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EPICOMPLETION OF ARCHIMEDEAN l-GROUPS AND VECTOR LATTICES WITH WEAK UNIT

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Abstract

In the category $\mathscr{W}$ of archimedean l-groups with distinguished weak order unit, with unit-preserving l-homomorphisms, let $\mathscr{B}$ be the class of $\mathscr{W}$-objects of the form $D(X)$, with X basically disconnected, or, what is the same thing (we show), the $\mathscr{W}$-objects of the form M/N, where M is a vector lattice of measurable functions and N is an abstract ideal of null functions. In earlier work, we have characterized the epimorphisms in $\mathscr{W}$, and shown that an object G is epicomplete (that is, has no proper epic extension) if and only if $G \in \mathscr{B}$. This describes the epicompletions of a given G (that is, epicomplete objects epically containing G). First, we note that an epicompletion of G is just a "$\mathscr{B}$-completion", that is, a minimal extension of G by a $\mathscr{B}$-object, that is, by a vector lattice of measurable functions modulo null functions. ($C[0, 1]$ has 2^{c} non-equivalent such extensions.) Then (we show) the $\mathscr{B}$-completions, or epicompletions, of G are exactly the quotients of the l-group $B(Y(G))$ of real-valued Baire functions on the Yosida space $Y(G)$ of G, by σ-ideals I for which G embeds naturally in $B(Y(G))/I$. There is a smallest I, called $N(G)$, and over the embedding $G \leq B(Y(G))/N(G)$ lifts any homomorphism from G to a $\mathscr{B}$-object. (The existence, though not the nature, of such a "reflective" epicompletion was first shown by Madden and Vermeer, using locales, then verified by us using properties of the class $\mathscr{B}$.) There is a unique maximal (not maximum) such I, called $M(Y(G))$, and $B(Y(G))/M(Y(G))$ is the unique *essential* $\mathscr{B}$-completion. There is an intermediate σ-ideal, called $Z(Y(G))$, and the embedding $G \leq B(Y(G))/Z(Y(G))$ is a σ-embedding, and functorial for σ-homomorphisms. The situation stands in strong analogy to the theory in Boolean algebras of free σ-algebras and σ-extensions, though there are crucial differences.

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1. Epicompletion and $\mathscr{B}$-completion

This section explains how, in $\mathscr{W}$ these two ideas are the same. After doing that, we proceed in the rest of the paper to a careful study of $\mathscr{B}$-completion, will hardly mention epimorphisms again, and the sequel can be read independently of [1], [2], and [19]. Our remarks now are quite "formal", and the few definitions in the abstract should suffice.

We write "$G \leq H$" to mean G is an l-subgroup of H, and "$G \leq H$ in $\mathscr{W}$" means also that G, $H \in \mathscr{W}$ and the embedding preserves the units.

1.1 THEOREM [2]. *In* $\mathscr{W}$

(a) *if* $H \in \mathscr{B}$, $G \leq H$ *in* $\mathscr{W}$, *and* G *has no epic extension within* H, *then* $G \in \mathscr{B}$,

(b) G *is epicomplete if and only if* $G \in \mathscr{B}$.

The condition "$G \in \mathscr{B}$" will be amplified in Section 3. Statement 1.1(a) is an explicit part of the proofs in [2] of both 1.1(b) and the fact that $\mathscr{B}$ is monoreflective in $\mathscr{W}$; and, on general grounds [16], monoreflectivity implies this. (We discuss monoreflectivity in Section 5 below.)

1.2 PROPOSITION. *Let* $\varphi\colon G \to E$ *be a* $\mathscr{W}$*-morphism, with* E *epicomplete. Then* φ *is epic if and only if* $\varphi(G) \leq E' \leq E$ *in* $\mathscr{W}$ *with* E' *epicomplete implies* $E' = W$.

PROOF. Only if statement. A final factor of an epic is always epic, so, if φ is epic, then $E' \leq E$ is epic. Thus, if E' is epicomplete, then $E' = E$.

IF STATEMENT. Factor φ as $G \xrightarrow{\varepsilon} E' \leq E$ with ε epic and E' having no epic extension within E. By 1.1, $E' = E$, whence $\varphi = \varepsilon$ is epic. (Such a factorization is possible on categorical grounds [2], granted some knowledge of the category $\mathscr{W}$, or with bare hands: let E' be the l-subgroup of E generated by $\bigcup\{H \leq E | \varphi(G) \leq H$ is epic$\}$ Then $G \xrightarrow{\varepsilon} E'$ is the "range-restriction" of φ. One checks that $\varphi(G) \leq E'$ is epic, whence ε is epic.)

1.3 DEFINITION. *Let* $\varphi\colon G \to E$ *be a* $\mathscr{W}$*-morphism, with* $E \in \mathscr{B}$.

(a) φ *is called* $\mathscr{B}$*-minimal* (*out of* G) *if* $\varphi(G) \leq E' \leq E$ *with* $E' \in \mathscr{B}$ *implies* $E' = E$.

(b) φ (*or* E, *or the pair* (φ, E)) *will be called a* $\mathscr{B}$*-completion of* G *if* φ *is an embedding which is* $\mathscr{B}$*-minimal.*

By 1.1 and 1.2, (a) says φ is epic and (b) says φ is an epicompletion of G. The sequel will treat epicompletion as $\mathscr{B}$-completion.

2. The Yosida functor

We review the Yosida representation of an archimedean l-group with a distinguished weak unit. This is part of our effort to make this paper readable independently of [1] and [2].

A central feature of this representation is that each $\mathscr{W}$-morphism $G \xrightarrow{\varphi} H$ is "realized" by a continuous map $Y(G) \xleftarrow{\tau} Y(H)$ of the Yosida spaces, as $\varphi(g) = g \circ \tau$ (as explained below). This is so like the situation for homomorphisms between $C(X)$'s and continuous maps of X's, as explained in [8] (especially Chapter 10), and the situation for homomorphisms of Boolean algebras and continuous maps of the Stone spaces (and indeed, the Yosida functor includes each of these as a special case), that the reader familiar with one of these theories, but largely ignorant of the special theory of l-groups, should be able to follow this paper by taking 2.2 below as an operational definition of "archimedean l-group with distinguished weak unit". If one doesn't want to do that, see [3].

The category $\mathscr{W}$ has as objects, archimedean l-groups G with a distinguished weak unit e_G; and as morphisms, l-homomorphisms $G \xrightarrow{\varphi} H$ with $\varphi(e_G) = e_H$. (By definition, a weak unit e in an l-group has $e \geq 0$, and $e \wedge |g| = 0$ implies $g = 0$.) We shall usually suppress explicit mention of the weak unit, and write things like "$G \in \mathscr{W}$", "$G \xrightarrow{\varphi} H$ is a $\mathscr{W}$-morphism" or even "$\varphi \in \mathscr{W}$", "$G \leq H \in \mathscr{W}$" (meaning G is embedded in H and the embedding is a $\mathscr{W}$-morphism), etc.

Let X be a topological space, always completely regular Hausdorff and usually compact; $C(X)$ is the $\mathscr{W}$-object of real-valued continuous functions on X (with pointwise addition and order), and unit the constant function 1; $D(X)$ is the *set* of continuous $f\colon X \to [-\infty, +\infty]$ for which $f^{-1}(R)$ is dense ($R = (-\infty, +\infty)$). For $f \in D(X), \infty(f) = f^{-1}\{\pm\infty\}$, $z(f) = f^{-1}\{0\}$ (the zero-set), $\operatorname{coz} f = X - Z(f)$ (the cozero set). In the pointwise order, $D(X)$ is a lattice, but usually fails to be a group. For $f, g, h \in D(X)$ we say "$f + g = h$ in $D(X)$" if $f(x) + g(x) = h(x)$ when $x \in f^{-1}(R) \cap g^{-1}(R) \cap h^{-1}(R)$ (which set is dense). It may well happen that, for particular $f, g \in D(X)$ there is *no* $h \in D(X)$ with $f + g = h$ in $D(X)$. However, it may well happen that a subset $G \subseteq D(X)$ has the property that for all $f, g \in G$ there exists $h \in G$ with $f + g = h$ in $D(X)$; if also $f, g \in G$ implies $f \vee g, f \wedge g, -f \in G$, then we say "$G$ is an l-group in $D(X)$". (For example, $C(X)$ is an l-group in $D(X)$.)

If also the constant function $1 \in G$, then 1 is a weak unit, and we invariably take 1 as the distinguished weak unit, making $G \in \mathscr{W}$.

