Envelope soliton solution for finite amplitude equatorial waves

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Abstract. Using shallow water equations on an equatorial beta plane, the nonlinear dynamics of the equatorial waves is investigated. A general mathematical procedure to study the nonlinear dynamics of these waves is developed using the asymptotic method of multiple scales. On faster temporal and spatial scales the equations describe the equatorial waves *viz*, the Rossby waves, Rossby gravity waves, the inertia gravity waves and the Kelvin waves. Assuming that the amplitude of these waves are functions of slower time and space scales, it is shown that the evolution of the amplitude of these waves is governed by the nonlinear Schrödinger equation. It is then shown that for the dispersive waves like Rossby waves and Rossby-gravity waves, the envelope of the amplitude of the waves has a 'soliton' structure.

Keywords. Shallow water equations ; finite amplitude solutions ; multiple time scale method ; envelope solitons.

1. Introduction

In the tropical and mid-latitude atmospheres, large (planetary) scale wave-like structures of the atmospheric flows are observed features of the general circulation and have been the subject of intense study among the theoretical and observational meteorologists. A large number of theoretical investigations on the linear stability analysis of atmospheric flows has been made with flows which are independent of time and longitude. Once the superimposed perturbations reach a finite amplitude or a fully developed eddy stage by deriving a substantial amount of energy from the zonal flow *via* either baroclinic or barotropic instability mechanisms or a combination of both, the total flow pattern becomes nonsteady and zonally non-uniform. Regardless of the energy

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sources of the atmospheric wave-like disturbances, their finite-amplitude existence is certain and therefore, nonlinear processes should play an important role in their dynamics.

For barotropic Rossby wave motion, instability has been established by Lorenz (1972). By employing linear perturbation analysis Lorenz showed that in the barotropic atmospheric flow, Rossby's (Rossby *et al* 1939) original solution (representing an east-west propagating planetary wave embedded in a constant westerly flow) is unstable provided the wave amplitude is sufficiently large or its wave number is sufficiently high. It is clearly of interest to us to determine how the nature of the instability is affected by the nonlinear processes which become increasingly important as the perturbations superimposed upon the basic state reach a finite amplitude regime. A few questions of fundamental importance arise here :

(i)   Given sufficient energy, how will the waves evolve in time and space? In other words, what is the structure and long time behaviour of the finite-amplitude waves ? ;

(ii) What are their propagation characteristics on slower time and space scales ? ;

(iii) How does the finite-amplitude perturbation state affect the basic flow in the long time ? ;

(iv) Are the fully developed zonally uniform or non-uniform and non-steady flows (basic zonal flow + finite amplitude waves) stable with respect to further perturbations. The present investigation addresses itself to these questions (or a few of them) which cannot be answered by a linear theory. The vast majority of theoretical investigations have been made to study the linear stability problem of unstable waves in the barotropic atmosphere on the equatorial and mid-latitude β-planes [Matsuno 1966; Longuet-Higgins and Gill 1967; Lorenz 1972; Duffy 1974 (to name a few) and references therein]. It was Phillips (1954) who first took account of some effects of nonlinearity and computed the second order changes in the basic current embedded in the unstable waves which exhibited exponential growth.

But this theory seems to have an unavoidable defect that each succeeding term in the amplitude expansion grows more rapidly than the previous term and therefore severely restricts the time for which the expansion remains valid. In recent past a number of nonlinear equations describing the evolution of finite-amplitude waves in dispersive media, have been shown to possess exact analytical solutions whose single most distinctive feature is the existence of solitary waves (solitons) [an excellent review on solitary solutions has been given by Scott *et al* (1973)].

Investigations on the nonlinear time evolution of finite-amplitude waves in the baroclinic atmosphere, have been made by several research workers (Long 1964; Larsen 1965; Benney 1966; Pedlosky 1970, 1971, 1972a, ₁b, 1977; Clarke 1971; Redekopp 1977; Redekopp and Weidmann 1978). But in the case of barotropic atmosphere with a zero basic zonal flow or a horizontally sheared zonal flow, only a few investigations have been made to study the nonlinear behaviour and time evolution of these waves (Gill 1974; Dommaracki

and Loesch 1977; Loesch 1978; Boyd 1980). Only a few of these research workers have shown the existence of "solitary" solution for large scale planetary waves in both the middle latitude and equatorial atmospheres.

The present investigation represents an extension of the linear analyses of Matsuno (1966) into the nonlinear regime, while the perturbation is finite but small. Under this assumption, the nonlinear analysis may be carried out using asymptotic expansions for perturbations and their derivatives and using the procedure of multiple scales. In this paper we have studied the long time evolution of finite-amplitude (*i. e.* weakly nonlinear) and slowly varying wave train propagating in the atmosphere with a *zero basic zonal flow*, using the divergent barotropic model on a β-plane.

In § 2, we have outlined our model and presented the governing equations and the formulation of the derivation of a modified nonlinear Schrödinger equation governing the amplitude modulation of the atmospheric waves. According to this equation, the waves have steady state "envelope soliton" solutions under certain conditions. Though dispersive waves in optics and in plasmas are known to have envelope soliton solutions, to our knowledge, this is a good theoretical evidence of the existence of 'envelope solitons' in any geophysical situation. We have also discussed briefly in this section the propagation characteristics of the linearized solutions [0(1) solutions] for these waves as the lowest order approximations of the model. In § 3, we have obtained the envelope solitary wave structures (solitons) as the stationary solutions of the modified nonlinear Schrödinger equation. In §4, we have discussed the results obtained. Finally, the results of our analysis are summarized in the concluding section.

2. The model and the governing equations

2.1 *Formulation*

The model we consider here, is the so called divergent barotropic model on a β-plane. We follow Matsuno's (1966) approach, retaining the nonlinear terms in the present analysis. The model consists of a single layer of homogeneous inviscid fluid of mean constant depth H, with a free surface under hydrostatic balance (figure 1). We take the local cartesian coordinate system shown in figure 1, where x' is positive in the eastward direction. y' is taken in northward direction and z' is antiparallel to the gravity vector and is thus vertically upward.

We assume that the free surface deformations 'h' are sufficiently small ($h < H$) so that these can be ignored when the effects of horizontal divergence are incorporated into the model. We make here β-plane approximation whereby the important dynamic effects of the earth's rotation and sphericity on large scale motions are considered by assuming the linear variation with latitude ϕ of the Coriolis parameter f' ($= 2\Omega \sin \phi$) as :

$$f' = f'_0 + \beta' y', \tag{1}$$

where $y' = a (\phi - \phi_0)$, *i.e.* the coordinate y' is centred at the latitude Φ_0 , a is the
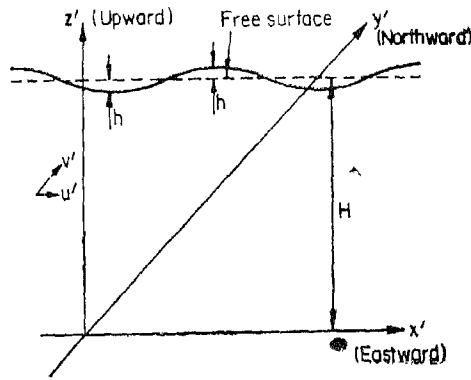
Figure 1. The model and the co-ordinate system.

mean earth radius, $f_0 = 2\Omega_0 \sin \phi_0$ (Ω being the earth's angular velocity and β' [$= (\partial f'/\partial \phi) \phi_0 / a$] is the Rossby parameter taken as constant.

The governing equations and the appropriate boundary conditions for the model subject to the above approximations are

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} - (f_0' + \beta' y') v' + \frac{\partial \Phi'}{\partial x'} = 0, \quad (2)$$

$$\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y} + (f_0' + \beta' y') u' + \frac{\partial \Phi'}{\partial y} = 0, \quad (3)$$

$$\frac{\partial \Phi'}{\partial t} + u' \frac{\partial \Phi'}{\partial x'} + v' \frac{\partial \Phi'}{\partial y'} - gH \left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right) = 0, \quad (4)$$

$$u' \, v', \, \Phi' \to 0 \text{ as } y' \to \pm \infty, \quad (5)$$

where prime denotes the dimensional quantities ; u' and v' are the components of horizontal velocity in zonal and meridional directions respectively; Φ' is the geopotential of the free surface. Assuming that there is no averaged basic flow, u', v' and Φ' here represent the perturbations around this state. Equation (4) is the integrated continuity equation assuming that u' and v' are independent of the vertical coordinate.

We now define non-dimensional variables as

$$(x, y) = (x', y')/L \; ; \; (u,v) = (u', v') \, / \, U$$

$$t = (\beta' \, L) \, t' \; ; \quad \Phi = \Phi' \, / \, (\beta' L^2 U), \quad (6)$$

where U is the horizontal characteristic velocity and L is the horizontal characteristic length defined by $L^2 = (gH)^{1/2}/\beta$, g being acceleration due to gravity. The set of the governing equations (2) — (4) are transformed, using (6), into nondimensional form as

$$u_t + \epsilon\,(uu_x + vu_y) - (f_0 + y)\,v + \Phi_x = 0, \tag{7}$$

$$v_t + \epsilon\,(uv_x + vv_y) + (f_0 + y)\,u + \Phi_y = 0, \tag{8}$$

$$\phi_t + \epsilon\,(u\phi_x + x\Phi_y) + u_x + v_y = 0, \tag{9}$$

with the boundary conditions

$$u, v, \phi \to 0 \; ; \quad \text{as } y \to \pm \infty, \tag{10}$$

where $f_0 = f_0'/\beta'L$ is the dimensionless planetary vorticity at the latitude ϕ_0 and is thus a constant parameter. A subscript represents differentiation with respect to that variable : $\epsilon = U/\beta'L^2$ is the equivalent Rossby number. For typical values of $U \simeq 10$ m/sec, $H \simeq 10$ km and $\epsilon \sim O\,(10^{-2})$ *i.e.* ϵ is a small parameter much less than unity.

