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Enumeration of unrooted maps of a given genus

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Abstract

Let $\mathcal{N}_g(f)$ denote the number of rooted maps of genus g having f edges. An exact formula for $\mathcal{N}_g(f)$ is known for $g = 0$ (Tutte, 1963), $g = 1$ (Arques, 1987), $g = 2, 3$ (Bender and Canfield, 1991). In the present paper we derive an enumeration formula for the number $\Theta_\gamma(e)$ of unrooted maps on an orientable surface S_γ of a given genus γ and with a given number of edges e. It has a form of a linear combination $\sum_{i,j} c_{i,j}\mathcal{N}_{g_j}(f_i)$ of numbers of rooted maps $\mathcal{N}_{g_j}(f_i)$ for some $g_j \leqslant \gamma$ and $f_i \leqslant e$. The coefficients $c_{i,j}$ are functions of γ and e. We consider the quotient S_γ/Z_ℓ of S_γ by a cyclic group of automorphisms Z_ℓ as a two-dimensional orbifold O. The task of determining $c_{i,j}$ requires solving the following two subproblems:

(a) to compute the number $Epi_o(\Gamma, Z_\ell)$ of order-preserving epimorphisms from the fundamental group Γ of the orbifold $O = S_\gamma/Z_\ell$ onto Z_ℓ;
(b) to calculate the number of rooted maps on the orbifold O which lifts along the branched covering $S_\gamma \to S_\gamma/Z_\ell$ to maps on S_γ with the given number e of edges.

The number $Epi_o(\Gamma, Z_\ell)$ is expressed in terms of classical number-theoretical functions. The other problem is reduced to the standard enumeration problem of determining the numbers $\mathcal{N}_g(f)$ for some $g \leqslant \gamma$ and $f \leqslant e$. It follows that $\Theta_\gamma(e)$ can be calculated whenever the numbers $\mathcal{N}_g(f)$ are known for $g \leqslant \gamma$ and $f \leqslant e$. In the end of the paper the above approach is applied to derive the functions $\Theta_\gamma(e)$ explicitly for $\gamma \leqslant 3$. We note that the function $\Theta_\gamma(e)$ was known only for $\gamma = 0$ (Liskovets, 1981). Tables containing the numbers of isomorphism classes of maps with up to 30 edges for genus $\gamma = 1, 2, 3$ are presented.

Keywords: Enumeration; Map; Surface; Orbifold; Rooted map; Unrooted map; Fuchsian group

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1. Introduction

By a map we mean a 2-cell decomposition of a compact connected surface. Enumeration of maps on surfaces has attracted a lot of attention in the last few decades. As shown in monograph [29] the enumeration problem was investigated for various classes of maps. Generally, problems of the following sort are considered:

Problem 1. How many isomorphism classes of maps with a given property $\mathcal{P}$ and given number of edges (vertices, faces) are there?

The beginnings of the enumerative theory of maps are closely related with the enumeration of plane trees considered in 1960s by Tutte [38] and Harary et al. [16] (see [15,28] as well). The foundations of the theory were built by Tutte in a series of "Census" papers published in years 1962–1963 [34–37]. Later many other distinguished classes of maps including triangulations, outerplanar, cubic, Eulerian, non-separable, simple, loopless, two-face maps, etc. were considered. Research in these areas until year 1998 is well represented in [29]. Although there are more than 100 published papers on map enumeration, see, for instance, [5,8,13,22,27,39,41,44], most of them deal with the enumeration of rooted maps of given property. In particular, there is a lack of results on enumeration of unrooted maps of genus $\geqslant 1$. The present paper can be viewed as an attempt to fill in this gap. A map on an orientable surface is called oriented if one of the two global orientations is specified. Isomorphisms between oriented maps preserve the chosen orientation. The problem considered in this paper reads as follows.

Problem 2. What is the number of isomorphism classes of oriented unrooted maps of given genus g and given number of edges e?

An oriented map is called rooted of one of the darts (arcs) is distinguished as a root. By a dart of a map we mean an edge endowed with one of the two possible orientations. Isomorphisms between oriented rooted maps take root onto root. A rooted variant of Problem 2 follows.

Problem 3. What is the number of isomorphism classes of oriented rooted maps of given genus g and given number of edges e?

The rooted version of the problem was first considered in 1963 by Tutte [37] for $g = 0$, i.e. for the planar case. A corresponding planar case of the unrooted version (Problem 2 for $g = 0$) was settled by Liskovets [23,24] and Wormald [43]. An attempt to enumerate rooted maps of given genus $g > 0$ and given number of edges was done by Walsh and Lehman in [40,41]. They derived an algorithm based on a recursion formula. The algorithm is applied to enumerate maps with small number of edges. An explicit formula for the number of rooted maps for $g = 1$ is obtained by Arquès [2].

In 1988 Bender et al. [6] derived an explicit enumeration formula for the number of rooted maps on the torus and projective plane. Three years later [4] Bender and Canfield determined the function $\mathcal{N}_g(e)$ of rooted maps of genus g with e edges for any genus g up to some constants. For $g = 2$ and $g = 3$ the generating functions are derived. Some refinement of these results can be found in [3].

In the present paper we shall deal with the problem of enumeration of oriented unrooted maps with given genus and given number of edges. Inspired by a fruitful concept of an orbifold used in low-dimensional topology and in the theory of Riemann surfaces we introduce a concept of a map on an orbifold. In the present paper, by an *orbifold* we will mean a quotient of a surface by a finite group of automorphisms. As it will become clear later, *cyclic orbifolds*, that is the quotients of the type S_γ/Z_ℓ, where S_γ is an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ, will play a crucial role in the enumeration problem. In order to establish an explicit enumeration formula we first derive a general counting principle which enables us to decompose the problem into two subproblems (see Theorem 3.1). First one requires an enumeration of certain epimorphisms defined on Fuchsian groups (or on F-groups) onto a cyclic group. This problem is completely solved in Section 4. The other requires enumerating rooted maps on cyclic orbifolds associated with the considered surface. Unfortunately, quotients of (ordinary) maps may have half-edges called semiedges here. In Section 5 we reduce this problem to the problem of enumeration of rooted maps without semiedges.

In order to formulate our main result we need to introduce some concepts.

Let S_γ be an orientable surface of genus γ and Z_ℓ a cyclic group of automorphisms of S_γ. Denote by $[g; m_1, m_2, \dots, m_r]$, $2 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r \leqslant \ell$, the *signature* of orbifold $O = S_\gamma/Z_\ell$. That is, the underlying space of O is an oriented surface of genus g and the regular cyclic covering $S_\gamma \to O = S_\gamma/Z_\ell$ is branched over r points of O with branch indexes $m_1, m_2, \dots, m_r$, respectively. In 1966 Harvey [17] derived necessary and sufficient conditions for the existence of a cyclic orbifold S_γ/Z_ℓ with signature $[g; m_1, m_2, \dots, m_r]$ (see Theorem 4.3).

Given an orbifold O of the signature $[g; m_1, m_2, \dots, m_r]$ define, the *orbifold fundamental group* $\pi_1(O)$ to be the F-group generated by the $2g$ generators $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ and the r generators e_j, $j = 1, \dots, r$, satisfying the relations

$$\prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{r} e_j = 1, \quad e_j^{m_j} = 1 \quad \text{for every } j = 1, \dots, r.$$

An epimorphism $\pi_1(O) \to Z_\ell$ onto a cyclic group of order ℓ is called *order-preserving* if it preserves the orders of generators e_j, $j = 1, \dots, r$. Equivalently, an order-preserving epimorphism $\pi_1(O) \to Z_\ell$ has a torsion-free kernel. We denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order-preserving epimorphisms $\pi_1(O) \to Z_\ell$.

For a technical reason it is convenient to modify the signature of $O = S_\gamma/Z_\ell$ as follows. Let

$$[g; m_1, m_2, \dots, m_r] = [g; \underbrace{2, \dots, 2}_{q_2 \text{ times}}, \underbrace{3, \dots, 3}_{q_3 \text{ times}}, \dots, \underbrace{\ell, \dots, \ell}_{q_\ell \text{ times}}].$$

Then we will write $[g; 2^{q_2}, 3^{q_3}, \dots, \ell^{q_\ell}]$ rather than $[g; m_1, m_2, \dots, m_r]$, listing only j^{q_j} with $j > 0$.

We denote by $Orb(S_\gamma/Z_\ell)$ set of ℓ-tuples $[g; 2^{q_2}, 3^{q_3}, \dots, \ell^{q_\ell}]$ formed by signatures of cyclic orbifolds of the type S_γ/Z_ℓ for some S_γ and Z_ℓ. By the definition, the fundamental group $\pi_1(O)$ is uniquely determined by the signature of the orbifold O. Hence, for any $O \in Orb(S_\gamma/Z_\ell)$, $O = [g; 2^{q_2}, 3^{q_3}, \dots, \ell^{q_\ell}]$, the group $\pi_1(O)$ is well defined. The main result of this paper follows.

Theorem 1.1. *The number $\Theta_\gamma(e)$ of unrooted oriented maps with e edges on an orientable surface of genus γ is*

$$\frac{1}{2e}\sum_{\ell|e}\sum_{\substack{O\in Orb(S_\gamma/Z_\ell)\\ O=[g;2^{q_2},3^{q_3},\ldots,\ell^{q_\ell}]}} Epi_0\big(\pi_1(O),Z_\ell\big)\sum_{s=0}^{q_2}\binom{2e/\ell}{s}\binom{\frac{e}{\ell}-\frac{s}{2}+2-2g}{q_2-s,q_3,\ldots,q_\ell}\mathcal{N}_g\left(\frac{e}{\ell}-\frac{s}{2}\right),$$

where $\mathcal{N}_g(n)$ denotes the number of rooted maps with n edges on an orientable surface of genus g with a convention that $\mathcal{N}_g(n)=0$ if n is not an integer.

