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Abstract

In this paper we describe a share package XMOD of functions for computing with finite, permutation crossed modules, cat^1-groups and their morphisms, written using the GAP group theory programming language. The category **XMod** of crossed modules is equivalent to the category **Cat1** of cat^1-groups and we include functions emulating the functors between these categories. The monoid of derivations of a crossed module $\mathcal{X}$, and the corresponding monoid of sections of a cat^1-group $\mathcal{C}$, are constructed using the Whitehead multiplication. The Whitehead group of invertible derivations, together with the group of automorphisms of X, are used to construct the actor crossed module of X which is the automorphism object in **XMod**. We include a table of the 350 isomorphism classes of cat^1-structures on groups of order at most 30.

1 Introduction

Our aim in this paper is to describe a share package XMOD [2] for the GAP [20] computational group theory language. This package implements calculations with finite, permutation crossed modules and their morphisms, and also with the equivalent cat^1-groups and their morphisms. We also present the results of the computation of all isomorphism classes of cat^1-group structures on groups of order at most 30.

This is the first computational contribution to the general programme of "higher-dimensional group theory" described, for example, by Brown in [4] and [5]. The 2-dimensional part of this programme is concerned with group objects in the category of groupoids, and these objects may equivalently be considered as crossed modules or cat^1-groups.

Whereas the theory of groups started with groups of permutations and developed into abstract groups and topological groups, the impetus for the study of "2-dimensional groups" comes from algebraic topology, in describing the homotopy double groupoid of a based pair of spaces. Nevertheless, the algebraic motivation is equally compelling: the automorphism group $\mathrm{Aut}(G)$ of a group G is most clearly exhibited as part of the automorphism crossed module $(\mathrm{inn} : G \to \mathrm{Aut}(G))$ determined by the inner automorphism map.

The term crossed module was introduced by J. H. C. Whitehead in [21]. See [6, 7, 8] and [12] for a variety of applications. In [16] Loday reformulated the notion of a crossed module as a cat^1-group (though he used the term 1-cat group) and showed that the category **XMod** is equivalent to the category **Cat1**. Loday also generalised the notion of cat^1-group to that of cat^n-group, for all $n \geq 1$. Crossed modules and their higher analogues were considered by Ellis in [13]. A proof that the categories of cat^n-groups and crossed n-cubes are equivalent was given by Ellis and Steiner in [14].

The category **XMod** is also equivalent to the category **GpGpd** of group-groupoids and to the category of 1-truncated simplicial groups with trivial Moore complex (see [3, 16]). These structures are not included in the current version of the package.

In section 2 we recall the basic properties of crossed modules and their derivations and of cat^1-groups and their sections. In section 3 we describe the implementation of these structures in GAP and include a short example illustrating how the package is used. In section 4 we discuss some of the algorithms used to compute lists of derivations, sections and cat^1-structures.

In section 5 we tabulate, for groups G of order at most 30, the order of $\mathrm{End}(G)$; the number of idempotent endomorphisms; the number of cat^1-structures on G; and the number of isomorphism classes of these structures.

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2 Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of three equivalent categories: **XMod**, the category of crossed modules and their morphisms; **Cat1**, the category of cat^1-groups and their morphisms; and **GpGpd**, the subcategory of groups in the category **Gpd** of groupoids. We also describe functors between these categories which exhibit the equivalences. We state the axioms using right actions, since this is the convention used by most computational group packages, but all functions will be written on the left.

A crossed module $\mathcal{X} = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary* of $\mathcal{X}$, together with an action $\alpha : R \to \mathrm{Aut}(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

$$\begin{array}{lrcl} \textbf{XMod 1:} & \partial(s^r) & = & r^{-1}(\partial s)r \\ \textbf{XMod 2:} & s^{\partial s'} & = & s'^{-1}ss'. \end{array}$$

The kernel of ∂ is abelian and the image of ∂ is normal in R.

Standard constructions for crossed modules include the following:

1. A *conjugation crossed module* is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.

2. An *automorphism crossed module* has as range a subgroup R of the automorphism group $\mathrm{Aut}(S)$ of S which contains the inner automorphism group $\mathrm{Inn}(S)$ of S. The boundary maps $s \in S$ to the inner automorphism of S by s.

3. An *R-Module crossed module* has an R-module as source and $\partial = 0$.

4. Any homomorphism $\partial : S \to R$, with S abelian and $\mathrm{im}\,\partial$ in the centre of R, provides a crossed module with R acting trivially on S.

5. A *central extension crossed module* has as boundary a surjection $\partial : S \to R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1}r$.

6. The *direct product* of $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ with R_1, R_2 acting trivially on $S_2,\ S_1$ respectively.

A morphism between two crossed modules $\mathcal{X}_1$ and $\mathcal{X}_2$ is a pair (σ, ρ), where $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$\partial_2\sigma = \rho\partial_1, \quad \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $\mathcal{X}_2 = \mathcal{X}_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphism of $\mathcal{X}_1$. The group of automorphisms is denoted by $\mathrm{Aut}(\mathcal{X}_1)$.

