of thermodynamics [25,26,27,28]. Landauer's principle can be derived from microscopic considerations [29] as well as well-established properties of Shannon–Gibbs–Boltzmann entropy [25]. Landauer's principle applies to both classical and quantum information. Landauer's principle has now been verified experimentally for classical bits and quantum qubits [29,30]. This important physical prediction that links information theory and thermodynamics was verified experimentally for the first time in 2012 [31]. The principle therefore appears fundamental and universal in its application. About these perspectives, the information is therefore directly related to the fundamental physics of nature.

Some equivalent formulations can be formulated as:

4.1.1. Entropy Boltzmann formulation

Regarding the relation Energy $=$ kT , this new entropy concept can take the form of

$$S = -kT\frac{k\ln(2)\,t}{h} \tag{14}$$

4.1.2. Entropy Einstein formulation

With Einstein mass energy equivalence, Energy $= mc^2$, this is:

$$S = -mc^2\frac{k\ln(2)\,t}{h} \tag{15}$$

4.1.3. Entropy Planck formulation

with Planck Einstein relation $E = h\,\nu$
this is with h$=\frac{E}{\nu}$

$$S = -h\nu\frac{k\ln(2)\,t}{\frac{E}{\nu}} \tag{16}$$

$$S = -k\ln(2)t\nu \tag{17}$$

4.1.4. Entropy Avogadro formulation

With kT $=\frac{RT}{N_A}$,

$$S = -\frac{RT}{N_A}\frac{k\ln(2)\,t}{h} \tag{18}$$

4.1.5. Entropy Fine Structure formulation

With $N_A = \frac{MuA_r(e)c\alpha^2}{2R_\infty h}$
N_A= Avogadro number

$$S = -\frac{RT}{\frac{MuA_r(e)c\alpha^2}{2R_\infty h}}\frac{k\ln(2)\,t}{h} \tag{19}$$

$$S = -\frac{2R\infty RTk\ln(2)\,t}{MuA_r(e)c\alpha^2} \tag{20}$$

As the Boltzmann constant may be used in place of the molar mass constant by working in pure particle count, N, rather than the amount of substance, n.

where:
$R\infty$: Rydberg constant
R molar gas constant
T is the temperature at which the bit of information is stored
k Boltzmann constant
t Time required to change the physical state of the information bit

Mu molar mass constant
Ar(e) relative atomic mass of the electron.
c speed of light in a vacuum,
α fine-structure constant,

Based on that view, the entropic information approach is founded on the bit of information such as the number of bits of the system, the number of bits necessary to specify the actual microscopic configuration among the total number of microstates allowed and thus characterize the macroscopic states of the system under consideration. The entropic information theory approach can formulate a set of five equivalent equations expressing entropy, Boltzmann, Einstein, Planck, Avogadro and fine structure formulation as seen in Figure 1.

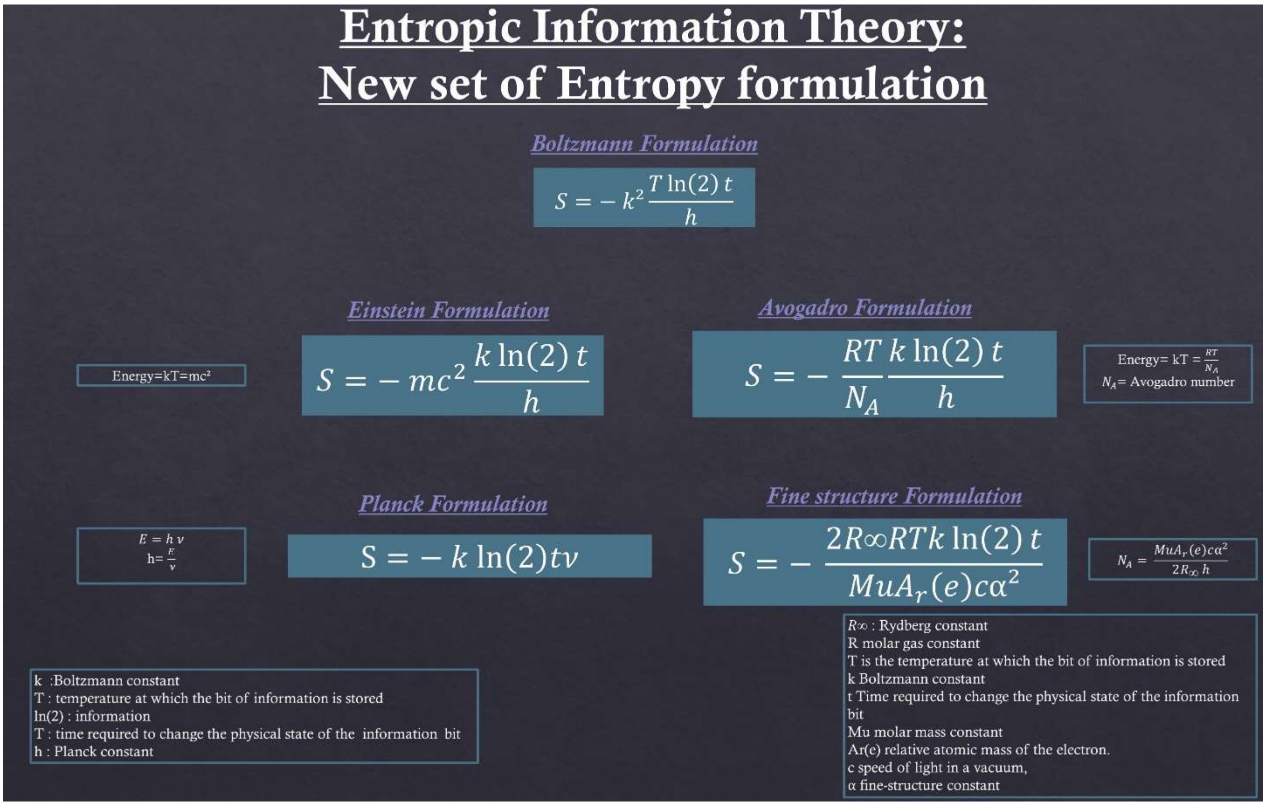

Figure 1. Set of new five equivalent formulas expressing the entropy based on formula of the mass of the bit of information.