An explicit formula to calculate $Epi_0(\pi_1(O),Z_\ell)$ is given in Section 4, Proposition 4.2. The number $Epi_0(\pi_1(O),Z_\ell)$ is expressed in terms of classical number-theoretical functions. In Sections 6 and 7, Theorem 1.1 is applied to derive explicit enumeration functions for $\gamma=0,1,2,3$. For $\gamma=0$ we have confirmed the result of Liskovets [23]; the enumeration formulas for $\gamma=1,2$ and 3 are original. To apply the theorem for $\gamma>1$ one needs to determine the elements of $Orb(S_\gamma/Z_\ell)$ for all admissible ℓ. Since the set of cyclic orbifolds coming from S_γ can be easily determined (see Section 4), oriented unrooted maps on S_γ can be enumerated using Theorem 1.1 provided that the numbers $\mathcal{N}_g(n)$ of rooted maps are known for $g\leqslant\gamma$.

2. Maps, coverings and orbifolds

In what follows we build a part of the theory of maps which reflects some well-known ideas from topology of low-dimensional manifolds.

Maps on surfaces. By *surface* we mean a connected, orientable surface without a border. A *topological map* is a 2-cell decomposition of a surface. Usually, maps on surfaces are described as 2-cell embeddings of connected graphs. A (combinatorial) graph is a 4-tuple (D,V,I,L), where D and V are disjoint sets of *darts* and *vertices*, respectively, I is an incidence function $I:D\to V$ assigning to each dart an initial vertex, and L is the dart-reversing involution. The *edges* of a graph are the orbits of L. In what follows we shall deal with the category of *oriented maps*, that means that one of the two global orientations of the underlying surface is fixed. A given oriented map M can be described combinatorially as a triple $M=(D,R,L)$, where D is the set of darts (edges endowed with an orientation), L is an involutory permutation of D (called the dart-reversing involution) permuting darts sharing the same edge, and R is a permutation of D permuting cyclically (following the global orientation) for each vertex v the darts whose initial vertex is v. By the connectivity of the underlying graph the group $\langle R,L\rangle$ acts transitively on D. Conversely, given an abstract *combinatorial map* (D,R,L), where $Mon(M)=\langle R,L\rangle$ is a transitive group of permutations of D and $L^2=1$, we can construct an associated topological map as follows: The orbits of R, L and RL give rise to the vertices, edges and boundary walks of faces of the map, respectively, and the incidence relations between vertices, edges and faces is given by non-empty intersections of the respective sets of darts. If x is a vertex, edge or face, the degree of x is the size of the respective orbit of R, L or RL. The degree of an edge is two or one. *Semiedges* are edges of degree one. Maps without semiedges will be called *ordinary* maps. The group $Mon(M)=\langle R,L\rangle$ will be called a *monodromy group*. Given an element $w(R,L)=R^{i_1}L^{j_1}R^{i_2}L^{j_2}\cdots R^{i_n}L^{j_n}\in Mon(M)$ and a dart x_0, there is an associated dart-walk formed by the darts x_0, $L^{j_n}(x_0)$, $RL^{j_n}(x_0),\ldots,R^{i_n}L^{j_n}(x_0),\ldots,w(R,L)(x_0)$. This walk can be topologically realized in the topological map associated with $(D;R,L)$ as a curve with the initial point at x_0 and terminal point at $w(R,L)(x_0)$. Thus the action of $Mon(M)$ has a topological meaning. In fact it gives information about the action of the fundamental groupoid of the surface restricted to a certain class of curves.

Given maps $M_i = (D_i, R_i, L_i)$, $i = 1, 2$, a *covering* $M_1 \to M_2$ is a mapping $\psi : D_1 \to D_2$ such that $\psi R_1 = R_2\psi$ and $\psi L_1 = L_2\psi$. Note that transitivity of the actions of the monodromy groups force ψ to be onto. In particular, two maps $M_i = (D, R_i, L_i)$, $i = 1, 2$, based on the same set of darts D are isomorphic if and only if there exists ψ in the symmetric group S_D such that $R_2 = R_1^{\psi}$ and $L_2 = L_1^{\psi}$. The coverings $M \to M$ form a group $Aut(M)$ of *automorphisms* of a map M. Since the monodromy group is transitive on the set of darts, the automorphism group acts with trivial stabilizers, i.e. the action of $Aut(M)$ is semi-regular. More information about combinatorial maps can be found in [19].

Regular coverings. Let $\psi : M \to N$ be a covering of maps. The covering transformation group consists of automorphisms α of M satisfying the condition $\psi = \psi \circ \alpha$. A covering $\psi : M \to N$ will be called *regular* if the covering transformation group acts transitively on a fibre $\psi^{-1}(x)$ over a dart x of N. The regular coverings can be constructed by taking a subgroup $G \leqslant Aut(M)$, $M = (D, R, L)$, and setting $\bar{D}$ to be the set of orbits of G, $\bar{R}[x] = [Rx]$, $\bar{L}[x] = [Lx]$. Then the natural projection $x \mapsto [x]$ defines a regular covering $M \to N$, where $N = (\bar{D}, \bar{R}, \bar{L})$. Regular coverings of maps are extensively used in many considerations on maps and graph embeddings (see, for instance, [14,30]).

Signatures of maps and orbifolds associated with maps. Given regular covering $\psi : M \to N$, let $x \in V(N) \cup F(N) \cup E(N)$ be a vertex, face or edge of N. The ratio of degrees $b(x) = \deg(\tilde{x})/\deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lift of x along ψ, will be called a *branch index* of x. It is a routine matter to show that a branch index is a well-defined positive integer not depending on the choice of the lift $\tilde{x}$. In some considerations, it is important to save information about branch indexes coming from some regular covering defined over a map N. This can be done by introducing a signature σ on M. A *signature* is a function $\sigma : x \in V(N) \cup F(N) \cup E(N) \to Z^+$ assigning a positive integer to each vertex, edge and face, with only the following restriction: If x is an edge of degree 2, then $\sigma(x) = 1$ and if it is of degree 1 then $\sigma(x) \in \{1, 2\}$. We say that a signature σ on N is *induced* by a covering $\psi : M \to N$ if it assigns to vertices, faces and edges of N their branch indexes with respect to ψ.

If a map $M = (D, R, L)$ is finite we can calculate the genus g of M by the well-known Euler–Poincaré formula: $v(M) - e_2(M) + f(M) = 2 - 2g$, where $v(M)$ is the number of vertices, $e_2(M)$ is the number of edges of degree two, and $f(M)$ is the number of faces. Given a couple (M, σ), where M is a finite map and σ is a signature, we define an *orbifold type* of (M, σ) to be an $(r + 1)$-tuple of the form $[g; m_1, m_2, \ldots, m_r]$, where g is the genus of the underlying surface, $1 < m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r$ are integers, and m_i appears in the sequence $s_i > 0$ times if and only if σ takes the value m_i exactly s_i times. The *orbifold fundamental group* $\pi_1(M, \sigma)$ of (M, σ) is an F-group

$$\begin{aligned}\pi_1(M, \sigma) &= F[g; m_1, m_2, \ldots, m_r] \\ &= \left\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r \;\middle|\; \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = 1, \ldots, e_r^{m_r} = 1 \right\rangle.\end{aligned} \tag{2.1}$$

Let $\psi : M \to N$ be a *regular* covering and σ be a signature defined on N. We say that ψ is σ*-compatible* if for each element $x \in V(N) \cup E(N) \cup F(N)$ the branch index $b(x)$ of x is a divisor of $\sigma(x)$. The signature σ defined on N lifts along a σ-compatible regular covering

$\psi : M \to N$ to a derived signature σ_ψ on M defined by the following rule: $\sigma_\psi(\tilde{x}) = \sigma(x)/b(x)$ for each $\tilde{x} \in \psi^{-1}(x)$ and each $x \in V(N) \cup E(N) \cup F(N)$. We note that if $\sigma(x) = 1$ for each $x \in V(N) \cup E(N) \cup F(N)$, then σ-compatible covers over M are just smooth regular covers over M. Such a signature will be called *trivial*.

Let $M \to M/G$ be a regular covering with a covering transformation group G and suppose M be finite. Denote the respective orbifold type of $N = M/G$ by $[g; m_1, m_2, \dots, m_r]$. Then the Euler characteristic of the underlying surface of M is given by the Riemann–Hurwitz equation:

$$\chi = |G|\left(2 - 2g - \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right).$$

A topological counterpart of a (combinatorial) map M with a signature σ can be established as follows. By an *orbifold* O we mean a surface S with a distinguished discrete set of points B assigned by integers $m_1, m_2, \dots, m_i, \dots$ such that $m_i \geqslant 2$, for $i = 1, 2, \dots$. The elements of B will be called *branch points*. If S is a compact connected orientable surface of genus g, then B is finite of cardinality $|B| = r$ and O is determined by its type $[g; m_1, m_2, \dots, m_r]$. Hence we write $O = O[g; m_1, m_2, \dots, m_r]$. The fundamental group $\pi_1(O)$ of O is an F-group defined by (2.1). A *topological map* on an orbifold O is a map on the underlying surface S_g of genus g satisfying the following properties:

(P1) if $x \in B$ then x is either an internal point of a face, or a vertex, or an end-point of a semiedge which is not a vertex;
(P2) each face contains at most one branch point;
(P3) the branch index of x lying at the free end of a semiedge is two.