Since this paper is all about "the class $\mathscr{B}$", we note that if X has the property that each dense cozero set is C^*-embedded [8], then X is called quasi-F [5] and $D(X) \in \mathscr{W}$ [15] (Proof: for $f, g \in D(X)$, $f^{-1}(R) \cap g^{-1}(R)$ is a dense cozero set C, $(f+g)|C \colon C \to (-\infty, +\infty)$ extends to $h \in D(X)$ and $f + g = h$ in $D(X)$.) We call X *basically disconnected* if each cozero set has open closure, and each basically disconnected space is quasi-F (because each b.d. space is F [8]). So we have

2.1 (a) PROPOSITION. *If X is (compact and) basically disconnected, then $D(X) \in \mathscr{W}$.*

(b) DEFINITION. *$\mathscr{B}$ is the subclass (or full subcategory) of $\mathscr{W}$, whose objects are $\mathscr{W}$-isomorphic to one of the form $D(X)$ for X compact and basically disconnected.*

We now describe the Yosida representation, and those of its features which we shall need. The proofs can be found in [3] and [17].

2.2 THEOREM. *Let $G \in \mathscr{W}$, with e_G the unit.*

(a) *There is a compact space $Y(G)$ (the Yosida space) and $\mathscr{W}$-isomorphism $G \ni g \mapsto \hat{g} \in \hat{G} \subseteq D(Y(G))$ onto an l-group $\hat{G}$ in $D(Y(G))$ (with $\hat{e}_G = 1$), with $\hat{G}$ separating the points of $Y(G)$.*

(b) *Let X be compact, and $G \ni g \mapsto \bar{g} \in \overline{G} \subseteq D(X)$ a $\mathscr{W}$-isomorphism onto an l-group $\overline{G}$ in $D(X)$ (with $\bar{e}_G = 1$) with $\overline{G}$ separating the points of X. Then there is a homeomorphism $\tau \colon X \to Y(G)$ for which $\bar{g} = \hat{g} \circ \tau$ for all $g \in G$.*

Statement 2.2(b) is used to recognize Yosida representations. For example, for $G \in \mathscr{W}$, let $G^* = \{g \in G \mid \text{for some } n \in N,\ |g| \leq ne_G\}$. This is the principal ideal in G generated by e_G, and in the Yosida representation of G, consists of all the $\hat{g}$ which are bounded. Give G^* the unit e_G, so $G^* \in \mathscr{W}$.

2.3 COROLLARY. *For $G \in \mathscr{W}$, $Y(G^*) = Y(G)$.*

PROOF. In the Yosida representation for G, G^* satisfies 2.2(b).

The following says that $Y(\ )$ is a functor from $\mathscr{W}$ to compact spaces.

2.4 THEOREM. *Let $G \xrightarrow{\varphi} H \in \mathscr{W}$. There is unique continuous $Y(G) \xleftarrow{\tau} Y(H)$ which "realizes" φ in the sense that $\varphi(g)^\frown = \hat{g} \circ \tau$ for all $g \in G$. And φ is one-to-one if and only if τ is onto, and if φ is onto then τ is one-to-one.*

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CONVENTION. For the sequel, for $G \in \mathscr{W}$, G and $\hat{G}$ are identified.

3. Baire functions and the class $\mathscr{B}$

In this section, we give several characterizations of $\mathscr{B}$-objects, one of which is the surjectivity of a certain natural embedding $G \leq \beta G$ of a $\mathscr{W}$-object into a $\mathscr{B}$-object. This is a variation on old ideas of Loomis, Sikorsky and Stone, about Boolean σ-algebras and basically disconnected spaces; see the remarks in 3.6 below.

It will develop in Section 5 that βG is the functorial $\mathscr{B}$-completion.

We begin with some preliminaries about measurable sets and functions.

Let Y be a set and $\mathscr{A}$ a σ-field of subsets of Y (meaning $A \in \mathscr{A}$ implies $Y - A \in \mathscr{A}$, and $A_1, A_2, \ldots \in \mathscr{A}$ implies $\bigcup_n A_n \in \mathscr{A}$). A function $f\colon Y \to \mathbf{R}$ is called $\mathscr{A}$-measurable if $f^{-1}(I) \in \mathscr{A}$ for every interval I of $\mathbf{R}$. Let $F(\mathscr{A})$, or just F, be $\{f | f \text{ is } \mathscr{A}\text{-measurable}\}$. With pointwise addition and order, and with designation of the constant function 1 as weak unit, we have $F \in \mathscr{W}$.

Let η be a σ-ideal in $\mathscr{A}$ (meaning $\mathscr{A} \ni A \subseteq N \in \mathscr{N}$ implies $A \in \mathscr{N}$, and $N_1, N_2, \cdots \in \mathscr{N}$ implies $\bigcup_n N_n \in \mathscr{N}$). We think of $\mathscr{N}$ as an abstract ideal of null sets, and define the corresponding l-group ideal in F of null functions $N(\mathscr{N})$, or just N, as $\{f \in F | \operatorname{coz} f \in \mathscr{N}\}$. Then, for the l-group quotient, $F/N \in \mathscr{W}$, where the weak unit is $1 + N$.

Notice that F and F/N can be made into vector lattices in the natural way. Thus, if $G \in \mathscr{W}$ is $\mathscr{W}$-isomorphic to some F/N, then G can be made into a vector lattice, and we say that the $\mathscr{W}$-object G "is a vector lattice of measurable functions modulo null functions".

An l-group, say archimedean, is called *conditionally σ-complete* if each countable family which is bounded above has a supremum, and *laterally σ-complete* if each countable pairwise disjoint family has a supremum.

An ideal I in an l-group G is called a σ-ideal if $f_1, f_2, \ldots \in I$ with $\bigvee_n f_n$ existing in G implies $\bigvee_n f_n \in I$. It is easily seen that, then, the quotient map $G \to B/I$ is a σ-homomorphism, that is, preserves all existing countable suprema and infima.

3.1 PROPOSITION. *Let $F(\mathscr{A}) = F$ and $N(\mathscr{N}) = N$ be as above.*

(a) *Let $f_1, f_2, \ldots \in F$. Then, $\bigvee_n f_n$ exists in F if and only if for all $x \in Y$, $\bigvee_n f_n(x)$ exists in $\mathbf{R}$; and then these are the same functions.*

(b) *F is conditionally and laterally σ-complete.*

(c) *N is a σ-ideal in F.*

(d) *If I is a σ-ideal in F, then $\operatorname{coz} I \equiv \{\operatorname{coz} f | f \in I\}$ is a Boolean σ-ideal, and $I = \{f | \operatorname{coz} f \in \operatorname{coz} I\}$ $(= N(\operatorname{coz} I))$.*

(e) *F/N is conditionally and laterally σ-complete.*

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PROOF. (a), (b), (c) are standard calculations; see [17] for example.

(d) $\operatorname{coz} I$ is a Boolean σ-ideal: $A \subseteq \operatorname{coz} f$, $f \in I^+$ implies $\chi_A = \bigvee_n \chi_A n f \in I$, which implies $A = \operatorname{coz} \chi_A \in \operatorname{coz} I$ (where χ_A is the characteristic function). And, $\bigcup_n \operatorname{coz} f_n = \operatorname{coz}(\bigvee_n |f_n| \wedge 1)$ since the sup is pointwise by (a). Now we show that $I = N(\operatorname{coz} I)$: $\subseteq$ is clear. For $\supseteq$, let $\operatorname{coz} g = \operatorname{coz} f$ with $f, g \geq 0$, and $f \in I$; then $g = \bigvee_n g \wedge nf \in I$.

(e) F/N is conditionally σ-complete because the quotient map is a σ-homomorphism. For the lateral σ-completeness, given pairwise disjoint $(f_n + N)$, one defines $f_1' = f_1$, f_2' to be 0 on $\operatorname{coz} f_1 \cap \operatorname{coz} f_2$ and f_2 elsewhere, f_3' to be 0 on $\operatorname{coz} f_1 \cap \operatorname{coz} f_3$, 0 on $\operatorname{coz} f_2 \cap \operatorname{coz} f_3$, and f_3 elsewhere, etc. Then, $f_n' + N = f_n + N$ for all n, (f_n') is pairwise disjoint, $\bigvee_n f_n'$ exists, and $\bigvee_n (f_n' + N) = \bigvee_n (f_n + N)$ because the quotient map is a σ-homomorphism.

Now suppose Y is a topological space. The *Baire field* $\mathscr{B}(Y)$ is the σ-field on Y generated by $\{Z(f) | f \in C(X)\}$, and the $\mathscr{W}$-object of *Baire functions* is $B(Y) = \{f | f \text{ is } \mathscr{B}(Y)\text{-measurable}\}$. We have $C(Y) \leq B(Y) \in \mathscr{W}$ (and, in fact, $\mathscr{B}(Y)$ is the least σ-field with respect to which continuous functions are measurable). One may see [14] and [17].