The equations (7) – (9) are nonlinear in nature, with ϵ as a ┌nondimensional smallness parameter representing a measure of the magnitude of the nonlinear products. We thus choose ϵ as an expansion parameter in the analysis developed here.

The derivative–expansion method suggests us to extend the independent variables x and t to the sets of independent variables

$$X_0, \; X_1, \; X_2, \ldots \text{ and } T_0, T_1, T_2 \ldots, \tag{11}$$

where long time and space scales (independent variables x and t) are defined as

$$X_i = \epsilon^i x \; ; \; T_i = \epsilon^i t \, , \; i = 0, 1, 2, 3, \ldots, \tag{12}$$

which are incorporated in (7) – (9) through the expansion of the derivative operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial t}$ as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial T_0} + \epsilon\,\frac{\partial}{\partial T_1} + \epsilon^2\,\frac{\partial}{\partial T_2} + \cdots$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial X_0} + \epsilon\,\frac{\partial}{\partial X_1} + {}^2\epsilon\,\frac{\partial}{\partial X_2} + \cdots \tag{13}$$

Accordingly, any dependent variable, a function of (x, y, t), is now regarded as a function of $(X_i, y, T_i, i = 0, 1, 2, \ldots)$. We assume further that the dependent variables, u, v, ϕ (perturbations from a uniform basic state under consideration) have the asymptotic expansions in ϵ of the following form :

$$(u, v, \Phi) = (u^{(0)}, v^{(0)}, \Phi^{(0)}) + \epsilon\,(u^{(1)}, v^{(1)}, \Phi^{(1)}) + \epsilon^2\,(u^{(2)}, v^{(2)}, \Phi^{(2)}) + \ldots \tag{14}$$

Substituting (13) and (14) into the governing equations (7) to (9) and equating coefficients of like powers of ϵ, we obtain a set of perturbation equations to the different orders in ϵ. These will then determine successively the perturbation quantities $(u^{(i)}, v^{(i)}, \Phi^{(i)} ; i = 0, 1, 2 \ldots$ to the different orders in ϵ as defined in (14). These dependent quantities are to be determined so as to be bounded ʿnon-secular) at each stage of perturbations.

2.2 O(1) problem : linear analysis

To the $O(1)$, the perturbation equations determining $v^{(0)}$, $u^{(0)}$ and $\Phi^{(0)}$ and obtained by equating the coefficients of ϵ^0 equal to zero are

$$Lv^{(0)} = 0, \tag{15}$$

$$Pu^{(0)} - Qv^{(0)} = 0, \tag{16}$$

$$\Phi_{T0}^{(0)} + u_{x0}^0 + v_Y^{0)} = 0 \tag{17}$$

where $L \equiv \partial_{T0T0T0} - \partial_{X0X0X0} - \partial_{yyT0} + y^x\partial_{T0} - \partial_{X0}$

$P \equiv \partial_{T0T0} - \partial_{X0X0} \; ; \; Q \equiv y\partial_{T0} + \partial_{yX0}$

This set of equations (15) — (17) constitutes the linear problem for $O(1)$ solutions as nonlinear terms are absent.

As we are concerned with the problem of amplitude modulation, we assume the linearized problem to have a dispersive wave solution of the form $\exp[i(kx_0 - \omega_{T0})]$, propagating in west-east direction, k being the wave number and ω the frequency. Let the solution to (15) be

$$v^{(0)} = \Sigma_j [A_j^{(0)}(X_1,T_1,...)V_j^{(o)}(y)e^{i(k_jX_0 - \omega_jT_0)} + c.c + \alpha_j(X_1, T_1, ...) \tag{18}$$

where j=0,1,2, . . . and $A_j^{(0)}$ are the complex wave amplitudes (constant with respect to X_0, T_0, y) which are assumed to be smooth slowly varying functions of slow time and space scales and satisfy $A_j^{(0)}(X_1,T_1, . .) = A_j^{(o)*}(-X_1,-T_1, ..)$, α_j are real constants, functions of higher order scales and can thus be termed as slow modes arising due to nonlinear interactions on slower time and space scales. Since we are concerned with the nonlinear modulation of the wave train represented by (18) and the nonlinear terms are absent in (15), we assume that $O(1)$ perturbation has the wave from which is devoid of the slower modes α_j. Therefore, $\alpha_j = 0$ is assumed here. If we take (18) as the $O(1)$ perturbation, we may treat interactions between the wave trains as slow modes. This case, however, is not treated in the present analysis.

Equations (15) to (17) are same as obtained by Matsuno (1966) who treated the $O(1)$ problem in detail and showed that the $O(1)$ problem is an eigen value problem. Following his approach, the orthogonal $O(1)$ eigen solutions belonging to eigenvalues ω_j can be obtained as (writing $\theta_j = k_jX_0 - \omega_jT_0$)

$$v^{(0)} = \sum_j [-i(\omega_j^2 - k_j^2)\psi_{nj} + c.c] \tag{19}$$

$$u^{(0)} = \sum_j [\tfrac{1}{2}(\omega_j + k_j)\psi_{n_{j+1}} + n_j(\omega_j - k_j)\psi_{n_{j-1}} + c.c] \tag{20}$$

$$\phi^{(0)} = \sum_j [\tfrac{1}{2}(\omega_j + k_j)\psi_{n_j+1} - n^j(\omega_j - k_j)\psi_{n_j-1} + c.c] \tag{21}$$

where $\psi_{nj} \neq A_j^{(0)}(X_1, T_1, \ldots)(\exp i\theta_j = \exp_a^{\cdot}(-y^2/2)H_{nj}(y)$; $n_j = 0, 1, 2. \cdots$

and H_{nj} is the Hermite polynomial of degree n_j provided that the following dispersion relation is satisfied

$$D\ (\omega_j,\ k_j)\ \equiv \omega_{jj}^3 - (k_j^2 + 2n_j + 1)\ \omega_j - k_j = 0. \tag{22}$$

Here n_j can be indentified as meridional mode of solutions. This is the cubic dispersion relation whose roots define the characteristic wave types of this model. This gives a relationship between the frequency and the longitudinal wave number for some definite meridional mode n_j. For $n_j \geqslant 1$ and using a trigonometric method of solution for cubic equation, the approximate values of three roots for ω_j are given as

$$\omega_{1j} \simeq (k_j + 2n_j + 1)^{1/2} + \tfrac{1}{2}k_j\ (k_j^2 + 2n_j + 1)^{-1} - \tfrac{3}{8}k_j^2\ (k_j^2 + 2n_j + 1)^{-5/2}, \tag{23a}$$

$$\omega_{2j} \simeq -(k_j + 2n^j + 1)^{1/2} + \tfrac{1}{2}k_j\ (k_j^2 + 2n_j + 1)^{-1} + \tfrac{3}{8}k_j^2(k_j^2 + 2n_j + 1)^{-5/2} \tag{23b}$$

$$\omega_{3j} \simeq -k_j(k_j^2 + 2n^j + 1)^{-1} \tag{23c}$$

The upper two frequencies ω_{1j} and ω_{2j} correspond to respectively the frequencies of east and west propagating inertial-gravity waves, while ω_{3} corresponds to the frequency of a westward propagating Rossby wave. The dispersion curves for these three frequencies for a few meridional modes n_j are shown in figure 2. Fo $n_j \geqslant 1$, these are completely separated from each other over the whole range of k_j.
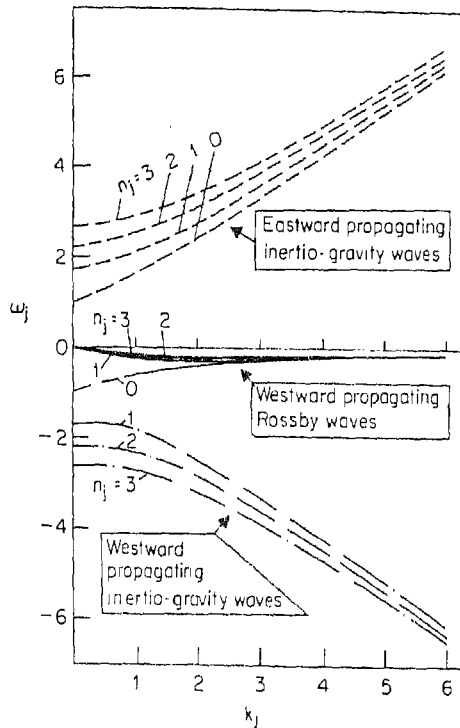

Figure 2. Frequencies as functions of wave number (dispersion curves).

For the lowest mode $n_j = 0$. The three roots of the dispersion relation (22) are

$$\omega_{1j} = -k_j \tag{24a}$$

$$\omega_{2j} = \tfrac{1}{2}k_j + (\tfrac{1}{4}k_j^2 + 1)^{1/2} \tag{24b}$$

$$\omega_{3j} = \tfrac{1}{2}k_j - (\tfrac{1}{4}k_j^2 + 1)^{1/2} \tag{24c}$$

The first root ω_{1j} gives a trivial solution, $u^{(0)} = v^{(0)} = \Phi^{(0)} = 0$ and hence, cannot be adopted as an eigenvalue of the O (1) equations (15) – (17). Thus in the case of the lowest mode $n_j = 0$, we have only two waves, the frequency of one of which (ω_{2j}) corresponds to an eastward propagating inertial-gravity wave and that of other (ω_{3j}) corresponds to a westward propagating mixed Rossby-gravity wave or Yanai wave. These are also shown in figure 2. As is obvious from figure 2, this mixed Rossby-gravity wave connects the two families of waves and its frequency ranges from a value which compared to the frequency of the inertial-gravity wave, to a value which is close to that of the Rossby wave.

