ENUMERATION UNDER TWO REPRESENTATIONS OF THE WREATH PRODUCT [1]

BY

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1. Introduction

Enumeration problems which can be solved by applying Pólya's Theorem [9] or Burnside's Lemma [1] always require a formula for $N(A)$, the number of orbits of group A, or a formula for its cycle index $Z(A)$. For example, Pólya [9] expressed the cycle index of the wreath product $A[B]$ of A around B in terms of the cycle indices $Z(A)$ and $Z(B)$. This result played a key role in the enumeration of k-colored graphs [13] and nonseparable graphs [14].

The exponentiation group $[B]^A$ of two permutation groups A and B was defined by Harary in [3]. It is abstractly isomorphic to the wreath product of A around B. But while $A[B]$ has as its object set the cartesian product $X \times Y$ of the object sets of A and B, $[B]^A$ acts on Y^X, the functions from X into Y. Formulas for $Z([S_n]^{S_2})$ and $Z([S_2]^{S_n})$ were found by Harary [2] and Slepian [16] respectively. Harrison and High [6] have constructed an algorithm for finding $Z([B]^{S_n})$ and have used their results to enumerate Post functions. In this paper we verify an explicit general formula for $Z([B]^A)$ in terms of $Z(A)$ and $Z(B)$ for any A and B. The result is easily obtained by substituting certain operators for the variables of $Z(A)$ and then letting them act on $Z(B)$. Several applications will then be sketched, including the enumeration of boolean functions, bicolored graphs, and Post functions.

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The matrix group $[A; B]$ introduced in [8] is another useful representation of the wreath product. It can be viewed as acting on classes of matrices with A permuting the rows among themselves while the row entires are permuted independently by elements of B. Our formula for the number $N[A; B]$ of orbits of this group generalizes Redfield's Enumeration Theorem [12] and enables us to enumerate a variety of interesting combinatorial structures. These include multigraphs or multidigraphs with a specified number of points and lines, and superpositions of interchangeable copies of a given graph or digraph.

For definitions and results not given here we refer to the books [4, 5].

2. Permutation groups

Let A be a permutation group with object set $X=\{1, 2, ..., m\}$. The order of A is denoted by $|A|$ and the *degree* of A is m. For any permutation α in A, we denote by $j_k(\alpha)$ the number of cycles of length k in the disjoint cycle decomposition of α. The *cycle type* $Z(\alpha)$ is the monomial in the variables $a_1, a_2, ..., a_m$ defined by $Z(\alpha)=\prod_{k=1}^{m} a_k^{j_k(\alpha)}$. The *cycle index* $Z(A)$ is

$$Z(A)=\frac{1}{|A|}\sum_{\alpha\in A} Z(\alpha).$$

It is often convenient to use the expression

$$Z(A)=Z(A; a_1, a_2, ..., a_m)$$

to display the variables used.

Let B be another permutation group of order $|B|$ and degree n with object set $Y=\{1, 2, ..., n\}$. The *wreath product* of A around B, denoted $A[B]$, is a permutation group with object set $X\times Y$. For each permutation α in A and each function τ from X into B there is a permutation in $A[B]$ denoted (α, τ) such that for every element (x, y) of $X\times Y$

$$(\alpha, \tau)(x, y) = (\alpha x, \tau(x)y).$$

It is easily checked that this is a collection of permutations closed under composition and hence forms a group.

For each integer $k\geqslant 1$, let

$$Z_k(B) = Z(B; b_k, b_{2k}, ..., b_{nk}).$$

Thus $Z_k(B)$ is the polynomial obtained from $Z(B)$ by multiplying each subscript by k. Pólya [9, p. 180] used his enumeration theorem to establish the following formula for $Z(A[B])$.

THEOREM 1 (Pólya). *The cycle index $Z(A[B])$ is obtained by replacing each variable a_k of $Z(A)$ by the polynomial $Z_k(B)$; symbolically*

$$Z(A[B]) = Z(A; Z_1(B), Z_2(B), \dots, Z_m(B)).$$

Our formulas for the cycle index of the exponentiation group and the number of orbits of the matrix group are considerably more complicated than that of Theorem 1 but are similar in that they involve the replacement of each variable a_k in $Z(A)$ by a suitable transformation of $Z(B)$ which depends on k.

A generalization of the wreath product is possible when A is intransitive. Suppose $X = \bigcup_{i=1}^t X_i$ and each X_i is a union of transitivity sets of A. Let $B_1, \dots, B_t$ be permutation groups with disjoint object sets $Y_1, \dots, Y_t$ respectively. The generalized wreath product, denoted $A[B_1, \dots, B_t]$, acts on $\bigcup_{i=1}^t X_i \times Y_i$. For each α in A and each sequence $\tau_1, \dots, \tau_t$ with each τ_i in $B_i^{X_i}$ there is an element denoted $(\alpha; \tau_1, \tau_2, \dots, \tau_t)$ in $A[B_1, \dots, B_t]$ defined as follows. For any (x, y) in $X_i \times Y_i$

$$(\alpha; \tau_1, \dots, \tau_t)(x, y) = (\alpha x, \tau_i(x) y).$$

To express the cycle index of this group we require the cycle index of A in the generalized form introduced by Pólya [9, p. 174]. For each α in A let

$$Z_{X_1, \dots, X_t}(\alpha) = \prod_{i=1}^{t} \prod_{s} a_{i,s}^{j(i,s)}$$

where $j(i, s)$ is the number of cycles of length s induced by α in X_i. Then let

$$Z_{X_1, \dots, X_t}(A) = \frac{1}{|A|} \sum_{\alpha \in A} Z_{X_1, \dots, X_t}(\alpha).$$

As asserted in [14, p. 336]

$$Z(A[B_1, \dots, B_t]) = Z_{X_1, \dots, X_t}(A)[a_{i,s} \to Z_s(B_i)]$$

where the arrow indicates substitution.

When $t = 1$, $X = X_1$ and $B = B_1$, this formula gives the same result as Theorem 1.

3. The exponentiation group

The permutation groups A and B have object sets $X = \{1, 2, \dots, m\}$ and $Y = \{1, 2, \dots, n\}$ respectively. Since the wreath product acts on $X \times Y$, it can be viewed as permuting the subsets of $X \times Y$ which correspond to functions from X into Y. This representation of the wreath product is called the *exponentiation* of A and B and is denoted by $[B]^A$. Thus each element (α, τ) of the wreath product $A[B]$ permutes the functions f in Y^X according to the rule

$$((\alpha, \tau)f)x = \tau(x)(f(\alpha^{-1}x))$$

for each x in X.

To state the theorem which expresses $Z([B]^A]$ in terms of $Z(A)$ and $Z(B)$ we require the next few definitions. Let $\mathbf{R} = \mathbf{Q}[b_1, b_2, \ldots]$ be the ring of polynomials in the commuting variables $b_1, b_2, \ldots$ over the ring $\mathbf{Q}$ of rational numbers.