4.2. Black Hole Entropy

The introduction of the Hawking Temperature permits to introduce to the problematic of black holes and to express it as being an informational process down to quantum level. Moreover, this new entropy formula provides a deeper and structurally different underlying theory that can explain the quantity describe by the entropy of black hole based on mass of the information bit.

We start from this equation obtained by the introduction of mass bits of information into the hidden thermodynamics of Louis de Broglie. See (Eq7):

$$\ln(\mathrm{W}) = -\frac{\mathrm{k\,T\,\ln(2)\,t}}{\mathrm{h}} \tag{21}$$

wherein we inject the Hawking Temperature represented by this formula:

$$\mathrm{T_H} = \frac{1}{\mathrm{k}} \frac{\hbar \mathrm{c}^3}{8\pi \mathrm{GM}} \tag{22}$$

$$\ln(W) = -\frac{k\frac{1}{k}\frac{\hbar c^3}{8\pi GM}\ln(2)\,t}{h} \tag{23}$$

With $\hbar = \frac{h}{2\pi}$

$$\ln(W) = -\frac{k\frac{1}{k}\frac{hc^3}{16\pi^2 GM}\ln(2)\,t}{h} \tag{24}$$

we obtain after simplification of k and h:

$$\ln(W) = -\frac{c^3\ln(2)\,t}{16\pi^2 GM} \tag{25}$$

With entropy according to Boltzmann's principle:

$$S = k\ln(W) \tag{26}$$

We obtain the black hole entropy formula from entropic information approach, at the hawking temperature, based on the mass of bit of information,

$$\boxed{S = k\ln(W) = -k\frac{c^3\ln(2)\,t_{evap}}{16\pi^2 GM}} \tag{27}$$

The ln 2 factor comes from defining the information as the logarithm to the base 2 of the number of quantum states [13]. The introduction of the Hawking temperature formula which is expressed by all the constants of modern physics combining: relativity, with c, the speed of light, gravitation, with the gravitational constant G, quantum physics with the reduced Planck constant and thermodynamics with the Boltzmann constant k, leads to new expressions of black hole entropy as seen in Figure 2. Using one of the five new equivalent equations expressing the notion of entropy, concerning the thermodynamics of black holes and the entropy of black holes, the equation of the "Boltzmann formulation" applied to the thermodynamics of black holes by the injection of the Hawking temperature, can solve the paradox of information and can express the gravitational fine-grained entropy of black holes. Let's see how.

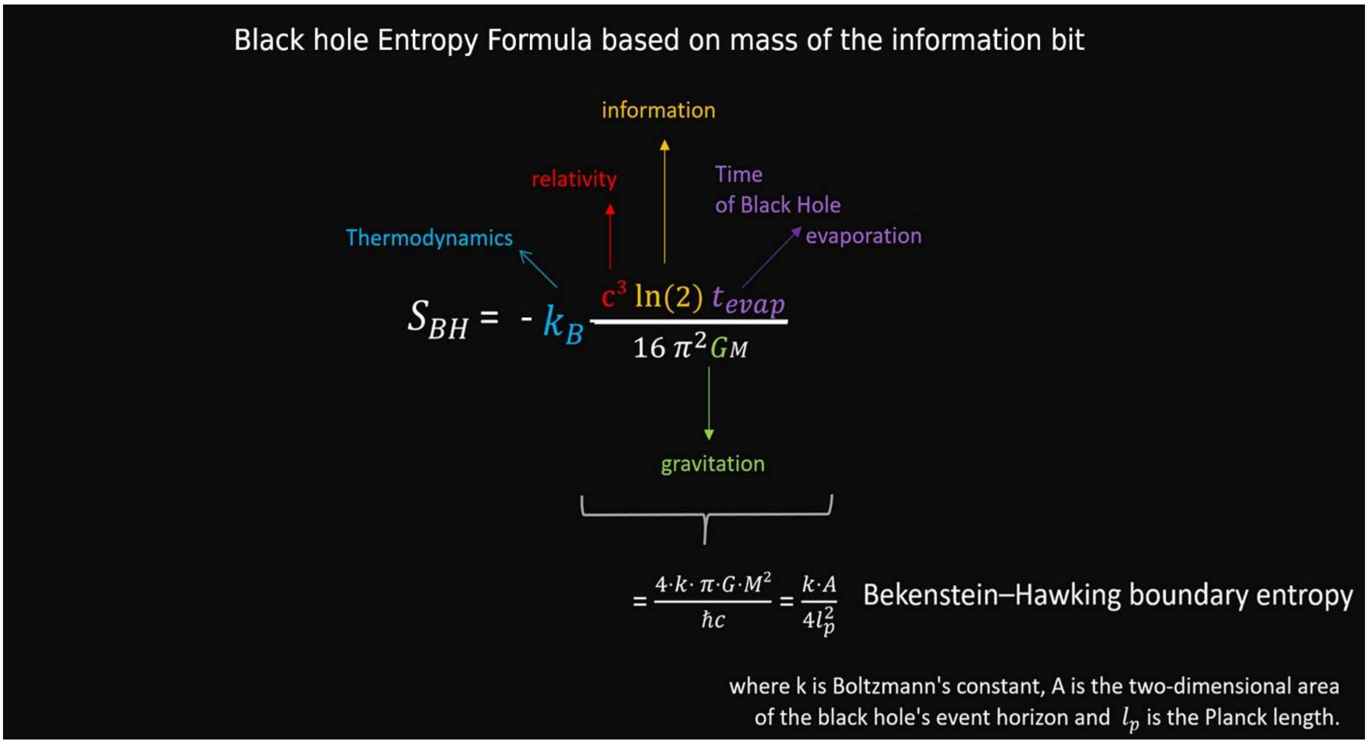

Figure 2. Black hole entropy formula based on mass of the bit of information from entropic information theory.