2.3 $O(\epsilon)$ problem: the finite-amplitude (non-linear stability analysis)

To the $O(\epsilon)$, the solutions $v^{(1)}$, $u^{(1)}$ and $\Phi^{(1)}$ can be found out from the $O(\epsilon)$ perturbation equations which are obtained by equating the coefficient of ϵ equal to zero and are

$$Lv^{(1)} = -L_1 v^{(0)} - P(u^{(0)}v_{x0}^{(0)} + v^{(0)}v_y^{(0)}) - M(u^{(0)}u_{x0}^{(0)} + v^{(0)}u_y^{(0)}) + N(u^{(0)}\Phi_{x0}^{(0)} + v^{(0)}\Phi_y^{(0)}), \tag{25}$$

$$Pu^{(1)} - Qv^{(1)} = -(P_1 u^{(0)} - Q_1 v^{(0)}) - (u^{(0)}u_{x0}^{(0)} + v^{(0)}u_y^{(0)})_{T0} - (u^{(0)}\Phi_{x0}^{(0)} + v^{(0)}\Phi_y^{(0)})_{x0} \tag{26}$$

$$\Phi_{T0}^{(1)} + u_{x0}^{(1)} + v_y^{(1)} = -\Phi_{T1}^{(0)} - u_{x1}^{(0)} - u^{(0)}\Phi_{x0}^{(0)} - v^{(0)}\Phi_y^{(0)}, \tag{27}$$

where $L_1 \equiv 3\partial_{T0T0T1} - \partial_{x0x0T1} - \partial_{yyT1} + y^2\partial_{T1} - \partial_{x1} - 2\partial_{x0T0x1}$;

$P_1 \equiv 2 (\partial_{T0T1} - \partial_{x0x1})$; $Q_1 \equiv y\partial_{T1} + \partial_{yx1}$;

$M \equiv (\partial_{x0y} - y\partial_{T0})$; $N \equiv \partial_{yT0} - y\partial_{x0}$

It is obvious that the solutions $v^{(1)}$, $u^{(1)}$ and $\Phi^{(1)}$ of (25) – (27) will be the linear combinations of the two solutions, namely (a) one solution of the homogeneous parts of (25) – (27) ; (b) another solution arising due to the inhomogeneity present on the r.h.s. of (25) – (27). As is clear from the above equations, knowing $v^{(1)}$ from (25) will enable us to solve for $u^{(1)}$ from (26) and finally $\Phi^{(1)}$ from (27). Since the operators appearing in l.h.s. of (25) – (27) are the same as those appearing in (15) – (17) which determine $v^{(0)}$, $u^{(0)}$ and $\Phi^{(0)}$, the solutions of the homogeneous parts of (25) - (27) (obtained by putting their r.h.s. = 0) will be similar to the $O(1)$ solutions described by (19)–(21) with the dispersion relation (22) being satisfied, in their (x_0, T_0, y) functional structures. Though the operators in l.h.s. of (26) and (27) are the same as those in l.h.s. of $O(1)$ equations (16) and (17), the solutions of the homogeneous parts of these equations may be slightly different from $O(1)$ solutions because these operators now operate on $v^{(1)}$

and $u^{(1)}$ as well. It can be noticed from the linear inhomogeneous terms of (25) – (27) that for every component of $v^{(1)}\Phi^{(1)}$ and $u^{(1)}$ there exists a corresponding resonant forcing arising from the introduction of multiple scales (T_1, X_1) which are involved in the operators L_1, P_1 and Q_1. Nonlinear inhomogeneous terms may also give rise to resonant interactions under special conditions. Both linear and inhomogeneous resonant forcings will produce higher harmonics in the fundamental wave on $O(\epsilon)$ space and time scales, X_1 and T_1. Thus, considering that on these scales the second harmonics are produced due to resonant forcings, we assume the $O(\epsilon)$ solutions of the following forms :

$$v^{(1)} = \sum_j [-i (\omega_j^2 - k_j^2) \psi_{nj}^{(1)} + B_{1j}^{(1)} (X_1, T_1, \ldots) \exp 2i\theta_j V_{nj}^{(1)} (y) + c.c + c_{1j}] \tag{28}$$

$$u^{(1)} \sum_j [A_{1j}^{(1)}(X_1, T_1, \ldots) \exp i\theta_j W_{1j}^{(1)} (y) + B_{2j}^{(1)} (X_1, T_1, \ldots)$$
$$\exp(2i\theta_j) W_{2j}^{(1)} (y) + c.c + c_{2j}], \tag{29}$$

$$\Phi^{(1)} = \sum_j [A_{2j}^{(1)} (X_1, T_1, \ldots) \exp i\theta_j Z_{1j}^{(1)} (y) + B_{3j}^{(1)} (X_1, T_1, \ldots)$$
$$\exp (2i\theta_j)Z_{2j}^{(1)} (y) + c.c + c_{3j}] \tag{30}$$

where $\psi_{nj}^{(1)} A_j^{(1)}(X_1, T_1, \ldots) \exp (i\theta^j) \exp (y^2/2)H_{nj}^n (y)$. Since we are interested in the nonlinear modulation of the amplitude $A_j^{(0)}$ of the wave train (described by the $O(1)$ solutions), we assume that $A_j^{(1)}$, $A_{(1,2)j}^{(1)}$ and $B_{(1,2,3j)}^{(1)}$ are functions of $A_j^{(0)}$ and $A_j^{(0)*}$ (asterisk denoting the complex conjugate) and thus are complex constants with respect to X_0, T_0 and y. $C_{(1,2,3)j}$ are real constants with respect to X_0, T_0 and y and assumed to be functions of $A_j^{(0)}$, $A_j^{(0)*}$. These constants are to be determined from the secularity removal conditions for higher order terms in ϵ. $V_{nj}^{(1)}$, $W_{(1,2)j}^{(1)}$, $Z_{(1,2)j}^{(1)}$ are functions of y, yet to be determined.

Substitution of (28) into (25) will give us an equation which will determine the solution $v^{(1)}$ completely. The structure of $v^{(1)}$ so obtained would contain resonant secular terms (proportional to $\exp (\pm i\theta_j)$ unless the coefficients of $\exp (\pm i\theta_j)$ vanish. Equating the coefficients of $\exp (2i\theta_j)$ in both sides of this equation will determine $B_{1j}^{(1)}$ and $V_{nj}^{(1)}$. In order that the solution for $v^{(1)}$ be non-secular, we must have

$$A_{1jT1}^{(o)} + V_g A_{jx1}^{(0)} = 0 , \tag{31}$$

which is the condition for non-secularity and thus removes the secular behaviour of $V^{(1)}$. By considering the complex conjugate terms, we shall get an exactly similar non-secular condition for $A_j^{(0)*}$. Here V_g is the group velocity of the waves defined by

$$V_g = - D_{kj} / D_{\omega j} = (1 + 2 \omega_j k_j)(3\omega_j^2 - k_j^2 - 2n - 1)^{-1} \tag{32}$$

D being the dispersion function defined by (22). Under these non-secular conditions (31) and equating the coefficients of $\exp (\pm 2i\theta_j)$, we obtain the secular-free solution $\nu^{(1)}$ as

$$v^{(1)} = \sum_j \left[-i (\omega_j^2 - k_j^2) \psi_{nj}^{(1)} + \frac{i}{2\omega_j} \psi_{nj}^{(1)} + c.c + c_{ij} \right] , \tag{33}$$

where $\psi_{nj}^{(1)} = KA_j^{(0)2} \exp(2i\theta_j) V_{nj}^{(i)}(y)$, K being the constant of proportionality and $V_{nj}^{(1)}$ is determined by solving the following differential equation

$$\frac{d^2 V_{nj}^{(1)}}{dy^2} + (4\omega_j^2 - 4k_j^2 - k_j/\omega_j - y^2)\, V_{nj}^{(1)} = K^{-1}\, F\,(\omega_j,\, k_j,\, y)\,, \tag{34}$$

where we have replaced the partial derivative $\partial^2/\partial y^2$ by the total derivation d^2/dy^2. The function $F\,(\omega_j,\, k_j,\, y)$ appearing in (34) is given by

$$F\,(\omega_j,\, k_j,\, y) = [\, F_{1j}\, H_{nj}^2 + F_{2j}\, H_{nj}\, H_{nj+1} + F_{3j}\, H_{nj+1}^2\,]\, \exp(-y^2) \tag{35}$$

where $F_{(1,2,3)j}$ are all functions of ω_j, k_j and y and are given by

$$F_{1j}\,(\omega_j,\, k_j, y) = 2\,[\, y^3\,(\omega_j - k_j)^2\,(2\omega_j^2 - k_j^2) + 2y^2\,k_j^2\,(\omega_j - k_j)^2 - y\,(\omega_j - k_j)$$

$$\{2\omega_j(\omega_j^2 - k_j^2)\,(\omega_j^2 - k_j^2 + \tfrac{1}{2}) + (\omega_j + k_j)\,(4n_j\,\omega_j^2 + 4k_j\,\omega_j - k_j^2)$$

$$-k_j\,\omega_j\,(n_j+1)\,(\omega_j - k_j)\} + (\omega_j - k_j)\,\{\omega_j\,(\omega_j+k_j)^2 + 2k_j(\omega_j^2 - k_j^2)$$

$$+ 4n_j\,\omega_j\,k_j^2\}\,]\,, \tag{36a}$$

$$F_{2j}\,(\omega_j,\, k_j,\, y) = 2\,[y^2\,(\omega_j - k_j)\,\{(\omega_j - k_j)^3 - \omega_j\,k_j\,(\omega_j - 5k_j)\}\, U -$$

$$y\{k_j^2\,(\omega_j^2 + k_j^2) - 2\omega_j\,k_j\} - 2\,(\omega_j^2 - k_j^2)^2\,(\omega_j^2 - 1) - \tfrac{1}{4}\,(\omega_j + k_j)$$

$$\{(\omega_j - k_j)\,(6\omega_j + 5k^j\,\omega_j) + 2k_j\,(\omega_j^2 - k_j^2)\,\} + 2n_j\,\{\,2\,(\omega_j^2 - k_j^2)^2$$

$$+ k_j^3\}\,]\,, \tag{36b}$$

$$F_{3j}\,(\omega_j, k_j,\, y) = y\omega_j\,\{\,2\,(\omega_j^2 - k_j^2) + k_j\,(3\omega_j^2 + k_j^2\} - 2\omega_j\,k_j^2\,. \tag{36c}$$

The secularity removal condition (31) essentially means that to the $O(\epsilon)$, the amplitude $A_j^{(0)}$ is constant in a frame of reference moving with the group velocity V_g. In other words, $A_j^{(0)}$ depends on X_1 and T_1 only through $\xi = X_1 - V_g\,T_1$.