Now we recall the cartesian product operation $\times$ on $\mathbf{R}$ introduced by Harary [2]. For two monomials in $\mathbf{R}$ we define

$$b_1^{j_1} b_2^{j_2} \ldots b_m^{j_m} \times b_1^{i_1} b_2^{i_2} \ldots b_n^{i_n} = \prod_{s=1}^{m} \prod_{t=1}^{n} b_{[s,t]}^{(s,t) j_s i_t} \tag{1}$$

where $[s, t]$ and (s, t) denote the l.c.m. and g.c.d. respectively. It is clear that this operation is associative for monomials. Then $\times$ is the unique $\mathbf{Q}$-bilinear operation on $\mathbf{R}$ which satisfies (1). We leave it in to the reader to check that $\times$ is associative.

Given any set S, we define scalar multiplication over $\mathbf{Q}$, addition and multiplication for the elements of R^S as follows. For every f and g in $\mathbf{R}^S$, λ in $\mathbf{Q}$ and P in S:

$$(\lambda f)P = \lambda(fP) \tag{2}$$

$$(f+g)P = fP + gP \tag{3}$$

$$(fg)P = fP \times gP. \tag{4}$$

With these operations $\mathbf{R}^S$ becomes a commutative ring over $\mathbf{Q}$, to be denoted by $S(+, \times)$.

For each positive integer r let I_r be the unique $\mathbf{Q}$-linear element of $\mathbf{R}(+, \times)$ which satisfies

$$I_r\left(\prod_{k=1}^{n} b_k^{j_k}\right) = \prod_{v=1}^{nr} b_v^{i_v} \tag{5}$$

where

$$i_v = \frac{1}{v} \sum_{w|v} \mu\left(\frac{v}{w}\right) \left(\sum k j_k\right)^{(r,w)} \tag{6}$$

the inside sum to be taken over all divisors k of $w/(r, w)$. From the $\mathbf{Q}$-linearity of I_r we have

$$I_r(Z(B)) = \frac{1}{|B|} \sum_{\beta \in B} I_r(Z(\beta)).$$

THEOREM 2. *The cycle index* $Z([B]^A)$ *is the image of* $Z(B)$ *under the function obtained by substituting the operator* I_r *for the variables* a_r *in* $Z(A)$; *symbolically*

$$Z([B]^A) = Z(A; I_1, \ldots, I_m) Z(B).$$

Before launching the proof of Theorem 2 we illustrate its use by finding the cycle index of a well known exponentiation group. Let $A = S_3$ and $B = S_2$, the symmetric groups of degree three and two respectively. We seek the cycle index of $[S_2]^{S_3}$, which is the group of the cube. First we substitute the operator I_r for each variable a_r in $Z(S_3)$:

$$Z(S_3, I_1, I_2, I_3) = \frac{1}{3!}(I_1^3 + 3I_1I_2 + 2I_3). \tag{7}$$

The terms of (7) act on $Z(S_2)$ as follows:

$$\begin{aligned} I_1^3(Z(S_2)) &= I_1(Z(S_2)) \times I_1(Z(S_2)) \times I_1(Z(S_2)) \\ I_1I_2(Z(S_2)) &= I_1(Z(S_2)) \times I_2(Z(S_2)). \end{aligned} \tag{8}$$

It follows from the definitions (5) and (6) that

$$I_1(Z(S_2)) = Z(S_2) = \tfrac{1}{2}(b_1^2 + b_2)$$

and

$$\begin{aligned} I_2(Z(S_2)) &= \tfrac{1}{2}(I_2(b_1^2) + I_2(b_2)) \\ &= \tfrac{1}{2}(b_1^2 b_2 + b_4) \end{aligned}$$

and

$$\begin{aligned} I_3(Z(S_2)) &= \tfrac{1}{2}(I_3(b_1^2) + I_3(b_2)) \\ &= \tfrac{1}{2}(b_1^2 b_3^2 + b_2 b_6). \end{aligned}$$

From (8) and the definition of the cartesian product $\times$ for polynomials, we find

$$I_1^3(Z(Z(S_2)) = \frac{1}{2^3}(b_1^8 + 7b_2^4)$$

$$I_1I_2(Z(S_2)) = \frac{1}{2^2}(b_1^4 b_2^2 + b_2^4 + 2b_4^2).$$

Having determined the images of $Z(S_2)$ under I_1^3, $I_1 I_2$ and I_3 we have by linearity its image under $Z(S_3; I_1, I_2, I_3)$:

$$Z([S_2]^{S_3}) = \frac{1}{3!\,2^3}(b_1^8 + 6b_1^4 b_2^2 + 8b_1^2 b_3^2 + 13b_2^4 + 8b_2 b_6 + 12b_4^2). \tag{9}$$

This result agrees pleasantly with the formula for the cycle index of the group of the cube worked out by Pólya [10].

The hardest part of these calculations occurs in the evaluation of $I_r(\prod_{k=1}^n b_k^{j_k})$ by formulas (5) and (6). But it is helpful to note that if $(r, v) = 1$, then $i_v = j_v$ and if p is prime, then

$$i_p = \begin{cases} j_p & \text{if } p \nmid r \\ (j_1^p - j_1)/p & \text{if } p \mid r. \end{cases}$$

Furthermore, with the aid of these observations, it can be seen that

$$I_2\left(\prod_{k=1}^{n} b_k^{j_k}\right) = \left(\prod_{2 \nmid k} b_k^{j_k}\right)\left(b_2^{(j_1^2-j_1)/2}\, b_4^{j_2(j_1+j_2)}\, b_6^{j_3(2j_1+3j_3-1)/2} \ldots\right)$$

and

$$I_3\left(\prod_{k=1}^{n} b_k^{j_k}\right) = \left(\prod_{3 \nmid k} b_k^{j_k}\right)\left(b_3^{(j_1^3-j_1)/3}\, b_6^{((j_1+2j_2)^3-2j_2-j_1^3)/6}\, b_9^{((j_1+3j_3)^3-j_1^3)/9} \ldots\right).$$

4. Proof of Theorem 2

Let A and B be permutation groups with object sets $X = \{1, \ldots, m\}$ and $Y = \{1, \ldots, n\}$ respectively.

For the first part of the proof assume α in A is the cycle $(1\ 2 \ldots m)$, fix β in B and consider any τ in B^X such that

$$\tau(m)\tau(m-1) \ldots \tau(2)\tau(1) = \beta. \tag{10}$$

We wish to determine the number of functions in Y^X left fixed by $(\alpha, \tau)^v$, where (α, τ) is viewed as a number of $[B]^A$. Equivalently, we want the number of functional subsets of $X \times Y$ left fixed by $(\alpha, \tau)^v$ where (α, τ) is considered as a member of $A[B]$. The latter viewpoint is the one taken in the sequel. For any y in Y we have

$$\begin{aligned}
(\alpha, \tau)(1, y) &= (\alpha 1, \tau(1)y) = (2, \tau(1)y)\\
(\alpha, \tau)^2(1, y) &= (3, \tau(2)\tau(1)y)\\
&\vdots\\
(\alpha, \tau)^m(1, y) &= (1, \tau(m) \ldots \tau(1)y) = (1, \beta y)\\
&\vdots\\
(\alpha, \tau)^{mq}(1, y) &= (1, \beta^q y)\\
&\vdots\\
(\alpha, \tau)^{mk}(1, y) &= (1, y)
\end{aligned}$$

where k is the least number such that $\beta^k y = y$. That is, k is the length of the cycle to which y belongs in the disjoint cycle decomposition of β.