The Whitehead monoid $\mathrm{Der}(\mathcal{X})$ of $\mathcal{X}$ was defined in [22] to be the monoid of all *derivations* from R to S, that is the set of all maps $R \to S$, with composition $\circ$, satisfying

$$\begin{array}{lrcl} \textbf{Der 1:} & \chi(qr) & = & (\chi q)^r\ (\chi r) \\ \textbf{Der 2:} & (\chi_1 \circ \chi_2)(r) & = & (\chi_1 r)(\chi_2 r)(\chi_1 \partial \chi_2 r). \end{array}$$

Invertible elements in the monoid are called *regular*. The Whitehead group $\mathcal{W}(\mathcal{X})$ is the group of $\mathrm{Der}(\mathcal{X})$. The *actor* of $\mathcal{X}$ is a crossed module $(\Delta : \mathcal{W}(\mathcal{X}) \to \mathrm{Aut}(\mathcal{X}))$ which was shown by Lue and Norrie, in [17, 18, 19], to be the automorphism object of $\mathcal{X}$ in the category **XMod**. Gilbert, in [15], has discussed a connection between derivations and group extensions.

In [16] Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of endomorphisms $t, h : G \to G$ having a common image R and satisfying certain axioms. We prefer a definition in which a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ has source group G, range group R, and three homomorphisms: two surjections $t, h : G \to R$ and an embedding $e : R \to G$ satisfying:

$$\textbf{Cat 1:} \quad te(r) = he(r) = r \ \text{ for all } \ r \in R,$$
$$\textbf{Cat 2:} \quad [\ker t, \ker h] = \{1_G\}.$$

The maps t, h are usually referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of $\mathcal{C}$, because *source* is the GAP term for the domain of a function.

A morphism $\mathcal{C}_1 \to \mathcal{C}_2$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$h_2\gamma = \rho h_1, \quad t_2\gamma = \rho t_1, \quad e_2\rho = \gamma e_1. \tag{1}$$

An arbitrary cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ is isomorphic to the cat^1-group $\mathcal{C}' = (e'; t', h' : R \ltimes S \to R)$ where $S = \ker t$, R acts on S by

$$s^r = s^{er} = (er)^{-1}s(er),$$

and the semidirect product $R \ltimes S$ has composition and inverse given by

$$(r_1, s_1)(r_2, s_2) \ = \ (r_1r_2, s_1^{r_2}s_2), \quad (r, s)^{-1} \ = \ (r^{-1}, (s^{-1})^{r^{-1}}).$$

The homomorphisms in $\mathcal{C}'$ are given by

$$t'(r, s) = r, \quad h'(r, s) = r(\partial s), \quad e'r = (r, 1) \tag{2}$$

and the isomorphism $(\theta, \ \mathrm{id}_R) \ : \ \mathcal{C}' \to \mathcal{C}$ is given by

$$\theta \ : \ R \ltimes S \to G, \quad (r, s) \mapsto (er)s$$

with inverse

$$\theta^{-1} \ : \ G \to R \ltimes S, \quad g \mapsto (tg, (etg^{-1})g).$$

The crossed module $\mathcal{X} = (\partial : S \to R)$ associated to $\mathcal{C}$ and $\mathcal{C}'$ has $\partial = h|_S$. The cat^1-group $\mathcal{C} = \mathcal{C}'$ associated to $\mathcal{X} = (\partial : S \to R)$ has $G = R \ltimes S$, where the action is that in $\mathcal{X}$, and homomorphisms given by (2). We denote by ϵ the inclusion of S in G, so that $\partial = h\epsilon$.

The construction for cat^1-groups equivalent to the derivation of a crossed module is the *section*, namely a group monomorphism $\xi : R \to G$ satisfying:

$$\textbf{Sect 1:} \quad t\xi(r) = r \ \text{ for all } \ r \in R.$$

The equation

$$\xi r = (er)(\epsilon\chi r) = (r, \chi r) \tag{3}$$

defines a section ξ of $\mathcal{C}$ in terms of a derivation χ of $\mathcal{X}$, and conversely. These sections form the monoid $\mathrm{Sect}(\mathcal{C})$ of $\mathcal{C}$, whose composition rule we determine from the rule **Der 2** for $\mathrm{Der}(\mathcal{X})$ by evaluating:

$$\begin{aligned}
(\xi_1 \circ \xi_2)r &= (er)(\epsilon(\chi_1 \circ \chi_2)r) \\
&= (er)(\epsilon\chi_1 r)(\epsilon\chi_2 r)(\epsilon\chi_1 h\epsilon\chi_2 r) \\
&= (\xi_1 r)(er^{-1})(\xi_2 r)(eh(\epsilon\chi_2 r)^{-1})(\xi_1 h\epsilon\chi_2 r) \\
&= (\xi_1 r)(er^{-1})(\xi_2 r)(eh((\xi_2 r)^{-1}(er)))(\xi_1 h((er^{-1})(\xi_2 r))) \\
&= ((er)(\xi_1 r^{-1}))^{-1}((\xi_2 r)(eh\xi_2 r^{-1}))((er)(\xi_1 r^{-1}))(\xi_1 h\xi_2 r).
\end{aligned}$$

Since $(er)(\xi_1 r^{-1})$ and $(\xi_1 h\xi_2 r)(eh\xi_2 r^{-1}) \in \ker t$ while $(\xi_2 r)(eh\xi_2 r^{-1}) \in \ker h$, this reduces to

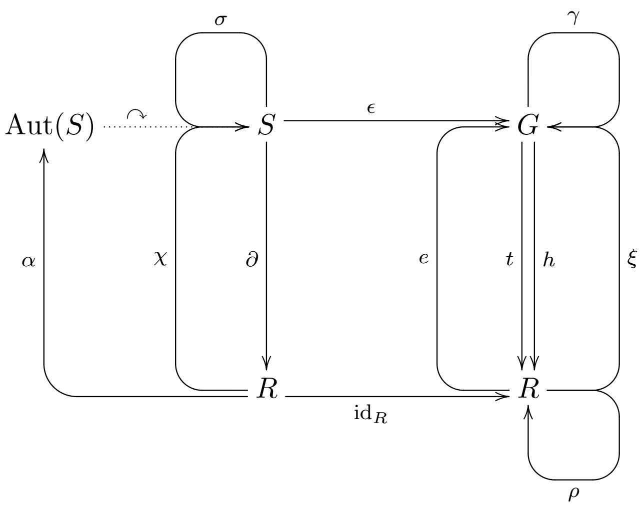
$$\textbf{Sect 2:} \quad (\xi_1 \circ \xi_2)r = (\xi_2 r)(eh\xi_2 r^{-1})(\xi_1 h\xi_2 r) = (\xi_1 h\xi_2 r)(eh\xi_2 r^{-1})(\xi_2 r).$$