A mapping $\psi : \tilde{O} \to O$ is *a covering* if it is a branched covering between underlying surfaces mapping the set of branch points $\tilde{B}$ of $\tilde{O}$ onto the set B of branch points of O and each $\tilde{x}_i \in \psi^{-1}(x_i)$ is mapped uniformly with the same branch index d dividing the prescribed index r_i of $x_i \in B$. The following result is a consequence of the well-known theorem of Koebe:

Theorem 2.1. (Koebe [42]) *Let O be a compact connected orbifold of type* $[g; m_1, m_2, \dots, m_r]$. *Then there is a universal orbifold $\tilde{O}$ covering O satisfying the following conditions*:

(a) *if there is a regular covering $\varphi : O_1 \to O$ then there is a regular covering $\psi : \tilde{O} \to O_1$;*
(b) *the covering $\Phi : \tilde{O} \to O$ is regular with the covering transformation group isomorphic to $F(g; m_1, m_2, \dots, m_r)$ and $\Phi = \psi \circ \varphi$.*

Remark. The reader familiar with Koebe's theorem may ask where the 'bad orbifolds' of type $[0; r]$ and $[0; r, q]$ $r \neq q$, $\gcd(r, q) = 1$ disappeared. They are included in the statement only that they give rise to trivial universal covers. Note that $F[0; r] = F[0; r, q] = 1$ is a trivial group in this case. The underlying surface of the universal cover is either a sphere or a plane, depending on whether the respective F-group is finite or infinite. In general, the universal cover of the orbifold $O = O[0; r, q]$, $r \neq q$ is $O = O[0; r/d, q/d]$, where $d = \gcd(r, q)$.

It is easy to see a bridge between maps with signatures and orbifolds. Indeed, a finite map M with signature σ of orbifold type $[g; m_1, m_2, \dots, m_r]$ determines an orbifold $O = O[g; m_1, m_2, \dots, m_r]$ with signature $[g; m_1, m_2, \dots, m_r]$ by taking the corresponding topological map and placing a branch point of index m_i inside the corresponding vertex, edge or face x

with $\sigma(x) = m_i$, for each $i = 1, \dots, r$. Moreover, σ-compatible covers over M are in correspondence with orbifolds covering O. Given the universal covering $\Phi : \tilde{O} \to O$ we can lift the map M to a map $\tilde{M}$ on $\tilde{O}$. The respective map $\tilde{M}$ will be called a *universal cover* with respect to (M, σ). As a consequence we have the following statement.

A homomorphism $\alpha : F[g; m_1, m_2, \dots, m_r] \to H$ is called order-preserving if it preserves the orders $m_1, m_2, \dots, m_r$ of generators $e_1, e_2, \dots, e_r$.

Theorem 2.2. *Let N be a finite map with signature σ induced by a regular covering $\varphi : M \to N$ with a group of covering transformations A. Let $\Phi : \tilde{N} \to N$ be the universal covering with respect to (N, σ). Then a regular covering $\psi : \tilde{N} \to M$ such that $\Phi = \psi \circ \varphi$ induces an order-preserving group epimorphism $\psi^* : \pi_1(N, \sigma) \to A$. Moreover, the monodromy action of $Mon(M)$ on a fibre $\varphi^{-1}(x)$, $x \in N$, is uniquely determined by ψ^*.*

Proof. Let $M = (D, R, L)$ and $\tilde{N} = (\tilde{D}, \tilde{R}, \tilde{L})$. Fix a dart $x_0 \in N$ and fibres $\Phi^{-1}(x_0)$, $\varphi^{-1}(x_0)$. In what follows all the considered darts will be elements of these two fibres. We show that every covering transformation $\tilde{\tau}$ of Φ projects onto some covering transformation $\tau \in A$. Choose a dart $\tilde{x} \in \Phi^{-1}(x_0)$.

Let $\tilde{\tau}$ take $\tilde{x} \mapsto \tilde{y}$. Let $x = \psi(\tilde{x})$ and $y = \psi(\tilde{y})$. By regularity of the action of A there is a unique covering transformation $\tau \in A$ taking $x \mapsto y$. For any $\tilde{z} \in \Phi^{-1}(x_0)$ there exists $w(\tilde{R}, \tilde{L}) \in Mon(\tilde{N})$ such that $w(\tilde{R}, \tilde{L})\tilde{x} = \tilde{z}$. We show that $\psi\tilde{\tau} = \tau\psi$. We have

$$\begin{aligned} \psi\tilde{\tau}(\tilde{z}) &= \psi\tilde{\tau}w(\tilde{R}, \tilde{L})(\tilde{x}) = \psi w(\tilde{R}, \tilde{L})\tilde{\tau}(\tilde{x}) = \psi w(\tilde{R}, \tilde{L})(\tilde{y}) \\ &= w(R, L)\psi(\tilde{y}) = w(R, L)\tau(x) = \tau w(R, L)(x) = \tau(z) = \tau\psi(\tilde{z}). \end{aligned}$$

Hence the mapping $\psi^* : \tilde{\tau} \mapsto \tau$ is a group homomorphism. Since for each $y \in \varphi^{-1}(x_0)$ there is a preimage $\tilde{y} \in \Phi^{-1}(x_0)$; it is an epimorphism.

By Theorem 2.1 N lifts to a map $\tilde{N} \to N$ on the universal orbifold with the group of covering transformations acting regularly on a fibre over a dart x. Moreover, this group is isomorphic to $\pi_1(N, \sigma)$. Thus ψ^* takes $\pi_1(N, \sigma)$ onto A. Furthermore, by regularity we may label darts of $\Phi^{-1}(x_0)$ by elements of $\pi_1(N, \sigma)$ and darts of $\varphi^{-1}(x_0)$ by elements of A. If ψ^* is determined then the covering ψ is determined on $\Phi^{-1}(x_0)$, and consequently, the action of $Mon(M)$ on $\varphi^{-1}(x_0)$ is prescribed by the projection of the action of $Mon(\tilde{N})$ along ψ.

The assumption that the derived signature σ_φ is trivial forces the covering $\Phi : \tilde{N} \to M$ to be smooth. Take an element $g \in \pi_1(N, \sigma)$ of finite order n. Then there exists an associated word $w(\tilde{R}, \tilde{L})$ taking a dart labelled by 1 onto a dart labelled by g. Then $w^j(\tilde{R}, \tilde{L})$ takes $1 \mapsto g^j$, and in particular, $w^n(\tilde{R}, \tilde{L})(1) = 1$. Thus it gives rise to a closed walk in $\tilde{N}$. The covering ψ takes $w^j(\tilde{R}, \tilde{L}) \mapsto w^j(R, L)$. The respective walk in M is closed if and only if $(\psi^*(g))^j = 1$. Since ψ is smooth, $w^j(R, L)$ is not closed for $1 \leqslant j < n$. Then $(\psi^*(g))^j \neq 1$ for $j = 1, \dots, n-1$. Hence ψ^* is order-preserving. □

Reconstruction of M. With the above notation, given $N = (\bar{D}; \bar{R}, \bar{L})$ on an orbifold $\bar{O}$ and an epimorphism $\psi^* : \pi_1(N, \sigma) \to A$ one may ask whether there is way to reconstruct the cover $M = (D; R, L)$ explicitly. To do this one can use the idea of ordinary voltage assignments used to describe regular covers of graphs [14] and modified in [30] to describe branched coverings of maps with branch points at vertices, faces and edges. Firstly we form a truncated map $T(N)$ whose vertices are darts of N and whose darts are ordered pairs of the form $x\bar{R}x$, $x\bar{R}^{-1}x$ and xLx. The dart-reversing involution of $T(N)$ interchanges the pairs $(x\bar{R}x, (\bar{R}x)x)$; and

$(x\bar{L}x, (\bar{L}x)x)$, while the rotation cyclically permutes $(x\bar{L}x, x\bar{R}x, x\bar{R}^{-1}x)$ for any $x \in \bar{D}$. We choose a spanning tree T of $T(N)$ and define an ordinary voltage assignment ν in A on darts of T to be 1. We fix a vertex $x_0 \in \bar{D}$ of $T(N)$. If z is a dart of $T(N)$ not belonging to T it creates (together with some paths of T joining x_0 to the initial and terminal vertex of z) a closed walk based at x_0. This closed walk corresponds to some word $w(\bar{R}, \bar{L})$ which lifts to $w(\tilde{R}, \tilde{L})$ taking $\tilde{x}_0$ onto $\tilde{y} = w(\tilde{R}, \tilde{L})x_0$. By regularity there is a unique element $h \in \pi_1(N, \sigma)$ such that $h(\tilde{x}_0) = \tilde{y}$. We set $\nu(z) = \psi^*(h)$. In this way the voltage assignment is defined at each dart of $T(N)$. We lift $T(N)$ using the definition of the derived graph and derived map (see [14, pp. 162–170]) onto a truncation $T(M)$ of a map M. Then we contract the faces of $T(M)$ which correspond to vertices of M to points, thus obtaining M. Taking different epimorphisms $\psi^*: \pi_1(N, \sigma) \to A$ we get all the σ-compatible regular covers over N with the covering transformation group isomorphic to A.