In physics, an upper limit on the thermodynamic entropy S, or Shannon entropy H, that can be contained within a given finite region of space which has a finite amount of energy is the Bekenstein bound (named after Jacob Bekenstein)—or conversely, the maximal amount of information required to perfectly describe a given physical system down to the quantum level [32]. Furthermore, generally, the entropy is proportional to the number of bits necessary to describe the state of the system considered. This result, which was demonstrated by Jacob Bekenstein corresponds to the interpretation in terms of bits of information. The universal bound originally founded by Jacob Bekenstein in 1981 as the inequality [33, 34].

$$S \leq \frac{2\pi kRE}{\hbar c} \tag{28}$$

It implies that the information of a physical system, or the information necessary to perfectly describe that system, must be finite if the region of space and the energy are finite.

In informational terms, the relation between thermodynamic entropy S and Shannon entropy H is given by:

$$S = k\,H\,\ln(2) \tag{29}$$

$$S = k\,H \ln(2) = \frac{2\pi kRE}{\hbar c} \tag{30}$$

$$S = k\frac{2\pi RE}{\hbar c \ln(2)}\ln(2) = \frac{2\pi kRE}{\hbar c} \tag{31}$$

where H is the Shannon entropy expressed in number of bits contained in the quantum states in the sphere. The ln 2 factor comes from defining the information as the logarithm to the base 2 of the number of quantum states [13].

Boltzmann ‘entropy formula can be derived from Shannon entropy formulae when all states are equally probable.

$$S = -k\sum p_i \ln p_i = k\sum\frac{\ln(W)}{W} = k\ln(W) \tag{32}$$

$$k\,\ln(W) = \frac{2\pi kRE}{\hbar c} \tag{33}$$

$$S = k\frac{2\pi RE}{\hbar c \ln(2)}\ln(2) = \frac{2\pi kRE}{\hbar c} \tag{34}$$

The black holes entropic information formula given as follows (see equation 27):

$$S = k\,\ln(W) = -\frac{kc^3 \ln(2) t_{evap}}{16\pi^2 GM} \tag{35}$$

As S, Boltzmann entropy can be derived from Shannon entropy H and following the relation between thermodynamic entropy and Shannon entropy.$S = k\,H\,\ln(2)$; we obtain:

$$S = k\ln(W) = -k\frac{c^3\ln(2)\,t_{evap}}{16\pi^2 GM} = k\frac{2\pi RE}{\hbar c} = k\,H\ln(2) \tag{36}$$

$$-\frac{c^3 t_{evap}}{16\pi^2 GM} = \frac{2\pi RE}{\hbar c \ln(2)} \tag{37}$$

where S is the entropy, k is Boltzmann's constant, R is the radius of a sphere that can enclose the given system, E (mc^2) is the total mass–energy including any rest masses, ħ is the reduced Planck constant, and c is the speed of light.

With $\frac{c^2}{2\,GM} = \frac{1}{R}$

$$-\frac{c t_{evap}}{8\pi^2 R} = \frac{2\pi cRM}{\hbar \ln(2)} \tag{38}$$

Simplification by c,

$$\frac{t_{evap}}{8\pi^2 R} = -\frac{2\pi RM}{\hbar \ln(2)} \tag{39}$$

We obtain,

$$t_{evap} = -\frac{16\pi^3 R^2 M}{\hbar \ln(2)} \tag{40}$$

With $\frac{2GM}{c^2} = R$

$$t_{evap} = -\frac{32\pi^3 GRM^2}{\hbar \ln(2)\, c^2} \tag{41}$$

With $\frac{2G}{c^2} = R$

$$t_{evap} = -\frac{64\pi^3 G^2 M^3}{\hbar c^4 \ln(2)} \tag{42}$$

With A, the area of the black hole horizon: $16\,\pi(\frac{GM}{c^2})^2$

$$\boxed{t_{evap} = -\frac{4\pi^2 M}{\hbar \ln(2)} A} \tag{43}$$

To inject in $k\frac{c^3\,ln(2)t_{evap}}{16\pi^2 GM}$,

$$\boxed{S = -k\frac{c^3 \ln(2)\, 4\pi^2 M}{16\pi^2 GM\, \hbar \ln(2)} A} \tag{44}$$

As a side note, it can also be shown that the Boltzmann entropy is an upper bound to the entropy that a system can have for a fixed number of microstates meaning:

$$S \leq k \ln(W) \tag{45}$$

With W reflecting the degree of freedom of a system.

We must take in account that the Bekenstein–Hawking boundary entropy of three-dimensional black holes exactly saturates the bound.

$$S = k \ln(W) \tag{46}$$

$$= -k\frac{c^3 \ln(2)\, 4\pi^2 M}{16\pi^2 GM\, \hbar \ln(2)} A$$

$$= \frac{4k\pi GM^2}{\hbar c}$$

$$= \frac{k}{4}\left(\frac{c^3}{\hbar G}\right) A = k\frac{A}{4l_p^2}$$

$$= k\frac{2\pi RE}{\hbar c}$$

$$= k\, H \ln(2)$$

The interpretation of negative time in the context of Hawking radiation is related to the mathematical description of the quantum fields near the event horizon of a black hole. Indeed, quantum field theory is a framework used to describe the behavior of subatomic particles and their interactions. In this theory, complex numbers and Fourier analysis are used to analyze the behavior of quantum fields near the event horizon. In this analysis, negative frequency solutions correspond to positive energy particles propagating forward in time outside the black hole and negative energy particles propagating backward in time inside the black hole. This means that, mathematically, negative time is used to describe the behavior of particles inside the black hole, while positive time is used to describe the behavior of particles outside the black hole.

It's important to note that negative time is not a physical quantity, but rather a mathematical result derived from certain calculations.

4.3. Casini's works

Casini proves the thermodynamics interpretation in the form of Bekenstein bound as valid. Indeed, we know following the work of Casini in 2008 [4] about the von Neumann entropy and the Bekenstein bound, that the proof of the Bekenstein bound is valid using quantum field theory [5-13].