The embedding e is the identity for this composition, and equation (3) determines a monoid isomorphism $\mathrm{Der}(\mathcal{X}) \cong \mathrm{Sect}(\mathcal{C})$. A section is *regular* when $h\xi$ is an automorphism and the group of regular sections is isomorphic to the Whitehead group.

Each χ or ξ determines endomorphisms of $R, S, G, \mathcal{X}$ and $\mathcal{C}$, namely

$$\begin{aligned}
\rho &: R \to R, & r &\mapsto r(\partial\chi r) = h\xi r, \\
\sigma &: S \to S, & s &\mapsto s(\chi\partial s) \;=\; s(e\partial s^{-1})(\xi\partial s), \\
\gamma &: G \to G, & g &\mapsto (eh\xi tg)(\xi tg^{-1})g(ehg^{-1})(\xi hg), \\
(\sigma, \rho) &: \mathcal{X} \to \mathcal{X}, \\
(\gamma, \rho) &: \mathcal{C} \to \mathcal{C},
\end{aligned}$$

and these assignments determine group homomorphisms from the Whitehead group to these five endomorphism groups. The accompanying diagram shows the relationship between the various groups and homomorphisms.



When axioms **XMod 2:** and **Cat 2:** are *not* satisfied, the corresponding structures are known as *pre-crossed modules* and *pre-cat^1-groups*. In this case the *Peiffer subgroup* P of S is the subgroup of $\ker(\partial)$ generated by *Peiffer commutators*

$$[\![s_1, s_2]\!] \;=\; (s_1^{-1})^{\partial s_2} s_2^{-1} s_1 s_2.$$

Then $\mathcal{P} = (0 : P \to \{1_R\})$ is a normal sub-pre-crossed module of $\mathcal{X}$ and $\mathcal{X}/\mathcal{P} = (\partial : S/P \to R)$ is a crossed module. The restriction of $\epsilon : S \to R \ltimes S$ to P is given by

$$\epsilon\, [\![s_1, s_2]\!] \;= [(1_R, s_1^{\partial s_2}), ((\partial s_2)^{-1}, s_2)] \in [\ker t, \ker h].$$

The image ϵP is the Peiffer subgroup $[\ker t, \ker h]$ of $R \ltimes S$ and, if ι is the inclusion $\{1_R\} \to R$, then $\mathcal{C}/(\epsilon, \iota)\mathcal{P} = (e; t, h : (R \ltimes S)/\epsilon P \to R)$ is the cat^1-group corresponding to $\mathcal{X}/\mathcal{P}$. This construction is used when implementing induced cat^1-groups.

The underlying groupoid $\mathcal{G}$ of a cat^1-group $\mathcal{C}$ has the elements of R as the set of objects and the elements of G as arrows. The identity arrow at r is er. For each arrow g the source(tail) is tg and the target(head) is hg. Arrows g, g' are composable only when $hg = tg'$, in which case the composite arrow is

$$g * g' = g(etg'^{-1})g' = g(ehg^{-1})g'$$

with tail tg and head hg'. The groupoid inverse $\tilde{g}$ of g for this composition is given by $\tilde{g} = (ehg)g^{-1}(etg)$ with $t\tilde{g} = hg$, $h\tilde{g} = tg$, $g * \tilde{g} = etg$ and $\tilde{g} * g = ehg$. The equivalent formulae in $R \ltimes S$ are $(r, s) * (r(\partial s), s') = (r, ss')$ and $\widetilde{(r, s)} = (r(\partial s), s^{-1})$.

Since $g^{-1}(etg) \in \ker t$ and $(ehg)g^{-1} \in \ker h$, the map $g \mapsto \tilde{g}$ is an automorphism of $\mathcal{G}$ which restricts to the identity map on eR and provides a cat^1-isomorphism from $\mathcal{C}$ to the *reverse* cat^1-group $\tilde{\mathcal{C}} = (e; h, t : G \to R)$ of $\mathcal{C}$. The set of arrows *out* from 1_R is $\ker t$ while the set of arrows *in* to 1_R is $\ker h$, so $\ker \partial$ is the set of loops at 1_R. The set of objects in the component of $\mathcal{G}$ connected to 1_R is the image of ∂, so $\mathcal{G}$ is discrete when $\partial = 0$.

3 GAP implementation

The group theory program GAP [20] is designed to facilitate the implementation of new structures as record types with their own output form. In version 3.4 of this package a separate operations record allows the overloading of functions such as kernel, centre and inner automorphism. We have developed a share package for GAP 3.4.4 containing some 160 functions for crossed modules, their morphisms and derivations; cat^1-groups, their morphisms and sections; and related constructions. All crossed modules and cat^1-groups require permutation groups as source and range, though groups of automorphisms, semidirect products and finitely presented groups are used by many of the functions. For each non-permutation group we find it convenient to set up a pairing with an isomorphic permutation group. Thus, if A is a group of automorphisms of a group G and if θ is an isomorphism from A to a permutation group P, an **AutoPair** for A is a record **pairA** with fields **pairA.auto := A**, **pairA.perm := P**, **pairA.a2p :=** θ, **pairA.p2a :=** θ^{-1} and **pairA.isAutoPair := true**. Such pairings are known in GAP4 as "nice isomorphisms".