The features of the following relatively obvious construction form the core of this paper.

3.2 CONSTRUCTION OF βG. Let $G \in \mathscr{W}$, consider the Yosida space $Y(G)$, the Baire field $\mathscr{B}(Y(G))$, and let $\mathscr{N}(G) = \{A \in \mathscr{B}(Y(G)) | A \subseteq \bigcup_n \infty(g_n)$ for some $g_1, g_2, \ldots \in G\}$; this is the σ-ideal in $\mathscr{B}(Y(G))$ generated by $\{\infty(g) | g \in G\}$. Then consider $B(Y(G)) \in \mathscr{W}$, let $N(G) \equiv \{f \in B(Y(G)) | \operatorname{coz} f \in \mathscr{N}(G)\}$ be the l-group σ-ideal of "null functions" associated with $\mathscr{N}(G)$ (previously called $N(\mathscr{N}(G))$), and finally, let $\beta G = B(Y(G))/N(G) \in \mathscr{W}$ be the l-group quotient.

Unlike $C(Y) \leq B(Y)$, we do not have $G \leq B(Y(G))$ because elements of G may take infinite values, while the Baire functions take only real values. Of course, in constructing βG, we factored out exactly that difficulty.

We define the $\mathscr{W}$-embedding $\beta_G \colon G \to \beta G$ as follows. Given $g \in G$, define $g' \colon Y(G) \to R$ as

$$g'(x) = \begin{cases} g(x) & \text{if } x \notin \infty(g), \\ 0 & \text{if } x \in \infty(g). \end{cases}$$

It is easily seen that $g' \in B(Y(G))$. Now let $\beta_G(g) = g' + N(G)$. It is easily seen that β_G is a $\mathscr{W}$-homomorphism, and β_G is one-to-one because $\beta_G(g) = 0$ means $g' \in N(G)$, which means $\operatorname{coz} g' \in \mathscr{N}(G)$, which in turn means $\operatorname{coz} g - \infty(g) \in \mathscr{N}(G)$, which means $\operatorname{coz} g \in \mathscr{N}(G)$, which finally

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means $\operatorname{coz} g = \varnothing$ (that is, $g = 0$), since $\mathscr{N}(G)$ consists of meagre sets, and no nonempty open set is meagre in $Y(G)$, by the Baire Category Theorem.

The notation "β" can suggest "Baire", of course, perhaps also "best", "biggest", and analogy with the Stone-Čech compactification functor in topology, as shall be explained in Sections 5 and 6. For now, we focus on the following theorem, especially condition (b). While the object βG seems new, the circle of ideas is hardly completely novel; see the proof and 3.6 below.

3.3 THEOREM. *For $G \in \mathscr{W}$, the following are equivalent.*

(a) *$G \in \mathscr{B}$, that is, $Y(G)$ is basically disconnected and $G = D(Y(G))$.*

(b) *The $\mathscr{W}$-embedding $\beta_G\colon G \to \beta G$ is onto.*

(c) *G "is" a vector lattice of measurable functions modulo null functions.*

(d) *G "is" a vector lattice which is conditionally and laterally σ-complete.*

PROOF. That (b) implies (c) is obvious, and that (c) implies (d) is 3.1(e).

3.4 For the other implications, we shall use the following standard notion: $G \in \mathscr{W}$, which "is" a vector lattice, is called *uniformly complete* if each sequence (g_n) in G which is Cauchy (in the sense that for all $\varepsilon > 0$ there exists n_0 such that $m, n \geq n_0$ implies $|g_m - g_n| \leq \varepsilon e_G$) converges in G (in the sense that there exists g such that for all $\varepsilon > 0$ there exists n_0 such that $n \geq n_0$ implies $|g_n - g| \leq \varepsilon e_G$). In the Yosida representation, this evidently translates to "each sequence in G which is uniformly Cauchy as a sequence of functions on $Y(G)$, converges uniformly on $Y(G)$ to an element of G."

Note that, if G is uniformly complete then so is G^*, and then $G^* = C(Y(G))$ by Section 2 and the Stone-Weierstrass Theorem.

Note also that, if (g_n) is Cauchy in G, then (g_n) is also bounded in G (by some $|g_{n_0}| + \varepsilon e_G$), and one can easily create (by standard methods of elementary analysis) a new Cauchy sequence (g'_n) in G with $g'_n \leq g'_{n+1}$ for all n, and such that if either sequence converges then so does the other, to the same limit.

Note finally that, if G is conditionally σ-complete, then so is G^*, and G is also uniformly complete (for a Cauchy sequence (g_n), take a (g'_n) described above, which will be bounded, so that $\bigvee_n g'_n$ exists in G, and $g'_n \to \bigvee_n g'_n$ follows, whence $g_n \to \bigvee_n g'_n$).

(d) implies (a). First, let G be a conditionally σ-complete vector lattice. Then by 3.4, so is G^*, G^* is uniformly complete, and $G^* = C(Y(G))$. By the Stone-Nakano Theorem (that $C(Y)$ is conditionally σ-complete if and only if Y is basically disconnected ([23], or see [8])), $Y(G)$ is basically disconnected.

Now suppose G is also laterally σ-complete. First, let $0 \leq f \in D(Y(G))$. For each $n \in N$, let $u_n \in C(X)$ have $0 \leq u_n \leq 1$ and satisfy

$$u_n(x) = \begin{cases} 1 & \text{if } n \leq f(x) \leq n+1, \\ 0 & \text{if } f(x) \leq n-1 \text{ or } n+2 \leq f(x). \end{cases}$$

Such u_n exists by 1.14 of [8] (or by Urysohn's Lemma). For each n, $u_n f \in C(X) = G^*$. Then, for $i = 0, 1$ and 2, let $g_i = \bigvee\{u_{3n+1} f | n \in N\}$; these sups exist in G because, for each i, $\{u_{3n+1} f\}$ is pairwise disjoint. And, $f = g_0 \vee g_1 \vee g_2 \in G$. For arbitrary $f \in D(X)$, $f = (f \vee 0) - (-f) \vee 0$, and $f \vee 0$, $(-f) \vee 0 \in G$ by the foregoing.

(a) implies (b). We are going to show that $G \in \mathscr{B}$ implies the embedding $G \leq B(Y(G))/N(G)$ of 3.2 is onto. We twice reduce the problem.

First, if $G = D(Y(G))$, if $G \leq H \in \mathscr{W}$, and if $G^* = H^*$, then $G = H$ follows: for $G^* = H^*$ implies $Y(G^*) = Y(H^*)$, whence $Y(G) = Y(H)$, and then $G = D(Y(G)) = D(Y(H)) \supseteq H$.

Second, if G is uniformly complete, if G is $\mathscr{W}$-embedded in any F/N, say $F = F(\mathscr{A})$ and $G \leq F/N$, and if for all $A \in \mathscr{A}$, $\chi_A + N \in G$ (χ_A being the characteristic function of A), then $G^* = (F/N)^*$ follows: for, $(F/N)^* = F^*/N \cap F^*$, and given $f \in F^*$, one easily finds a sequence $f_n \to f$, each f_n being a linear combination of "χ_A's". By the assumption each $f_n + N \in G$. It is automatic that $f_n + N \to f + N$. Thus $f + N \in G$, because G is uniformly complete.

So, it suffices to prove that, for every Baire set $A \in \mathscr{B}(Y(G))$, $\chi_A + N(G) \in G$. This condition says that given A, there is a $g \in G$ such that $g' - \chi_A \in N(G)$, that is $\{y \in Y(G) | g'(y) \neq \chi_A(y)\} \in \mathscr{N}(G)$. Such a g is χ_C for the clopen C in statement (2) of

3.5. Let Y be compact basically disconnected, and let $\mathscr{Z}(Y)$ be the σ-ideal in $\mathscr{B}(Y)$ generated by the nowhere dense zero-sets. Then

(1) $\mathscr{Z}(Y) = \mathscr{N}(D(Y))$, and

(2) for each $A \in \mathscr{B}(Y)$ there exists a clopen C with $A - C$ and $C - A \in \mathscr{Z}(Y)$.

PROOF. (1) For any space at all, the collection of nowhere dense zero-sets coincides with $\{\infty(f) | f \in D(Y)\}$ *via* inversion of functions.