3. A formula for counting maps of given genus

In this section we shall deal with the problem of enumeration of oriented unrooted maps of given genus γ. Recall that a map is called *rooted* if it has one distinguished dart x_0 called a *root*. A morphism between rooted maps takes root onto root. A map is called *labelled* if all its darts are distinguished by some labelling. Since the automorphism group of a rooted map as well as that of a labelled map is trivial, each rooted map with n darts gives rise to $(n-1)!$ labelled maps. Moreover, if (M, x) and (M, y) are two rooted maps based on the same map with a dart-set D then the number of isomorphism classes for (M, x) and (M, y) is the same. We note that there is a 1–1 correspondence between isomorphism classes of rooted (and labelled) maps defined in the category of oriented maps and isomorphism classes of rooted (and labelled) maps in the category of maps on orientable surfaces as they are defined, for example in monograph [29, p. 7].

To be more precise, we fix the set of darts D and consider different maps based on D. We want to determine the number of isomorphism classes of (unrooted) maps based on n darts and of a given genus γ. This number will be denoted by $NUM_\gamma(n)$. Denote by $\mathcal{M} = \mathcal{M}(n)$ the set of all (labelled) maps on D of a given genus. The symmetric group S_n, $|D| = n$, acts on $\mathcal{M}$ by conjugation as follows: $M = (D; R, L) \mapsto M^\psi = (D; R^\psi, L^\psi)$. By definition ψ is a map isomorphism taking $M \mapsto M^\psi$. Then the number of orbits $\mathcal{M}/S_n = NUM_\gamma(n)$ and the number of orbits of the stabilizer S_{n-1} of a dart $x_0 \in D$ is equal to the number of rooted maps: $NRM_\gamma(n) = \mathcal{M}/S_{n-1}$.

By Burnside's lemma [11, pp. 494–495]

$$NUM_\gamma(n) = \sum_{\alpha \in S_n} \frac{|Fix_{\mathcal{M}}(\alpha)|}{n!},$$

where $Fix_{\mathcal{M}}(\alpha)$ is the set of maps on D fixed by the action of α. Since the set of darts is fixed, each such a map is determined by a pair of permutations (R, L) acting on D such that $\langle R, L\rangle$ is transitive and $L^2 = 1$. In what follows we shall concentrate on $Fix_{\mathcal{M}}(\alpha)$.

Hall's result, see [25,26], implies the following assertion:

If $Fix_{\mathcal{M}}(\alpha) \neq \emptyset$ then α is a regular permutation; that means that α can be expressed as a product of m (disjoint) cycles of the same length ℓ, say $\alpha = C_1C_2\dots C_m$, where $\ell m = n$.

Thus we may reduce our investigation to regular permutations. Since all permutations with a prescribed cyclic structure are conjugate in S_n, the size of sets $Fix_{\mathcal{M}}(\alpha)$ depends only on the decomposition $n = \ell m$. Denote by $[\ell^m]$ the conjugacy class of regular permutations of order ℓ. By a well-known formula $|[\ell^m]| = \frac{n!}{m!\ell^m}$. Hence Burnside's formula specializes to

$$NUM_\gamma(n) = \sum_{\ell|n, n=\ell m} \frac{|Fix_{\mathcal{M}}[\ell^m]|}{\ell^m m!},$$

where $Fix_{\mathcal{M}}[\ell^m]$ is the set of maps in $\mathcal{M}$ fixed by some regular permutation α with cycle structure ℓ^m.

Since $Fix_{\mathcal{M}}(\alpha) = \{(D, R, L) \in \mathcal{M} \mid R^\alpha = R,\ L^\alpha = L\}$, $\langle\alpha\rangle$ is a cyclic group of map automorphisms for each $M = (D, R, L) \in Fix_{\mathcal{M}}(\alpha)$. Take the quotient $N = M/\langle\alpha\rangle = (\bar{D}, \bar{R}, \bar{L})$. The covering $\varphi : M \mapsto N$ determines the signature σ on N assigning to vertices, faces and edges their branch index with respect to φ. Denote by O the respective orbifold associated with (N, σ). By Theorem 2.2 there is a covering $\psi : \tilde{N} \to M$ which induces an order-preserving epimorphism $\psi^* : \pi_1(N, \sigma) \to Z_\ell$. The map N is a labelled map on the orbifold O whose darts are assigned by $C_1, C_2, \ldots, C_m$. Since for given $N = (\bar{R}, \bar{L})$ every monodromy action on the cycle C_1 is determined by an epimorphism from the orbifold fundamental group into the cyclic group $Z_\ell \cong \langle\alpha\rangle$, we have $Epi_0(\pi_1(N, \sigma), Z_\ell) = Epi_0(\pi_1(O), Z_\ell)$ possibilities to reconstruct the action of $Mon(M)$ on C_1. Here O denotes the orbifold associated with (N, σ). Now in each cycle $C_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,\ell}\}$, $i \neq 1$, we choose one dart. We have ℓ^{m-1} such choices. In this way the labelling of the darts of M is determined by the following rule: $v_{i,x} \in C_i$ $(i \neq 1)$ has the second coordinate $x = j$ if and only if a monodromy transformation τ taking $v_{1,1} \mapsto v_{1,j}$ maps $v_{i,1} \mapsto v_{i,x}$. Thus the permutations (R, L) are completely determined by the action of $\langle\bar{R}, \bar{L}\rangle$ and by the action of the set-wise stabilizer of C_1.

Denote by $Orb(S_\gamma/Z_\ell)$ the set of all orbifolds arising as cyclic quotients by some action of Z_ℓ from a surface of genus γ and by $NLM_O(m)$ the number of labelled quotient maps for a given orbifold type O which lift onto maps on a surface of genus γ, having $n = \ell m$ darts.

We have proved that

$$\begin{aligned} NUM_\gamma(n) &= \sum_{\ell|n, n=\ell m} \frac{|Fix_{\mathcal{M}}[\ell^m]|}{\ell^m m!} \\ &= \sum_{\ell|n, n=\ell m} \sum_{O\in Orb(S_\gamma/Z_\ell)} \frac{Epi_0(\pi_1(O), Z_\ell)\ell^{m-1} NLM_O(m)}{\ell^m m!}. \end{aligned}$$

Denote by $NRM_O(m)$ the number of rooted maps on a cyclic orbifold S_γ/Z_ℓ which lift to maps without semiedges possessing exactly $m\ell$ darts. Since $NLM_O(m) = (m-1)!NRM_O(m)$ we get the following theorem.

Theorem 3.1. *With the above notation the following enumeration formula holds*:

$$NUM_\gamma(n) = \frac{1}{n} \sum_{\ell|n, n=\ell m} \sum_{O\in Orb(S_\gamma/Z_\ell)} Epi_0(\pi_1(O), Z_\ell) NRM_O(m).$$

Remark 1. The above theorem establishes a general counting principle which makes it possible to reduce the problem of enumerating of maps of given genus γ sharing a certain map property $\mathcal{P}$ to a problem of enumerating the rooted maps on associated cyclic orbifolds which lift to maps of genus γ sharing the property $\mathcal{P}$. In this paper we are interseted in enumeration of ordinary maps of genus γ; so $\mathcal{P}$ means here: no semiedges in M. Generally, by a map property we mean a property preserved by isomorphisms of unrooted maps. Checking the proof of Theorem 3.1 one can see that its proof is independent of the choice of $\mathcal{P}$; hence one can apply this counting principle for more restricted families of maps such as one-face maps, loopless maps, non-separable maps, etc. It remains, however, to solve the problem of determining the numbers $Epi_0(\pi_1(O), Z_\ell)$ and $NRM_O(\mathcal{P}, m)$, where $NRM_O(\mathcal{P}, m)$ denotes the number of rooted maps on a cyclic orbifold S_γ/Z_ℓ which lift to maps with $m\ell$ darts sharing property $\mathcal{P}$. In what follows we shall deal with both problems.

Remark 2. As noted by Liskovets (personal communication) using results of the following sections one can prove that the above formula derived in Theorem 3.1 agrees with the general reductive formula derived in [25, Theorem 2.8] (see [26] as well).

4. The number of epimorphisms from an F-group onto a cyclic group

As one can see from Theorem 3.1 to derive an explicit formula for the number of unrooted maps of a given genus and given number of edges one needs to deal with the numbers $Epi_0(\Gamma, Z_\ell)$ of order-preserving epimorphisms from an F-group Γ onto a cyclic group Z_ℓ. The aim of this section is to calculate these numbers.

Denote by $\mathrm{Hom}_0(\Gamma, Z_\ell)$ the set of order-preserving homomorphisms from the group Γ into Z_ℓ. Let

$$\begin{aligned}\Gamma &= F[g; m_1, \ldots, m_r] \\ &= \left\langle \mathsf{a}_1, \mathsf{b}_1, \ldots, \mathsf{a}_g, \mathsf{b}_g, \mathsf{x}_1, \ldots, \mathsf{x}_r \colon \prod_{i=1}^{g} [\mathsf{a}_i, \mathsf{b}_i] \prod_{j=1}^{r} \mathsf{x}_j = 1, \mathsf{x}_1^{m_1} = 1, \ldots, \mathsf{x}_r^{m_r} = 1 \right\rangle\end{aligned}$$

be an F-group of signature $(g; m_1, \ldots, m_r)$.