Also included are functions to compute the actor crossed module of a crossed module; and other crossed modules and morphisms in the actor square (see Norrie, [18, 19]); and functions to calculate induced crossed modules (see Brown and Wensley, [9, 10, 11]) and induced cat^1-groups ([1], Chapter 3).

A revised version of the package using GAP4 syntax is in preparation, in which the basic data structures are pre-crossed modules and pre-cat^1-groups. Functions for constructing Peiffer subgroups; the underlying groupoid $\mathcal{G}$; crossed squares and their morphisms will be included.

We implement a crossed module $\mathcal{X} = (\partial : S \to R)$ as a record **X** with fields:

X.source,	the source group S of ∂,
X.boundary,	the homomorphsim ∂,
X.range,	the range group R of ∂,
X.aut,	a group of automorphisms of S,
X.action,	the homomorphism α from R to **X.aut**,
X.isXMod,	a boolean flag, normally **true**,
X.isDomain,	always true,
X.operations,	a special set of operations **XModOps**,
X.name,	a concatenation of the names of S and R.

Further fields, such as **.isConjugationXMod**, are added where appropriate.

The operations record **XModOps** includes functions for equality; size; list of elements; a special output form; and various functions for the actor square.

A morphism (σ, ρ) of crossed modules is implemented as a record **mor** with fields:

mor.source,	the source crossed module **X**,
mor.range,	the range crossed module **Y**,
mor.sourceHom,	the homomorphism σ from **X.source** to **Y.source**,
mor.rangeHom,	the homomorphism ρ from **X.range** to **Y.range**,
mor.isXModMorphism,	a Boolean flag, normally **true**,
mor.operations,	a special set of operations **XModMorphismOps**,
mor.name,	a concatenation of the names of **X** and **Y**.

The operations record **XModMorphismOps** includes functions for equality; kernel and image; composite and inverse morphism; and tests such as **IsEpimorphism**.

A derivation $\chi : R \to S$ is defined in the same way that a group homomorphism is defined, by specifying a list of images for the generators of R:

chi := XModDerivationByImages(R, S, R.generators, genimages); .

If $w = r_1 r_2 \ldots r_k$ is an element of R expressed as a word in the generators, then axiom **Der 1:** gives

$$\chi w = (\chi r_1)^{r_2 \ldots r_k} (\chi r_2)^{r_3 \ldots r_k} \ldots (\chi r_{k-1})^{r_k} (\chi r_k). \tag{4}$$

The function **IsDerivation** (see section 4.1) is used to test that such a χ is indeed a derivation. We store χ as a record **chi** with fields: source; range; generators; genimages; xmod; operations; isDerivation.

We implement a cat^1-group $\mathcal{C} = (e; t, h : G \to R)$ as a record **C** with fields: source; range; tail; head; embedRange; kernel; embedKernel; boundary; isDomain; operations; name; isCat1. A morphism **mor** $= (\gamma, \rho)$ of cat^1-groups is a record with fields similar to those of a morphism of crossed modules.

The functors providing the equvalence between the categories **Cat1** and **XMod** are implemented as functions

- **XModCat1(C)**, **XModMorphismCat1Morphism(mor)**,
- **Cat1XMod(X)**, **Cat1MorphismXModMorphism(mor)**.

The second of these calculates the semidirect product $R \ltimes S$ and then finds a suitable isomorphic permutation group G to act as the source, producing a **SemidirectPair**. In order to minimise the degree of G it is preferable to start with $\mathcal{C}$, when a representation for $\mathcal{C}$ is known, and then construct $\mathcal{X}$.

There are two functions to determine the elements of the Whitehead group and the Whitehead monoid of $\mathcal{X}$, namely **RegularDerivations** and **AllDerivations**. If the whole monoid is needed at some stage, then the latter function should be used. A sub-record **D = X.derivations** of **X** is created which stores all the required information.

The functions **WhiteheadMonoidTable** and **WhiteheadGroupTable** calculate the multiplication tables of the monoid or group using the Whitehead multiplication, while **WhiteheadPermGroup** constructs a faithful, regular permutation representation of the group of regular derivations from the multiplication table.

The corresponding functions for sections are **RegularSections** and **AllSections**. Both create or modify a sub-record **C.sections** .