Thus $(1, y)$ falls in a cycle of length mk in the cycle decomposition of (α, τ). Call this cycle C. The cycle into which $(1, y)$ falls in the cycle decomposition of $(\alpha, \tau)^v$ is found by taking every v'th member of C, starting with $(1, y)$. Call this cycle C_v. The situation is illustrated in Figure 1 for the case $m = 10$, $k = 3$, and $v = 12$.

Let s be the length of C_v. The v'th power of any cycle of length mk consists of (v, mk) cycles of length $mk/(v, mk)$. Hence $s = mk/(v, mk)$. Now a necessary condition for a function f containing $(1, y)$ to be fixed by $(\alpha, \tau)^v$ is that f also contain all the other pairs in C_v. In

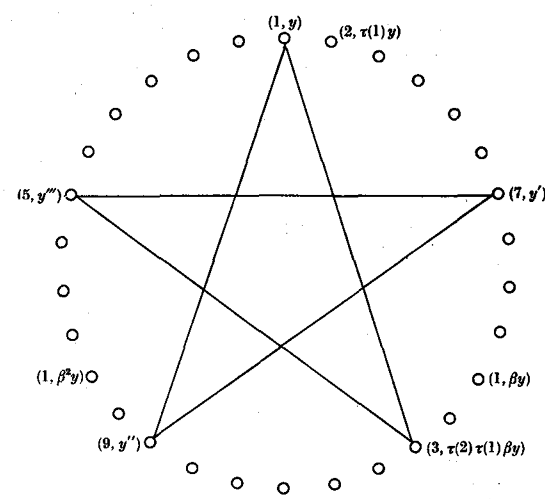

Figure 1. Diagram of C and C_v with $k=3$, $m=10$, $v=12$.

particular, C_v must not contain any pair of the form $(1, y')$ with $y' \neq y$. This means that for $1 \leqslant p < s$, vp must *not* be a multiple of m. Therefore $s \leqslant m/(m, v)$. But sv is a multiple of m and hence we also have $s = m/(m, v)$. But $m/(m, v) = mk(mk, v)$ just if $k \mid (v/(v, m))$.

Conversely, it is easily seen that if y is in a cycle of length k in the cycle decomposition of β and $k \mid (v/(v, m))$, then $(1, y)$ is in a cycle C_v of length $m/(m, v)$ induced by $(\alpha, \tau)^v$. Moreover C_v is functional when viewed as a set of pairs, since there is nothing special about 1 in the preceding analysis. The domain of C_v contains j for $1 \leqslant j \leqslant m$ just if

$$j + mq = 1 + rv$$

for some integers q, r. This implies that $j \equiv 1$ modulo (m, v), a condition satisfied by exactly $m/(m, v)$ integers between 1 and m. Since $m/(m, v)$ is the length of C_v and C_v is functional, the domain of C_v must contain all of these numbers. That is, the domain of C_v is exactly

$$\{i \mid 1 \leqslant i \leqslant m \quad \text{and} \quad i \equiv 1 \text{ modulo } (m, v)\}.$$

The pairs in C_v are determined by $(1, y)$ and $(\alpha, \tau)^v$, and if $f(1) = y$ and $(\alpha, \tau)^v f = f$ they must all appear in f. This determines f on the domain of C_v. All that is needed to determine any f left fixed by $(\alpha, \tau)^v$, then, are the values $f(1), f(2), \ldots, f((m, v))$ since there is nothing special about 1 in the above analysis.

Recall that the cycle type $Z(\beta)$ of β is $b_1^{j_1} b_2^{j_2} \ldots b_n^{j_n}$. For any integer i between 1 and (m, v) the number of choices available for $f(i)$ where $(\alpha, \tau)^v f = f$ is $\sum_{k=1}^{*n} k j_k$; the asterisk represents the restriction of the summation index k to divisors of $v/(m, v)$. Since the (m, v) choices for $f(1), \ldots, f((m, v))$ are independent, there are a total of

$$\left(\sum_{k=1}^{n}{}^{*} kj_k\right)^{(m,v)}$$

functions left fixed by $(\alpha, \tau)^v$.

Now let i_w be the number of cycles of length w in the cycle decomposition of (α, τ) viewed now as acting on Y^X. Then

$$\sum_{w|v} wi_w = (\sum{}^{*} kj_k)^{(v,m)}.$$

An explicit formula for i_v is obtained by an application of möbius inversion, giving the formula (6) for the definition of I_m. Consequently the cycle type $Z(\alpha, \tau)$ of (α, τ) acting on Y^X is just $I_m(Z(\beta))$. There are $|B|^{m-1}$ functions τ in B^X which satisfy (10) since $\tau(m), \ldots, \tau(2)$ may be chosen from B arbitrarily, and then $\tau(1)$ is uniquely determined. Summing over all τ satisfying (10) we have

$$\frac{1}{|B|^m}\sum_{\tau} Z(\alpha, \tau) = \frac{1}{|B|^m}|B|^{m-1} I_m(Z(\beta)) = \frac{1}{|B|} I^m(Z(\beta)).$$

Summing over all β in B, which allows τ to run through all of B^X, and applying the linearity of I_m, we find

$$\frac{1}{|B|^m}\sum_{\tau\in B^X} Z(\alpha, \tau) = I_m(Z(B)). \tag{11}$$

Now consider the case when α is a product of disjoint cycles α_1 and α_2 of lengths m_1 and m_2 respectively. We can view (α, τ) for τ in B^X as the product of (α_1, τ_1) and (α_2, τ_2) where τ_1 and τ_2 are the restrictions of τ to the elements permuted by α_1 and α_2. If f_1 and f_2 are the restrictions to α_1 and α_2 of a function f in Y^X, then we have $f = f_1 \cup f_2$ and $(\alpha, \tau)f = (\alpha_1, \tau_1)f_1 \cup (\alpha_1, \tau_2)f_2$, the unions being disjoint. Thus if f_1 is in a cycle C_1 of length p induced by (α_1, τ_2) and f_2 is in a cycle C_2 of length q induced by (α_2, τ_2), then f is in a cycle of length $[p, q]$ induced by (α, τ). The total pq of functions obtained by pairing one from C_1 with one from C_2 must be divided into (p, q) cycles of length $[p, q]$. This corresponds to taking a factor b_p from $Z(\alpha_1, \tau_1)$ and b_q from $Z(\alpha_2, \tau_2)$ and finding $b_p \times b_q = b_{[p,q]}^{(p,q)}$ in $Z(\alpha, \tau)$. These factors may be chosen independently, and so using the associativity of the cartesian product operation $\times$ we find that

$$Z(\alpha, \tau) = Z(\alpha_1, \tau_1) \times Z(\alpha_2, \tau_2).$$

Applying (11) to the cycles α_1 and α_2 we have for $i = 1, 2$

$$\frac{1}{|B|^{m_i}} \sum_{\tau_i} Z(\alpha_{i}, \tau_i) = I_{m_i}(Z(B))$$

where the sum is over all τ_i from the set of elements permuted by α_i into B. Consequently

$$\frac{1}{|B|^m} \sum_{\tau \in B^X} Z(\alpha, \tau) = I_{m_1}(Z(B)) \times I_{m_2}(Z(B)) = I_{m_1} I_{m_2}(Z(B)),$$

the second step in view of the fact that I_{m_1} and I_{m_2} belong to the ring $\mathbf{R}(+, \times)$ for all algebraic purposes.