Following the arguments used by Jones in [18] we obtain

$$Epi_0(\Gamma, Z_\ell) = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) \left|\mathrm{Hom}_0(\Gamma, Z_d)\right|,$$

where $\mu(\frac{\ell}{d})$ denotes the Möbius function. Set $m = lcm(m_1, \ldots, m_r)$ to be the least common multiple of $m_1, m_2, \ldots, m_r$. We note that if $r = 0$ then the group $F[g; \emptyset] = F[g; 1]$; so, we set $m = 1$ for $r = 0$. Since $\mathrm{Hom}_0(\Gamma, Z_d)$ is empty if at least one of $m_1, \ldots, m_r$ is not a divisor of d, we also have

$$Epi_0(\Gamma, Z_\ell) = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) \left|\mathrm{Hom}_0(\Gamma, Z_d)\right|. \tag{4.1}$$

We suppose that the numbers $m_1, \ldots, m_r$ are divisors of d. Identify the group Z_d with the additive group of residues $\{1, \ldots, d\}$ mod d. Since the group Z_d is abelian, there is a one-to-one correspondence between order-preserving epimorphisms from $\mathrm{Hom}_0(\Gamma, Z_d)$ and the elements of the set

$$\begin{aligned}&\big\{(a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r) \in Z_d^{2g+r}: \\ &\ x_1 + \cdots + x_r = 0 \bmod d, (x_1, d) = d_1, \ldots, (x_r, d) = d_r\big\},\end{aligned}$$

where (x, d) is the greatest common divisor of x and d (well defined in the group Z_d). Set $d_1 = \frac{d}{m_1}, \ldots, d_r = \frac{d}{m_r}$.

Hence

$$\left|\mathrm{Hom}_0(\Gamma, Z_d)\right| = d^{2g} \cdot E_d(m_1, \ldots, m_r), \tag{4.2}$$

where $E_d(m_1, \ldots, m_r)$ is the number of solutions of the equation $x_1 + \cdots + x_r = 0 \bmod d$, $(x_1, d) = d_1, \ldots, (x_r, d) = d_r$.

Denote by $\mu(n), \phi(n)$ and $\Phi(x, n)$ the Möbius, Euler and von Sterneck functions, respectively. The relationship between them is given by the formula

$$\Phi(x, n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\mu\left(\frac{n}{(x, n)}\right),$$

where (x, n) is the greatest common divisor of x and n. It was shown by Hölder that $\Phi(x, n)$ coincides with the Ramanujan sum $\sum_{1 \leqslant k \leqslant n,\, (k,n)=1} \exp(\frac{2ikx}{n})$. For the proof, see Apolstol [1, p. 164] and [31].

Lemma 4.1. *Let $m_1, \ldots, m_r$ be divisors of d and $d_1 = \frac{d}{m_1}, \ldots, d_r = \frac{d}{m_r}$. Then the number $E = E_d(m_1, \ldots, m_r)$ of solutions $(x_1, x_2, \ldots, x_r)$, $x_j \in Z_d$ for $j = 1, 2, \ldots, r$, of the system of the equations*

$$x_1 + \cdots + x_r = 0 \mod d, \qquad (x_1, d) = d_1, \ldots, (x_r, d) = d_r$$

is given by the formula

$$E = \frac{1}{d}\sum_{k=1}^{d} \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r).$$

Proof. Consider the polynomial

$$P(z) = \sum_{\substack{1 \leqslant x_1, \ldots, x_r \leqslant d \\ (x_1,d)=d_1, \ldots, (x_r,d)=d_r}} z^{x_1 + \cdots + x_r}.$$

Then the number of solutions E coincide with the sum of the coefficients of $P(z)$ whose exponents are divisible by d. Hence

$$E = \frac{1}{d}\sum_{k=1}^{d} P\left(\varepsilon^k\right), \quad \text{where } \varepsilon = e^{\frac{2\pi i}{d}}.$$

We have

$$\begin{aligned}
P\left(\varepsilon^k\right) &= \sum_{\substack{1 \leqslant x_1 \leqslant d \\ (x_1,d)=d_1}} \sum_{\substack{1 \leqslant x_2 \leqslant d \\ (x_2,d)=d_2}} \cdots \sum_{\substack{1 \leqslant x_r \leqslant d \\ (x_r,d)=d_r}} (\varepsilon^k)^{x_1 + \cdots + x_r} \\
&= \sum_{\substack{1 \leqslant x_1 \leqslant d \\ (x_1,d)=d_1}} \varepsilon^{kx_1} \cdot \sum_{\substack{1 \leqslant x_2 \leqslant d \\ (x_2,d)=d_2}} \varepsilon^{kx_2} \cdot \ldots \cdot \sum_{\substack{1 \leqslant x_r \leqslant d \\ (x_r,d)=d_r}} \varepsilon^{kx_r} \\
&= \sum_{\substack{1 \leqslant x_1 \leqslant d \\ (x_1,d)=d_1}} e^{\frac{2\pi ikx_1}{d}} \cdot \sum_{\substack{1 \leqslant x_2 \leqslant d \\ (x_2,d)=d_2}} e^{\frac{2\pi ikx_2}{d}} \cdot \ldots \cdot \sum_{\substack{1 \leqslant x_r \leqslant d \\ (x_r,d)=d_r}} e^{\frac{2\pi ikx_r}{d}} \\
&= \sum_{\substack{1 \leqslant y_1 \leqslant m_1 \\ (y_1,m_1)=1}} e^{\frac{2\pi iky_1}{m_1}} \cdot \sum_{\substack{1 \leqslant y_2 \leqslant m_2 \\ (y_2,m_2)=1}} e^{\frac{2\pi iky_2}{m_2}} \cdot \ldots \cdot \sum_{\substack{1 \leqslant y_r \leqslant m_r \\ (y_r,m_r)=1}} e^{\frac{2\pi iky_r}{m_r}}
\end{aligned}$$

$$= \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r).$$

Hence

$$E = \frac{1}{d} \sum_{k=1}^{d} \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r). \qquad \square$$

As was observed by Liskovets (personal communication) $E_d(m_1, m_2, \ldots, m_r) = E_m(m_1, m_2, \ldots, m_r)$ for any d, $m|d$. Thus the function $E_d(m_1, m_2, \ldots, m_r)$ does not depend on d and we set

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r), \tag{4.3}$$

where $m = lcm(m_1, m_2, \ldots, m_r)$. Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined as (for more information see [12, p. 199], [20,33])

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

From the above arguments we obtain the following proposition.

Proposition 4.2. *Let $\Gamma = F[g; m_1, \ldots, m_r]$ be an F-group of signature $(g; m_1, \ldots, m_r)$. Denote by $m = lcm(m_1, \ldots, m_r)$ the least common multiple of $m_1, \ldots, m_r$ and let $m|\ell$. Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula*

$$Epi_0(\Gamma, Z_\ell) = m^{2g} \phi_{2g}(\ell/m) E(m_1, m_2, \ldots, m_r),$$

where

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r),$$

$\phi_{2g}(\ell)$ is the Jordan multiplicative function of order $2g$, and $\Phi(k, m_j)$ is the von Sterneck function.

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g, then we have

$$Epi_0(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

Proof. By (4.1) and (4.2)

$$Epi_0(\Gamma, Z_\ell) = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) \left|\mathrm{Hom}_0(\Gamma, Z_d)\right| = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2g} \cdot E_d(m_1, \ldots, m_r).$$

By Lemma 4.1

$$E_d(m_1, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k, m_1) \cdot \Phi(k, m_2) \cdot \ldots \cdot \Phi(k, m_r).$$

Hence

$$Epi_0(\Gamma, Z_\ell) = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2g} \cdot E(m_1, \ldots, m_r).$$

Inserting $d = d_1 m$ and $\ell = \ell_1 m$ we get

$$Epi_0(\Gamma, Z_\ell) = m^{2g} \sum_{d_1|\ell_1} \mu\left(\frac{\ell_1 m}{d_1 m}\right) d_1^{2g} \cdot E(m_1, \dots, m_r)$$
$$= m^{2g} \phi_{2g}(\ell/m) E(m_1, m_2, \dots, m_r). \qquad \square$$

We note that the condition $m|\ell$ in the above proposition gives no principal restriction, since $Epi_0(\Gamma, Z_\ell) = 0$ by the definition, provided that m does not divide ℓ. An orbifold $O = O[g; m_1, \dots, m_r]$ will be called γ-*admissible* if it can be represented in the form $O = S_\gamma / Z_\ell$, where S_γ is an orientable surface of genus γ and Z_ℓ is a cyclic group of automorphisms of S_γ. By the Koebe's theorem there is an orbifold $O = S_\gamma / Z_\ell$ with signature $[g; m_1, m_2, \dots, m_r]$ if and only if there exists ℓ such that the number $Epi_0(\pi_1(O), Z_\ell) \neq 0$ and the numbers γ, g, $m_1, \dots, m_r$ and ℓ are related by the Riemann–Hurwitz equation $2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$. Although the condition $Epi_0(\pi_1(O), Z_\ell) \neq 0$ can be checked using Proposition 4.2, for practical use it is more convenient to employ the following result by Harvey [17], see [7,9] as well. The Wiman theorem [10, p. 131] ensures us that $1 \leqslant \ell \leqslant 4\gamma + 2$ for $\gamma > 1$.

Theorem 4.3. [17] *Let $O = O[g; m_1, \dots, m_r]$ be an orbifold. Then O is γ-admissible if and only if there exists an integer ℓ such that following conditions are satisfied*:

(1) *$m = lcm(m_1, m_2, \dots, m_r)$ divides ℓ and $m = \ell$ if $g = 0$*;
(2) *$2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$ (Riemann–Hurwitz equation)*;
(3) *$lcm(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_r) = lcm(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_r)$ for each $i = 1, 2, \dots, r$*;
(4) *if $m = lcm(m_1, m_2, \dots, m_r)$ is even, then the number of m_j divisible by the maximal power of 2 dividing m is even*;
(5) *if $\gamma \geqslant 2$, then $r \neq 1$ and $r \geqslant 3$ for $g = 0$; if $\gamma = 1$, then $r \in \{0, 3, 4\}$; if $\gamma = 0$, then $r = 2$ or $r = 0$.*

If $\gamma > 1$, then the integer ℓ is bounded by $1 \leqslant \ell \leqslant 4\gamma + 2$.