Example 3.1 *Let R be the symmetric group S_3 and S its normal subgroup C_3. The conjugation crossed module $\mathcal{X} = (\iota : C_3 \to S_3)$, the associated cat^1-group, the derivation monoid of $\mathcal{X}$ and the actor crossed module are obtained as follows.*

```
gap> X := ConjugationXMod( s3, c3 );
Crossed module [c3->s3]
gap> XModPrint( X );
Crossed module [c3->s3] :-
:  Source group has parent ( s3 ) and has generators:
   [ (1,2,3) ]
:  Range group has parent ( s3 ) and has generators:
   [ (1,2), (2,3) ]
:  Boundary homomorphism maps source generators to:
   [ (1,2,3) ]
:  Action homomorphism maps range generators to automorphisms:
   (1,2) --> ( source gens --> [ (1,3,2) ] )
   (2,3) --> ( source gens --> [ (1,3,2) ] )
   These 2 automorphisms generate the group of automorphisms.
gap> C := Cat1XMod( X );
cat1-group [Perm(s3 |X c3) ==> s3]
gap> C.source.generators;
```

```
[ (2,3)(4,5), (2,3)(5,6), (1,2,3) ]
gap> D := AllDerivations( X );
AllDerivations record for crossed module [c3->s3]
:  9 derivations found but unsorted.
gap> DerivationsSorted( D );
true
gap> D.regular;
6
gap> PrintList( WhiteheadMonoidTable( X ) );
[ 1, 2, 3, 4, 5, 6, 7, 8, 9 ],
[ 2, 1, 6, 5, 4, 3, 7, 9, 8 ],
[ 3, 4, 1, 2, 6, 5, 8, 7, 9 ],
[ 4, 3, 5, 6, 2, 1, 8, 9, 7 ],
[ 5, 6, 4, 3, 1, 2, 9, 8, 7 ],
[ 6, 5, 2, 1, 3, 4, 9, 7, 8 ],
[ 7, 7, 7, 7, 7, 7, 7, 7, 7 ],
[ 8, 8, 8, 8, 8, 8, 8, 8, 8 ],
[ 9, 9, 9, 9, 9, 9, 9, 9, 9 ] ]
gap> W := WhiteheadPermGroup( X );
WG([c3->s3])
gap> A := AutomorphismPermGroup( X );
PermAut([c3->s3])
gap> Act := Actor( X );;
gap> XModPrint( Act );
Crossed module Actor[c3->s3]
:  Source group WG([c3->s3]) has generators:
   [ (1,2)(3,4)(5,6), (1,3)(2,6)(4,5) ]
:  Range group has parent ( PermAut(c3)xPermAut(s3) ) and has generators:
   [ (3,4,5), (1,2)(4,5) ]
:  Boundary homomorphism maps source generators to:
   [ (1,2)(3,5), (1,2)(4,5) ]
:  Action homomorphism maps range generators to automorphisms:
   (3,4,5) --> ( source gens --> [ (1,5)(2,4)(3,6), (1,2)(3,4)(5,6) ] )
   (1,2)(4,5) --> ( source gens --> [ (1,5)(2,4)(3,6), (1,3)(2,6)(4,5) ] )
   These 2 automorphisms generate the group of automorphisms.
```

4 Outline algorithms

In this section we comment on the algorithms used to construct derivations, sections, and the set of cat^1-structures on a group G.

4.1 IsDerivation

This function tests that a chosen set of images for the generators of R *does* define a derivation. Let $genR = \{r_1, r_2, \ldots, r_m\}$ be the generating set of R, and let $\{s_1, s_2, \ldots, s_m\}$ be the chosen images for χ. First calculate the images $\chi(r_i^{-1}) = ((\chi r_i)^{r_i^{-1}})^{-1}$. Construct an **FpPair** *pairR* for R with finitely presented group $F = pairR.fp$ having generating set $\{f_1, f_2, \ldots, f_m\}$ and $pairR.f2p$ mapping f_i to r_i, $(1 \leq i \leq m)$. For each relator *rel* in the presentation, with w the corresponding word in $genR$, check that $\chi w = (\ )$ using (4). When this is true, χr is well-defined for all $r \in R$. Note that *pairR* is stored as a field in the record R, so the pairing only has to be set up once.

4.2 RegularDerivations and AllDerivations

The default method for calculating all the derivations of $\mathcal{X}$ is a simple backtrack algorithm. Let $\{R_1, R_2, \ldots, R_m = R\}$ be the sequence of subgroups of R in which R_i is generated by the first i generators. The backtracking procedure constructs m-tuples of elements of S as potential images for *genR*. As each $s_i = \chi r_i$ is chosen, with order k_i, the following are checked:

- $\rho_i : R_i \to R,\ \ r_j \mapsto r_j(\partial s_j)\ \ (1 \leq j \leq i)$ is a homomorphism,
- $s_i^{r_i^{k_i-1}} \ \ldots \ s_i^{r_i} \ s_i = (\ )$, which is the special case of (4) when $w = r_i^{k_i}$.

If either test fails, that part of the backtrack tree is discarded. The **IsDerivation** function is called when a full set of images has been found. If only the regular derivations are needed, the corresponding $\rho : R \to R$ is tested to see whether it is an automorphism. Note that *all* the derivations are found, even when not required.

4.3 RegularSections and AllSections

If **X.derivations** already exists when **C.sections** is required (and conversely) the sections are quickly obtained using $\xi r = (er)(\epsilon\chi r)$.

When **X.derivations** does not already exist, a different method is used by default to calculate sections of $\mathcal{C}$. A section ξ is determined by the choice of $s_i = \chi r_i$ for each r_i in *genR*. Since $r^{-1}(\rho r) = \partial\chi r$ it follows that $\chi r \in \partial^{-1}(r^{-1}(\rho r))$. In order to find all regular sections, we use the standard GAP function **AutomorphismGroup** to obtain $\mathrm{Aut}(R)$. For each $\rho \in \mathrm{Aut}(R)$, lists of preimages

$$[\partial^{-1}(r_1^{-1}(\rho r_1)),\ \partial^{-1}(r_2^{-1}(\rho r_2)),\ \ldots,\ \partial^{-1}(r_m^{-1}(\rho r_m))]$$

are constructed, and a backtrack procedure is used to select $s_1, s_2, \ldots, s_m$ from these lists, with each selection being tested to see whether it provides a partial homomorphism $R \to G$. Only the *regular* sections are found by this method.

A similar strategy is used to find *all* the sections, replacing $\mathrm{Aut}(R)$ by the endomorphism monoid $\mathrm{End}(R)$. Since no standard GAP function yet exists for computing $\mathrm{End}(R)$, we have added a function **EndomorphismClasses**.