For example, given a spatial region V, Casini defines the entropy on the left-hand side of the Bekenstein bound as:

$$\mathrm{Sv} = S(\rho_V) - S(\rho_V^0) = -\mathrm{tr}(\rho_V \log \rho_V) + \mathrm{tr}(\rho_V^0 \log \rho_V^0) \tag{47}$$

S_V where S (ρ_V) is the von Neumann entropy of the reduced density matrix ρ_V associated with V, V in the excited state ρ, and S (ρ_V^0) is the corresponding von Neumann entropy for the vacuum state ρ^0.

Casini defines the right-hand side of the Bekenstein bound as the difference between the expectation value of the modular Hamiltonian in the excited state and the vacuum state,

$$K_V = \mathrm{tr}(K\, \rho_V) - \mathrm{tr}(K\rho_V^0) \tag{48}$$

With these definitions, the bound reads SV ≤ KV, which can be rearranged to give:

$$\mathrm{tr}(\rho_V \log \rho_V) - \mathrm{tr}(\rho_V \log \rho_V^0) \geq 0 \tag{49}$$

This is simply the statement of positivity of quantum relative entropy, which proves the Bekenstein bound.

Diving into Casini's work with the black hole's entropic information formula, we obtain new enlightening about black hole fine-grained entropy.

The ingenious proposal of Casini [4] is to replace 2 π R E, by:

$$K_V = \mathrm{tr}(K\, \rho_V) - \mathrm{tr}(K\, \rho_V^0) \tag{50}$$

Indeed, in Casini's work, on the right-hand side of the Bekenstein bound, a difficult point is to give a rigorous interpretation of the quantity 2 π R E, where R is a characteristic length scale of the system and E is a characteristic energy. This product has the same units as the generator of a Lorentz boost, and the natural analog of a boost in this situation is the modular Hamiltonian of the vacuum state $K = -\log \rho_V^0$.

With these definitions, the bound reads

$$\mathrm{Sv} \le \mathrm{K_V} \tag{51}$$

The version of the Bekenstein bound is Sv $\le K_V$, namely:

$$\mathrm{S}(\rho_\mathrm{V}) - \mathrm{S}(\rho_\mathrm{V}^0) \le \mathrm{tr}(\mathrm{K}\ \rho_\mathrm{V}) - \mathrm{tr}(\mathrm{K}\, \rho_\mathrm{V}^0) \tag{52}$$

is equivalent to:

$$\mathrm{Sv} \equiv \mathrm{S}(\rho_\mathrm{V} \mid \rho_\mathrm{V}^0) \equiv \mathrm{tr}(\rho_\mathrm{V}\,(\log \rho_\mathrm{V} - \log \rho_\mathrm{V}^0)) \ge 0 \tag{53}$$

As black holes entropic information formula is equal to Bekenstein universal bound.

$$-\frac{\mathrm{kc^3 ln(2) t_{evap}}}{16\pi^2 \mathrm{GM}} = \frac{2\pi \mathrm{kRE}}{\hbar c} \tag{54}$$

As the difference between the expectation value of the modular Hamiltonian in the excited state and the vacuum state $K_V = tr(K\ \rho_V) - \mathrm{tr}(\mathrm{K}\, \rho_V^0)$ is equal to Bekenstein universal bound.
We obtain:

$$\begin{aligned}
\mathrm{Sv} &\equiv \mathrm{S}(\rho_\mathrm{V} \mid \rho_\mathrm{V}^0) = \mathrm{S}(\rho_\mathrm{V}) - \mathrm{S}(\rho_\mathrm{V}^0) \\
&= -\mathrm{tr}(\rho_\mathrm{V} \log \rho_\mathrm{V}) + \mathrm{tr}(\rho_\mathrm{V}^0 \log \rho_\mathrm{V}^0) \\
&= \mathrm{tr}(\rho_\mathrm{V}\,(\log \rho_\mathrm{V} - \log \rho_\mathrm{V}^0)) \\
&= \mathrm{K_V} = \mathrm{tr}(\mathrm{K}\, \rho_\mathrm{V}) - \mathrm{tr}(\mathrm{K}\, \rho_\mathrm{V}^0) \\
&= \frac{2\pi \mathrm{kRE}}{\hbar c} \\
&= -\frac{\mathrm{kc^3 ln(2) t_{evap}}}{16\pi^2 \mathrm{GM}}
\end{aligned} \tag{55}$$

4.4. Global formulation

Starting from the hidden thermodynamics of Louis de Broglie, where we inject the mass of information bit of Vopson's mass-energy-information equivalence principle, proposing that a bit of information is not only physical, as already demonstrated, but that it has a finite and quantifiable mass, leading to the expression of a new formulation of entropy based on the degrees of freedom, on the number of bits of the system. Several formulations of this entropy formula based on different relationships, based on Einstein's mass-energy equivalence or on the Einstein-Planck relation, or based on the Avogadro number, or on the fine structure relation have been formulated, leading to a set of five new equivalent entropy formulas. After that, we inject the Hawking temperature, into one of these formulas leading to a new formulation of the Bekenstein-Hawking entropy formula where a new formulation based on an evaporation time of black holes and the degrees of freedom of black holes seen as an entire quantum system was calculated. The entropic black holes information formula can calculate the entropy of the entangled Hawking radiation up to the quantum system, describing black holes.

The entropic information formula of black holes saturates exactly the universal bound, the entropic information formula of black holes is equal to the universal bound originally found by Jacob Bekenstein [34] which is equal by Casini's work [4] to the difference between the expectation value of the modular Hamiltonian in the excited state and in the vacuum state. Naive definitions of entropy and energy density in quantum field theory suffer from ultraviolet divergences.

In the case of the Bekenstein bound, ultraviolet divergences can be avoided by taking the differences between the quantities calculated in the excited state and the same quantities calculated in the vacuum state [35]. The entropic information formula for black holes can calculate Bekenstein–Hawking entropy; as the entropy of the black hole saturates exactly the Bekenstein bound so that it is equal to the Bekenstein bound which is itself according to Casini's work equal to the von Neumann entropy, itself equal to that of the Hawking radiation, which with the degrees of freedom of black holes produces a pure state, while Hawking radiation being entangled with fields inside black holes, allowing us to extract information that resides from the semi-classical point of view, inside the black hole; at the end of evaporation, the complete von Neumann entropy is again equal to 0, so no information is lost!