Using Theorem 4.3, see [7,9,21] as well, we derive the following lists of γ-admissible orbifolds, for $\gamma = 0, 1, 2, 3$. Employing Proposition 4.2 the numbers $Epi_0(\pi_1(O), Z_\ell)$ are calculated for each orbifold in the list.

Corollary 4.4. *The 0-admissible orbifolds are $O = O[0; \ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ for any positive integer ℓ.*

Corollary 4.5. *Let $O = O[g; m_1, m_2, \dots, m_r] = S_1/Z_\ell$ be a 1-admissible orbifold. Then one of the following statements holds*:

$$O = O[1; \emptyset], \quad \textit{with } Epi_0\big(\pi_1(O), Z_\ell\big) = \sum_{k|\ell} \mu(\ell/k) k^2 = \phi_2(\ell) \quad \textit{for any } \ell,$$
$$\ell = 2 \quad \textit{and} \quad O = O\big[0; 2^4\big], \quad \textit{with } Epi_0\big(\pi_1(O), Z_\ell\big) = 1,$$
$$\ell = 3 \quad \textit{and} \quad O = O\big[0; 3^3\big], \quad \textit{with } Epi_0\big(\pi_1(O), Z_\ell\big) = 2,$$
$$\ell = 4 \quad \textit{and} \quad O = O\big[0; 4^2, 2\big], \quad \textit{with } Epi_0\big(\pi_1(O), Z_\ell\big) = 2,$$

$$\ell = 6 \quad and \quad O = O[0; 6, 3, 2], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 2.$$

Corollary 4.6. *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_2/Z_\ell$ be a 2-admissible orbifold. Then one of the following statements holds*:

$$\ell = 1 \quad and \quad O = O[2; \emptyset], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 1,$$
$$\ell = 2 \quad and \quad O = O[1; 2^2] \quad or \quad O[0; 2^6],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 4, 1,\ respectively,$$
$$\ell = 3 \quad and \quad O = O[0; 3^4], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 6,$$
$$\ell = 4 \quad and \quad O = O[0; 2^2, 4^2], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 2,$$
$$\ell = 5 \quad and \quad O = O[0; 5^3], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 12,$$
$$\ell = 6 \quad and \quad O = O[0; 2^2, 3^2] \quad or \quad O = O[0; 3, 6^2],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 2, 2,\ respectively,$$
$$\ell = 8 \quad and \quad O = O[0; 2, 8^2], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 4,$$
$$\ell = 10 \quad and \quad O = O[0; 2, 5, 10], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 4.$$

Corollary 4.7. *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_3/Z_\ell$ be a 3-admissible orbifold. Then one of the following statements holds*:

$$\ell = 1 \quad and \quad O = O[3; \emptyset], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 1,$$
$$\ell = 2 \quad and \quad O = O[2; \emptyset], \quad O[1; 2^4] \quad or \quad O[0; 2^8],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 15, 4, 1,\ respectively,$$
$$\ell = 3 \quad and \quad O = O[1; 3^2] \quad or \quad O[0; 3^5],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 18, 10,\ respectively,$$
$$\ell = 4 \quad and \quad O = O[1; 2^2], \quad O[0; 2^3, 4^2] \quad or \quad O[0; 4^4],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 12, 2, 8,\ respectively,$$
$$\ell = 6 \quad and \quad O = O[0; 2, 3^2, 6] \quad or \quad O[0; 2^2, 6^2],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 2, 2,\ respectively,$$
$$\ell = 7 \quad and \quad O = O[0; 7^3], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 30,$$
$$\ell = 8 \quad and \quad O = O[0; 4, 8^2], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 8,$$
$$\ell = 9 \quad and \quad O = O[0; 3, 9^2], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 12,$$
$$\ell = 12 \quad and \quad O = O[0; 2, 12^2] \quad or \quad O[0; 3, 4, 12],$$
$$with\ Epi_0(\pi_1(O), Z_\ell) = 4, 4,\ respectively,$$
$$\ell = 14 \quad and \quad O = O[0; 2, 7, 14], \quad with\ Epi_0(\pi_1(O), Z_\ell) = 6.$$

5. The numbers of rooted maps on cyclic orbifolds

Notation. Let M be a rooted map on an orbifold O such that $M = \tilde{M}/Z_\ell = (D; R, L)$ is a quotient of an ordinary finite map $\tilde{M}$ on a surface S. Thus $O = S/Z_\ell$. It follows that each branch index is a divisor of ℓ and we can write $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$, where $q_i \geqslant 0$ denotes the number of branch points of index i, for $i = 2, \dots, \ell$. In order to shorten the length of expressions, given an orbifold $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$ we denote the number of rooted maps with m darts embedded in O such that each semiedge is endowed with a branch point of index two by $\nu_O(m) = \nu_{[g;2^{q_2},\dots,\ell^{q_\ell}]}(m) = NRM_O(m)$. Also we use the convention $\nu_g(m) = \nu_{[g;\emptyset]}(m)$ denoting the number of rooted maps with m darts on a surface of genus g. We note that in this case m is necessarily even and $\nu_g(m) = \mathcal{N}_g(m/2)$, where $\mathcal{N}_g(m/2)$ denotes the number of rooted maps with $m/2$ edges on a surface of genus g.

We denote by v, f, m and s the number of vertices, faces, darts and semiedges of a map M on an orbifold O, respectively. Since we are primarily interested in enumeration of maps without semiedges we assume that a free-end of each semiedge is incident with a branch point of index two. Hence $0 \leqslant s \leqslant q_2$. Moreover, by the Euler–Poincaré formula $v - \frac{m-s}{2} + f = 2 - 2g$. By $Cor(M)$ we denote an ordinary rooted map on S_g which arises from M by using the following rules:

(1) delete all semiedges of M;
(2) if the root of M occupies a semiedge x in M, we choose a root of $Cor(M)$ to be the first dart following x in the local rotation of M sharing an edge of degree 2;
(3) if $Cor(M)$ is a map without darts we consider it as a unique rooted map.

Given integers $x_1, x_2, \dots, x_q$ and $y \geqslant x_1 + x_2 + \cdots + x_q$ we denote by

$$\binom{y}{x_1, x_2, \dots, x_q} = \frac{y!}{x_1! x_2! \cdots x_q! (y - \sum_{j=1}^{q} x_j)!},$$

the multinomial coefficient. Note that the meaning of the symbol consistently extends also to the case of non-negative y satisfying $y < x_1 + x_2 + \cdots + x_q$. In this case the multinomial coefficient takes value 0.

Reconstruction of M from $Cor(M)$. We start from the map $Cor(M)$ which is an ordinary rooted map with $\frac{m-s}{2}$ edges. How many different rooted maps M on the orbifold O come from a fixed ordinary rooted map $Cor(M)$? We split the discussion into three subcases.

Case 1. Number of distributions of branch points which are not attached to semiedges.

We have to find the number of divisions of the set $V(M) \cup F(M)$ of cardinality $v + f = e + 2 - 2g$ into disjoint subsets of cardinalities $q_1, q_2 - s, \dots, q_\ell$. This is just the number

$$\binom{\frac{m-s}{2} + 2 - 2g}{q_2 - s, q_3, \dots, q_\ell}$$

(see, for instance, [11, p. 62]).

Case 2. Number of distributions of semiedges if the root of M is not located at a semiedge.

The family of semiedges of $M = (D, R, L)$ splits into families S_i, $i = 1, 2, \dots$, defined by the following rule: A semiedge determined by a unique dart x belongs to S_i if and only if x belongs to a sequence of darts $x_0, x_1, \dots, x_i, x_{i+1}$ satisfying

(i) $x_j = R(x_{j-1})$ for $j = 1, \dots, i+1$;
(ii) for the initial and terminal darts we have $L(x_0) \neq x_0$ and $L(x_{i+1}) \neq x_{i+1}$, or $R = (x_1, x_2, \dots, x_i)$;
(iii) for the internal darts $x_j = L(x_j)$, $j = 1, 2, \dots, i$.

Set $c_i = \frac{|S_i|}{i}$. Clearly, c_i is the number of sequences of darts satisfying the above conditions (i)–(iii).

We have $s = \sum i c_i \leqslant m - s$, because a position of such a sequence in M is uniquely determined by choosing its initial dart x_0, which is a dart of $Cor(M)$ as well. Note that $c_j = 0$ if $j > s$. Given partition $s = c_1 + 2c_2 + \cdots + sc_s$ we have

$$\binom{m-s}{c_1, c_2, \dots, c_s}$$

choices to distribute the respective sequences of semiedges in $Cor(M)$. Denote by $Par(s)$ the set of partitions of s. In what follows we write a partition of s in the exponential form as $1^{c_1} 2^{c_2} \cdots s^{c_s}$.

It follows that the total number of distributions of semiedges is

$$\sum_{Par(s)} \binom{m-s}{c_1, c_2, \dots, c_s},$$

where the sum runs through all non-negative solutions $(c_1, c_2, \dots, c_s)$ of the equation $x_1 + 2x_2 + \cdots + sx_s = s$.

In fact it makes sense to consider only partitions satisfying $c_1 + c_2 + \cdots + c_s \leqslant m - s$ but in view of the remark after the definition of the multinomial coefficient the expression is correct even if we do not write this condition in the subscript of the sum.