4.4 EndomorphismClasses

An endomorphism ε of R is determined by

- a normal subgroup N of R and a faithful permutation representation of the quotient $\theta : R/N \to Q$, giving a projection $\theta \circ \nu : R \to Q$ where $\nu : R \to R/N$ is the natural homomorphism;
- an automorphism α of Q;
- a subgroup H' in a conjugacy class $[H]$ of subgroups of R isomorphic to Q having representative H; an isomorphism $\phi : Q \cong H$; and a conjugating element $c \in R$ such that $H^c = H'$.

Then ε takes values $\varepsilon r \;=\; (\phi\alpha\theta\nu r)^c$.

Endomorphisms are placed in the same class if they have the same choice of N and $[H]$, so the number of endomorphisms is

$$|\mathrm{End}(R)| \;=\; \sum_{classes} |\mathrm{Aut}(Q)|\; |[H]|.$$

The function returns a record **E = R.endomorphismClasses** having a subfield **.classes** which is a list of records with fields:

.quotient,	the group $Q \cong R/N$,
.projection,	the homomorphism $\theta \circ \nu$,
.autoGroup,	a permutation representation of the automorphism group of Q,
.rangeNumber,	the position of $[H]$ in **R.lattice.classes**
.isomorphism,	the isomorphism $\phi \circ \theta \circ \nu$,
.conj,	the list of conjugating elements c.

4.5 AllCat1s

A list A of cat^1-groups with source G is initialised, containing $(\mathrm{id};\mathrm{id},\mathrm{id} : G \to G)$ and $(1;0;0 : G \to \{1_G\})$. A list of representatives $\{H_1, H_2, \ldots, H_m\}$ of the non-trivial conjugacy classes of subgroups of G is selected. (It would be more efficient to choose a set of representatives from the poset of automorphism classes of subgroups, rather than the conjugacy poset - particularly when G is abelian.) For each H_i, all the idempotent endomorphisms $\phi : G \to H_i$ are constructed, and the images ϕg_k of the generators g_k of G stored in a list L_i. These ϕ are candidates for the tail and head maps, so from each ordered pair of images in L_i homomorphisms t, h are constructed. If $[\ker t, \ker h]$ is trivial, then t, h determine a cat^1-group $\mathcal{C}$. This $\mathcal{C}$ is compared with the entries in A already obtained, using an **AreIsomorphicCat1s** function, and is added to A if no isomorphism is found.

4.6 Comparative timings

We now present average times, in seconds, for the calculation of derivations and sections for six crossed modules, using the six standard constructions listed in section 2, each having $\mid G = R \ltimes S \mid = 288$. Computations were performed on a DEC3000 Model 300LX Digital Alpha 64-bit workstation running **xgap** with 20M memory. Details of the six examples used are shown in the following table.

For each of the six crossed modules $\mathcal{X}$ three calls were made to each of the following functions.

- **AllDerivations(X)**, creating a field **X.derivations**.
- **X.derivations** deleted, then **RegularDerivations(X)** executed. This takes a little longer since the irregular derivations are constructed and discarded.
- **C := Cat1XMod(X)** called to construct the cat^1-group $\mathcal{C}$.
- **X.derivations** deleted, then **RegularSections(C)** executed.

Table 1: Six examples of crossed modules

No.	Type	Source	Range	im∂	degree of S, R, G
1	Conjugation	a4	s4	a4	$4, 4, 12$
2	Inner automorphism	q24	d12	d12	$11, 8, 24$
3	RModule	c6^2	d8	I	$12, 4, 12$
4	Trivial action	c12	c8 $\ltimes$ c3	c4	$12, 11, 23$
5	Central extension	sl(2,3)	a4	a4	$11, 4, 24$
6	Direct product	c3 $\times$ c8	s3 $\times$ c2	c3	$11, 5, 29$

- **RegularDerivations(X)** called again. This time, the existing list of sections is converted to a list of derivations.
- **X.derivations**, **C.sections** and **C.range.automorphismGroup** deleted, then **AllSections(C)** executed.

In order to eliminate one cause of fluctuating times, a garbage collection was performed before each call. Execution times were recorded, and the following averages noted.

Table 2: Execution times in seconds

No.	$[S \to R]$	All Der.	Reg. Der.	Cat1 XMod	Reg. Sec.	Reg. Der.	All Sec.
1	$[a4 \to s4]$	40	44	34	8	0.2	18
2	$[q24 \to d12]$	35	40	26	7.5	0.4	29
3	$[c6^2 \to d8]$	298	292	42	89	0.8	95
4	$[c12 \to c8 \ltimes c3]$	2.4	2.1	0.3	6.8	0.04	9.5
5	$[sl(2,3) \to a4]$	35	40	46	8	0.2	10
6	$[c3 \to s3] \times [c8 \to c2]$	275	283	42	23	1.2	42

Note that, given $\mathcal{C}$, the algorithm of 4.3 is quicker that that of 4.2 except in the fourth example. However in Examples 1,2,5, starting with $\mathcal{X}$, algorithm 4.2 finds all the derivations more quickly, due to the time required to construct a permutation representation of $G = R \ltimes S$.