This line of reasoning works as well when α is any product of disjoint cycles and so in general

$$\frac{1}{|B|^m} \sum_{\tau \in B^X} Z(\alpha, \tau) = I_1^{u_1} I_2^{u_2} \dots I_m^{u_m}(Z(B)) \tag{12}$$

where $Z(\alpha) = \prod_{k=1}^m a_k^{u_k}$. The proof is concluded by summing (12) over all α in A, and dividing by $|A|$.

The generalized wreath product $A[B_1, \dots, B_t]$ acting on $\bigcup_{i=1}^t X_i \times Y_i$ induces a group $[B_1, \dots, B_t]^A$ which acts on $Y_1^{X_1} \times \dots \times Y_t^{X_t}$. This induced group is a generalized exponentiation group whose cycle index we shall now express.

For any t-tuple $(P_1, \dots, P_t)$ in $\mathbf{R}^t$, any $i = 1$ to t and any positive integer s, let

$$I_{i,s}(P_1, \dots, P_t) = I_s(P_i)$$

On viewing the operators $I_{i,s}$ as belonging to the ring $\mathbf{R}^t(+, \times)$, the cycle index formula is given by

$$Z([B_1, \dots, B_t]^A) = Z_{X_1, \dots, X_t}(A)\,[a_{i,s} \to I_{i,s}]\,(Z(B_1), \dots, Z(B_t)). \tag{13}$$

The proof of (13) requires only straightforward modification of the proof of Theorem 2.

5. Applications of Theorem 2

We shall now outline a few of the results which require the cycle index of an exponentiation group.

A boolean function of n variables can be regarded as a mapping from the set of all n-sequences of zeros and ones into $\{0, 1\}$. Hence it corresponds to a subset of the points of the n-cube Q_n. Pólya [10] regarded two such subsets as equivalent if an automorphism of Q_n takes one to the other. Denoting the group of the n-cube by $\Gamma(Q_n)$, he used his enumeration theorem to obtain the following result: the number $N(n, r)$ of boolean functions of n variables which have exactly r nonzero values is the coefficient of x^r in $Z(\Gamma(Q_n), 1+x)$.

As observed in [2], $\Gamma(Q_n)$ and $[S_2]^{S_n}$ are identical and hence Theorem 2 can be used to complete this enumeration problem.

On substituting $1+x$ in $Z([S_2]^{S_3})$, given by formula (9), we have

$$1+x+3x^2+3x^3+6x^4+3x^5+3x^6+x^7+x^8.$$

Then, for example, there are 6 boolean functions with 4 nonzero values. The 6 cubes which correspond to these functions are shown in Figure 2 where dark points represent the nonzero values.

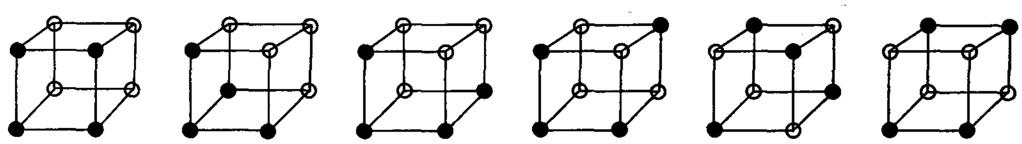

Figure 2. The 6 cubes with 4 points of each value.

Pólya calculated $Z(\Gamma(Q_n))$ for $n \leqslant 4$ and Slepian [16] found a general method for calculating this cycle index and applied it for $n=5$ and 6.

A *Post function* of n variables can be defined as a mapping from the set of all n-sequences of the numbers $0, 1, 2, \ldots, m-1$ into the set $\{0, 1, \ldots, m-1\}$. When $m=2$, these are just boolean functions and their total number, when equivalence is determined by the group $[S_2]^{S_n}$ of the n-cube, is $Z([S_2]^{S_n}, 2)$. When m variables are present, the number of Post functions is $Z([S_m]^{S_n}, m)$ as mentioned in [6]. Harrison and High used their method for deriving the cycle index of the exponentiation group to calculate some of the values of $Z([S_m]^{S_n}, m)$. They also found the number of Post functions under different equivalences determined when S_m is replaced by the cyclic or dihedral groups of degree m.

The exponentiation group was also used by Harary [2] to count bicolored graphs: the number of bicolored graphs with r lines and n points of each color is the coefficient of x^r in $Z([S_n]^{S_2}, 1+x)$.

An explicit formula for $Z([S_n]^{S_2})$ was found in [2] but our general formula also applies. For example, Theorem 2 can be used to find that

$$Z([S_3]^{S_2}) = \frac{1}{72}(b_1^9 + 12\, b_1^3 b_2^3 + 8\, b_3^3 + 9\, b_1 b_2^4 + 18\, b_1 b_4^2 + 24\, b_3 b_6).$$

Then the polynomial which counts bicolored graphs with 3 points of each color is

$$1+x+2x^2+4x^3+5x^4+5x^5+4x^6+2x^7+x^8+x^9.$$

The coefficient of x^3 is illustrated in Figure 3.

We conclude by mentioning some results from [7] concerned with determining the cycle index of the group of a graph.

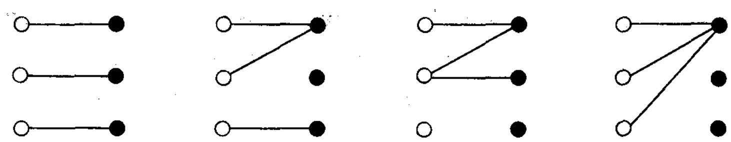

Figure 3. The 4 bicolored graphs with 3 lines and 3 points of each color.

Sabidussi [15] introduced a binary operation $\times$ on graphs and showed that with respect to $\times$ every nontrivial connected graph has a unique factorization into prime graphs. From his results it also follows that if G is a connected prime graph then the group of the cartesian product of n copies of G is precisely the exponentiation group $[\Gamma(G)]^{S_n}$ where $\Gamma(G)$ is the group of G. Thus Theorem 2 can be used to calculate $Z(\Gamma(G \times ... \times G))$ when $Z(\Gamma(G))$ is known. This in turn provides a basis for applying Polya's counting theorem to problems involving $G \times ... \times G$, for instance to find the number of ways to color the points of this graph with a given number of colors.

6. The matrix group

As before the permutation groups A and B have object sets $X = \{1, ..., m\}$ and $Y = \{1, ..., n\}$ respectively, so that the wreath product $A[B]$ acts on $X \times Y$. A partition of $X \times Y$ is called *functional* if each subset of $X \times Y$ in the partition is a function from X to Y. We have viewed the wreath product as acting on functions from X to Y and next shall regard it as permuting the $(n!)^{m-1}$ functional partitions of $X \times Y$. Thus any element (α, τ) of $A[B]$ sends the functional partition $F = \{f_1, f_2, ..., f_n\}$ to the set of functions which are the images of the f_i under (α, τ) viewed as a member of $[B]^A$. It is obvious that this new set of functions is again a functional partition of $X \times Y$, and we denote this new representation of the wreath product by $[A; B]$.