Finally, we obtain:

$$\boxed{\begin{aligned}
\mathrm{Sv} &\equiv \mathrm{S}(\rho_V \mid \rho_V^0) = \mathrm{S}(\rho_V) - \mathrm{S}(\rho_V^0) \\
&= -\mathrm{tr}(\rho_V \log \rho_V) + \mathrm{tr}(\rho_V^0 \log \rho_V^0) \\
&= \mathrm{tr}(\rho_V (\log \rho_V - \log \rho_V^0)) \\
&= \mathrm{K}_V = \mathrm{tr}(\mathrm{K}\, \rho_V) - \mathrm{tr}(\mathrm{K}\, \rho_V^0) \\
&= \frac{2\pi k R E}{\hbar c} \\
&= \frac{\mathrm{k}}{4}\left(\frac{\mathrm{c}^3}{\hbar \mathrm{G}}\right)\mathrm{A} = \frac{4\mathrm{k}\pi \mathrm{G}\mathrm{M}^2}{\hbar \mathrm{c}} = \mathrm{k}\,\frac{\mathrm{A}}{4\mathrm{l}_\mathrm{p}^2} \\
&= -\frac{kc^3 \ln(2) t_{evap}}{16\pi^2 GM} = -\,k\,\frac{c^3\, ln(2)\, 4\boldsymbol{\pi}^2 M}{16\pi^2 GM\, \hbar\, ln(2)} A \\
&= \mathrm{k}\ \ln(\mathrm{W}) \\
&= k\, H \ln(2)
\end{aligned}} \tag{56}$$

with $t_{evap} = -\frac{4\pi^2 M}{\hbar\, ln(2)}\mathrm{A}$

with A, the area of the black hole horizon: $16\,\pi(\frac{GM}{c^2})^2$

with l_p: Planck length, $\sqrt{\frac{\hbar G}{c^3}}$

The Page curve is typically applying to an evaporating black hole system, describing the von Neumann entropy of the evaporated Hawking radiation as a function of time. According to Anna Karlsson in the "Replica wormhole and island incompatibility with monogamy of entanglement" article [36], "the replica wormhole derivation and the island conjecture rely on density matrix theory e.g. for the expression for the radiation entropy. Since entanglement is monogamous in density matrix theory, it is problematic to employ that theory in the presence of non-monogamous entanglement. Effectively, the two methods introduce new physics (non-monogamy, contrary to the present claim) and the treatment risks being inconsistent". The formula about black holes entropic information is a new formulation of Bekenstein-Hawking entropy, new formulation based on information; it is equal to Bekenstein bound, as Bekenstein-Hawking entropy saturates exactly the Bekenstein bound and equal to von Neumann entropy which is the fine-grained entropy, as in Casini's work, the von Neumann entropy calculates the Bekenstein bound, moreover the black holes entropy horizon law turns out to be a special case of the Ryu–Takayanagi conjecture. See Figure 3. We must take note that the first version of the fine-grained entropy formula discovered by Ryu and Takayanagi is a general formula for the fine-grained entropy of quantum systems coupled to gravity [14]. At this level we can relate the entropic information of black holes with the Ryu-Takayanagi formula which is a general formula for the fine-grained entropy of quantum systems coupled to gravity. As the black holes entropy horizon law which turns out to be a special case of the Ryu–Takayanagi conjecture, we can put emphasis on the process of emergence of quantum gravity through the fundamentality of entangled quantum information.