The y–structure of the $O(\epsilon)$ perturbation in v is governed by (34) which is an inhomogeneous, second-order ordinary differential equation. Using power series and green's function methods (Margenau and Murphy 1966) the solution of (34) can be formally written as

$$V_{nj}^{(1)} = \frac{1}{k}\int_{-L_y}^{L_y}\, G\,(\delta, y)\, F\,(\omega_j,\, k^j,\, \delta)\, d\delta\,, \tag{37}$$

satisfying the boundary conditions. δ is a point which divides the range of y from $-L_y$ to L_y into two parts (i) $-L_y < y < \delta$ and (ii) $\delta < y < L_y$. $G(\delta, y)$ is the Green's function for the homogeneous differential equation (34), with boundary conditions same as those for $V_{nj}^{(1)}$. The integrals in (37) can be evaluated analytically or numerically so that $V_{nj}^{(1)}$ is known. The structure of $V_{nj}^{(1)}$ for a special case corresponding to $n_j = 1$ is shown in figure 3. Once $V_{nj}^{(1)}$ is known $v^{(1)}$ described by (33) is determined.
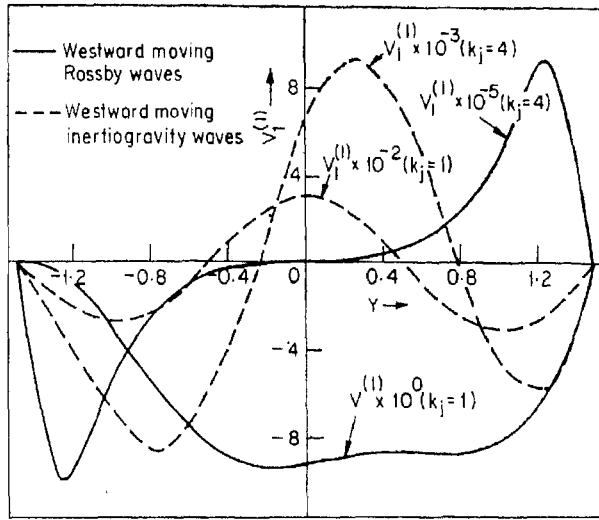
Figure 3. $V_{nj}^{(1)}$ as functions of $Y\,(=f_0+y)$ for the meridional mode $nj=1$.

Now substituting $v^{(1)}$ and $u^{(1)}$ represented by (29) into (26) which governs $u'^{(1)}$, subsequently removing the secular terms (*i.e.*, equating the coefficients of $\exp\,(\pm\,i\theta_j)$ to zero) which will determine $A_{1j}^{(1)}$, $W_{1j}^{(1)}$ (and its c. c. equivalent) and equating the coefficients of $\exp|(\pm 2i\theta_j)$ which will determine $B_{2j}^{(1)}$ $W_{2j}^{(1)}$ (and its c.c. equivalent), we get the secular-free solution $u^{(1)}$ in terms of $V_{nj}^{(1)}$ as

$$u^{(1)} = \sum_j [\, y\,(\omega_j - k_j)\,\psi_{nj}^{(1)} + k_j\,\psi_{nj+1}^{(1)} - i\,(\omega_j^2 - k_j^2)^{-1}\,\{\,[\,y\,(\omega_j^2 - k_j^2)\,\psi_{nj}$$

$$- 2\omega_j\,k_j\,\psi_{nj+1}\,]\;A_{jT1}^{(0)}/A_j^{(0)} - [\,y\,(\omega_j^2 - k_j^2)\,\psi_{nj} + (\omega_j^2 - k_j^2)\,\psi_{nj+1}\,] \times$$

$$A_{jx1}^{(0)}\,/A_j^{(o)}\,\} + \tfrac{1}{4}\,(\omega_j^2 - k_j^2)^{-1}\,\{\,y\,\psi_{nj}^{(1)} + (k_j/\omega_j)\,\psi_{nj}^{(1)}\,V_{nj}^{(1)\prime}/\,V_{nj}^{(1)}\,\}$$

$$- \tfrac{1}{2}\,\{\,[\,y^2\,(\,(\omega_j^2 - k_j^2)\omega_j + (\omega_j^2 - k_j^2)\,]\;\psi_{nj}^2 - y\,\omega_j^2\,\psi_{nj}\,\psi_{nj+1}\,\}$$

$$+ c.\,c + c_{2j}\,]\;. \tag{38}$$

Similarly, the solution of (27) which governs $\Phi^{(1)}$ can be obtained as

$$\Phi^{(1)} = \sum_j [-y\,(\omega_j - k_j)\,\psi_{nj}^{(1)} + \omega_j\,\psi_{nj+1}^{(1)} + i\,\{\,y\,(1 - 2k_j/\omega_j)\psi_{nj}$$

$$- (\omega_j^2 - 3k_j^2)\,(\omega_j^2 - k_j^2)^{-1}\,(\psi_{nj+1})\,(A_{jT1}^{(0)}/A_j^{(0)}) - (y(1 - 2k_j/\omega_j)\psi_{nj}$$

$$- 2k_j^3\,\omega_j^{-1}\,(\omega_j^2 - k_j^2)^{-1}\,\psi_{nj+1}\,)\,(A_{jx1}^{(0)}/A_j^{(0)})\,\} + \tfrac{1}{4}\,(\omega_j^2 - k_j^2)^{-1}$$

$$(y\,k_j/\omega_j + V_{nj}^{(1)\prime}/V_{nj}^{(1)})\,\psi_{nj}^{(1)} + \tfrac{1}{2}\,\{\,(y^2\,\omega_j\,(\omega_j - k_j) - (\omega_j^2 - k_j^2)$$

$$(1 + 2k_j/\omega_j + 2n_j))\,\psi_{nj}^2 + y\,k_j\,(\omega_j + 2)\,\psi_{nj}\,\psi_{nj+1} + k_j^2\,\psi_{nj+1}^2$$

$$+ c.\,c. + C_{3j}\,]\,, \tag{39}$$

where primes in (38) and (39) denote derivative with respect to y.   The first two terms on the r. h. s. of (38) and (39) are equivalent to the first two terms in the expressions for $u^{(0)}$ and $\Phi^{(0)}$ ((20) and (21) respectively) when the recurrence relations for the Hermite polynomial are made use of.   We observe here that for knowing $v^{(1)}$, $u^{(1)}$ and $\Phi^{(1)}$ in terms of known quantities, we have to still determine the complex constant $A_j^{(1)}$ and real constants C_{ij}, $i = 1, 2, 3, \ldots$.   We shall show in the next section that these constants can be determined from the secularity removal conditions for the equations corresponding to higher order in ϵ.   [Following Bogoliubov and Mitropolsky (1961), we may set $A_j^{(1)}$ and $A_j^{(1)*}$ equal to zero.   However, as will be seen later, it is not necessary to do so within the order of approximations considered here].

2.4 $O(\epsilon^2)$ problem: Derivation of non-linear Schrödinger equation

The $O(\epsilon^2)$ solutions $v^{(2)}$, $u^{(2)}$ and $\Phi^{(2)}$ can be found out from the $O(\epsilon^2)$ perturbation equations which are obtained by equating the coefficient of ϵ^2 equal to zero and are given by

$$Lv^{(2)} = -L_1 v^{(1)} - L_2 v^{(0)} - P \left(u^{(0)} v_{x0}^{(1)} + u^{(1)} v_{x0}^{(0)} + u^{(0)} v_{xj}^{(0)} + v^{(0)} v_y^{(1)} + v^{(1)} v_y^{(0)} \right)$$

$$- M \left(u^{(0)} u_{x0}^{(1)} + u^{(1)} u_{x0}^{(0)} u^{(0)} + u_{x1}^{(0)} + v^{(0)} u_y^{(1)} + v^{(1)} u_y^{(0)} \right) + N \left(u^{(0)} \phi_{x0}^{(1)} + u^{(1)} \phi_{x0}^{(0)} \right)$$

$$+ u^{(0)} \phi_{x1}^{(0)} + v^{(0)} \phi_y^{(1)} + v^{(1)} \phi_y^{(0)}) - P_1 \left(u^{(0)} v_{x0}^{(1)} + v^{(0)} v_y^{(0)} \right) - M_1 \left(u^{(0)} u_{x0}^{(0)} \right.$$

$$+ v^{(0)} u_y^{(0)}) + N_1 \left(u^{(0)} \phi_{x0}^{(0)} + v^{(0)} \phi_y^{(0)} \right), \tag{40}$$