This representation was called the *matrix group* in [8] because each functional partition F corresponds in a natural fashion to an equivalence class of $m \times n$ matrices. For this purpose two $m \times n$ matrices are *equivalent* if they have the same set of columns. Then if $F = \{f_1, ..., f_n\}$, a correspondent to F is the matrix M for which the i, j entry is $f_j(i)$. Thus the images of the jth function determine the entries in the jth column of M.

The action of $[A; B]$ on the $(n!)^{m-1}$ functional partitions is equivalent to its action on these $(n!)^{m-1}$ classes of matrices. Specifically, (α, τ) can be regarded as sending the class of matrices to which M belongs to the class to which M' belongs, where M' has as its i, j entry $\tau(\alpha^{-1}i)f_j(\alpha^{-1}i)$. Thus $\tau(k)$ permutes each entry in the kth row of M and then the rows are permuted by α to get M'. This interpretation of the object set of $[A; B]$ will be useful to us later.

Each functional partition $F = \{f_1, ..., f_n\}$ has associated with it a permutation group whose object set is F. Suppose (α, τ) in the exponentiation group $[B]^A$ fixes F setwise. Then the restriction of (α, τ) to F is regarded as an automorphism of F and the totality of different restrictions make up *the group of* F. We denote the cycle index of this group by $Z(F)$.

We now illustrate some of these concepts with $A = S_2$ and $B = \{(1)(2)(3)(4), (13)(24)\}$. We shall soon see that the matrix group $[S_2; B]$ has 7 orbits. Each of the seven 2×4 matrices in Table 1 corresponds to a functional partition, one from each of these orbits. Next to each matrix is the cycle index of the corresponding functional partition.

Table 1. *Cycle indices of 7 functional partitions*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \tfrac{1}{2}(b_1^4 + b_2^2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \tfrac{1}{4}(b_1^4 + 3\,b_2^2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \tfrac{1}{4}(b_1^4 + b_2^2 + 2\,b_4)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad \tfrac{1}{2}(b_1^4 + b_1^2 b_2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \tfrac{1}{2}(b_1^4 + b_1^2 b_2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \tfrac{1}{4}(b_1^4 + 2\,b_1^2 b_2 + b_2^2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \quad b_1^4.$$

The next theorem provides a formula for the sum of the cycle indices of the groups of any set of distinct representatives of the orbits of $[A; B]$. This formula depends only on $Z(A)$ and $Z(B)$. To state the result we require a few preliminary definitions.

The operation ☋ introduced by Redfield [12] is defined for monomials in $\mathbf{R}$ as follows:

$$(b_1^{i_1} b_2^{i_2} \dots b_n^{i_n}) ☋ (b_1^{j_1} b_2^{j_2} \dots b_n^{j_n}) = \prod_k (kb_k)^{j_k} j_k! \tag{14}$$

if $i_k = j_k$ for all k and is zero otherwise.(¹) Then ☋ is the unique $\mathbf{Q}$-bilinear operation on $\mathbf{R}$ which satisfies (14). Clearly ☋ is associative.

(¹) The figure ☋ used by Redfield is the astronomical symbol for the "descending node of the moon or a planet" (cf. Webster's unabridged dictionary).

For any set S let $S(+, \mho)$ be the ring with elements from $\mathbf{R}^S$, and operations defined as for $S(+, \times)$ except to replace $\times$ by $\mho$ in equation (4).

For each positive integer r, let J_r be the unique $\mathbf{Q}$-linear operation in $\mathbf{R}(+, \mho)$ which satisfies the two following equations.

$$J_r(b_k^j) = j!\, k^j Z(S_j; d_1, d_2, \dots, d_j) \tag{15}$$

$$J_r\left(\prod_{k=1}^{n} b_k^{j_k}\right) = \prod_{k=1}^{n} J_r(b_k^{j_k}). \tag{16}$$

Here for each i between 1 and j we let

$$d_i = \begin{cases} b_{ki}/k & \text{if } \; i|r \quad \text{and} \quad (r/i, k) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since J_r is linear we have

$$J_r(Z(B)) = \frac{1}{|B|} \sum_{\beta \in B} J_r(Z(\beta)).$$

THEOREM 3. *Let F_k be a functional partition in the k'th orbit of the matrix group $[A; B]$ for $k = 1, 2, \dots, N[A; B]$. The sum of the cycle indices of the F_k is the image of $Z(B)$ under the function obtained by substituting the operators J_r for the variables a_r in $Z(A)$; symbolically*

$$\sum_k Z(F_k) = Z(A : J_1, \dots, J_m)\, Z(B).$$

To illustrate the theorem we again take $A = S_2$ and $B = \{(1)(2)(3)(4), (13)(24)\}$ so that

$$Z(A; J_1, J_2) = \tfrac{1}{2}(J_1^2 + J_2),$$

and

$$Z(B) = \tfrac{1}{2}(b_1^4 + b_2^2).$$

We seek

$$\tfrac{1}{2}(J_1^2 + J_2)\,(Z(B)) = \tfrac{1}{2}\{J_1^2(Z(B)) + J_2(Z(B))\}. \tag{17}$$

Since J_1 is by definition the identity operator

$$J_1^2(Z(B)) = J_1(Z(B)) \mho J_1(Z(B)) = Z(B) \mho Z(B).$$

By the definition of $\mho$.

$$Z(B) \mho Z(B) = \tfrac{1}{4}(b_1^4 \mho b_1^4 + b_2^2 \mho b_2^2) = \tfrac{1}{4}(4!\, b_1^4 + 2^2\, 2 b_2^2) = 6 b_1^4 + 2 b_2^2. \tag{18}$$

At this point it is helpful to observe that for any prime p, formula (15) for $J_p(b_k^j)$ can be written:

$$J_p(b_k^j) = \begin{cases} 0, & \text{if } p|k \text{ but } p\nmid j \\ (j!\,k^{j(p-1)/p} b_{pk}^{j/p})/((j/p)!\,p^{j/p}), & \text{if } p|k \text{ and } p|j \\ \sum_{s=0}^{[j/p]} (j!\,k^{(p-1)s} b_{kp}^{s} b_k^{j-sp})/((j-sp)!\,s!\,p^s) & \text{if } p\nmid k. \end{cases}$$

The linearity of J_2 and the previous formula imply

$$J_2(Z(B)) = \tfrac{1}{2}(J_2(b_1^4) + J_2(b_2^2)) = \tfrac{1}{2}((b_1^4 + 6b_1^2b_2 + 3b_2^2) + 2b_4). \tag{19}$$

Substituting (18) and (19) in the right side of (17) yields

$$\tfrac{1}{2}(J_1^2 + J_2)(Z(B)) = \tfrac{1}{2}\{6b_1^4 + 2b_2^2 + \tfrac{1}{2}(b_1^4 + 6b_1^2b_2 + 3b_2^2 + 2b_4)\}. \tag{20}$$

The reader can verify that the right side of (20) is indeed the cycle index sum for the 7 functional partitions listed in Table 1.