5 Table of cat^1-structures

In the following table the 92 groups of size ≤ 30 are ordered by their GAP number. For each group G we list the size of End(G); the size of the set IE(G) of idempotents in End(G), which are candidates for t and h; the size of $\mathcal{C}(G)$, the set of all cat^1-structures on G; and the number of isomorphism classes of cat^1-structures. For each G the first cat^1-structure is $\mathcal{C} = (\mathrm{id}; \mathrm{id}, \mathrm{id} : G \to G)$, corresponding to $\mathcal{X} = (0 : I \to G)$, and we omit this from the list. These are the *only* cat^1-structures for the trivial group; the quaternion groups $q8,\ q16$; and the special linear group $sl(2,3)$. For each of the remaining isomorphism classes we list the names of S, R and, when $t \neq h$ and $\partial \neq 0$, the kernel of the boundary. Just 52 of these 350 structures have $\partial \neq 0$.

Table 3: Isomorphism classes of cat[1]-structures

GAP#	G	$\lvert\mathrm{End}(G)\rvert$	$\lvert\mathrm{IE}(G)\rvert$	$\lvert\mathcal{C}(G)\rvert$	$\lvert\mathcal{C}/\cong\rvert$	Names of S, R and $\ker\partial$
1/1	I	1	1	1	1	
2/1	c2	2	2	2	2	[c2,I]
3/1	c3	3	2	2	2	[c3,I]
4/1	k4	16	8	14	4	[k4,I], [c2,c2], [c2,c2,I]
4/2	c4	4	2	2	2	[c4,I]
5/1	c5	5	2	2	2	[c5,I]
6/1	c6	6	4	4	4	[c6,I], [c3,c2], [c2,c3]
6/2	s3	10	5	4	2	[c3,c2]
7/1	c7	7	2	2	2	[c7,I]
8/1	$c2^3$	512	58	226	6	[$c2^3$,I], [k4,c2], [k4,c2,c2], [c2,k4], [c2,k4,I]
8/2	c4c2	32	10	18	6	[c4c2,I], [c4,c2], [c4,c2,c2], [c2,c4], [c2,c4,I]
8/3	c8	8	2	2	2	[c8,I]
8/4	d8	36	10	9	3	[c4,c2], [k4,c2]
8/5	q8	28	2	1	1	
9/1	$c3^2$	81	14	38	4	[$c3^2$,I], [c3,c3], [c3,c3,I]
9/2	c9	9	2	2	2	[c9,I]
10/1	c10	10	4	4	4	[c10,I], [c5,c2], [c2,c5]
10/2	d10	26	7	6	2	[c5,c2]
11/1	c11	11	2	2	2	[c11,I]
12/1	c6c2	48	16	28	8	[c6c2,I], [c6,c2], [c6,c2,c3], [k4,c3], [c3,k4], [c2,c6], [c2,c6,I]
12/2	c12	12	4	4	4	[c12,I], [c4,c3], [c3,c4]
12/3	d12	64	21	12	4	[c6,c2], [c3,k4], [c2,s3]
12/4	q12	20	5	4	2	[c3,c4]
12/5	a4	33	6	5	2	[k4,c3]
13/1	c13	13	2	2	2	[c13,I]
14/1	c14	14	4	4	4	[c14,I], [c7,c2], [c2,c7]
14/2	d14	50	9	8	2	[c7,c2]
15/1	c15	15	4	4	4	[c15,I], [c5,c3], [c3,c5]
16/1	$c2^4$	65536	382	4162	9	[$c2^4$,I], [$c2^3$,c2], [$c2^3$,c2,k4], [k4,k4], [k4,k4,c2], [k4,k4,I], [c2,$c2^3$], [c2,$c2^3$,I]
16/2	c4k4	1024	82	322	12	[c4k4,I], [c4c2,c2], [c4c2,c2,c4], [c4c2,c2,k4], [c4,k4], [c4,k4,c2], [k4,c4], [k4,c4,c2], [c2,c4c2], [c2,c4c2,I], [c2,c4c2,I]
16/3	c8c2	64	10	18	6	[c8c2,I], [c8,c2], [c8,c2,c4], [c2,c8], [c2,c8,I]
16/4	$c4^2$	256	26	98	5	[$c4^2$,I], [c4,c4], [c4,c4,c2], [c4,c4,I]
16/5	c16	16	2	2	2	[c16,I]
16/6	d8c2	1088	82	97	9	[c4c2,c2], [$c2^3$,c2], [c4,k4], [c4,k4,c2], [k4,k4], [k4,k4,c2], [c2,d8], [c2,d8,I]
16/7	q8c2	448	18	17	3	[c2,q8], [c2,q8,I]
16/8	d8y4	224	26	13	2	[c4c2,c2]
16/9	c2⋉c4c2	128	18	25	4	[c4c2,c2], [k4,c4], [k4,c4,c2]
16/10	c4⋉c4	96	10	17	3	[c4,c4], [c4,c4,c2]
16/11	c2⋉c8	48	6	5	2	[c8,c2]
16/12	d16	100	18	9	2	[c8,c2]
16/13	qd16	52	10	5	2	[c8,c2]
16/14	q16	36	2	1	1	
17/1	c17	17	2	2	2	[c17,I]
18/1	c6c3	162	28	76	8	[c6c3,I], [$c3^2$,c2], [c6,c3], [c6,c3,c2], [c3,c6], [c3,c6,I], [c2,$c3^2$]
18/2	c18	18	4	4	4	[c18,I], [c9,c2], [c2,c9]
18/3	d18	82	11	10	2	[c9,c2]