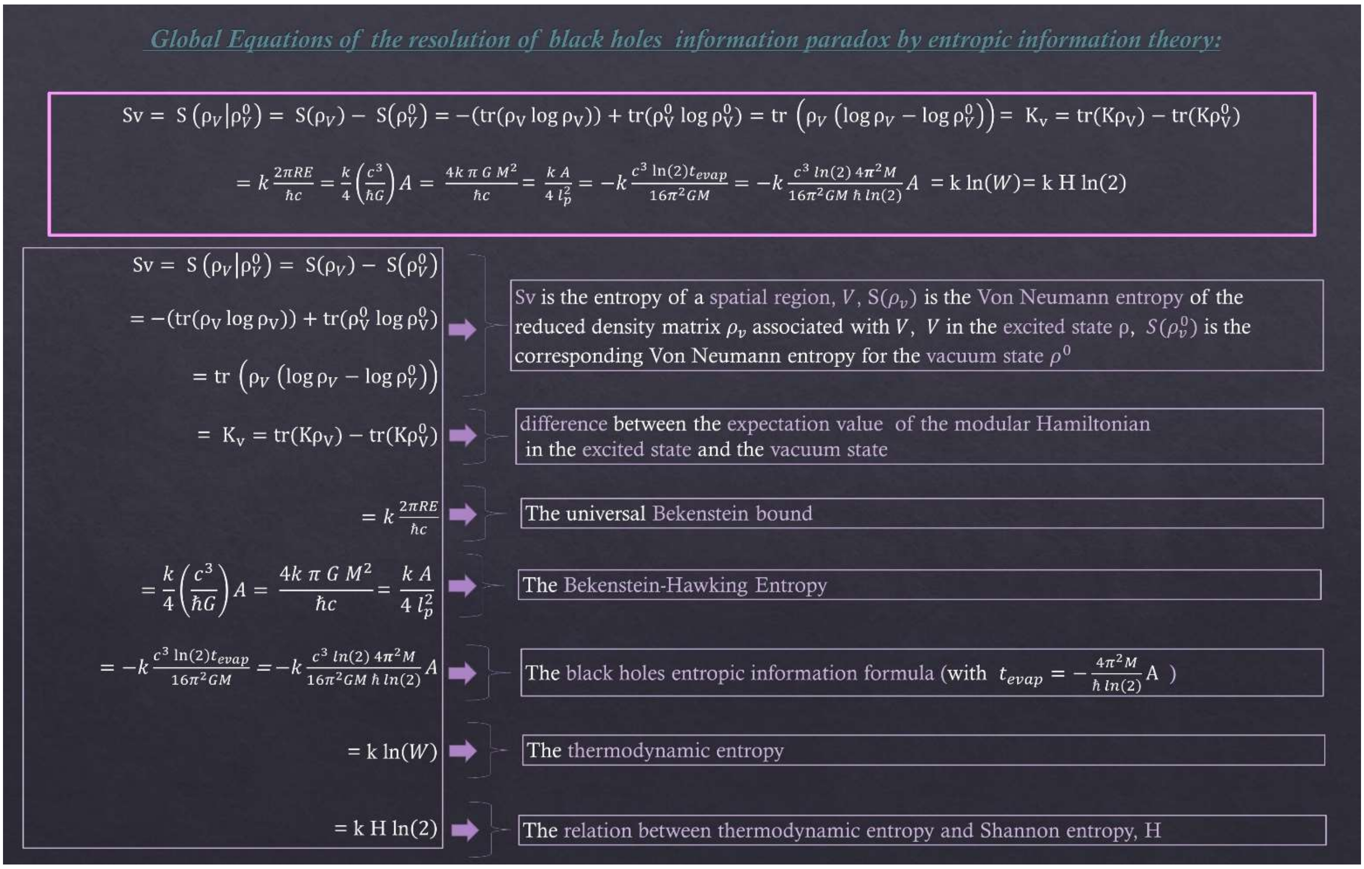

Figure 3. Global equations of the resolution of black holes information paradox by entropic information theory

5. Ryu–Takayanagi conjecture

The Ryu-Takayanagi conjecture is a fundamental result in the field of quantum gravity and holography. It establishes a connection between the entanglement entropy in a quantum field theory and the geometry of its dual gravitational theory in one higher dimension. The conjecture was proposed by Shinsei Ryu and Tadashi Takayanagi in 2006 and has since been supported by a large body of evidence.

The Ryu–Takayanagi formula calculates the entropy of quantum entanglement in conformal field theories on

Bekenstein-Hawking entropy of black holes in the context of Juan Martín Maldacena's holographic principle, in which conformal field theories on a surface form a gravitational theory in a closed volume. The first version of the fine-grained entropy formula was discovered by Ryu and Takayanagi [37]. It was subsequently refined and generalized by several authors [38-45]. Originally, the Ryu-Takayanagi formula was proposed to calculate holographic entanglement entropy in anti-de Sitter spacetime, but the present understanding of the formula is much more general. It requires neither holography, nor entanglement, nor anti-de Sitter spacetime. Rather it is a general formula for the fine-grained entropy of quantum systems coupled to gravity [14]. The black-hole entropy is proportional to the area of its event horizon. The black holes entropy horizon law turns out to be a special case of the Ryu–Takayanagi conjecture. The black-holes entropy area relationship was generalized to arbitrary regions via the Ryu–Takayanagi formula, which relates the entanglement entropy of a boundary conformal field theory to a specific surface in its dual gravitational theory [46], but the current understanding of the formula is much more general. As being a general formula for the fine-grained entropy of quantum systems coupled to gravity [14].
Casini's work on von Neumann entropy and the Bekenstein bound, gives the proof that the Bekenstein bound is valid using quantum field theory. Bekenstein Hawking entropy saturates exactly the Bekenstein bound, which is equal to von Neuman entropy, according to the works of Casini and Bousso.

Bekenstein Hawking entropy = Bekenstein bound= von Neumann entropy

"In fact, this is precisely what happens with Hawking radiation. The radiation is entangled with the fields living in the black hole interior [14]."
"In other words, if the black hole degrees of freedom together with the radiation are producing a pure state, then the fine-grained entropy of the black hole should be equal to that of the radiation $S_{Black\ Hole} = S_{rad}$ [14]."

Bekenstein Hawking entropy = Bekenstein bound= von Neumann entropy = entropy of entangled Hawking radiation.
With Ryu-Takayanagi formula (black holes entropy horizon law being a special case):

entropic information formula for black holes entropy
=Bekenstein Hawking entropy
= Bekenstein bound
= von Neumann entropy
= entropy of entangled Hawking radiation
= fine-grained entropy of quantum systems coupled to gravity
=gravitational fine-grained entropy of black holes

In the context of quantum systems coupled to gravity, the fine-grained entropy refers to the measure of the information content or uncertainty associated with the quantum state of the system when gravity is considered.

6. Conclusion

After presenting, describing, and representing the concept of black holes evaporation, we show that we can calculate the entropy of the entangled Hawking radiation of black holes, based on a new informational approach built on the mass of the information bit as a finite and quantifiable mass. With the help of one of the five new equivalent equations of entropy formulated by the entropic information approach and by introducing the Hawking temperature into one of these equations, we can express the entropy of black holes and express it as an informational process up to the quantum level. This new informational entropy of the black hole calculates the Bekenstein Hawking entropy that saturates the Bekenstein bound, which according to Casini and Bousso is equal to the von Neumann entropy, and since the combined degrees of freedom of a black hole and the radiation it produces form a pure state, then the fine-grained entropy of the black hole should be equal to the entropy of the radiation. This radiation is entangled with the fields inside black holes (GHZ states), which allows us to extract information that resides from the semi-classical point of view, inside black holes. In addition, the Ryu–Takayanagi conjecture as a general formula for the fine-grained entropy of gravity-coupled quantum systems, which is equal to the Bekenstein Hawking entropy, makes it possible to calculate the entropy of the entangled Hawking radiation as the gravitational fine-grained entropy of black holes. Entropic information theory can describe in an informational way the phenomena of black holes evaporation and solve the information paradox while calculating the entropy of the entangled Hawking radiation as the gravitational fine-grained entropy of black holes.