(2) Consider the σ-ideal $\overline{\mathscr{Z}}$ in the power set algebra, generated by $\mathscr{Z}$, that is, $\overline{\mathscr{Z}} = \{S \subseteq Y |$ there exists $Z \in \mathscr{Z}$ with $S \subseteq Z\}$, and let $\mathscr{A} \equiv \{A \subseteq Y |$ there exists a clopen C with $A - C$ and $C - A \in \overline{\mathscr{Z}}\}$. Now $\mathscr{A}$ is a σ-field: for complements, one sees easily that if C "works" for A, then $Y - C$ "works" for $Y - A$. For countable unions, if $A_1, A_2, \ldots \in \mathscr{A}$ then there are clopen $C_1, C_2, \ldots$ with $A_n - C_n$ and $C_n - A_n \in \overline{\mathscr{Z}}$ for all. Since Y is basically disconnected, $C = \overline{\bigcup_n C_n}$ is clopen, then, $\bigcup_n A_n - C \subseteq \bigcup_n A_n - \bigcup_n C_n \subseteq$

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$\bigcup_n(A_n - C_n) \in \overline{\mathscr{Z}}$, and

$$C - \bigcup_n A_n = \left[\left(C - \bigcup_n C_n\right) - \bigcup_n A_n\right] \cup \left[\bigcup_n C_n - \bigcup_n A_n\right]$$
$$\subseteq \left(C - \bigcup_n C_n\right) \cup \left(\bigcup_n (C_n - A_n)\right) \in \overline{\mathscr{Z}}$$

because $C - \bigcup_n C_n$ is a nowhere dense zero-set.

Clearly, $\mathscr{A}$ contains the clopen sets, thus $\mathscr{A} \supseteq \mathscr{B}(Y)$ since $\mathscr{A}$ is a σ-field, and that proves (b) because $\overline{\mathscr{Z}} \cap \mathscr{B}(Y) - \mathscr{Z}$.

That concludes the proof of 3.5, and hence of 3.3.

3.6. Remarks. As we said above, 3.3 is not completely novel. A version of 3.3 for l-algebras is in [9] (and that was not completely novel then either). There, the analogue of 3.3(b) has the ideal $\gamma(G)$ replaced by the ideal $\mathscr{M}(Y(G))$ of meagre Baire sets, and the proof was reduced, in a somewhat similar way, to 3.5(2) with $\mathscr{Z}(Y)$ replaced by $\mathscr{M}(Y)$. That last statement ("3.5(2), using $\mathscr{M}(Y)$") is exactly the Loomis-Sikorsky-Stone device as presented in [13, page 102]. Stone's version of this [23, Theorems 9 and 15] is that on a compact basically disconnected space, each bounded Baire function differs from some continuous function only on a meagre set; this is closer to our proof that (a) implies (b), and we have, of course, proved a bit more than this above.

The equivalence of (a) and (d) in 3.3 is just a simple extension of the Stone-Nakano Theorem quoted in our proof that (d) implies (c). Exactly this was noted long ago by Vulikh (see [27]).

For our purposes, the ideal $\mathscr{N}(G)$ is crucial, as the sequel, especially Sections 5 and 6, shows. This is why we wrote out the proof that (a) implies (b) in such detail. Still, the ideals $\mathscr{Z}(Y(G))$ and $\mathscr{M}(Y(G))$ have important places in the theory, as we shall see in Sections 7, 8 and 9 below.

4. Preservation of certain countable suprema

The details of this section will be important to the sequel, and also the consequence (which is known; see 4.7 below) that whenever $G \xrightarrow{\varphi} H \in \mathscr{W}$, with $G \in \mathscr{B}$, then φ is a σ-homomorphism and $\varphi(G) \in \mathscr{B}$. Our treatment is heavily dependent on the Yosida representation.

We emphasize here that any $G \in \mathscr{W}$ is identified with its Yosida representation; 1_G denotes the weak unit (the constantly 1 function $Y(G)$).

4.1 LEMMA. *Let $G \in \mathscr{W}$.*

(a) *$g \in G$, and $\{g_\alpha\} \subseteq G$. Then $\bigvee_\alpha g_\alpha = g$ in G if and only if $\{x \mid \bigvee_\alpha g_\alpha(x) = g(x)\}$ is dense in $Y(G)$. And, if $\bigvee_\alpha g_\alpha = 1_G$ in G, then $\bigcup_\alpha \operatorname{coz} g_\alpha$ is dense in $Y(G)$.*

(b) *If U is a dense open set in $Y(G)$, then there is $\{g_\alpha\} \subseteq G^+$ with $\bigcup_\alpha \operatorname{coz} g_\alpha = U$ and $\bigvee_\alpha g_\alpha = 1_G$ in G. If U is a cozero set, then $\{g_\alpha\}$ may be chosen to be countable.*

PROOF. (a) Let $P = \{x \mid \bigvee_\alpha g_\alpha(x) = g(x)\}$.

IF STATEMENT. Suppose P is dense. Since $g_\alpha \leq g$ on P, we have $g_\alpha \leq g$, by density and continuity. If there were $h < g$ with $g_\alpha \leq h$ for all α, there would be $x_0 \in P$ with $h(x_0) < g(x_0)$, which would contradict $\bigvee g_\alpha(x_0) = g(x_0)$.

ONLY IF STATEMENT. Suppose $\bigvee_\alpha g_\alpha = g$. Let $U_n = \{x \mid g_\alpha(x) > g(x) - 1/n$ for some $\alpha\}$. We have $\bigcap_n U_n = P \cap g^{-1}(R)$, and this set will be dense if and only if P is dense. So it suffices that each U_n be dense, by the Baire Category Theorem.

Suppose U_n is not dense. We then have nonvoid open V missing U_n, and we can suppose g is bounded by $1/2n$ on V. It follows that

$$\max\{g_\alpha(x) \mid x \in \overline{V}\} \leq \min\{g(x) \mid x \in \overline{V}\} - 1/2n \quad \text{for all } \alpha,$$

and that

$$\max\{2ng_\alpha(x) \mid x \in \overline{V}\} \leq \min\{2ng(x) \mid x \in \overline{V}\} - 1 \quad \text{for all } \alpha.$$

Now choose $h \in G$ with $\operatorname{coz} h \subseteq V$ and $0 \leq h \leq 1$. We then have $2ng_\alpha \leq 2ng - h$ for all α. But $\bigvee_\alpha 2ng_\alpha = 2ng$ for any n (since $\bigvee_\alpha g_a = g$), and we have a contradiction.

(b) For each $p \in U$, find $g_p \in G$ with $0 \leq g_p \leq 1_G$, with $g_p = 1$ on some neighborhood of p and $g_p = 0$ off U. Clearly, $\bigvee_{p \in U} g_p = 1_G$. In case U is a cozero set, it is F_σ in compact $Y(G)$, and so has the Lindelöf property. Then, U is the union of countably many of the sets $\{x \mid g_p(x) = 1\}$, which produces $g_{p_1}, g_{p_2}, \ldots$, clearly with $\bigvee_n g_{p_n} = 1_G$.

4.2 PROPOSITION. *Let $G \xrightarrow{\varphi} H \in \mathscr{W}$, with $Y(G) \xleftarrow{\tau} Y(H)$ the continuous map realizing φ (as per Section 2, as $\varphi(g) = g \circ \tau$). The following are equivalent:*

(a) *φ is a σ-homomorphism;*

(b) *$\bigvee g_n = 1_G$ in G^+ implies $\bigvee_n \varphi(g_n) = 1_H$ in H;*

(c) *C a dense cozero set in $Y(G)$ implies $\tau^{-1}(C)$ is dense (cozero) in $Y(H)$ (or dually, with nowhere dense zero sets).*

PROOF. That (a) implies (b) is obvious.

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(b) implies (a). It is well known that, in any l-group, for any h, the map $g \mapsto g+h$ is a lattice isomorphism. This implies that $\bigvee_\alpha(g_\alpha+h) = (\bigvee_\alpha g_\alpha)+h$ for any $\{g_\alpha\}$, in the sense that one exists if and only if the other does, and then they are equal.

Thus, if $\bigvee_n g_n = g$, then $\bigvee_n(g_n - g + 1_G) = 1_G$. If (b) holds, then $\bigvee_n(\varphi(g_n)-\varphi(g)+1_H) = \bigvee_n \varphi(g_n-g+1_G) = 1_H$, and the previous paragraph yields $\bigvee_n \varphi(g_n) = \varphi(g)$.

(b) implies (c). Given C, choose g as in 4.1(b). By (b), $\bigvee_n \varphi(g_n) = 1_H$, so by 4.1(a), $\bigvee_n \operatorname{coz}\varphi(g_n)$ is dense. Since $\varphi(g_n) = g_n \circ \tau$, we have $\tau^{-1}(C) = \tau^{-1}(\bigcup_n \operatorname{coz} g_n) = \bigcup_n \tau^{-1}(\operatorname{coz} g_n) = \bigcup_n \operatorname{coz} g_n \circ \tau = \bigcup_n \operatorname{coz}\varphi(g_n)$.