$$Pu^{(2)} - Qv^{(2)} = - (P_1 u^{(1)} - Q_1 v^{(1)}) - (P_2 u^{(0)} - Q_2 v^{(0)}) - (u^{(0)} u_{x0}^{(1)} + u^{(1)} u_{x0}^{(0)}$$

$$+ u_{x1}^{(0)} u^{(0)} + v^{(0)} u_y^{(1)} + v^{(1)} u_y^{(0)})_{T0} - (u^{(0)} \Phi_{x0}^{(1)} + u^{(1)} \Phi_{x1}^{(0)} + u^{(0)} \Phi_{x1}^{(0)}$$

$$+ v^{(0)} \Phi_y^{(1)} + v^{(1)} \Phi_y^{(0)})_{x0} - (u^{(0)} u_{x0}^{(0)} + v^{(0)} u_y^{(0)})_{T1} - (u^{(0)} u_{x0}^{(0)} + v^{(0)} u_y^{(0)})_{T1}$$

$$- (u^{(0)} \phi_{x0}^{(0)} + v^{(0)} \phi_y^{(0)})_{x1} \qquad , \tag{41}$$

$$\phi_{T0}^{(2)} + u_{x0}^{(2)} + v_y^{(2)} = - \phi_{T1}^{(1)} - u_{x1}^{(1)} - \Phi_{T2}^{(0)} - u_{x2}^{(0)} - u^{(0)} \phi_{x0}^{(1)} - u^{(1)} \phi_{x0}^{(0)}$$

$$- u^{(0)} \Phi_{x1}^{(0)} - {}^{(0)} \Phi_y^{(1)} - v^{(1)} \Phi_y^{(0)}, \tag{42}$$

where $L_2 \equiv 3\partial_{T0T0T2} - \partial_{x0x0T2} - \partial_{yyT2} + y^2 \partial_{T2} + 3\partial_{T0T1T1} - 2\partial_{x0x1T1}$

$- \partial_{x2} - 2\partial_{x0T0x2} - \partial_{T0x1x1}$;

$M_1 \equiv \partial_{yx1} - y\partial_{T1}$; $N_1 \equiv \partial_{yT1} - y\partial_{x1}$; $Q_2 \equiv y\partial_{T2} + \partial_{yx2}$;

$P_2 \equiv 2 (\partial_{T0T2} - \partial_{x0x2}) + \partial_{T1T1} - \partial_{x1x1}$.

As is obvious from the above equations, the solutions $v^{(2)}$, $u^{(2)}$ and $\phi^{(2)}$ will now contain additional linear resonant forcings arising from the introduction of higher space and time scales $(X_2 T_2)$, giving rise to higher harmonics in the fundamental wave. The nonlinear resonant forcings due to the inhomogeneous terms containing the nonlinear products of $O(1)$ and $O(\epsilon)$ quantities, will also produce further higher harmonics produced by the nonlinear terms involving the products of $O(1)$ quantities. Considering the fact that these forcings might give rise to the second and third harmonics in the fundamental wave, we assume the solutions $v^{(2)}$, $u^{(2)}$ and $\phi^{(2)}$ of the following forms (as was done in the case of $v^{(1)}$, $u^{(1)}$ and $\phi^{(1)}$) :

$$v^{(2)} = \sum_j \left[-i\left(\omega_j^2 - k_j^2 \right) \psi_{nj}^{(2)} + B_{1j}^{(2)} \exp(2i\theta_j) \, V_{1j}^{(2)} + C_{1j}^{(2)} \exp(3i\theta_j) \, V_{2j}^{(2)} + c.c + d_{1j} \right] \tag{43}$$

$$u^{(2)} \quad \sum_j \left[A_{1j}^{(2)} \exp(i\theta_j) \, W_{1j}^{(2)} + B_{2j}^{(2)} \exp(2i\theta_j) \, W_{2j}^{(2)} + C_{2j}^{(2)} \exp(3i\theta_j) \, W_{3j}^{(2)} + c.c + d_{2j} \right], \tag{44}$$

$$\phi^{(2)} = \sum_j \left[A_{2j}^{(2)} \exp(i\theta_j) \, Z_{1j}^{(2)} + B_{3j}^{(2)} \exp(2i\theta_j) \, Z_{2j}^{(2)} + C_{3j}^{(2)} \exp(3i\theta_j) \, Z_{3j}^{(2)} + c.c + d_{3j} \right], \tag{45}$$

where $\psi_{nj}^{(2)} = A_j^{(2)} \exp(i\theta_j - y^2/2) H_{nj}(y)$; $A_j^{(2)}$, $A_{(1,2)j}^{(2)}$, $B_{(1,2,3)j}^{(2)}$, $C_{(1,2,3)j}^{(2)}$ are functions of $A_j^{(0)}$ and $A_j^{(0)*}$ and are thus complex constants with respect to X_0, T_0, and y. d_{ij}, $i = 1, 2, 3$ are real constants with respect to X_0, T_0, y and functions of $A_j^{(0)}$ and $A_j^{(0)*}$. These constants are to be determined from the non-secular conditions in higher order in ϵ. The functions $V_{(1,2)j}^{(2)}$, $W_{(1,2,3)j}^{(2)}$ and $Z_{(1,2,3)j}^{(2)}$ are functions of y.

The solutions $v^{(2)}$, $u^{(2)}$ and $\Phi^{(2)}$ in terms of known quantities can be found out in the same manner as the solutions $v^{(1)}$, $u^{(1)}$ and $\Phi^{(1)}$ were obtained. In order to have the $O(\epsilon^2)$ perturbations completely known, we must determine all the constants and functions just defined above. We are, however, not proposing to do so in this paper. We shall only assume that these $O(\epsilon^2)$ solutions are well-behaved. This means that all the secular terms appearing in the equations governing these solutions, should vanish. The equations governing these solutions will now contain two types of secular terms, (a) secular constant terms and (b) secular resonant terms proportional to $\exp(\pm i\theta_j)$. Therefore, we will now have two secular-free conditions in order that the $O(\epsilon^2)$ solutions are bounded (well-behaved), one for constant terms and another for resonant terms.

Substitutions of (43), (33), (38), (19), (20) and (21) into (40) which governs the evolution of $v^{(2)}$, yields

$$2i\omega_j \left(4\omega_j^2 - 4k_j^2 - k_j/\omega_j - y^2\partial_{yy}\right) B_{1j}^{(2)} \, V_{1j}^{(2)} \exp(i2\theta_j) + 3i\omega_j \left(9\omega_j^2 - 9k_j^2\right.$$

$$\left. - k_j/\omega_j - y^2 + \partial_{yy}\right) C_{1j}^{(2)} \, V_{2j}^{(2)} \exp(3i\theta_j) + c.c$$

$$= (y^2 V_g + 1) \, \partial C_{1j}/\partial_{x1} - (\omega_j^2 - k_j^2) \, (D\omega_j H_{nj}) \, \{\exp(-y^2/2)\} \, [i(A_{jT2}^{(0)} + V_g A_{jx2}^{(0)})$$

$$+ i \, (A_{jT1}^{(1)} + V_g A_{jx1}^{(1)}) - \omega_j D_{\omega j}^{-1} \, (3A_{jT1T1}^{(0)} - A_{jx1x1}^{(0)} + 2k_j\omega_j^{-1} A_{jx1T1}^{(0)})$$

$$+f_j\, A_j^{(0)2}\, A_j^{(0)*}(\omega_j^2 - k_j^2)^{-1}D^{-1}\omega_j + (ih'_{1j}c_{1j} - h'_{2j}c_{2j})\, (\omega_j^2 - k_j^2)^{-1}D^{-1}\omega_j A_j^{(0)}]\, \exp(i\theta_j)$$

$$+ \text{terms proportional to } \exp(2i\theta_j) \text{ and } \exp(3i\theta_j) + c.c \tag{46}$$

where $D\omega_j = \partial D/\partial\omega_j = 3\,\omega_j^2 - k_j^2 - 2n_j - 1$ and V_g is the group velocity defined by (32) and

$$f_j = (f_{1j} - f_{2j} - f_{3j})\, H_{nj}^{-1}\, \exp\,(+y^2/2)\ ;\quad h'_{1j} = h_{1j}\, H_{nj}^{-1}\, \exp\,(y^2/2);\quad h'_{2j} = h_{2j}\, H_{nj}^{-1}$$

$$\exp\,(y^2/2) \tag{47}$$

with $f_{(1,2,3)j}$ and $h_{(1,2)j}$ being functions of ω_j, k_j, y, which are given in the appendix I. Equating the coefficients of $\exp\,(\pm 2i\theta_j)$ and $\exp\,(\pm 3i\theta_j)$ in both sides of (46), we will get the expressions for $B_{1j}^{(2)}$, $V_{1j}^{(2)}$ and $C_{1j}^{(2)}$, $V_{2j}^{(2)}$ while the constant $[d_{1j}]$ can be determined from the higher order equations in ϵ. This will determine the solution $v^{(2)}$ completely. But, as discussed earlier, we are interested only in knowing under what conditions the solution $v^{(2)}$ remains well behaved (bounded) and this can be done by removing the secular constant and resonant terms appearing in (46). Therefore, we have not presented here the terms proportional to $\exp\,(\pm 2i\theta_j)$ and $\exp\,(\pm 3i\theta_j)$ in (46).