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References

[1] Vopson M.M., The mass-energy-information equivalence principle. AIP Adv. 2019;9(9):095206. DOI:https://doi.org/10.1063/1.5123794,
[2] Denis , O. (2023). The Emergent Entangled Informational Universe. Physical Science International Journal, 27(1), 54–81. https://doi.org/10.9734/psij/2023/v27i1777.
[3] Hawking SW. (1974-03-01). Black hole explosions?". Nature. 1974;248(5443):30–31. DOI:10.1038/248030a0. ISSN1476-4687. S2CID4290107.
[4] Casini H. Relative entropy and the Bekenstein bound. Class Quantum Grav. 2008;25(20):205021. DOI: 10.1088/0264-9381/25/20/205021. arXiv:0804.2182
[5] *Bousso, Raphael (1999). "Holography in general space-times". Journal of High Energy Physics.* ***1999*** *(6): 028. arXiv:hep-th/9906022. Bibcode:1999JHEP...06..028B. doi:10.1088/1126-6708/1999/06/028. S2CID 119518763.*
[6] *Bousso, Raphael (1999). "A covariant entropy conjecture". Journal of High Energy Physics.* ***1999*** *(7): 004. arXiv:hep-th/9905177. Bibcode:1999JHEP...07..004B. doi:10.1088/1126-6708/1999/07/004. S2CID 9545752.*
[7] *Bousso, Raphael (2000). "The holographic principle for general backgrounds". Classical and Quantum Gravity.* ***17*** *(5): 997–1005. arXiv:hep-th/9911002. doi:10.1088/0264-9381/17/5/309. S2CID 14741276.*
[8] *Bekenstein, Jacob D. (2000). "Holographic bound from second law of thermodynamics". Physics Letters B.* ***481*** *(2–4): 339–345. arXiv:hep-th/0003058. doi:10.1016/S0370-2693(00)00450-0. S2CID 119427264.*
[9] *Bousso, Raphael (2002). "The holographic principle" (PDF). Reviews of Modern Physics.* ***74*** *(3): 825–874. arXiv:hep-th/0203101. doi:10.1103/RevModPhys.74.825. S2CID 55096624. Archived from the original (PDF) on 2011-08-12. Retrieved 2010-05-23.*
[10] Jacob D. Bekenstein, "Information in the Holographic Universe: Theoretical results about black holes suggest that the universe could be like a gigantic hologram", *Scientific American*, Vol. 289, No. 2 (August 2003), pp. 58-65. Mirror link.
[11] *Bousso, Raphael; Flanagan, Éanna É.; Marolf, Donald (2003). "Simple sufficient conditions for the generalized covariant entropy bound". Physical Review D.* ***68*** *(6): 064001. arXiv:hep-th/0305149. doi:10.1103/PhysRevD.68.064001. S2CID 119049155.*
[12] *Bekenstein, Jacob D. (2004). "Black holes and information theory". Contemporary Physics.* ***45*** *(1): 31–43. arXiv:quant-ph/0311049. doi:10.1080/00107510310001632523. S2CID 118970250.*
[13] *Tipler, F. J. (2005). "The structure of the world from pure numbers" (PDF). Reports on Progress in Physics.* ***68*** *(4): 897–964. arXiv:0704.3276. doi:10.1088/0034-4885/68/4/R04. S2CID 119620977.*. Tipler gives a number of arguments for maintaining that Bekenstein's original formulation of the bound is the correct form. See in particular the paragraph beginning with "A few points ..." on p. 903 of the *Rep. Prog. Phys.* paper (or p. 9 of the *arXiv* version), and the discussions on the Bekenstein bound that follow throughout the paper.Barrau et Grain 2016] Aurélien Barrau et Julien Grain, *Relativité générale : cours et exercices corrigés*, Paris, Éditions Dunod, coll. « Sciences Sup / Physique », 17 août 2016, 2e éd. (1re éd. 24 août 2011), 1 vol., VIII-231, 17 × 24 cm (ISBN 978-2-10-074737-5, EAN 9782100747375, OCLC 958388884, BNF 45101424, SUDOC 195038134, présentation en ligne [archive], lire en ligne [archive]).E. Sawaguchi, Journal of the Physical Society of Japan, vol. 8, No. 5, pp. 615-629 (1953).
[14] Almheiri A, Hartman T, Maldacena J, Shaghoulian E, Tajdini A. The entropy of Hawking radiation. Available:https://arxiv.org/abs/2006.06872v1.
[15] Bousso, Penington Entanglement Wedge For Gravitating Regions; Sept 2022.Available: https://arxiv.org/abs/2208.04993
[16] Ryu S, Takayanagi T. Aspects of holographic entanglement entropy. J High Energy Phys.2006-08-21;2006(8). Available:arXiv:hep-th/0605073:045-. DOI: 10.1088/1126-6708/2006/08/045. DOI: 10.1088/1126-6708/2006/08/045. ISSN 1029-8479. S2CID 14858887.
[17] Stanford Institute for Theoretical Physics. Gravity and entanglement, [retrieved 2017-5-7]; 2015-10-15.
[18] Aurélien Barrau et Julien Grain, *Relativité générale : cours et exercices corrigés*, Paris, Éditions Dunod, coll. « Sciences Sup / Physique », 17 août 2016, 2e éd. (1re éd. 24 août 2011), 1 vol., VIII-231, 17 × 24 cm (ISBN 978-2-10-074737-5, EAN 9782100747375, OCLC 958388884, BNF 45101424, SUDOC 195038134, présentation en ligne [archive], lire en ligne [archive]).
[19] [Luminet 2011] Jean-Pierre Luminet, *Illuminations : cosmos et esthétique*, Paris, Éditions Odile Jacob, coll. « Sciences », 22 septembre 2011, 1re éd., 1 vol., 487-[16], 15,5 × 24 cm (ISBN 978-2-7381-2562-0 et 2-7381-2562-X, EAN 9782738125620, OCLC 780211696, BNF 42279998, SUDOC 155577441, présentation en ligne [archive], lire en ligne [archive]).
[20] [Penrose 2007] Roger Penrose (trad. de l'anglais par de l'anglais par Céline Laroche), *À la découverte des lois de l'univers : la prodigieuse histoire des mathématiques et de la physique* [« The road to reality : a complete guide to the laws of the universe »], Paris, Éditions Odile Jacob, coll. « Sciences », 30 août 2007 (réimpr. 2008 et 2010), 1re éd., 1 vol., XXII-1061, 15,5 × 24 cm (ISBN 978-2-7381-1840-0, EAN 9782738118400, OCLC 209307388, BNF 41131526, SUDOC 118177311, présentation en ligne [archive], lire en ligne [archive])
[21] *"Top Cited Articles of All Time (2014 edition)". INSPIRE-HEP. Retrieved 26 December 2015.*
[22] physics.aps.org Available:https://physics.aps.org/articles/v9/49(accessed on 02 02 2023).
[23] Denis, O. (2022). The Dark Side of the Entangled Informational Universe. *Physical Science International Journal*, *26*(6), 39–58. https://doi.org/10.9734/psij/2022/v26i6750
[24] Vopson M.M., Experimental protocol for testing the mass–energy–information equivalence principle. AIP Advances. 2022;12:035311. DOI:https://doi.org/10.1063/5.0087175
[25] Daffertshofer A, Plastino AR. Landauer's, principle and the conservation of information. Phys Lett A. 2005;342(3):213-6, DOI: 10.1016/j.physleta.2005.05.058.
[26] Plenio MB, Vitelli V. The physics of forgetting: Landauer's erasure principle and information theory. Contemp Phys. 2001;42(1):25-60. DOI: 10.1080/00107510010018916.
[27] Ladyman J, Presnell S, Short AJ, Groisman B. The connection between logical and thermodynamic irreversibility. Stud Hist Philos Mod Phys. 2007;38(1): 58-79. DOI: 10.1016/j.shpsb.2006.03.007.
[28] Barbara Piechocinska, Information erasure, Phys. Rev. A 61, 062314 – Published 17 May 2000, DOI: https://doi.org/10.1103/PhysRevA.61.062314.
[29] Braunstein SL, Pati AK. Quantum information cannot be completely hidden in correlations: Implications for the black-hole information paradox. Phys Rev Lett. 2007;98(8):080502. Available: gr-qc/0603046 DOI:10.1103/PhysRevLett.98.080502, PMID 17359079
[30] Lee J-W, Lee J, Kim HC. Quantum informational dark energy: Dark energy from forgetting. arXiv E-Print, 2008;8. [arXiv/0709.0047]
[31] Bérut A, Arakelyan A, Petrosyan A, Ciliberto S, Dillenschneider R, Lutz E. Experimental verification of Landauer's principle linking information and thermodynamics. Journal Nature on March 8. Nature. 2012;483(7388):187-9. DOI:10.1038/nature10872, PMID 22398556
[32] Jacob D Bekenstein. Black holes and entropy, Phys. Rev. D 7, 2333 – Published 15 April 1973 An article within the collection: 2015 - General Relativity's Centennial and the Physical Review D 50th Anniversary Milestones; 2019
[33] Bekenstein JD. How does the Entropy / Information Bound Work? Found Phys. 2005;35(11):1805-23. Available:arXiv:quant-ph/0404042. DOI: 10.1007/s10701-005-7350-7. Bibcode. 2005FoPh... 35.1805B. DOI: 10.1007/s10701-005-7350-7.S2CID. 118942877.
[34] Available online at Jacob B. Bekenstein bound. Scholarpedia. 2008;3(10):7374 (accessed: 12-06-2023)
[35] Wikipedia Available online: https://en.wikipedia.org/wiki/Bekenstein_bound.(accessed on 02 02 2023).
[36] Anna Karlsson, Replica wormhole and island incompatibility with monogamy of entanglement, https://arxiv.org/abs/2007.10523