(c) implies (b). Let $\bigvee_n g_k = 1_G$. We want $\bigvee_k g_k \circ \tau = 1_H$, and shall use 4.1(a). By that proof, $U_n = \{x | g_k(x) > 1 - 1/n \text{ for some } k\}$ is dense. This set equals $\bigcup_k\{x|g_k(x) > 1 - 1/n\}$, which is the union of a sequence of cozeroes, and is thus cozero. By (c), $\tau^{-1}(U_n)$ is dense. Now, $\tau^{-1}(U_n) \subseteq \{y|g_k(y)) > 1 - 1/n \text{ for some } k\}$, so the latter is dense. As in the proof of 4.1(a), $\{y| \bigvee_k g_k(\tau(y)) = 1\}$ is dense, so by 4.1 $\bigvee_k g_k \circ \tau = 1_H$.

4.3 Lemma. *Let $G \xrightarrow{\varphi} H \in W$, with $Y(G) \xleftarrow{\tau} Y(H)$ the continuous map realizing φ (as per Section 2, as $\varphi(g) = g \circ \tau$). Then, for each $g \in G$*

(a) *$g^{-1}(R)$ is a dense cozero set, and $\infty(g)$ is a nowhere dense zero set, in $Y(G)$, and*

(b) *$\tau^{-1}g^{-1}(R)$ is a dense cozero set, and $\tau^{-1}(\infty(g))$ is a nowhere dense zero set, in $Y(H)$.*

Proof. (a) $g^{-1}(R) = \operatorname{coz}(1/|g| \vee 1_G)$.

(b) $\tau^{-1}g^{-1}(R) = \varphi(g)^{-1}(R)$.

The above is noted at the risk of belabouring the obvious. At any rate, from 4.3 and 4.2 we now have

4.4 Corollary. *Let $G \in \mathscr{W}$. Then every $\mathscr{W}$-morphism out of G is a σ-homomorphism if every dense cozero set in $Y(G)$ is of the form $g^{-1}(R)$ for some $g \in G$ (or dually, every nowhere dense zero set is an $\infty(g)$).*

As noted before, for any space Y, each dense cozero set is of the form $f^{-1}(R)$ for some $f \in D(Y)$ ($\operatorname{coz} h = f^{-1}(R)$, with $f = 1/|h|$ extended to be $+\infty$ on $Z(h)$), whether or not $D(Y)$ is an l-group. When $D(Y)$ is an l-group, that is, Y is quasi-F, or in particular, if Y is compact basically disconnected, then 4.4 applies:

4.5 Corollary. *If $G \in \mathscr{B}$, then every $\mathscr{W}$-morphism of G is a σ-homomorphism.*

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4.6 Corollary. *Let $G \xrightarrow{\varphi} H \in W$ be surjective. If $G \in \mathscr{B}$, then $H \in \mathscr{B}$.*

Proof. With $G \in \mathscr{B}$, we have $B(Y(G)) \xrightarrow{q} B(Y(G))/N(G) = \beta G = G \xrightarrow{\varphi} H$ by 3.3. We have $B(Y(G)) \in \mathscr{B}$ and $G \in \mathscr{B}$, so q and φ are σ-homomorphisms, by 4.5. Thus φq is a σ-homomorphism, so $\ker \varphi q$ is a σ-ideal in $B(Y(G))$. Thus, by 3.1(a), $\ker \varphi q = \{f \in B(Y(G)) | \operatorname{coz} f \in \mathscr{N}\}$, for some Boolean σ-ideal $\mathscr{N}$ in the Baire field, and thus $H = B(Y(G))/\ker \varphi q \in \mathscr{B}$, by 3.3(c).

4.7 Remark. Corollaries 4.5 and 4.6 are due to Tzeng and Veksler; see [24] and Theorem 0 of [25]. They reappear in [7], now without units.

5. βG is the functorial $\mathscr{B}$-completion

We now prove this theorem, in the exact sense of the following.

Let $\mathscr{C}$ be a category and $\mathscr{R}$ a subcategory, assumed full for simplicity. $\mathscr{R}$ is said to be *reflective* in $\mathscr{C}$ if for each object $C \in \mathscr{C}$ there is $r_c\colon C \to rC$, with $rC \in \mathscr{R}$, with this universal mapping property: for each $\varphi\colon C \to R$, with $R \in \mathscr{R}$, there is a unique $\overline{\varphi}\colon rC \to R$ with $\overline{\varphi} r_c = \varphi$. Then the pair (rC, r_c), or sometimes just the object rC, is called the *reflection* of C into $\mathscr{R}$, and r_c is called the *reflection morphism*; it is easy to show that (rC, r_c) is essentially unique, and that we have a functor $\mathscr{C} \xrightarrow{r} \mathscr{R}$ left-adjoint to the inclusion $\mathscr{C} \hookleftarrow \mathscr{R}$ whose action on objects is $C \to rC$, and whose action on morphisms is $G \xrightarrow{\varphi} H \mapsto rG \xrightarrow{r\varphi} rH$, where $r\varphi = \overline{(r_H \varphi)}$. When every reflection morphism r_c is monic, we say that $\mathscr{R}$ is *monoreflective*; for that it suffices that ($\mathscr{R}$ be reflective and) each $C \in \mathscr{C}$ embed into some $\mathscr{R}$-object. (see [16].)

Madden and Vermeer [19] first showed that, in $\mathscr{W}$ the class of epicomplete objects is monoreflective, using locales, without the explicit identification of the epicomplete objects as the $\mathscr{B}$-objects, and without an identification within $\mathscr{W}$ of the epicomplete monoreflection (what we have been calling, somewhat inexactly, the functorial epicompletion). At that point, we knew that "epicomplete $= \mathscr{B}$" (that is, 1.1(b), here), and shortly verified that $\mathscr{B}$ is monoreflective using a well-known adjoint functor theorem; these results are presented in [2].

We now give an explicit and independent proof that the construct $(\beta_G, \beta G)$ of 3.2 is the reflection of G into $\mathscr{B}$. This, with "epicomplete $= \mathscr{B}$" thus reproves, in a concrete way, the monoreflectivity of epicompleteness.

This section takes place in $\mathscr{W}$: all the l-groups are $\mathscr{W}$-objects and all the homomorphisms are $\mathscr{W}$-homomorphisms.

The following is the main result of the paper. (The terminology in (d) comes from Section 1.)

5.1 Theorem. *For each* G, $\beta_G\colon G \to \beta G$ *has these properties*:

(a) $\beta G \in \mathscr{B}$;

(b) *If* $E \in \mathscr{B}$ *and* $\varphi\colon G \to E$ *is a homomorphism, then there is a homomorphism* $\overline{\varphi}\colon \beta G \to E$ *with* $\overline{\varphi}\beta_G = \varphi$;

(c) β_G *is epic; thus the* $\overline{\varphi}$ *in* (b) *is unique*;

(d) β_G *is* $\mathscr{B}$*-minimal: thus* $\beta_G\colon G \to \beta G$ *is a* $\mathscr{B}$*-completion.*

Proof. (a) is part of 3.3.

(b) and (c). We proceed a bit more generally. Let $\varphi\colon G \to H$ be a homomorphism, and consider the diagram

$$\begin{array}{ccccccc} G \geq & G^* \leq & B(Y(G^*)) = B(Y(G)) & \xrightarrow{q_G} & B(Y(G))/N(G) = \beta G \\ \varphi\downarrow & \downarrow\varphi^* & \downarrow B(\varphi^*) = B(\varphi) & & \downarrow\overline{\overline{\varphi}} \\ H \geq & H^* \leq & B(Y(H^*)) = B(Y(H)) & \xrightarrow[q_H]{} & B(Y(H))/N(H) = \beta H \end{array} \tag{5.2}$$

in which $G^* = \{g \in G | g \text{ is bounded on } Y(G)\}$. Clearly, $\varphi(G^*) \subseteq H^*$, and $\varphi^* \equiv \varphi|G^*\colon G^* \to H^*$. We have $Y(G^*) = Y(G)$. (See Section 2.) There is continuous $\tau\colon Y(H) \to Y(G)$ for which $\varphi(g) = g \circ \tau$, for all $g \in G$. Being continuous, τ is Baire, and we define a homomorphism $B(\varphi^*)\colon B(Y(G^*)) \to B(Y(H^*))$ be $B(Y(H^*))$ by $B(\varphi^*)(f) = f \circ \tau$. Since $Y(G^*) = Y(G)$, etc., we may as well, and do, rename this as $B(\varphi)\colon B(Y(G)) \to B(Y(H))$. Obviously, if $g \in G^*$, then $B(\varphi)(g) = \varphi(g)$.