GAP#	G	$\lvert\text{End}(G)\rvert$	$\lvert\text{IE}(G)\rvert$	$\lvert\mathcal{C}(G)\rvert$	$\lvert\mathcal{C}/\cong\rvert$	Names of S, R and $\ker\partial$
18/4	s3c3	36	12	8	4	[c3^2,c2], [c3,c6], [c3,s3]
18/5	c2$\ltimes$c3^2	730	47	118	4	[c3^2,c2], [c3,s3], [c3,s3,I]
19/1	c19	19	2	2	2	[c19,I]
20/1	c10c2	80	16	28	8	[c10c2,I], [c10,c2], [c10,c2,c5], [c5,k4], [k4,c5], [c2,c10], [c2,c10,I]
20/2	c20	20	4	4	4	[c20,I], [c5,c4], [c4,c5]
20/3	d20	144	31	18	4	[c10,c2], [c5,k4], [c2,d10]
20/4	q20	52	7	6	2	[c5,c4]
20/5	c4$\ltimes$c5	36	7	6	2	[c5,c4]
21/1	c21	21	4	4	4	[c21,I], [c7,c3], [c3,c7]
21/2	c3$\ltimes$c7	57	9	8	2	[c7,c3]
22/1	c22	22	4	4	4	[c22,I], [c11,c2], [c2,c11]
22/2	d22	122	13	12	2	[c11,c2]
23/1	c23	23	2	2	2	[c23,I]
24/1	c6k4	1536	116	452	12	[c6k4,I], [c6c2,c2], [c6c2,c2,c6], [c2^3,c3], [c6,k4], [c6,k4,c3], [k4,c6], [k4,c6,c2], [c3,c2^3], [c2,c6c2], [c2,c6c2,I]
24/2	c12c2	96	20	36	12	[c12c2,I], [c12,c2], [c12,c2,c6], [c4c2,c3], [c6,c4], [c6,c4,c3], [c4,c6], [c4,c6,c2], [c3,c4c2], [c2,c12], [c2,c12,I]
24/3	c24	4	4	4	4	[c24,I], [c8,c3], [c3,c8]
24/4	d8c3	108	20	18	6	[c12,c2], [c6c2,c2], [c4,c6], [k4,c6], [c3,d8]
24/5	q8c3	84	4	2	2	[c3,q8]
24/6	s3k4	1792	157	116	8	[c6c2,c2], [c6,k4], [c6,k4,c3], [k4,s3] [c3,c2^3], [c2,d12], [c2,d12,I]
24/7	s3c4	128	27	12	4	[c12,c2], [c4,s3], [c3,c4c2]
24/8	q12c2	160	25	36	6	[c6,c4], [c6,c4,c3], [c3,c4c2], [c2,q12], [c2,q12,I]
24/9	c8$\ltimes$c3	40	5	4	2	[c3,c8]
24/10	a4c2	72	15	10	4	[c2^3,c3], [k4,c6], [c2,a4]
24/11	d8$\ltimes$c3	124	23	12	4	[c6c2,c2], [k4,s3], [c3,d8]
24/12	d24	196	33	20	4	[c12,c2], [c4,s3], [c3,d8]
24/13	q24	124	5	4	2	[c3,q8]
24/14	sl(2,3)	33	6	1	1	
24/15	s4	58	12	5	2	[k4,s3]
25/1	c5^2	625	32	152	4	[c5^2,I], [c5,c5], [c5,c5,I]
25/2	c25	25	2	2	2	[c25,I]
26/1	c26	26	4	4	4	[c26,I], [c13,c2], [c2,c13]
26/2	d26	170	15	14	2	[c13,c2]
27/1	c3^3	19683	236	2108	6	[c3^3,I], [c3^2,c3], [c3^2,c3,c3], [c3,c3^2], [c3,c3^2,I]
27/2	c9c3	243	20	56	6	[c9c3,I], [c9,c3], [c9,c3,c3], [c3,c9], [c3,c9,I]
27/3	c27	27	2	2	2	[c27,I]
27/4	c3$\ltimes$c3^2	729	38	37	2	[c3^2,c3]
27/5	c3$\ltimes$c9	135	11	10	2	[c9,c3]
28/1	c14c2	112	16	28	8	[c14c2,I], [c14,c2], [c14,c2,c7], [c7,k4], [k4,c7], [c2,c14], [c2,c14,I]
28/2	c28	28	4	4	4	[c28,I], [c7,c4], [c4,c7]
28/3	d28	256	41	24	4	[c14,c2], [c7,k4], [c2,d14]
28/4	q28	100	9	8	2	[c7,c4]
29/1	c29	29	2	2	2	[c29,I]
30/1	c30	30	8	8	8	[c30,I], [c15,c2], [c10,c3], [c6,c5], [c5,c6], [c3,c10], [c2,c15]
30/2	d10c3	78	14	12	4	[c15,c2], [c5,c6], [c3,d10]
30/3	d6c5	50	10	8	4	[c15,c2], [c5,s3], [c3,c10]
30/4	d30	226	25	24	4	[c15,c2], [c5,s3], [c3,d10]

Example 5.1 The data in the table is stored as a list **Cat1List**, so any cat^1-group in the table may be selected using the **Cat1Select** function. In the first call only the group G is specified, while in the second call a third parameter is supplied.

```
gap> C := Cat1Select( 12, 5 );
There are 2 cat1-structures for the group a4.
[ [range gens], source & range names, [tail.genimages], [head.genimages] ] :-
[ [ (1,2,3),(2,3,4) ], tail = head = identity mapping ]
[ [ (2,4,3),(2,3,4) ], "k4", "c3", [ (2,4,3),(2,3,4) ], [ (2,4,3),(2,3,4) ] ]
Usage:  Cat1Select( size, gpnum, num )
Group has generators [ (1,2,3), (2,3,4) ]
gap> C := Cat1Select( 12, 5, 2 );
cat1-group [a4 ==> c3]
gap> X := XModCat1( C );
Crossed module [k4->c3]
```

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