If only $N[S_2; B]$ is desired, it can be found by summing the coefficients of the right side of (20). This follows from the fact that the coefficient sum of any cycle index is 1.

COROLLARY. *The number of orbits $N[A; B]$ of the matrix group $[A; B]$ is the coefficient sum of $Z(A; J_1, \ldots, J_m)Z(B)$.*

7. Proof of Theorem 3

For each functional partition F of $X \times Y$ let T_F be the subgroup of $[A; B]$ consisting of all elements which leave F fixed. For each (α, τ) in $[A; B]$ let

$$O(\alpha, \tau) = \{F \mid (\alpha, \tau) \in T_F\}.$$

If $F \in O(\alpha, \tau)$ let

$$Z((\alpha, \tau); F) = \prod_{v=1}^{n} a_v^{i_v},$$

where i_v is the number of cycles of functions in F of length v induced by (α, τ), viewed as being in $[B]^A$. Thus

$$Z(F) = \frac{1}{|T_F|} \sum_{(\alpha,\tau)\in T_F} Z((\alpha, \tau); F).$$

Let R be a set of distinct representatives for the equivalence classes induced by $[A; B]$ on all the functional partitions of $X \times Y$. By an extension of Burnside's lemma due to one of the authors [14, equation (2) on p. 329]

$$\sum_{F\in R} Z(F) = \frac{1}{|A|\,|B|^m} \sum_{(\alpha,\tau)\in A\times B^X} \sum_{F\in O(\alpha,\tau)} Z((\alpha, \tau); F). \tag{21}$$

Direct evaluation of the sum on the right will be the basic task of this proof.

The use of this extension of Burnside's lemma is not justified unless

$$Z((\gamma, \sigma)^{-1}(\alpha, \tau)(\gamma, \sigma); (\gamma, \sigma)^{-1}F) = Z((\alpha, \tau); F)$$

for all (α, τ) in T_F and (γ, σ) in $[A; B]$. To see this, view (α, τ) and (γ, σ) as being in $[B]^A$ and note that $(f_1 f_2 \dots f_k)$ is a cycle of (α, τ) in F just if $((\gamma, \sigma)^{-1}f_1 \dots (\gamma, \sigma)^{-1}f_k)$ is a cycle of $(\gamma, \sigma)^{-1}(\alpha, \tau)(\gamma, \sigma)$ in $(\gamma, \sigma)^{-1}F$.

First suppose that $\alpha = (1\ 2 \dots m)$, fix any $\tau \in B^X$ and let $\beta = \tau(m)\tau(m-1) \dots \tau(2)\tau(1)$. As shall be seen,

$$\sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F)$$

depends only on m and $Z(\beta)$.

Take any y in Y and let k be the length of the cycle in β to which y belongs. We are going to make use of the following two observations from the proof of Theorem 2.

We have seen that $(1, y)$ is taken through a cycle C of length mk by (α, τ). As before let C_v be the cycle in which $(1, y)$ is permuted by $(\alpha, \tau)^v$. Then

(i) C_v is functional if and only if $k \mid (v/(m, v))$,

and

(ii) when $k \mid (v/(m, v))$ the domain of C_v is

$$\{s \mid 1 \leqslant s \leqslant m \quad \text{and} \quad s \equiv 1 \pmod{(m, v)}\}.$$

Suppose F is some functional partition of $X \times Y$ left fixed by (α, τ). Let f be the element of F such that $f(1) = y$. Let $v \geqslant 1$ be minimal so that $(\alpha, \tau)^v f = f$. Let $i = (m, v)$. By fact (i) we can write $v = rik$ for some r. Now $(m, ik) = i$ since $(m, rik) = i$. Clearly C_{rik} is contained in C_{ik}. But $k \mid (ik/(m, ik))$ and, so by fact (ii) C_{rik} and C_{ik} have the same domain. Thus they are equal. Thus $(\alpha, \tau)^{ik}(1, y)$ is in C_{rik}, hence is in f since $(\alpha, \tau)^{rik} f = f$. But also $(\alpha, \tau)^{ik}(1, y)$ is in $(\alpha, \tau)^{ik} f$. Since f and $(\alpha, \tau)^{ik} f$ are members of a partition, they must be equal. So the minimality of v requires $r = 1$.

To summarize our findings: if (α, τ) maps $f \in F$ into a cycle of length v then $v = ik$ where $i \mid m$ and $(k, m/i) = 1$. Now it follows that k is the length of the cycle which β induces on *any* element of the range of f. For if $i'k' = ik$, $i' \mid m$ and $(k', m/i') = 1$ then it is easy to see that $i = i'$ and $k = k'$. For each $k \geqslant 1$ let

$$D_k = \{y \mid 1 \leqslant y \leqslant n \quad \text{and} \quad y \text{ is in a cycle of length } k \text{ in } \beta\}.$$

What we have seen is that if $(f_1 \dots f_v)$ is a cycle of functions induced on F by (α, τ) then the ranges of $f_1, \dots, f_v$ all lie in a single set D_k, and $v = ik$ where $i \mid m$ and $(k, m/i) = 1$.

Now consider the problem of how many functional partitions F are left fixed by (α, τ) and have a particular cycle type induced by (α, τ). Pick $y \in D_k$ and a function f containing

$(1, y)$. Then f must lie in a cycle of length ik for some i as above in order for f to be in a functional partition fixed by (α, τ). So fix such an i, and consider how many ways there are to form such a cycle of functions. Since f is fixed by $(\alpha, \tau)^{ik}$ (viewed as a member of $[B]^A$), f must contain all of the pairs $(\alpha, \tau)^{rik}(1, y)$ (viewing $(\alpha, \tau)^{rik}$ as a member of $A[B]$) for $r = 1, 2, \ldots$. By fact (ii) this means that f is determined for those arguments $s \equiv 1$ modulo i. Moreover f cannot contain any pair $(\alpha, \tau)^w(1, y)$ if $ik \nmid w$. For then as before if f is to be contained in some partition left fixed by (α, τ) we would have $(\alpha, \tau)^w f = f$. This contradicts our assumption that f is to be permuted in a cycle of length ik by (α, τ), which implies that $(\alpha, \tau)^v f = f$ just if $ik | v$. Now $(\alpha, \tau)^w(1, y)$ for $w = 0, 1, 2, \ldots$ runs through all the pairs (s, y') for $1 \leqslant s \leqslant m$ and y' in the same cycle of β as y. Thus, the different equivalence classes modulo i of $\{1, \ldots, m\}$ must be sent into distinct cycles of β, each of length k. Thus we must choose $f(1), \ldots, f(i)$ to be in distinct cycles of D_k. Then by our facts (i) and (ii) f is completely determined, and is permuted in a cycle of length ik which is a functional partition of $X \times D$, where D is the union of the cycles of D_k which contain $f(1), \ldots, f(i)$. Fixing D, there are exactly $k^i i!$ ways to choose such an f. For there are i cycles to choose $f(1)$ from and k elements in each, $i-1$ cycles left to choose $f(2)$ from and k elements in each, etc.