We have now proved (a) in

5.3 Lemma. *Let* $\varphi\colon G \to H$ *be a homomorphism.*

(a) *There is a homomorphism* $B(\varphi)\colon B(Y(G)) \to B(Y(H))$ *for which* $B(\varphi)|G^* = \varphi^*$.

(b) $G^* \leq B(Y(G))$ *is epic; thus, the* $B(\varphi)$ *in* (a) *is unique.*

(c) β_G *is epic.*

(d) *There is a homomorphism* $\overline{\overline{\varphi}}\colon \beta G \to \beta H$ *for which* $\overline{\overline{\varphi}}\beta_G = \beta_H\varphi$.

Upon proving the rest of 5.3, we have 5.1(c) (=5.3(c)) and 5.1(b): if in 5.3(d), $H \in \mathscr{B}$, then by 3.3(b), β_H is an isomorphism, and we put $\overline{\varphi} = \beta_H^{-1}\overline{\overline{\varphi}}$. We turn to the rest of 5.3.

We first construct the $\overline{\overline{\varphi}}$ in (d) (which appears in 5.2). In 5.2, note that q_G and q_H are just the projections onto the indicated quotients. In order that $B(\varphi)$ "drop" to a homomorphism $\overline{\overline{\varphi}}$ of the quotients, of necessity given by

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$\bar{\bar{\varphi}}(f + N(G)) = B(\varphi)(F) + N(H)$, it is exactly required that $B(\varphi)(N(G)) \subseteq N(H)$, that is, if $f \in B(Y(G))$ has $\operatorname{coz} f \subseteq \bigcup_n \infty(g_n)$ for some sequence $(g_n) \subseteq G$, then there is $(h_n) \subseteq H$ such that $\operatorname{coz} f \circ \tau \subseteq \bigcup_n \infty(h_n)$. But $\operatorname{coz} f \circ \tau = \tau^{-1}(\operatorname{coz} f) \subseteq \tau^{-1}(\bigcup_n \infty(g_n)) = \bigcup_n \tau^{-1}(\infty(g_n)) = \bigcup_n \infty(g_n \circ \tau)$; thus put $h_n = g_n \circ \tau$. We have $g_n \circ \tau = \varphi(g_n)$, so $h_n \in H$.

Thus we have $\bar{\bar{\varphi}}\colon \beta G \to \beta H$, and we now verify that $\bar{\bar{\varphi}}\beta_G = \beta_H\varphi$. Recall that, for $g \in G$, $\beta_G(g) = g' + N(G)$ where g' is g redefined as 0 on $\infty(g)$. So, $\beta_G(g) = g' + N(G) = q_G(g')$ so $\bar{\bar{\varphi}}\beta_G(g) = \bar{\bar{\varphi}}q_G(g') = B(\varphi)(g') + N(H) = g' \circ \tau + N(H)$. On the other hand, for $h \in H$, $\beta_H(h) = h' + N(H)$, so $\beta_H\varphi(g) = \varphi(g)' + N(H) = (g \circ \tau)' + N(H)$. However, $g' \circ \tau = (g \circ \tau)'$ since $\infty(g \circ \tau) = \tau^{-1}(\infty(g))$. So $\bar{\bar{\varphi}}\beta_G(g) = \beta_H\varphi(g)$ for all $g \in G$.

Now we prove that $G^* \leq B(Y(G))$ epic implies β_G is epic: for, q_G is a surjection, hence epic, and thus the composition $e\colon G^* \leq B(Y(G)) \xrightarrow{q_G} \beta G$ is epic. But e is also the composition $G^* \leq G \xrightarrow{\beta_G} \beta G$. Thus β_G is a "final factor" of an epic, and such a thing is always epic.

Finally, we show that $G^* \leq B(Y(G))$ is epic: let $\varphi_1, \varphi_2\colon B(Y(G)) \to K$ be homomorphisms with $\varphi_1|G^* = \varphi_2|G^*$. Let $f \in B(Y(G))$. For $k \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, $f^{-1}([k/n, (k+1)/n))$ is a Baire set; let χ_n^k be its characteristic function, which is Baire. Then f is the pointwise supremum, $f = \bigvee_n \bigvee_k (k/n)\chi_n^k$, and this supremum is also the supremum in the l-group $B(Y(G))$. The φ_i are σ-homomorphisms by 4.5, so that $\varphi_i(f) = \bigvee_n \bigvee_k (k/n)\varphi_i(\chi_n^k)$ (the suprema now being in K). For $\varphi_1(f) = \varphi_2(f)$, it is enough now to show that $\varphi_1(\chi) = \varphi_2(\chi)$ for each characteristic function $\chi = \chi_F$ of a Baire set F.

We shall prove that by transfinite induction using the Baire classification $\mathscr{B}(Y) = \bigcup_{\alpha<\omega_1} \mathscr{B}_\alpha$, in which $\mathscr{B}_0$ is all cozero sets of Y, $\mathscr{B}_{\alpha+1}$ consists of all countable intersections (respectively unions) when α is even (respectively odd), and $\mathscr{B}_\beta = \bigcup_{\alpha<\beta} \mathscr{B}_\alpha$ for limit β. (See [17].) Here, for $Y = Y(G)$, each cozero set C of $Y(G)$ is an open F_σ, and hence Lindelöf, and the $\operatorname{coz} g$ ($g \in G$) form a base for the topology (Section 2), and so a covering argument yields $C = \bigcup_n \operatorname{coz} g_n$ for some $g_1, g_2, \ldots \in G^*$.

Now, if $F \in \mathscr{B}_0$, then $F = \bigcup_n \operatorname{coz} g_n$ for g_n's $\in G^*$, and we can easily redefine the g_n's so that $\chi_F = \bigvee_n g_n$. Then, $\varphi_1(\chi_F) = \bigvee_n \varphi_1(g_n) = \bigvee_n \varphi_2(g_n) = \varphi_2(\chi_F)$, again since the φ_i are σ-homomorphisms (4.5) with suprema in $B(Y(G))$ being pointwise (3.1(a)), and since $\varphi_1|G^* = \varphi_2|G^*$. Similarly, one shows that $\varphi_1(\chi_F) = \varphi_2(\chi_F)$ for all $F \in \mathscr{B}_\beta$, for all $\beta < \alpha$, implies $\varphi_1(\chi_F) = \varphi_2(\chi_F)$ when $F \in \mathscr{B}_\alpha$. Induction completes the proof of 5.3(c).

The proof of 5.3 is complete.

Proof of 5.1(d). (Note that (c) and 1.2 immediately yield (d). However, we are eschewing Section 1 here, so make a separate argument.)

We have shown that $\beta_G\colon G \to \beta G$ is the $\mathscr{B}$-reflection morphism. It then follows on general grounds that β_G is $\mathscr{B}$-minimal, by 2 of [11]. We indicate a version of the argument: suppose $G \overset{e}{\leq} E \overset{f}{\leq} \beta G$ with $E \in \mathscr{B}$ (e, f being labels for the postulated inclusions, with $fe = \beta_G$). By (b), there is $\bar{e}\colon \beta G \to E$ with $\bar{e}\beta_G = e$. Let i denote the identity morphism on βG. We have $i\beta_G = \beta_G = fe = f(\bar{e}\beta_G) = (f\bar{e})\beta_G$. Since β_G is epic (by (c)), $i = f\bar{e}$; so f is a retraction. But f is also one-to-one, and thus onto. So $E = \beta G$.

The proof of 5.1 is complete.

6. Arbitrary $\mathscr{B}$-completions

We now show that the $\mathscr{B}$-completions of G are exactly the quotients over G of βG. This is readily at hand from the universal mapping property of βG (5.1), and the fact that a $\mathscr{W}$-quotient of a $\mathscr{B}$-object is in $\mathscr{B}$ (4.6). A bit more generally, we have

6.1 Proposition. (a) *If $\varphi\colon G \to E$ is $\mathscr{B}$-minimal, then the $\overline{\varphi}$ with $\overline{\varphi}\beta_G = \varphi$ (from* 5.1) *is a surjection.*

(b) *If $s\colon \beta G \to E$ is a surjective homomorphism, then $s\beta_G$ is $\mathscr{B}$-minimal.*

Proof. (a) We have $\varphi(G) \leq \overline{\varphi}(\beta G) \leq E$. By 4.6, $\overline{\varphi}(\beta G) \in \mathscr{B}$, so if φ is $\mathscr{B}$-minimal, then $\overline{\varphi}(\beta G) = E$.