The condition that the constant terms appearing in (46) be non-secular so as to get secular-free solution $v_{(2)}$, requires that

$$(y^2 V_g + 1)\frac{\partial c_{1j}}{\partial x_1} = 0. \tag{48}$$

As $(y^2 V_g + 1) = 0$ gives dependence of V_g on y which is not permitted, we have $y^2 V_g + 1 \neq 0$. Therefore we must have

$$\frac{\partial c_{1j}}{\partial x_1} = 0, \text{ or } \left(\frac{1}{V_g}\ \frac{\partial c_{1j}}{\partial T_1}\right) = 0,$$

[by virtue of (31)] $\tag{49}$

which means that C_{1j} are constant with respect to X_1 and T_1. However, C_{1j} could be variable in higher order time and space scales. Nevertheless, as far as time scale T_1 and space scale x_1 are concerned, C_{1j} is a constant which means that to this order the meridional velocity has a constant ;component. Physically, this is unrealistic because it will mean constant piling up of momentum and vorticity at high latitudes. Therefore, from physical consideration we shall put $C_{1j} = 0$

Similarly, the condition that the resonant terms proportional to $\exp\,(\pm i\theta_j)$ should be non-secular, requires that

$$i(A_{jT2}^{(0)} + V_g A_{jx2}^{(0)}) + \tfrac{1}{2}\frac{dv_g}{dk_j}\, A_{jx1x1}^{(0)}$$

$$= -\,(\omega_j^2 - k_j^2)^{-1}\, D^{-1}\omega_j\, [A_j^{(0)2}\, A_j^{(0)*}f_j + (ih'_{1j}\, c_{1j} - h'_{2j}\, c_{2j})\, A_j^{(0)}], \tag{50}$$

together with its complex conjugate relation. In deriving (50) we have made use of the following relations

$$A_{jT1}^{(1)} + V_g A_{jX1}^{(1)} = \frac{\partial A_j^{(1)}}{\partial A_j^{(0)}} (A_{jT1}^{(0)} + Vg A_{jX1}^{(0)}) + \frac{\partial A_j^{(1)}}{\partial A_j^{(0)*}} (A_{jT1}^{(0)*} + V_g A_{jX1}^{(0)*}) = 0, \quad (51)$$

and

$$-\frac{\omega_j}{D\omega_j} \left(3A_{jT1T1}^{(0)} - A_{jX1X1}^{(0)} - 2\frac{k_1}{\omega_j} A_{jX1T1}^{(0)} \right) = \frac{1}{2} \frac{dV_g}{dk_j} A_{jX1X1}^{(0)}, \quad (52)$$

where

$$\frac{dV_g}{dk_j} = \frac{\partial V_g}{\partial k_j} + V_g \frac{\partial V_g}{\partial \omega_j} = [2\omega_j (1 + 3V_g^2) + 4k_j V_g] / D\omega_j, \quad (53)$$

and the fact that $A_{jT1}^{(0)} + V_g A_{jX1}^{(0)} = 0$. Equation (43) and its complex conjugate relation have been made use of.

Note that the equation (50) does not contain the arbitrary constants (with respect to X_0, T_0, Y) $A_j^{(1)}$ and $A_j^{(1)*}$ so that we need not determine them so far as (50) for the time evolution of the wave amplitude $A_j^{(0)}$ is concerned. It does contain as yet undetermined constants C_{2j}. It can be shown that the substitution of (43)–(45) into (41) and (42) and then imposition of the condition that the constant terms be secular-free, still leave the constants C_{2j} undetermined. However, these constants can be determined from the next higher order equations in ϵ, *i. e.* $O(\epsilon^2)$ set of equations.

Proceeding exactly in a manner described in this section, it can be shown that by having the requirement that the constant terms appearing in the set of $O(\epsilon^3)$ perturbation equations be secular free, one gets the following non-secular conditions for constant terms :

$$- (y^2 V_g + 1) \frac{\partial d^1_j}{\partial x_1} = [2(V_i^2 - 1)\alpha_4 + V_g \alpha_2 + \alpha_3] \frac{\partial^2}{\partial x_1^2} (A_j^{(0)} A_j^{(0)*}) \quad (54)$$

and

$$y^2 V_g \frac{\partial d_{1j}}{\partial x_1} + y(V_g^2 - 1) \frac{\partial^2 c_{2j}}{\partial x_1^2} = (V_g \alpha_2 - \alpha_3) \frac{\partial^2}{\partial x_1^2} (A_j^{(0)} A_j^{(0)*}). \quad (55)$$

The functions α_2, α_3 and α_4 appearing in (54) and (55) are functions of ω_j, k_j and y and are defined in appendix 2. Eliminating d_{1j} between (54) and (55), we get

$$(V_g^2 - 1) \frac{\partial^2 C_{2j}}{\partial x_1^2} = g \frac{\partial^2}{\partial x_1^2} (|A^{(0)}|^2), \quad (56)$$

where

$$g = [V_g \alpha_2 - \alpha_3 - 2y^2 V_g (\alpha_3 + \alpha_4 V_g^2 - \alpha_4)]/[y (y^2 V_g + 1)],$$

Integrating (56) twice with respect to X_1 we get

$$C_{2j} = g |A^{(0)}|^2/(V_g^2 - 1) + \delta_j (X_2, T_2, ...) X_1 + \beta_j (X_2, T_2, ...), \quad (57)$$

where δ_j and β_j are two constants of integration and functions of X_2, T_2, ... Using the relation (31) and its complex conjugate in (56) the second term on r.h.s. of (57) will be replaced by $\delta_j T_1$. As these terms give rise to the secularity in C_{2j} with respect to X_1 (or T_1) we assume that for secular-free

C_{2j}, $\delta_j = 0$. The function β_j has to be evaluated on physical grounds from the appropriate initial or boundary conditions. For example, if we assume that $C_{2j} \to 0$ at $\xi = X_1 - V_{\text{g}} T_1 \to \pm \infty$ which means that there is no steady part at infinity, we get

$$\beta_j = -g \mid A_{j\infty} \mid^2/(V_{\text{g}}^2 - 1) \qquad \text{where} \quad A_{j\infty} = A_j^{(0)}(\xi = \pm\infty), \tag{58}$$

we note that this solution is valid only for $V_{\text{g}}^2 \neq 1$. This may always be the case for dispersive waves under consideration for which the group velocity is different from the phase velocity.

Substituting for C_{2j} in (50), we get a nonlinear equation for the evolution of complex amplitude $A_j^{(0)}$, that is,

$$i\left(\frac{\partial A_j^{(0)}}{\partial T_2} + V_{\text{g}}\frac{\partial A_j^{(0)}}{\partial X^2}\right) + M_j \frac{\partial^2 A_j^{(0)}}{\partial X_1^2} = N_j \mid A_j^{(0)} \mid^2 A_j^{(0)} + R_j A_j^{(0)}, \tag{59}$$

together with its complex conjugate relation, where $A_j^{(0)}$ now depends upon ξ, $X_2, T_2, \ldots$, while R_j (through B_j) depends upon X_2, T_2, X_3, T_3, $\ldots$ and is constant with respect to ξ ($= X_1 - V_g T_1$) and

$$M_j = \frac{1}{2}\frac{dV_{\text{g}}}{dk_j} ; \quad N_j = -[f_j - h'_{2j}\nu \ (V_{\text{g}}^2 - 1)^{-1}] \ (\omega_j^{2'} - k_j^2)^{-1} \ D^{-1}\omega_j ;$$

$$R_j = \beta_j \, h'_{2j} \ (\omega_j^2 - k_j^2)^{-1} \ D\omega_j^{-1} . \tag{60}$$

The second term in (59) is the dispersion term and the third term is the non linear term. This equation shows that, in a frame of reference moving with the group velocity of the waves, the modulation amplitude $A_j^{(0)}$ is determined by the nonlinear interaction (N_j term), the dispersive term (M_j term) and the linear interaction term R_j.

Now, introducing a coordinate transformation such as

$$\tau = T_2 ; \quad \zeta - X_2 - V_g T_2, \tag{61}$$

(59) can be written as

$$i\frac{\partial A_j^{(0)}}{\partial \tau} + M_j \frac{\partial^2 A_j^{(0)}}{\partial \zeta^2} = N_j \mid A_j^{(0)} \mid^2 A_j^{(0)} + R_j A^{(0)}, \tag{62}$$

where $A_j^{(0)}$ is a function of ξ, ζ, τ, X_3, T_3 and R_j is that of τ, ζ, X_3, T_3. This equation, in a frame of reference moving with the group velocity of the waves, is a nonlinear equation which governs the evolution in slow time and space scales T_2 and X_2, of the waves propagating in a barotropic atmosphere with a zero basic state. When we consider the variation of $A_j^{(0)}$ with respect to τ and ζ (whose order can be regarded as $O(1)$ with respect to (62), then X_3, T_3 become of higher orders and in this sense they can be considered as slow parameters. Therefore, upto second order variation, $A_j^{(0)}$ can be regarded as a complex function of τ and ζ, while R_j as a real function of β_j. If boundary or initial conditions

are such that   (and hence R_j) is constant with respect to τ, (62) is called the "modified nonlinear Schrödinger equation", because of the linear terms appearing on the right hand side of (62) as a modification to the usual nonlinear Schrödinger equation. Whether R_j is a constant or varies with τ, the linear interaction $R_j A_j$ term in (62) is not very important because it causes only a phase shift in the solution A_j.   Making a transformation such that

$$A_j^{(0)} = \widetilde{A}_j^{(0)} (\zeta, \tau) \exp \left[- i \int^{\tau} R (\tau') \, d\tau' \right],$$ (63)

the $R_j A_j$ (0) term can be formally eliminated from (62). With the transformation (63), (62) can be rewritten as

$$i \frac{\partial \widetilde{A}_j^{(0)}}{\partial \tau} + M_j \frac{\partial^2 \widetilde{A}_j^{(0)}}{\partial \xi^2} = N_j \mid \widetilde{A}_j^{(0)} \mid^2 \widetilde{A}_j^{(0)}.$$ (64)

It should be noted that the modified nonlinear Schrödinger equation (64) describing amplitude modulation of the wave trains represented by (19), (20) and (21), has been derived from the non-secular conditions (boundness condition for the wave solutions) for $O(\epsilon^2)$ perturbation equations and therefore, is valid for scales as large as $O(\epsilon^{-2})$.

