Envelope Theorems for Arbitrary Choice Sets

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Introduction

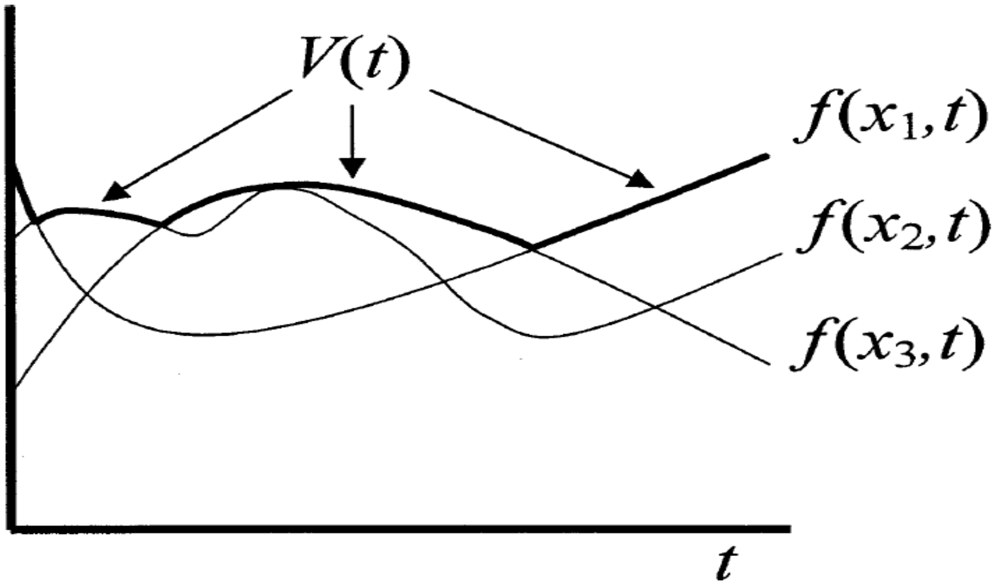
- Envelope theorems do the following two:
  - Conditions for a differentiable value function (in the parameter)
  - A formula for this value function

- Classically to guarantee differentiability:
  - Convexity and a topological structure on the choice set.

- Applications:
  - Concave optimization in demand theory
  - Analysis of Incentive Constraints on Contract Theory
  - Non-convex production Problems
  - The theory of *monotone* and *robust* comperative statics.

Introduction

- Choice sets often lack the convexity or the topological structure.
- This paper do not assume anything on the choice sets.



Results

- X denotes the choice set.
- $t \in [0,1]$ denotes the relevant parameter.
- $f : X \times [0,1] \to R$ denotes the objective function.
- $V(t) = \sup_{x \in X} f(x,t)$ is the value function.
- $X^*(t) = \{x \in X : f(x,t) = V(t)\}$ is the optimal choice set.

Results

- **Theorem 1:** Take $t \in [0,1]$ and suppose $f_t(x^*, t)$ exists. If $t > 0$ and V is left-hand differentiable at $t,$ then $V'(t-) \leq f_t(x^*, t)$. If $t < 1$ and V is right-hand differentiable, then $V'(t+) \geq f_t(x^*, t)$. If $t \in [0,1]$ and V is differentiable at t then, $V'(t) = f_t(x^*, t)$.

- Theorem 1 is useful when the value function is sufficiently well-behaved.

- **Theorem 2.a:** Suppose $f(x,.)$ is absolutely continuos for all x. Suppose also that there is an integrable function $b:[0,1] \to R_+$ such that $|f_t(x,t)| \leq b(t)$ for all x and almost all t. Then, V is absolutely continuous.

Results

- **Definition :** f is absolutely continuos if f' exists almost everywhere and it is Lebesque integrable and for any $x \in [a,b]$

$$f(x) = f(a) + \int_a^x f'(t)dt$$

Example : $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x/\sin(x) & \text{ow} \end{cases}$ is not absolutely continuous.