(b) By 4.6, $E \in \mathscr{B}$. Suppose we have $H \in \mathscr{B}$ with $s\beta_G(G) \leq H \leq E$. Then $G \leq s^{-1}(G) \leq \beta G$. Let $s_1 = s|s^{-1}(H)\colon s^{-1}(H) \to H$. Let $\beta'_G\colon G \to s^{-1}(H)$ be the range-restriction of βG. By 5.1, since $H \in \mathscr{B}$, there is $\bar{s}_1\colon \beta G \to H$ with $\bar{s}_1\beta_G = s_1\beta'_G$. For clarity, attach the label $H \overset{m}{\leq} E$; so $ms_1\beta'_G = s\beta_G$. Thus, $m\bar{s}_1\beta_G = ms_1\beta'_G = s\beta_G$, which implies $m\bar{s}_1 = s$, since β_G is epic (8.2(c)). Since s is a surjection, so is m, and this says $H = E$.

(Using Section 1, we also can argue like this: since $E \in \mathscr{B}$, E is epicomplete, by 1.1; and $s\beta_G$ is epic, as the composition of two epics, and hence is $\mathscr{B}$-minimal by 1.2.)

6.2 Theorem. (a) *If $\varphi\colon G \to E$ is a $\mathscr{B}$-completion of G, then there is a (unique) surjective homomorphism $\overline{\varphi}\colon \beta G \to E$ with $\overline{\varphi}\beta_G = \varphi$.*

(b) *If $s\colon \beta G \to E$ is a surjective homomorphism with $s\beta_G$ one-to-one, then $s\beta_G\colon G \to E$ is a $\mathscr{B}$-completion of G.*

This is just the specialization of 6.1 to embeddings.

It is now clear that, for a given G, there is an association between $\mathscr{B}$-completions, certain ideals in βG, certain ideals in $B(Y(G))$, and certain

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ideals in $\mathscr{B}(Y(G))$. We postpone the formal discussion to Section 8, and first examine a subclass of the $\mathscr{B}$-completions with another canonical $\mathscr{B}$-completion of G.

7. $\mathscr{B}$-completions in which G is σ-embedded

A σ-embedding is an embedding which is a σ-homomorphism. A $\mathscr{B}$-completion $\varphi: G \to E$ for which φ is a σ-embedding will be called a $\mathscr{B}_\sigma$-completion. We now shall construct the $\mathscr{B}_\sigma$-completion which is functorial for σ-homomorphisms. The development relies heavily on Section 4.

7.1 Construction of μG. Let $G \in \mathscr{W}$, and let $\mathscr{Z}(Y(G)) \equiv \{A \in \mathscr{B}(Y(G)) | A \subseteq \bigcup_n Z_n$, for some sequence $Z_1, Z_2, \ldots$ of nowhere dense zero-sets of $Y(G)\}$. This is the σ-ideal in $\mathscr{B}(Y(G))$ generated by the collection of nowhere dense zero-sets; it depends only on the Yosida space $Y(G)$. Then let $Z(Y(G)) \equiv \{f \in B(Y(G)) | \operatorname{coz} f \in \mathscr{Z}(Y(G))\}$ be the associated l-group σ-ideal of "null functions", and let $\mu G \equiv B(Y(G))/Z(Y(G)) \in \mathscr{W}$ be the l-group quotient. It is clear that there is an embedding of G into μG; it can be described like this: since $N(G) \subseteq Z(Y(G))$, there is the quotient $\bar{\mu}: \beta G \twoheadrightarrow \mu G$. Set $\mu_G = \bar{\mu}_G \beta_G: G \to \mu G$; then $\bar{\mu}_G$ is the extension of μG over βG provided by 5.1. And μ_G is one-to-one because $\mathscr{Z}(Y(G))$ consists of meagre sets, just as in 3.2, β_G was shown one-to-one because $N(G)$ consists of meagre sets.

By 6.2, $\mu_G: G \to \mu_G$ is a $\mathscr{B}$-completion of G.

From 3.3, $G \in \mathscr{B}$ implies β_G is an isomorphism, and then that μ_G is an isomorphism, in particular, $Z(Y(G)) = N(G)$ for $G \in \mathscr{B}$; note that we explicitly proved that in 3.5.

7.2 Theorem. *For each G, $\mu_G: G \to \mu G$ has these properties:*

(a) *it is a $\mathscr{B}_0$-completion of G;*

(b) *if $E \in \mathscr{B}$, and $\varphi: G \to E$ is a σ-homomorphism, then there is a* (σ-) *homomorphism $\varphi^0: \mu G \to E$ with $\varphi^0 \mu_G = \varphi$;*

(c) *μ_G is epic; thus the φ^0 in* (b) *is unique.*

Remark. 7.2 describes a situation of monoreflectivity, analogous to that of 5.1, exactly as follows. Let $\mathscr{W}_\sigma$ denote the category with $\mathscr{W}$-objects, and morphisms the $\mathscr{W}$-morphisms which are σ-homomorphisms. Let $\mathscr{B}_\sigma$ be the full subcategory of $\mathscr{B}$-objects; since every $\mathscr{W}$-morphism from a $\mathscr{B}$-object is a σ-homomorphism (4.6), we really have $\mathscr{B}_\sigma = \mathscr{B}$.

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Then, 7.2 says that $\mathscr{B}_\sigma$ is monoreflective in $\mathscr{W}_\sigma$, with $\mu_G\colon G \to \mu G$ being the reflection of G; that is, $\mathscr{W}_\sigma \xrightarrow{\mu} \mathscr{B}_\sigma$ is in fact a functor, which is left-adjoint to the inclusion $\mathscr{W}_\sigma \hookleftarrow \mathscr{B}_\sigma$. (In (c), "$\mu_G$ is epic" is meant in $\mathscr{W}$; by 4.6, this is the same as "μ_G is $\mathscr{W}_\sigma$-epic".)

Since on general grounds, reflections are unique (noted in Section 5) it follows that $\mu_G\colon G \to \mu G$ is essentially unique for the properties listed in 7.2.

Proof of 7.2. (a) The construction of μG showed it is a $\mathscr{B}$-completion. We show μ_G is a σ-homomorphism by verifying 4.2(b).

Let $\bigvee g_n = 1_G$ in G^+. Let $q_G\colon B(Y(G)) \to B(Y(G))/N(G) = \beta G$ be the quotient. We have the g_n's $\in G^*$, and $G^* \leq B(Y(G))$, so we view each g_n as an element $g_n' \in B(Y(G))$, and we have $\beta_G(g_n) = q_G(g_n)$, whence $\mu_G(g_n') = \bar{\mu}_G\beta_G(g_n) = \bar{\mu}_G q_G(g_n')$. Then $\bigvee_n \mu_G(g_n) = \bigvee_n \bar{\mu}_G q_G(g_n') = \bar{\mu}_G q_G(\bigvee_n g_n')$, since $\bar{\mu}_G$ and q_G are σ-homomorphisms by 4.5. But $\mu_G q_G(\bigvee_n g_n') = 1_{\mu G}$: by 4.1(a)n and its proof, $P = \{x \mid \bigvee_n g_n(x) = 1\} = \bigcap_k U_k$ is dense, and each U_k is cozero (as noted in the proof of 4.2.) Thus $Y(G) - P \in \mathscr{Z}(Y(G))$. Since $(\bigvee_n g_n')(x) = \bigvee_n g_n'(x) = 1$ holds for $x \in P$, we have $1_{B(Y(G))} - \bigvee_n g_n' \in Z(Y(G))$, as desired.

(c) Now $\mu_G = \mu_G\beta_G$ is the composition of two epics, and hence is epic.

(b) We shall prove the following

7.3 **Lemma.** *If $\varphi\colon G \to H$ is a σ-homomorphism, then there is a unique $(\sigma$-$)$ homomorphism $\varphi^{00}\colon \mu G \to \mu H$ with $\varphi^{00}\mu_G = \mu_H\varphi$.*

This immediately implies 7.2(b): if, in 7.3, $H \in \mathscr{B}$, then μ_H is an isomorphism onto, and we put $\varphi^0 = \mu_H^{-1}\varphi^{00}$.