3. Solutions of modified nonlinear schrödinger equation (64)

If now, we look for a plane wave solution to (64) for the amplitude $\widetilde{A}_j$ (0), of the form

$$\widetilde{A}_j^{(0)} \rightarrow \widetilde{A}_j^{(0)} \exp \left[i (\widetilde{K}_j \, \xi - \widetilde{\Omega}_j \tau \right],$$ (65)

we get following nonlinear dispersion relation

$$\widetilde{\Omega}_j = M_j \, \widetilde{K}_j^2 + N_j \mid \widetilde{A}_j^{(0)} \mid^2 \quad,$$ (66)

where $\widetilde{K}_j$ and $\widetilde{\Omega}_j$ are respectively the wave number and frequency of the envelope of the wave trains represented by (19), (20) and (21).

3.1 Stationary solitary wave solution

Let us look for a localized solution of (64) for $\widetilde{A}_j^{(0)}$ satisfying the boundary conditions ; (a) $\mid \widetilde{A}_j^{(0)} \mid^2$ is bounded between two limits A_{max} and A_{min}. (b) at $\mid \widetilde{A}_j^{(0)} \mid^2 = A_{max}$, an extremum *i.e.* $\delta \mid \widetilde{A}_j^{(0)} \mid^2 / \delta \xi = 0$ but $\delta^2 \mid \widetilde{A}_j^{(0)} \mid^2 / \partial \xi^2 \neq 0$ and (c) A_{min} is the asymptotic value of $\mid \widetilde{A}_j^{(0)} \mid^2$ at $\xi \rightarrow \pm \infty$.

introducing two real variables $\overline{A}_j$ and θ which represent the real and imaginary parts of $\widetilde{A}_j^{(0)}$ we can write $\widetilde{A}_j^{(0)}$ as

$$\widetilde{A}_j^{(0)} (\xi, \tau) = \sqrt{A_j (\xi, \tau)} \, \exp \{ i\theta_j (\xi, \zeta) / 2M^j \}$$ (67)

Substituting (67) into (64) and separating real and imaginary parts, we get after simplification.

$$\frac{\partial A_j}{\partial \tau} + \frac{\delta}{\delta \xi}\left(\overline{A}\ \frac{\partial}{\partial \xi}\right) = 0 \quad , \tag{68}$$

$$\frac{\partial \Omega}{\partial \tau} + \tfrac{1}{2}\left(\frac{\partial \theta}{\partial \xi}\right)^2 = -2M_j N_j \overline{A}_j + \tfrac{1}{2} M_j^2 \frac{\partial}{\partial A_{j'}}\left\{\overline{A}_j^{-1}\left(\frac{\partial \overline{A}_j}{\partial \zeta}\right)^2\right\} \tag{69}$$

It can be seen from (68) and (69) that $M_j = 0$, the envelope will steepen in the course of time evolution and finally break down. On the other hand, if $M_j \neq 0$, the dispersive term

$$\left[\tfrac{1}{2} M_j^2 \frac{\partial}{\partial \overline{A}}\left\{\overline{A}_j^{-1}\left(\frac{\partial A_j}{\partial \xi}\right)^2\right\}\right]$$

begins to play a role in checking the steepening of the envelope. Therefore, one may expect an equilibrium stationary state resulting from the balance between nonlinear steepening and the dispersive effects. Following Hasegawa (1975), the stationary solution (*i.e.* when $\partial/\partial\tau \to 0$) to the modified nonlinear Schrödinger equation (64) for the case when $M_j N_j < 0$ and subject to the boundary conditions (a), (b), (c), mentioned above, can be written as

$$A_j^{(0)}(\xi, \tau) = \triangle_j \operatorname{sech}(\xi / W_j)\exp\{ i\ (\overline{\Omega}_j / 2M_j)\ \tau\} \tag{70}$$

where

$$\triangle_j = A_{\max}^{*1/2} = \sqrt{-\overline{\Omega}_j / M_j N_j}, \quad W_j = (-2M / N_j)^{1/2}/\triangle_j \text{ and } \Omega_j > 0$$

and is a constant. This solution has structure similar to the "soliton soluton" for weakly nonlinear waves in a dispersive medium and hence, is called "envelope soliton" which moves at a speed equal to V_g *i.e.* group velocity of the waves. It is seen that the amplitude of envelope soliton $\triangle_j$ (also called the height) is inversely proportional to its width W_j. As is obvious, the constant parameter $\overline{\Omega}_j$ in (70) is a phase factor and will thus determine the phase of envelope soliton. When $M_j N_j > 0$, the solution of (64) has a "envelope hole" structure and has not been discussed here for its lack of relevance in geophysical fluid dynamics.

4. Results and discussions

The various asymptotic behaviours of the atmospheric waves have been discussed, using multiple time and space scales and the derivative expansion method. This method has, as pointed out by Kawahara (1973), an advantage over the other methods in the sense that the dependence of the parameters on the independent variables need not be specified *a priori* and also that the perturbation analysis to the different orders could be carried out systematically. Based upon the divergent barotropic model on a β–plane, we have investigated the long time evolution of finite amplitude (*i.e.* weakly nonlinear) and slowly varying wave trains propagating in the atmosphere with a zero basic zonal flow. The present analysis is basically an extension of the linear stability analysis of the atmospheric wave motions in the equatorial region (Matsuno 1966) into the nonlinear regimes for these waves.

In the lowest order approximations of our model, *i.e.* to the $O(1)$, the results of the linear theory of Matsuno (1966) are retrieved. The propagation characteristics of the linearized solutions for these atmospheric waves are such that there exist three distinct classes of planetary scale waves for the meridional modes $n_j \geqslant 1$. The first two types are the eastward and westward propagating inertial-gravity waves while the third is the westward propagating Rossby waves. They are distinguished from each other by the fact that the frequencies of the inertial-gravity waves are much larger than that of Rossby waves (figure 2). For the particular lowest meridional mode $n_j = 0$, the existence of only two types of waves is possible; one is eastward propagating inertial-gravity waves and the other describes the westward propagating mixed Rossby-gravity or 'Yanai waves' (figure 2). The dispersion relation for the latter type resembles that of the Rossby waves for the wavelength much smaller than the meridional extent. For the case where the wavelength is longer than the meridional extent, however, the frequency approaches that of the gravity waves of the same wavelength. In other words, the mixed Rossby-gravity wave connects two families of waves and its frequency ranges from a value which is comparable to the frequency of the inertial-gravity waves to a value which is close to that of the Rossby waves.

Using multiple time scale method to solve the weakly nonlinear equations, it is shown that the evolution of the wave amplitude on slowly varying time and space scales is governed by the modified nonlinear Schrödinger equation. The stationary solutions to the modified nonlinear Schrödinger equation are obtained and found to have the structure of 'envelope solitary waves (envelope solitons or wave packets)' moving at a speed equal to the group velocity of waves, which results from the balance between the nonlinear steepening and the dispersive effects. We consider this result as an indication of the possible existence of the envelope solitary waves in the atmosphere. Though the dispersive waves in optics and plasmas are known to have envelope soliton solutions, this is, to our knowledge a very good theoretical evidence of the existence of 'envelope solitons' in a geophysical situation.

The methodology presented in §2 for deriving (59) [or (62)] governing the the evolution of the amplitude is quite general in the sense that this equation governs the amplitude evolution for all the waves described by the Shallow Water Equations. However, the envelope soliton solution presented in § 3 is valid only for the dispersive waves. For example, Rossby waves and Rossby-gravity waves will have such nonlinear solution. On the other hand the Kelvin waves will not have soliton solution. We note that for the Kelvin waves $\omega_j = -k_j$ and $n_j = -1$. Therefore, the group velocity V_g for the Kelvin waves given by (32) is a constant. Thus M_j in (62) is given by $M_j = \frac{1}{2}\frac{dV_g}{dk_j} = 0$ [see (60)]. In the absence of the dispersive term containing M_j, (62) represents steepening of the amplitude of the Kelvin waves in the nonlinear regime.

We note here that the modified nonlinear Schrodinger equation describing the dependence of the wave amplitude on slower time and space scales (T_2 and X_2), has been derived from the non-secular conditions (boundedness conditions) for the wave solutions) for $O(\epsilon^2)$ perturbation equations and therefore, is valid for scales as large as $O(\epsilon^{-2})$. Although the dependence of the wave amplitude on

$O(\epsilon^3)$ and higher order scales should be determined up to this order of perturbations we have however not considered in the model the variation in the change of the wave amplitude upto $O(\epsilon^3)$ and higher order scales which may be considered in a sense as a kind of slow parameters since $\epsilon \ll 1$. In other words, the evolution of the wave amplitude in the present analysis is accurate up to $O(\epsilon^2)$ scales and will have an error of $O(\epsilon^3)$.

5. Conclusions

In conclusion, our model predicts that the long time evolution of finite-amplitude waves generated in a barotropic single layer model atmosphere on a β-plane can be described by a modified nonlinear Schrödinger equation. The model further predicts that these finite–amplitude waves finally evolve in time in the shape of envelope solitary waves or envelope solitons (*i. e.* in the shape of wave packets) and this provides us theoretical evidence for the existence of 'envelope solitons' in a barotropic atmosphere, in particular and in any geophysical situation, in general. These envelope solitons propagate with the group velocity of the wave.