Case 3. The root of M lies on a semiedge.

By the definition of $Cor(M)$ the position of the root of M is determined by the position of the root of $Cor(M)$ up to its position in the internal part of a sequence $x_0, x_1, \dots, x_i, x_{i+1}$ satisfying (i)–(iii). We use one semiedge z_0 for the root. The remaining $s-1$ semiedges have to be distributed in $m-(s-1) = m-s+1$ places which are given by darts of $Cor(M)$ and by z_0. Similar arguments as in Case 2 apply. We get

$$\sum_{Par(s-1)} \binom{m-s+1}{c_1, c_2, \dots, c_{s-1}}$$

distributions in this case.

The number s of semiedges takes only values of the same parity as m, since we are assuming that $\frac{m-s}{2}$ is the number of edges of $Cor(M)$ which is an integer. Denote by $p(m,s)$ the parity function taking the value 0 if the numbers have different parity and 1 otherwise.

Given the integers n and s let

$$\beta(n,s) = \sum_{Par(s)} \binom{n}{c_1, c_2, \dots, c_s}.$$

We set $\beta(n,-1) = 0$ and $\beta(n,0) = 1$ as well.

Summarizing all the above calculations we finally get:

$$\nu_O(m) = \sum_{s=0}^{q_2} p(m,s)\big(\beta(m-s,s) + \beta(m-s+1,s-1)\big) \\ \times \binom{\frac{m-s}{2}+2-2g}{q_2-s, q_3, \ldots, q_\ell} \mathcal{N}_g\left(\frac{m-s}{2}\right).$$

The following lemma significantly simplifies the computation of $\beta(n,s)$.

Lemma 5.1.

$$\beta(n,s) = \binom{n+s-1}{s}.$$

Proof. By the multinomial formula [11, p. 123] we have

$$\big(1+x+x^2+x^3+\cdots\big)^n = \sum_{s=0}^{\infty} \sum_{c_1+2c_2+\cdots+sc_s=s} \binom{n}{c_1,\ldots,c_s} x^{c_1}x^{2c_2}\cdots x^{sc_s} \\ = \sum_{s=0}^{\infty} \beta(n,s)x^s.$$

On the other hand,

$$\big(1+x+x^2+x^3+\cdots\big)^n = \frac{1}{(1-x)^n} = \binom{n-1}{0} + \binom{n}{1}x + \cdots + \binom{n+s-1}{s}x^s + \cdots.$$

Comparing the coefficients of x^s we get the result. □

Since $\beta(m-s,s)+\beta(m-s+1,s-1) = \binom{m-1}{s} + \binom{m-1}{s-1} = \binom{m}{s}$ we have proved the following statement.

Proposition 5.2. *Let $O = O[g; 2^{q_2}, \ldots, \ell^{q_\ell}]$ be an orbifold, $q_i \geqslant 0$ for $i = 2, \ldots, \ell$. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is*

$$\nu_O(m) = \sum_{s=0}^{q_2} \binom{m}{s} \binom{\frac{m-s}{2}+2-2g}{q_2-s, q_3, \ldots, q_\ell} \mathcal{N}_g\big((m-s)/2\big), \tag{5.1}$$

with the convention that $\mathcal{N}_g(n) = 0$ if n is not an integer.

6. Counting unrooted maps on the sphere

In this section we apply the above results to calculate the number of unrooted maps with given number of edges on the sphere. These numbers were derived by Liskovets in [23,24].

First we deal with the numbers $\nu_O(m)$ where O is one of the spherical orbifolds $O = O[0; \ell^2]$.

If $\ell > 2$ then the number s of semiedges of a rooted map M which are lifted to a spherical map with $m\ell$ darts is equal to 0. By Proposition 5.2 we have

$$\nu_{[0;\ell^2]}(m) = \binom{\frac{m}{2}+2}{2} \mathcal{N}_0(m/2), \quad \ell > 2 \text{ and } m \text{ even},$$

and

$$\nu_{[0;\ell^2]}(m) = 0, \quad \ell > 2 \text{ and } m \text{ odd.}$$

If $\ell = 2$, then $O = O[0; 2^2]$ and the number of semiedges is $s = 0$ or $s = 2$ for m even and it is $s = 1$ for m odd.

By Proposition 5.2

$$\nu_{[0;2^2]}(m) = \binom{\frac{m}{2}+2}{2}\mathcal{N}_0(m/2) + \binom{m}{2}\mathcal{N}_0(m/2-1), \quad \text{if } m \text{ is even,}$$

and

$$\nu_{[0;2^2]}(m) = m\left(\frac{m-1}{2}+2\right)\mathcal{N}_0\big((m-1)/2\big), \quad \text{if } m \text{ is odd.}$$

Now we are ready to apply our formula to express the number of ordinary unrooted maps on the sphere with e edges in terms of the Tutte numbers $\mathcal{N}_0(e)$ denoting the number of rooted ordinary maps with e edges on the sphere [37].

We distinguish two cases.

Case 1. The number of edges e is even. Note that $n = 2e$ and $n \equiv 0 \bmod 4$.

We have

$$\Theta_0(e) = NUM(n) = \frac{1}{n}\sum_{\substack{\ell|n \\ n=\ell m}}\sum_{O\in Orb(S_0/Z_\ell)} Epi_0\big(\pi_1(O), Z_\ell\big)\nu_O(m).$$

Writing the terms for $\ell = 1$ and $\ell = 2$ separately and using the fact that given $\ell > 1$ there is only one 0-admissible orbifold, namely $O[0; \ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ (see Theorem 4.4), we get

$$\Theta_0(e) = \frac{1}{n}\left(\nu_0(n) + \binom{\frac{n}{4}+2}{2}\nu_0(n/2) + \binom{\frac{n}{2}}{2}\nu_0(n/2-2) + \sum_{\substack{\ell|n,\ell>2 \\ n=\ell m, m \text{ even}}}\phi(\ell)\binom{\frac{m}{2}+2}{2}\nu_0(m)\right).$$

Using $e = 2n$, $\nu_0(n) = \mathcal{N}_0(\frac{n}{2}) = \mathcal{N}_0(e)$ and $\nu_0(m) = \mathcal{N}_0(\frac{m}{2})$ we rewrite it as follows:

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + \binom{e}{2}\mathcal{N}_0(e/2-1) + \sum_{\substack{\ell|e \\ \ell\geqslant 2}}\phi(\ell)\binom{\frac{e}{\ell}+2}{2}\mathcal{N}_0(e/\ell)\right).$$

Setting $d = \frac{e}{\ell}$ we have

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + \binom{e}{2}\mathcal{N}_0(e/2-1) + \sum_{\substack{d|e \\ d<e}}\phi(e/d)\binom{d+2}{2}\mathcal{N}_0(d)\right), \tag{6.1}$$

where e is an even number.

Assume now that $n \equiv 2 \bmod 4$. Then extracting the first two terms from the sum and inserting $n = 2e$ we get

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + e\left(\frac{e-1}{2} + 2\right)\mathcal{N}_0\big((e-1)/2\big) + \sum_{\substack{\ell|n=2e,\ell>2\\ n=\ell m, m \text{ even}}} \phi(\ell)\binom{\frac{m}{2}+2}{2}\nu_0(m)\right).$$

All the conditions in the sum are satisfied if and only if $m = 2d$ for some $d|e$. Hence we have

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + e\left(\frac{e-1}{2} + 2\right)\mathcal{N}_0\big((e-1)/2\big) + \sum_{\substack{d|e\\ d<e}} \phi(e/d)\binom{d+2}{2}\mathcal{N}_0(d)\right) \quad (6.2)$$

for e odd. Hence we have proved the following result of Liskovets [23]. Recall that $\mathcal{N}_0(e)$ denotes the number of rooted planar maps with e edges and is given by $\mathcal{N}_0(e) = \frac{2(2e)!3^e}{e!(e+2)!}$ (Tutte [37]).

Theorem 6.1. [23] *The number of oriented spherical unrooted maps with e edges is given by* (6.1) *if e is even, and* (6.2) *if e is odd.*

7. Counting unrooted maps on surfaces of genus 1, 2 and 3

The aim of this section is to derive a more explicit formula for counting unrooted maps on surfaces of genera 1, 2 and 3. The list of 1-admissible orbifolds and the respective numbers $Epi_o(\pi_1(O), Z_\ell)$ were derived in Theorem 4.5. Rooted toroidal maps were enumerated in [2]. It was proved that

$$\mathcal{N}_1(e) = \sum_{k=0}^{e-2} 2^{e-3-k}\big(3^{e-1} - 3^k\big)\binom{e+k}{k}.$$

Following Theorem 1.1 and taking into account Corollary 4.5 we have

$$\Theta_1(e) = NUM_1(n) = \frac{1}{n}\Bigg(\nu_{[0;2^4]}(n/2) + 2\nu_{[0;3^3]}(n/3) + 2\nu_{[0;2,4^2]}(n/4) + 2\nu_{[0;2,3,6]}(n/6)$$
$$+ \sum_{\substack{\ell|n\\ n=\ell m}} \sum_{k|\ell} \mu(\ell/k)k^2\nu_1(n/\ell)\Bigg). \quad (7.1)$$

Since $\nu_1(n/\ell) = \mathcal{N}_1(e/\ell)$ for $e = n/2$, it remains to calculate the numbers of rooted maps on orbifolds $O[0; 2^4]$, $O[0; 3^3]$, $0[2; 4^2]$ and $O[0; 2, 3, 6]$.