[37] Ryu S, Takayanagi T. Holographic derivation of entanglement entropy from AdS/CFT. Phys Rev Lett. 2006, Available:arXiv:hep-th/0603001;96(18):181602. DOI: 10.1103/PhysRevLett.96.181602, PMID 1671235
[38] Penington G. Entanglement wedge reconstruction and the information paradox. J High Energ Phys; 2020. Available:arXiv:1905.08255 [hep-th];2020(9). DOI: 10.1007/JHEP09(2020)002.
[39] Almheiri A, Engelhardt N, Marolf D, Maxfield H. The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole. J High Energ Phys; 2019.Available: arXiv:1905.08762[hep-th] ;2019(12). DOI: 10.1007/JHEP12(2019)063.
[40] Hubeny VE, Rangamani M, Takayanagi T. A Covariant holographic entanglement entropy proposal. J Hepatol; 2007.Available:arXiv:0705.0016[hep-th];07:062
[41] Lewkowycz A, Maldacena J. Generalized gravitational entropy. J Hepatol; 2013. Available:arXiv:1304.4926[hep-th];08:090.
[42] Barrella T, Dong X, Hartnoll SA, Martin VL. Holographic entanglement beyond classical gravity. J Hepatol; 2013. Available:arXiv:1306.4682 [hep-th];09:109.
[43] Faulkner T, Lewkowycz A, Maldacena J. Quantum corrections to holographic entanglement entropy. J Hepatol; 2013. Available:arXiv:1307.2892 [hep-th];11:074.
[44] Engelhardt N, Wall AC. Quantum extremal surfaces: holographic entanglement entropy beyond the classical regime. J Hepatol. 2015, Available:arXiv:1408.3203 [hep-th];01:073
[45] Almheiri A, Mahajan R, Maldacena J, Zhao Y. The page curve of hawking radiation from semiclassical geometry. J High Energ Phys; 2020. Available:arXiv:1908.10996 [hep-th]. 2020;(3). DOI: 10.1007/JHEP03(2020)149.
[46] Van Raamsdonk M. Lectures on gravity and entanglement. New Front Fields Strings. ISBN 978-981-314-943-4. S2CID 119273886. August 31 2016:297-351. Available: arXiv:1609.00026. DOI: 10.1142/9789813149441_0005.