Results

- **Theorem 2.b :** Suppose the hypothesis of Theorem 2a holds and in addition $f(x,.)$ is differentiable for all x and $X^*(t) \neq \emptyset$ almost everywhere on $[0,1]$. Then, for any selection $x^*(t)$,

$$V(t) = V(0) + \int_0^t f_t(x^*(s), s)\, ds$$

- **Definition :** The family of functions $\{f(x,.)\}_{x \in X}$ is *equidifferentiable* at t, if $(f(x,t') - f(x,t))/(t'-t)$ converges uniformly as $t' \to t$.

Results

- **Theorem 3:** Suppose the family of functions $\{f(x,.)\}_{x\in X}$ is equidifferentiable at t_0, $\sup_{x\in X}\left|f_t(x,t_0)\right|<\infty$, and $X(t)\neq\varnothing$ for all t. Then, V is left and right differentiable at t_0. For any $x^* \in X^*(t)$, the directional derivatives are

$$V'(t_0+)=\lim_{t\to t_0+} f_t(x^*(t),t_0) \text{ for } t_0<1.$$

$$V'(t_0-)=\lim_{t\to t_0-} f_t(x^*(t),t_0) \text{ for } t_0>0.$$

V is differentiable at $t_0 \in (0,1)$ *iff* $f(x^*(t),t_0)$ is continuous in t at $t=t_0$.

- Note that all the assumptions are tight.

Application from Decision Theory

- Dekel-Lipman Self-Control and Random Strotz Representations

- More of a note on Gul-Pesendorfer(2001) on temptation

- Self-Control representation over the set of menus vs.
  Expected future choices by a future-self whose preferences are not known with certainity

Application from Decision Theory

- Let B be a compact set and Δ(B) denote the set of lotteries over B.
- $X \subseteq \Delta(B)$ denotes the set of menus. Compact and non-empty.
- The preference $\succ$ is defined over X.
- $w : \Delta(B) \to \Re$ is an expected utility if it is continuous and linear.
- Random Strotz representation:

  Let u be an expected utility function and μ be a measure over the set of expected utility functions such that $\succ$ is represented by:

$$V_{RS}(x) = \int \max_{\beta \in B_w(x)} u(\beta)\mu(dw)$$

Best elements in B wrt w.

Application from Decision Theory

- Self Control Representation:

  Let (u, v) be a pair of expected utility functions such that $\succ$ is represented by :

$$V_{SC}(x) = \max_{\beta \in x}[u(\beta) + v(\beta)] - \max_{\beta \in x} v(\beta)$$

- Re - write as :

$$V_{SC}(x) = \max_{\beta \in x}[u(\beta) - c(\beta; x)]$$

- $c(\beta; x)$ is the cost of resisting temptation.

- **Theorem :** Fix any self-control representation (u, v) and the corresponding V_{SC}. Then, there is a random Strotz representation V_{RS} such that for every menu x, $V_{SC}(x) = V_{RS}(x)$.

Application from Decision Theory

- **Proof :** Let W denote the set of expected utility preferences such that $w \in W$ *iff* there is $A \in [0,1]$ with $w = v + Au$. Define μ over W by taking s.t

$$\mu(E) = \Pr[\{A \in [0,1] : v + Au \in E\}],$$

where $\Pr$ is the uniform distribution. Let V_{RS} be the corresponding representation.

Now, fix any menu x. Then,

$$V_{RS}(x) = \int_0^1 \max_{\beta \in B_{v+Au}(x)} u(\beta)\, dA = \int_0^1 \max_{\beta \in B_{v+Au}(x)} \hat{u}(A)\, dA$$

Define $U(A) = v(\max_{\beta \in B_{v+Au}(x)} u(\beta)) + A\hat{u}(A) = \hat{v}(A) + A\hat{u}(A)$

$$= \max_{\bar{A} \in [0,1]} \hat{v}(\bar{A}) + A\hat{u}(\bar{A})$$

Now, by Theorem 2,

$$U(1) - U(0) = \int_0^1 U'(A)\, dA = \int_0^1 \hat{u}(A)\, dA = V_{RS}(x)$$

However, $U(1) = \max_{\beta \in x}[u(\beta) + v(\beta)]$ and $U(0) = \max_{\beta \in x} v(\beta)$.

Therefore, $U(1) - U(0) = V_{SC}(x)$.