By Proposition 5.2 we have

$$\nu_{[0;3^3]}(m) = \binom{\frac{m}{2}+2}{3}\mathcal{N}_0(m/2), \quad (7.2)$$

for m even, and it is 0 for m odd.

For the orbifold $O = O[0; 2, 4^2]$ we have

$$\nu_{[0;2,4^2]}(m) = \binom{\frac{m}{2}+2}{1,2}\mathcal{N}_0(m/2), \quad m \text{ even}, \quad (7.3)$$

and

$$\nu_{[0;2,4^2]}(m) = m\binom{\frac{m-1}{2}+2}{2}\mathcal{N}_0\big((m-1)/2\big), \quad m \text{ odd.} \tag{7.4}$$

For the orbifold $O = O[0; 2, 3, 6]$ we get

$$\nu_{[0;2,3,6]}(m) = \binom{\frac{m}{2}+2}{1,1,1}\mathcal{N}_0(m/2), \quad m \text{ even,} \tag{7.5}$$

and

$$\nu_{[0;2,3,6]}(m) = m\binom{\frac{m-1}{2}+2}{1,1}\mathcal{N}_0\big((m-1)/2\big), \quad m \text{ odd.} \tag{7.6}$$

And finally, by Proposition 5.2 we get

$$\begin{aligned}\nu_{[0;2^4]}(m) = {} & \binom{\frac{m}{2}+2}{4}\mathcal{N}_0(m/2) + \binom{m}{2}\binom{\frac{m-2}{2}+2}{2}\mathcal{N}_0\big((m-2)/2\big) \\ & + \binom{m}{4}\mathcal{N}_0\big((m-4)/2\big)\end{aligned} \tag{7.7}$$

for m even.

Table 1
The numbers of rooted and oriented unrooted maps on the torus with at most 30 edges

No. edges	No. rooted maps	No. unrooted maps
02	1	1
03	20	6
04	307	46
05	4280	452
06	56914	4852
07	736568	52972
08	9370183	587047
09	117822512	6550808
10	1469283166	73483256
11	18210135416	827801468
12	224636864830	9360123740
13	2760899996816	106189359544
14	33833099832484	1208328304864
15	413610917006000	13787042250528
16	5046403030066927	157700137398689
17	61468359153954656	1807893066408464
18	747672504476150374	20768681225892328
19	9083423595292949240	239037464947999900
20	110239596847544663002	2755989928117365244
21	1336700736225591436496	31826208029615881656
22	16195256987701502444284	368074022535205870382
23	196082659434035163992720	4262666509741017440552
24	2372588693872584957422422	49428931123444048643388
25	28692390789135657427179680	573847815786545413529104
26	346814241363774726576771244	6669504641624799675973078
27	4190197092308320889669166128	77596242450201993985513136
28	50605520500653135912761192668	903670008940406050891508432
29	610946861846663952302648987552	10533566583563768540393559344
30	7373356726039234245335035186504	122889278767322703855171530872

For m odd Proposition 5.2 implies

$$\nu_{[0;2^4]}(m) = m\binom{\frac{m-1}{2}+2}{3}\mathcal{N}_0\big((m-1)/2\big) + \binom{m}{3}\left(\frac{m-3}{2}+2\right)\mathcal{N}_0\big((m-3)/2\big) \tag{7.8}$$

for odd m.

Now we are ready to formulate the statement establishing the number of unrooted toroidal maps with given number of edges.

Theorem 7.1. *The number of oriented unrooted toroidal maps with e edges is*

$$\frac{1}{2e}\left(\alpha(e) + \sum_{\ell|e}\phi_2(\ell)\mathcal{N}_1(e/\ell)\right),$$

where

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv 0 \bmod 12,$$
$$\alpha(e) = \nu_{[0;2^4]}(e), \quad \textit{if } e \equiv \pm1, \pm5 \bmod 12,$$
$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2), \quad \textit{if } e \equiv \pm2 \bmod 12,$$
$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv \pm3 \bmod 12,$$
$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2), \quad \textit{if } e \equiv \pm4 \bmod 12,$$
$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv 6 \bmod 12.$$

Let us remark that $\phi_2(\ell)$ denotes the Jordan function of order 2 *and the other functions used in the statement are defined by* (7.2)–(7.8).

Let us remark that the initial values confirm the available data for $e \leqslant 6$ obtained by Walsh [41] (the sequence M4253 in [32]).

The statements establishing $\Theta_\gamma(e)$ for genus two and genus three surfaces follow.

Theorem 7.2. *The number of oriented unrooted maps on genus two surface with e edges is given by the formula*

$$\frac{1}{2e}\big(\mathcal{N}_2(e) + 4\nu_{[1;2^2]}(e) + \nu_{[0;2^6]}(e) + 6\nu_{[0;3^4]}(2e/3) + 2\nu_{[0;2^2,4^2]}(e/2) + 12\nu_{[0;5^3]}(2e/5)$$
$$+ 2\nu_{[0;2^2,3^2]}(e/3) + 2\nu_{[0;3,6^2]}(e/3) + 4\nu_{[0;2,8^2]}(e/4) + 4\nu_{[0;2,5,10]}(e/5)\big),$$

where $\nu_O(m)$ is defined in (5.1) *and $\mathcal{N}_g(e)$ is the number of rooted maps of genus g.*

Theorem 7.3. *The number of oriented unrooted maps on genus three surface with e edges is given by the formula*

$$\frac{1}{2e}\big(\mathcal{N}_3(e) + 15\mathcal{N}_2(e/2) + 4\nu_{[1;2^4]}(e) + \nu_{[0;2^8]}(e) + 18\nu_{[1;3^2]}(2e/3) + 10\nu_{[0;3^5]}(2e/3)$$
$$+ 12\nu_{[1;2^2]}(e/2) + 2\nu_{[0;2^3,4^2]}(e/2) + 8\nu_{[0;4^4]}(e/2)$$
$$+ 2\nu_{[0;2,3^2,6]}(e/3) + 2\nu_{[0;2^2,6^2]}(e/3) + 30\nu_{[0;7^3]}(2e/7)$$
$$+ 8\nu_{[0;4,8^2]}(e/4) + 12\nu_{[0;3,9^2]}(2e/9) + 4\nu_{[0;2,12^2]}(e/6)$$
$$+ 4\nu_{[0;3,4,12]}(e/6) + 6\nu_{[0;2,7,14]}(e/7)\big),$$

where $\nu_O(m)$ is defined in (5.1) *and $\mathcal{N}_g(e)$ is the number of rooted maps of genus g.*

Table 2
The numbers of rooted and oriented unrooted maps of genus two with at most 30 edges

No. edges	No. rooted maps	No. unrooted maps
04	21	4
05	966	106
06	27954	2382
07	650076	46680
08	13271982	830848
09	248371380	13804864
10	4366441128	218353000
11	73231116024	3328822880
12	1183803697278	49325772812
13	18579191525700	714586880940
14	284601154513452	10164338225482
15	4272100949982600	142403410942816
16	63034617139799916	1969831979334086
17	916440476048146056	26954132420126920
18	13154166812674577412	365393525753591368
19	186700695099591735024	4913176199287631232
20	2623742783421329300190	65593569635906036912
21	36548087103760045010148	870192550284377429780
22	505099724454854883618924	11479539192932030062066
23	6931067091334952379275496	150675371553731499821264
24	94498867785495807431128548	1968726412209522334197356
25	1280884669005154962723094680	25617693380147483835449016
26	17269149245085316894987194432	332099023944121243161761560
27	231687461653506761485020818832	4290508549139665515691123744
28	3094389154894054750463387898444	55256949194539206365604601052
29	41156529959321075124439691833704	709595344126234852207569048760
30	545290525617230994007326084007416	9088175426953885980802745018758

Tables 1–3 were computed using MATHEMATICA, Ver. 4. The input numbers of rooted maps come from [6] for genus 1, and from [4] for genus 2 and 3.

Concluding remarks. There was some progress since the manuscript of this paper was submitted. Recently, we successfully employed the method presented in this paper to derive a formula for the number of unrooted hypermaps of a given genus with a given number of darts; equivalently, we can enumerate the numbers of conjugacy classes of subgroups of the free group of rank 2 of a given genus and index. Liskovets investigated properties of the function $E(m_1, m_2, \dots, m_r)$ defined by (4.3) and found its expression in a multiplicative form.

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Table 3
The numbers of rooted and oriented unrooted maps of genus three with at most 30 edges

No. edges	No. rooted maps	No. unrooted maps
06	1485	131
07	113256	8158
08	5008230	313611
09	167808024	9326858
10	4721384790	236095958
11	117593590752	5345316004
12	2675326679856	111472798586
13	56740864304592	2182345314816
14	1137757854901806	40634231364914
15	21789659909226960	726322104184848
16	401602392805341924	12550075287918360
17	7165100439281414160	210738250570954064
18	124314235272290304540	3453173212810875280
19	2105172926498512761984	55399287587418128520
20	34899691847703927826500	872492296405529104608
21	567797719808735191344672	13518993329700676078500
22	9084445205688065541367710	206464663769623968602698
23	143182713522809088357084720	3112667685295345475820652
24	2226449757923955373340520612	46384369956820665320587902
25	34199303698053326789771187600	683986073961364663577206704
26	519494783678325912052481379156	9990284301507510446092217236
27	7811251314435936176791882965696	144652802119189104865404688680
28	11635901795255222876280159315184	2077839606295596379211506191640
29	1718465311469518829323877355423840	29628712266715926913818949155968
30	25178356967150456246664822271180140	419639282785841282782195528667536

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