In all there are $(k^i i!)/(ki)$ ways to obtain a cycle of length ki induced on a functional partition of $X \times D$ by (α, τ), since it makes no difference which of the ki members of the cycle is considered to be the first one.

Suppose now that D_k contains exactly j cycles. There will be a functional partition of $X \times D_k$ fixed by (α, τ) with cycle type $\prod_i b_{ki}^{q_i}$ just if

(a) $q_i = 0$ unless $i | m$ and $(k, m/i) = 1$,

and

(b) $\sum_i iq_i = j$.

In that case we claim that there are exactly

$$\frac{j!}{\prod q_i!\,(i!)^{q_i}} \prod_i \left(\frac{k^i i!}{ki}\right)^{q_i} \tag{22}$$

ways to choose a functional partition. The left factor is the number of ways to arrange the j cycles into disjoint groups, q_i groups of size i for each i. Now each group of size i must be the range of a cycle of functions of length ik induced by (α, τ), the choice of function cycle being independent for each group. So the right factor gives the total number of ways to complete the functional partition.

The term in $j!k^j Z(S_j)$ corresponding to the sequence $q_1, q_2, \ldots$ where $\sum_i iq_i = j$ is just

$$\frac{k^j j!}{\prod_i q_i! \, i^{q_i}} \prod_i b_i^{q_i}.$$

Observe that (22) times $\Pi_i b_{ik}^{q_i}$ is obtained by substituting b_{ik}/k for b_i in this term. Refering to the definition (15) of J_m, we have shown that if $Y = D_k$ then

$$\sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F) = j! \, k^j Z(S_j; d_1, d_2, \ldots, d_j) = J_m(b_k^j). \tag{23}$$

It was seen earlier that if $F \in O(\alpha, \tau)$ then F is the union of functional partitions of $X \times D_k$, $k = 1, 2, \ldots, n$, each left fixed by (α, τ). Since the choices for these partitions are independent for different k, we can apply (23) repeatedly, obtaining

$$\sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F) = J_m(b_1^{j_1}) \, J_m(b_2^{j_2}) \ldots J_m(b_n^{j_n}) = J_m(b_1^{j_1} b_2^{j_2} \ldots b_n^{j_n}) \tag{24}$$

if $Z(\beta) = b_1^{j_1} b_2^{j_2} \ldots b_n^{j_n}$. This is under the original hypothesis that α is a single cycle of length m and

$$\beta = \tau(m)\tau(m-1) \ldots \tau(1). \tag{25}$$

Now, as seen in the proof of Theorem 2 there are just $|B|^{m-1}$ functions τ in B^X which satisfy (25). Summing (24) over this set of functions gives

$$\frac{1}{|B|^m} \sum_{\tau} \sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F) = \frac{1}{|B|^m} |B|^{m-1} J_m(Z(\beta)) = \frac{1}{|B|} J_m(Z(\beta)). \tag{26}$$

Summing (24) over all $\tau \in B^X$ corresponds to summing (26) over all $\beta \in B$, which gives

$$\frac{1}{|B|^m} \sum_{\tau \in B^X} \sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F) = J_m(Z(B)) \tag{27}$$

since J_m is $\mathbf{Q}$-linear.

The assumption that α is a single cycle is now dropped. Instead, let α be any element of A and suppose that X is the disjoint union of X_1, X_2 where each is a union of cycles of α. Then $\alpha(X_1) = X_1$ and $\alpha(X_2) = X_2$. Let $\alpha_1 = \alpha|_{X_1}$ and $\alpha_2 = \alpha|_{X_2}$. Similarly for any f in Y^X or τ in B^X, we can split these into disjoint parts f_1 and f_2 or τ_1 and τ_2, by considering the restrictions to X_1 and X_2. Functional partitions of $X \times Y$ correspond in a natural way to triples $\langle F_1, F_2, \varphi \rangle$ where F_1 is a functional partition of $X_1 \times Y$, F_2 is a functional partition of $X_2 \times Y$, and φ is a $1-1$ map from F_1 onto F_2. With the triple $\langle F_1, F_2, \varphi \rangle$ corresponds the partition

$$\{f \cup \varphi(f) \mid f \in F_1\}.$$

This correspndence is easily seen to be $1-1$ and onto. A necessary and sufficient set of conditions for $\langle F_1, F_2, \varphi \rangle$ to correspond to a partition in $O(\alpha, \tau)$ is:

1. $F_1 \in O(\alpha_1, \tau_1)$
2. $F_2 \in O(\alpha_2, \tau_2)$
3. If $(f_1 \ldots f_k)$ is a cycle induced on F by (α_1, τ_1) then $(\varphi(f_1) \ldots \varphi(f_k))$ is a cycle induced on F_2 by (α_2, τ_2).

Condition 3 implies

4. $Z((\alpha_1, \tau_1);\ F_1) = Z((\alpha_2, \tau_2);\ F_2)$.

Given F_1, F_2 satisfying 1, 2, and 4 where the common cycle type is $\prod_{i=1}^m b_i^{j_i}$, there are exactly

$$\prod_{i=1}^m i^{j_i} j_i!$$

ways to choose a $1-1$ correspondence φ satisfying 3. To see this note that for each i there are $j_i!$ ways to match up the i cycles of length i in F_1 with the j_i cycles of length i in F_2. For any two particular cycles of length i there are just i different ways to match them up.

Refering to the definition of $\mho$ (14) we have shown that

$$Z((\alpha_1, \tau_1);\ F_1) \mho Z((\alpha_2, \tau_2);\ F_2) = \sum_F Z((\alpha, \tau);\ F) \tag{28}$$

for any F_1 and F_2 satisfying 1 and 2, the sum on the right to be taken over all $F \in O(\alpha, \tau)$ corresponding to $\langle F_1, F_2, \varphi\rangle$ for some φ. Summing (28) over all τ_1 in B^{X_1}, all F_1 in $O(\alpha_1, \tau_1)$, all τ_2 in B^{X_2} and all F_2 in $O(\alpha_2, \tau_2)$ gives

$$\Big(\sum_{\tau_1 \in B^{X_1}} \sum_{F_1 \in O(\alpha_1, \tau_1)} Z((\alpha_1, \tau_1); F_1)\Big) \mho \Big(\sum_{\tau_2 \in B^{X_2}} \sum_{F_2 \in O(\alpha_2, \tau_2)} Z((\alpha_2, \tau_2); F_2)\Big) = \sum_{\tau \in B^X} \sum_{\tau \in O(\alpha, \tau)} Z((\alpha, \tau); F), \tag{29}$$

in light of the $\mathbf{Q}$-linearity of $\mho$.