Acknowledgements

Appendix 1. Expressions for $h_{(1,2)\,j}$ and f_{ij}, $i=1, 2, 3$ appearing in (47).

$$h_{1j} = (\omega_j^2 - k_j^2)^2 \, (yH_{nj} - H_{nj+1}) \exp - (y^2/2) + (y\omega_j + k_j \, \partial/\partial y)$$

$$[\exp (-y^2/2) \{(y^2(\omega_j - k_j) + \omega_j + (2n_j + 1)k_j) H_{nj} - y\omega_j H_{nj+1}\}]$$

$$-(yk_j + \omega_j \, \partial/\partial y \, ([\exp (-y^2/2) \{(y^2(\omega_j - k_j) - k_j - \omega_j (2n_j + 1))H_{nj}$$

$$+yk_j H_{nj+1}\}], \tag{A1a}$$

$$h_{2j} = k_j \, (\omega_j^2 - k_j^2)^2 \, H_{nj} \exp (-y^2/2) + k_j \, (y\omega_j + k_j \, \partial/\partial y) [\exp (-y^2/2)$$

$$\{y(\omega_j - k_j) H_{nj} + k^j H_{nj+1}\}] - k_j \, (yk_j + \omega \, \partial/\partial y) [\exp(-y^2/2) \{y(\omega_j - k_j)$$

$$H_{nj} - \omega_j H_{nj+1}\}], \tag{A1b}$$

$$f_{1j} = \tfrac{1}{2} \, (\omega_j^2 - k_j^2) \, [\{y(\tfrac{3}{2}k_j + \omega_j^{-1} \, (\omega_j^2 - 3k_j^2)) \, V_{nj}^{(1)} + (2\omega_j^2 - 3k_j^2) \cdot V_{nj}^{(1)\prime}/2\omega_j\}$$

$$H_{nj} \exp (-y^2/2) - (\omega_j - 3k_j^2/\omega_j) \, V_{nj}^{(1)} H_{nj+1} \exp (-y^2/2) + k_j \, (\omega_j^2 - k_j^2)$$

$$H_{nj} \exp (-\tfrac{3}{2}y^2) \{(y^2\omega_j(\omega_j - k_j) + \omega_j^2 - k_j^2) H_{nj}^2 - y\omega_j^2 H_{nj} H_{nj+1}\}], \tag{A2a}$$

$$f_{2j} = (y\omega_j + k_j\, \partial/\partial y)\, [\exp(-y^2/2)\{k_j(y(\omega_j - k_j)\, H_{nj} + k_j H_{nj+1})\, (\tfrac{1}{4}(\omega_j^2 - k_j^2)^{-1}$$

$$(yV_{nj}^{(1)} + V_{nj}^{(1)\prime}\, k_j/\omega_j) + (y^2\omega_j\, (\omega_j - k_j) + \omega_j^2 - k_j^2)H_{nj}^2\, \exp\, (-y^2) - y\omega_j^2 H_n$$

$$\times\, H_{nj+1}\, \exp\, (-y^2)) + \tfrac{1}{4}H_{nj}\, (V_{nj}^{(1)} + yV_{nj}^{(1)\prime} + k_j V_{nj}^{(1)\prime\prime}/\omega_j - 4\, (\omega_j^2 - k_j^2)$$

$$\{(y^2\omega_j(\omega_j - k_j) + \omega_j^2 + \omega_j\, (\omega_j - k_j) - n_j\omega_j^2 - k_j^2)\, y\, H_{nj}^2 - (y^2\omega_j\, (\omega_j - k_j)$$

$$+\tfrac{1}{2}\omega_j^2 - k_j^2)\, H_{nj}\, U_{nj+1} + \tfrac{1}{2}y\omega_j^2\, H_{nj+1}\}\, \exp(-y^2)) + \tfrac{1}{2}\omega_j^{-1}V_{nj}^{(1)}\, (H_{nj}\, [\, y^2(\omega_j$$

$$- k_j) + \omega_j + (2n_j + 1)\, k_j] - y\omega_j H_{nj+1})\}], \tag{A2b}$$

$$f_{3j} = k_j(yk_j + \omega\, \partial/\partial y)\, [\exp - y^2/2\{\tfrac{1}{4}(\omega_j^2 - k_j^2)^{-1}\, [y(\omega_j - k_j)\, H_{nj} - \omega_j H_{nj+1}]$$

$$(yV_{nj}^{(1)} + k_j V_{nj}^{(1)}/\omega_j) + \tfrac{1}{2}(\omega_j^2 - k_j^2)[y(\omega_j - k_j)\, H_{nj} + k_j\, H_{nj+1}](k_j y V_{nj}^{(1)}/\omega_j$$

$$+ V_{nj}^{(1)\prime}) - V_{nj}^{(1)}\{(y^2(\omega_j - k_j) - k_j + (2n_{j+1})\, \omega_j)\, H_{nj} + yk_j\, H_{nj+1}\}/2\omega_j k_j$$

$$+ H_{nj}(V_{nj}^{(1)} + yV_{nj}^{(1)\prime} + \omega_j V_{nj}^{(1)\prime\prime}/k_j)/4\omega_j + \exp\, (-y^2)(\{y\, (\omega_j - k_j)\, H_{nj} + k_j$$

$$H_{nj+1}\}\, \{(y^2(\omega_j - k_j)\, \omega_j + (\omega_j^2 - k_j^2)\, (1 + 2k_j/\omega_j) + 2n_j\, (\omega_j^2 - k_j^2)\,)$$

$$H_{nj}^2 + yk_j\, (\omega_j + 2)\, H_{nj}\, H_{nj+1} + k_j^2\, H_{nj+1}\} - \tfrac{1}{2}\{y(\omega_j - k_j)\, H_{nj} - \omega_j$$

$$H_{nj+1}\}\, \{(y^2(\omega_j - k_j)\omega_j + \omega_j^2 - k_j^2)\, H_{nj}^2 - y\omega_j^2 H_{nj}\, H_{nj+1}\} - \tfrac{1}{2}H_{nj}\, (\omega_j^2$$

$$- k_j^2)\, [y\, (\omega_j\, (\omega_j - k_j)\, (y^2 + 1) + \omega_j^{-1}\, (\omega_j^2 - k_j^2)\, (\omega_j + 2k_j) - 2k_j^2 + k_j$$

$$(n_j + 1)\, (\omega_j + 2) + 2n_j\, (\omega_j^2 - k_j^2))\, H_{nj}^2 - yk_j\, (1 + k_j + \omega_j/2)\, H_{nj+1}^2$$

$$-(y^2\omega_j\, (\omega_j - k_j) + (\omega_j^2 - k_j^2)\, (\omega_j + 2k_j)/\omega_j - 2k_j^2 - k_j\, (1 + \omega_j/2)$$

$$+ 2n_j(\omega_j^2 - 2k_j^2)\,)\, H_{nj}\, H_{nj+1}])\}], \tag{A2c}$$

where a prime denotes the derivative with respect to Y.

Appendix 2. Expressions for α_2, α_3 and α_4 appearing in (54) and (55).

$$\alpha_2 = y\, \exp\, (-y^2)\, [(y\, (\omega_j - k_j)\, H_{nj} + k_j\, H_{nj+1})^2 - \{(V_g + 1)\, (y^2 + 1)$$

$$(\omega_j^2 - k_j^2) - 2\, (n_j + 1)\, (2\omega_j k_j V_g - \omega_j^2 - k_j^2)\}\, H_{nj}^2 + y\{(V_g + 1)$$

$$(\omega_j^2 - k_j^2) - 2\omega_j k_j V_g + \omega_j^2 + k_j^2\}H_{nj}\, H_{nj+1}],$$

$$\alpha_3 = y^1\, \exp\, (-y^2)\, [\{yk_j(V_g + 1)\, (1 - 2k_j/\omega_j)\, H_{nj} - k_j\omega_j^{-1}\, (\omega_j^2 - k_j^2)^{-1}$$

$$[\omega_j\, (\omega_j^2 - 3k_j^2)V_g + 2k_j^3]\, H_{nj+1}\}\{y\, (\omega_j - k_j)\, H_{nj} + k_j\, H_{nj+1}\} + \{yk_j$$

$$(V_g + 1)\, H_{nj} - k_j\, (\omega_j^2 - k_j^2)^{-1}\, (2\omega_j k_j V_g - \omega_j^2 - k_j^2)\, H_{nj+1}\}\, \{y\, (\omega_j - k_j)$$

$$H_{nj} - \omega_j H_{nj+1}\} + (\omega_j^2 - k_j^2)\, H_{nj}\{(\, (V_g + 1)\, (y^2 + 1)$$

$$(\omega_j - 2k_j)/\omega_j - 2\omega_j^{-1}\, (\omega_j^2 - k_j^2)^{-1}\, (n_j + 1)\, [\omega_j V_g\, (\omega_j^2 - 3k_j^2) + 2k_j^3])$$

$$H_{nj} - y \left((V_g + 1) (\omega_j - 2k_j)/\omega_j - \omega_j^{-1} (\omega_j^2 - k_j^2)^{-1} [\omega_j V_g (\omega_j^2 - 3k_j^2) \right.$$

$$\left. + 2k_j^2]) H_{nj+1} \right\} - (\omega_j^2 - k_j^2) (y^2 H_{nj}^2 - y H_{nj} H_{nj+1}) + \omega_j k_j H_{nj+1}^2],$$

$$\varsigma_4 = 2(\omega_j^2 - k_j^2) H_{nj} \exp(-y^2) \{ y (\omega_j^2 - 2k_j^2 + \omega_j k_j) H_{nj} - (\omega_j^2 - 2k_j^2) H_{nj+1} \}.$$

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