PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

ENUMERATION OF POSETS GENERATED BY DISJOINT UNIONS AND ORDINAL SUMS[1]

RICHARD P. STANLEY

ABSTRACT. Let f_n be the number of n-element posets which can be built up from a given collection of finite posets using the operations of disjoint union and ordinal sum. A curious functional equation is obtained for the generating function $\Sigma f_n x^n$. Using a result of Bender, an asymptotic estimate can sometimes be given for f_n. The analogous problem for labeled posets is also considered.

Let P and Q be partially ordered sets (or posets). Regard P and Q as being relations on two disjoint sets T and T', respectively. The *disjoint union* $P+Q$ is defined to be the partial ordering on $T \cup T'$ satisfying: (1) If $x \in T$, $y \in T$, and $x \leq y$ in P, then $x \leq y$ in $P+Q$; (2) if $x \in T'$, $y \in T'$, and $x \leq y$ in Q then $x \leq y$ in $P+Q$. The *ordinal sum* $P \oplus Q$ is defined to be the partial ordering on $T \cup T'$ satisfying (1), (2), and the additional condition (3) if $x \in T$ and $y \in T'$, then $x \leq y$ in $P \oplus Q$. Hence $+$ is commutative but $\oplus$ is not.

The question we consider is the following. Let S be a set of nonvoid isomorphically distinct finite posets, such that no element P of S is a disjoint union or an ordinal sum of two nonvoid posets. (We say that P is $(+, \oplus)$*-irreducible*.) How many isomorphically distinct posets of cardinality n can be built up from the elements of S by the operations of disjoint union and ordinal sum? Call a poset that can be obtained in this way an *S-poset*. Hence if P and Q are S-posets, then so are $P+Q$ and $P \oplus Q$. Moreover, the $(+, \oplus)$-irreducible S-posets are simply the members of S. For instance, if S consists of a single one-element poset, then there are two 2-element S-posets and five 3-element S-posets, viz., 21, 2, 31, $2+1$, $(21) \oplus 1$, $1 \oplus (21)$, and 3. Here $\mathbf{n}$ denotes an n-element chain and nP a disjoint union $P + \cdots + P$ of n copies of P.

Received by the editors April 23, 1973.

AMS (MOS) subject classifications (1970). Primary 05A15; Secondary 06A10.

Key words and phrases. Poset, partially ordered set, disjoint union, ordinal sum, generating function, Pólya's enumeration theorem, functional equation.

[1] The research was supported by a Miller Research Fellowship at the University of California at Berkeley.

Let f_n denote the number of S-posets of cardinality n. We set $f_0 = 1$. Define the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$. We shall determine a functional equation for $F(x)$. The technique used is analogous to that appearing in [3, Chapter 6, § 10] for the enumeration of series-parallel networks. However, instead of a duality principle allowing us to obtain an explicit functional equation for $F(x)$, we instead are helped by the triviality of one of the two groups arising from the enumeration.

If P is an S-poset and P can be written $P_1 + P_2$ where neither P_i is void, then we say P is *essentially* $+$. Similarly if $P = P_1 \oplus P_2$ where neither P_i is void, we say P is *essentially* $\oplus$. Every S-poset not a member of S is either essentially $+$ or essentially $\oplus$, but not both. By convention we agree that every member of S is both essentially $+$ and essentially $\oplus$.

Let a_n be the number of n-element members of S, so $a_0 = 0$ since we are assuming the members of S to be nonvoid. Let u_n be the number of n-element essentially $+$ S-posets and v_n the number of n-element essentially $\oplus$ S-posets. We define $u_0 = v_0 = 0$. Hence $f_n = u_n + v_n - a_n$ if $n \geq 1$. Define the generating functions

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad U(x) = \sum_{n=0}^{\infty} u_n x^n, \quad V(x) = \sum_{n=0}^{\infty} v_n x^n.$$

It follows that

$$F(x) = U(x) + V(x) - A(x) + 1. \tag{1}$$

Now every essentially $+$ S-poset P not belonging to S can be written *uniquely* as a disjoint union $P_1 + P_2 + \cdots + P_m$ of $m \geq 2$ essentially $\oplus$ S-posets P_i, where the order of the P_i's is immaterial. Let u_{mn} be the number of essentially $+$ S-posets of cardinality n which are the disjoint union of m essentially $\oplus$ S-posets. Define for $m \geq 2$ the generating functions $U_m(x) = \sum_{n=1}^{\infty} u_{mn} x^n$.

Since the order of addition in a disjoint union is immaterial, it follows immediately from Pólya's theorem, as expounded in [3, Chapter 6], that

$$U_m(x) = Z(\mathfrak{S}_m | V(x), V(x^2), V(x^3), \ldots), \qquad m \geq 2, \tag{2}$$

where $Z(\mathfrak{S}_m | z_1, z_2, z_3, \cdots)$ is the cycle index polynomial of the symmetric group $\mathfrak{S}_m$ of degree m. Now $A(x) + \sum_{m=2}^{\infty} U_m(x) = U(x)$. Hence from (2) we obtain

$$U(x) = A(x) + \sum_{m=0}^{\infty} Z(\mathfrak{S}_m | V(x), V(x^2), V(x^3), \ldots) - V(x) - 1, \tag{3}$$

since $Z(\mathfrak{S}_0 | V(x), V(x^2), \ldots) = 1$ and $Z(\mathfrak{S}_1 | V(x), V(x^2), \ldots) = V(x)$.