Now we claim that in general

$$\frac{1}{|B|^m} \sum_{\tau \in B^X} \sum_{F \in O(\alpha, \tau)} Z((\alpha, \tau); F) = Z(\alpha; J_1, \ldots, J_m)\, Z(B), \tag{30}$$

and proceed by induction on the number of cycles of α. If α is a single cycle this reduces to (27). If α has more than one cycle then X is the disjoint union of sets X_1, X_2 which are unions of cycles of α, and have cardinalities m_1, m_2 respectively with $m_1, m_2 \geqslant 1$. Then with α_1, α_2 as before note that each has fewer cycles than α, and in fact

$$Z(\alpha) = Z(\alpha_1) Z(\alpha_2).$$

Also $|B|^{m_1} |B|^{m_2} = |B|^m$. By the induction hypothesis we assume (30) for α_1, α_2 in place of α. With these relations and (29) we obtain

$$\frac{1}{|B|^m}\sum_{\tau\in B^X}\sum_{F\in O(\alpha,\tau)} Z((\alpha,\tau);F)$$

$$=(Z(\alpha_1;J_1,\ldots,J_m)Z(B))\mho(Z(\alpha_2;J_1,\ldots,J_m)Z(B))$$
$$=Z(\alpha;J_1,\ldots,J_m)Z(B)).$$

Here it is important to recall that $J_1, \ldots, J_m$ are members of $\mathbf{R}(+,\mho)$ for algebraic purposes. Thus (30) is proved by induction.

Finally, the theorem follows from (21) and the result of summing (30) over all $\alpha\in A$ and dividing by $|A|$. This concludes the proof of Theorem 3.

At the end of section 2 a generalized wreath product $A[B_1, \ldots, B_t]$ acting on $\bigcup_{i=1}^t X_i\times Y_i$ was introduced. This induces a generalization of the matrix group which is denoted $[A; B_1, \ldots, B_t]$. The object set of $[A; B_1, \ldots, B_t]$ is the set of partitions F of $\bigcup_{i=1}^t X_i\times Y_i$ into subsets S which have the property that for each $x\in X_i$ there is exactly one $y\in Y_i$ such that (x, y) is in S. For any such partition F we denote by $Z(F)$ the cycle index of the subgroup of $[A; B_1, \ldots, B_t]$ which leaves F fixed, with F itself as the object set. If F_k ranges over some selection of distinct representatives of the orbits of $[A; B_1, \ldots, B_t]$ then an expression for $\Sigma_k Z(F_k)$ can be found which is a generalization of Theorem 3. For each $1\leqslant i\leqslant t$, all $s\geqslant 1$, and any $P_1, \ldots, P_t$ in $\mathbf{R}$ let

$$J_{i,s}(P_1,\ldots,P_t)=J_s(P_i).$$

The operators $J_{i,s}$ are to be viewed as members of the ring $\mathbf{R}^t(+,\mho)$. Then

$$\sum_k Z(F_k)=Z_{X_1,\ldots,X_t}(A)\,[a_{i,s}\to J_{i,s}]\,(Z(B_1),\ldots,Z(B_t)). \tag{31}$$

In case $t=1$ and $B_1=B$ this gives the same result as Theorem 3. In case A is the identity group E_t and $X_i=\{i\}$ for $1\leqslant i\leqslant t$ this gives Redfield's Decomposition Theorem [12, p. 445]. It should be noted that the object set of $[A; B_1, \ldots, B_t]$ is empty if any of the object sets Y_i of B_i have different cardinalities. It follows from the definition of $\mho$ that in this case (31) gives the value 0 for $\Sigma_k Z(F_k)$.

8. Applications of Theorem 3

The *superposition* of a set of graphs $G_1, \ldots, G_m$ all on the same set of n points is the union of their sets of lines, multiplicity included. Furthermore, in this union the lines of G_i are assumed to have color c_i different from color c_j for $j\neq i$. All eight superpositions of two paths P_4 of order 4 are shown in Figure 4.

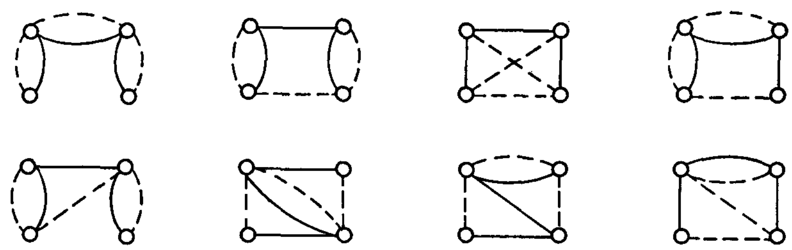

Figure 4. All eight superpositions of two paths of order 4.

Read [11] and Redfield [12] were able to calculate the total number of superpositions of $G_1, \ldots, G_m$ as a function of the cycle indices of the groups $\Gamma(G_i)$ of these m graphs. In fact Redfield showed that this number is the coefficient sum of

$$Z(\Gamma(G_1)) \mho \ldots \mho Z(\Gamma(G_m)). \tag{32}$$

Now suppose all the graphs $G_1, \ldots, G_m$ are isomorphic to G with point set $Y = \{1, \ldots, m\}$ and let E_m be the identity group on $X = \{1, \ldots, m\}$. Then it can be seen that each functional partition of $X \times Y$ corresponds to a superposition of m copies of G, and furthermore the number of superpositions is the number of orbits of the matrix group $[E_m; \Gamma(G)]$. From Theorem 3 it quickly follows that this number is the coefficient sum of

$$Z(\Gamma(G)) \mho \ldots \mho Z(\Gamma(G))$$

which agrees with Redfield's result (32). For example, if G is the path of order 4, its cycle index is $\frac{1}{2}(b_1^4 + b_2^2)$ and hence the number of superpositions of 2 copies of G is the coefficient sum of $\frac{1}{2}(b_1^4 + b_2^2) \mho \frac{1}{2}(b_1^4 + b_2^2)$ which is 8 (compare Figure 4).

When dealing with superpositions of m copies of a given graph G, however, we can ask for the number obtained when specified copies are allowed to be permuted among themselves. Thus if we allow the 2 paths of order 4 to be interchangeable, then the last 2 graphs in Figure 4 are to be identified. This simply amounts to using the matrix group $[S_2; \Gamma(P_4)]$ instead of $[E_2; \Gamma(P_4)]$. In general we have the following result.

The number of superpositions of m interchangeable copies of the graph G is $N[S_m; \Gamma(G)]$. Redfield used his enumeration theorem to calculate superpositions of cycles of order n, whose group is the dihedral group D_n. We have used Theorem 3 to compute the corresponding number of superpositions of interchangeable copies of cycles. The results are summarized in Table 2.

We can also apply Theorem 3 to enumerate multigraphs with a given number m of lines and n points. Let G be the graph of order n with exactly one line. Then the cycle index of its group $Z(\Gamma(G))$ is $Z(S_2)Z(S_{n-2})$. Each superposition of m interchangeable copies of G

Table 2. *The number of superpositions of cycles of order $n \leqslant 6$*

n	$N[E_2; D_n]$	$N[S_2; D_n]$	$N[E_3; D_n]$	$N[S_3; D_n]$
3	1	1	1	1
4	2	2	5	3
5	4	4	24	9
6	12	10	391	89
7	39	28	9 549	1 705
8	208	130	401 691	67 774

constitutes a multigraph of order n with m lines. Hence the total number is $N[S_m; \Gamma(G)]$, and the only cycle indices involved are those of the symmetric groups S_2, S_{n-2} and S_m.

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