Now it is well known that

$$\sum_{m=0}^{\infty} Z(\mathfrak{S}_m | z_1, z_2, z_3, \cdots)t^m = \exp(z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots)$$

(cf. [3, p. 133]). Thus from (3) we get

$$U(x) = \exp\left[\sum_{k=1}^{\infty} \frac{V(x^k)}{k}\right] - V(x) + A(x) - 1. \tag{4}$$

Similarly every essentially $\oplus$ S-poset P not belonging to S can be written *uniquely* as an ordinal sum $P_1 \oplus P_2 \oplus \cdots \oplus P_m$ of $m \geq 2$ essentially $+$ S-posets P_i, where now the order of the P_i's is fixed. Define v_{mn} and $V_m(x)$ for $m \geq 2$ in analogy to u_{mn} and $U_m(x)$. Since now the order of summing cannot be altered, we get from Pólya's theorem that

$$V_m(x) = Z(G_m | U(x), U(x^2), U(x^3), \cdots), \qquad m \geq 2, \tag{5}$$

where $Z(G_m | z_1, z_2, z_3, \cdots)$ is the cycle index polynomial of the trivial group G_m of order one acting on an m-set. Since $Z(G_m | z_1, z_2, \ldots) = z_1^m$, we get

$$V_m(x) = U(x)^m, \qquad m \geq 2. \tag{6}$$

Of course (6) can be easily obtained directly, but we wanted to make clear the similarity of (2) and (5).

Now

$$V(x) = A(x) + \sum_{m=2}^{\infty} V_m(x) = \sum_{m=0}^{\infty} U(x)^m - U(x) + A(x) - 1,$$

so

$$V(x) = 1/(1 - U(x)) - U(x) + A(x) - 1. \tag{7}$$

Eliminating $U(x)$ from (1) and (7) yields

$$V(x) = F(x) + 1/F(x) - 2 + A(x). \tag{8}$$

Thus by (4) and (8),

$$\begin{aligned} F(x) &= U(x) + V(x) - A(x) + 1 \\ &= \exp\left[\sum_{k=1}^{\infty} \frac{1}{k}\left(F(x^k) + \frac{1}{F(x^k)} - 2 + A(x^k)\right)\right]. \end{aligned}$$

We have obtained

Theorem 1. *$F(x)$ satisfies the functional equation*

$$F(x) = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k}\left(F(x^k) + \frac{1}{F(x^k)} - 2 + A(x^k)\right)\right]. \tag{9}$$

Since $\exp(\Sigma_{k=1}^{\infty} ax^k/k) = (1-x)^{-a}$, (9) may be rewritten in the rather curious form

$$F(x) = \prod_{i=1}^{\infty} (1 - x^i)^{-(f_i + g_i + a_i)},$$

where $1/F(x) = 1 + \Sigma_1^{\infty} g_i x^i$.

For instance, if S consists of a single one-element poset, then $A(x) = x$ and

$$F(x) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 48x^5 + 167x^6 + \cdots.$$

Another case of interest is when S consists of all isomorphically distinct finite $(+, \oplus)$-irreducible posets. Then f_n is the number of isomorphically distinct posets of cardinality n. It can be shown (cf. [4] for the numbers f_n) that here

$$A(x) = x + x^4 + 12x^5 + 104x^6 + 956x^7 + \cdots,$$

$$F(x) = 1 + x + 2x^2 + 5x^3 + 16x^4 + 63x^5 + 318x^6 + 2045x^7 + \cdots.$$

A theorem of Bender [1, Theorem 5] allows the determination of an asymptotic formula for f_n when $A(x)$ is a polynomial. We shall spare the reader the details of the calculations and merely state that when $A(x) = x$, we get $f_n \sim Cn^{-3/2}\alpha^{-n}$, where α is the unique positive root of $F(\alpha) - (1 + \sqrt{5})/2$, and C is a constant given by

$$C = \left(\frac{1}{\pi(3\sqrt{5} - 5)}\left[\frac{\alpha}{1-\alpha} + \sum_{k=2}^{\infty} \alpha^k F'(\alpha^k)\left(1 - \frac{1}{F(\alpha^k)^2}\right)\right]\right)^{1/2}.$$

Note that not surprisingly f_n is very much smaller than the *total* number p_n of posets of cardinality n, which by [2] is given by $p_n = 2^{n^2/4 + o(n^2)}$.

One can also ask analogous questions for *labeled* posets. A *labeling* of a finite poset P is an injection $\phi: P \to \mathbf{Z}$. Two labelings, ϕ and ψ, of P are *equivalent* if there is an automorphism $\rho: P \to P$ and an order-preserving injection $\sigma: \mathbf{Z} \to \mathbf{Z}$ such that $\phi = \sigma\psi\rho$. Let S be a set of nonvoid inequivalent $(+, \oplus)$-irreducible labeled finite posets. Let h_n be the number of n-element inequivalent labeled posets (P, ϕ) that can be built up from the members of S by the operations of $+$ and $\oplus$, such that the restriction of ϕ to any $(+, \oplus)$-irreducible component of P is equivalent to the labeling of a member of S. (We set $h_0 = 1$ by convention.) Let b_n be the number of n-element members of S (so $b_0 = 0$). Define the exponential generating functions

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \qquad H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}.$$

Then by an argument analogous to that used to prove Theorem 1, which we omit, one obtains

Theorem 2. *$H(x)$ satisfies the functional equation*

$$H(x) = \exp\left[H(x) + 1/H(x) - 2 + B(x)\right].$$

REFERENCES

1. E. A. Bender, *Asymptotic methods in enumeration*, SIAM Rev. (to appear).
2. D. Kleitman and B. Rothschild, *The number of finite topologies*, Proc. Amer. Math. Soc. 25(1970), 276–282. MR 40 #7157.
3. J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York; Chapman & Hall, London, 1958. MR 20 #3077.
4. J. Wright, *Cycle indices of certain classes of quasiorder types or topologies*, Dissertation, University of Rochester, Rochester, N. Y., 1972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139