Proof of 7.3. Consider the diagram

$$\begin{tikzcd}
 & B(Y(G)) \arrow[d, "B(\varphi)"] \arrow[dr] & \\
G \arrow[rr, "\mu_G"'] \arrow[dd, "\varphi"'] & & B(Y(G))/Z(Y(G)) = \mu G \arrow[dd, dashed, "\varphi^{00} \equiv B(\varphi) + Z(Y(H))"] \\
 & B(Y(H)) \arrow[dr] & \\
H \arrow[rr, "\mu_H"'] & & B(Y(H))/Z(Y(H)) = \mu H
\end{tikzcd}$$

where $B(\varphi)$ is the construct in 5.2: $B(\varphi)(f) = f \circ \tau$, where $\tau\colon Y(H) \to Y(G)$ is the continuous map for which $\varphi(g) = g \circ \tau$. In order that $B(\varphi)$ "drop" over the quotients to $\varphi^{00} = B(\varphi) + Z(Y(H))$ it is exactly required that $\mathscr{B}(\varphi)Z(Y(G)) \subseteq Z(Y(H))$. In view of the definition of $B(\varphi)$, this is the

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statement that $\tau^{-1}\mathscr{Z}(Y(G)) \subseteq \mathscr{Z}(Y(H))$, which is immediate from 4.2(c) when φ is a σ-homomorphism.

The verification that $\varphi^{00}\mu_G = \mu_H\varphi$ is similar to 5.3(d). Now φ^{00} is unique, since we already know μ_G is epic. That completes the proof of 7.3, and thus of 7.2.

For what it's worth, 7.3 and 7.2(b) have the following "converses."

7.4. *Let $\varphi: G \to H$ be a homomorphism.*

(a) *If there is a (σ-)homomorphism $\psi: \mu G \to \mu H$ for which $\psi\mu_G = \mu_H\varphi$, then φ was already a σ-homomorphism.*

(b) *If there is a (σ-)homomorphism $\psi: \mu G \to H$ for which $\psi\mu_G = \varphi$, then φ was already a σ-homomorphism.*

PROOF. μ_G is a σ-homomorphism, by 7.2, and so is the composition $\psi\mu_G$ in either (a) or (b). So in (b), φ is a σ-homomorphism. Likewise in (a) (for if φ were not, then μ_H would not be).

Finally, μG is to the class of $\mathscr{B}_\sigma$-completions of G just as βG is to the class of $\mathscr{B}$-completions:

7.5 THEOREM. (a) *If $\varphi: G \to E$ is a $\mathscr{B}$-minimal σ-homomorphism (in particular, if it is a $\mathscr{B}_0$-completion), then the $\varphi^0: \mu G \to E$ with $\varphi^0\mu_G = \varphi$ (from 7.2) is a surjection.*

(b) *If $s: \mu G \to E$ is a surjective (σ-)homomorphism then $s\mu_G$ is a $\mathscr{B}$-minimal σ-homomorphism, and if $s\mu_G$ is also one-to-one, then $s\mu_G: G \to E$ is a $\mathscr{B}_\sigma$-completion.*

PROOF. This follows from 6.1 and 7.2.

8. The partially ordered set of $\mathscr{B}$-completions

Throughout this section $G \in \mathscr{W}$ is fixed.

Let $\mathscr{B}\mathscr{C}(G)$ denote the class of all $\mathscr{B}$-completions $\varphi: G \to E$ of G; we shall write $(\varphi, E) \in \mathscr{B}\mathscr{C}(G)$, or even $\varphi \in \mathscr{B}\mathscr{C}(B)$; such φ is epic (either by Section 1 or since, as per 6.2, $\varphi = \overline{\varphi}\beta_G$, the composition of epics).

If $\varphi_1, \varphi_2 \in \mathscr{B}\mathscr{C}(G)$, we write $\varphi_1 \geq \varphi_2$ if there is h with $h\varphi_1 = \varphi_2$; such an h is unique (since φ_1 is epic) and a surjection (by 4.6 and $\mathscr{B}$-minimality of φ_2). Then $\mathscr{B}\mathscr{C}(G)$ is quasi-ordered by $\geq$, and β_G is a maximal element (6.2).

Now, $\varphi_1 \sim \varphi_2$ means $\varphi_1 \geq \varphi_2$ and $\varphi_2 \geq \varphi_1$. Then $\varphi_2 = h\varphi_1 = k\varphi_2$, and we see that $h = k^{-1}$ is an isomorphism. Then $\sim$ is an equivalence relation, $\mathscr{B}\mathscr{C}(G)/\sim$ is a partially ordered set, and now the equivalence class of β_G is the maximum element. (The issue of minimal and minimum elements will be taken up to Section 9.)

We translate the partial order into inclusion of l-group ideals in $\beta G = B(Y(G))/N(G)$, and in $B(Y(G))$, and into inclusion of Boolean σ-ideals in $\mathscr{B}(Y(G))/\mathscr{N}(G)$, and in $\mathscr{B}(Y(G))$. It will suffice to display the translation procedure, fix some notation, and state the result. Consider

($\mathscr{W}$) ("Bool")

$$
(8.1)\qquad
\begin{tikzcd}
& B(Y(G)) \arrow[d, "q_G"] \\
G \arrow[r, "\beta_G"] \arrow[dr, "\varphi"'] & \beta G = B/N \arrow[d, "\overline{\varphi}"] \\
& E \sim (B/N)/\ker\overline{\varphi}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{B}(Y(G)) \arrow[d] & \\
\mathscr{B}/\mathscr{N} \arrow[d] & \operatorname{coz} G \arrow[l] \arrow[ul] \arrow[dl] \\
(\mathscr{B}/\mathscr{N})\mathscr{K}(\varphi) &
\end{tikzcd}
$$

in which for the diagram in $\mathscr{W}$, on the left, $B = B(Y(G)), N = N(G)$, and $\varphi \in \mathscr{B}\mathscr{C}(G)$, etc; for the diagram on the right, in which $\mathscr{B} = \mathscr{B}(Y(G))$, $\mathscr{N} = \mathscr{N}(G)$, etc; the vertical part is in Boolean σ-algebras, $\operatorname{coz} G$ denotes $\{\operatorname{coz} g | g \in G\}$, which is at least a lattice, and the arrows out of $\operatorname{coz} G$ are lattice embeddings. In (8.1)(Bool), $\mathscr{K}(\varphi)$ is the σ-ideal in $\mathscr{B}/\mathscr{N}$ which corresponds to $\ker\overline{\varphi}$ in a manner which we now describe.

In (8.1)($\mathscr{W}$), $\ker\overline{\varphi}$ is a σ-ideal in B/N with $\beta_G(G) \cap \ker\overline{\varphi} = (0)$. And, of course, if J is a σ-ideal in B/N with $\beta_G(G) \cap J = (0)$, then let $\overline{\varphi}_J \colon B/N \twoheadrightarrow (B/N)/J$ be the quotient, let $\varphi_J = \overline{\varphi}_J\beta_G$, and we have $\varphi_J \in \mathscr{B}\mathscr{C}(G)$. In this notation, if $\varphi \in \mathscr{B}\mathscr{B}(G)$, then $\varphi \sim \varphi_{\ker\overline{\varphi}}$; and $\beta_G \sim \varphi(0)$.

We thus have an order-reversing surjection,

(8.2) $\mathscr{B}\mathscr{C}(G) \ni \varphi \mapsto \ker\overline{\varphi} \in$ "σ-ideals of the form J"; and $\varphi_1 \sim \varphi_2$ if and only if $\ker\overline{\varphi}_1 = \ker\overline{\varphi}_2$,

and the J's correspond one-to-one and order-preserving with Boolean σ-ideals $\mathscr{J}$ in $\mathscr{B}/\mathscr{N}$ which have $\mathscr{J} \cap \operatorname{coz} G = \varnothing$. Then, in (8.1)(Bool), $\mathscr{K}(\varphi) = \mathscr{J}$ corresponds to $\ker\overline{\varphi} = J$.

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We prefer to focus on the correspondence induced between the associated σ-ideals in B and $\mathscr{B}$: first, we have (implicitly from Section 3) one-to-one order-preserving mutually inverse mappings

$$\text{(8.3)} \qquad \sigma\text{-ideals in } B \underset{n}{\overset{\mathrm{coz}}{\rightleftarrows}} \sigma\text{-ideals in } \mathscr{B}$$

defined by $\operatorname{coz} I = \{\operatorname{coz} f | f \in I\}$ and $n\mathscr{I} = \{f | \operatorname{coz} f \in \mathscr{I}\}$ ("n" for "null"). The following is clear.

8.4 Proposition. *Let I be a σ-ideal in $B(Y(G))$, and $\mathscr{I} = \operatorname{coz} I$ the associated σ-ideal in $\mathscr{B}(Y(G))$ (so $n\mathscr{I} = I$). The following are equivalent:*
(a) *there is $\varphi \in \mathscr{B}\mathscr{C}(G)$ with $I = q_G^{-1}(\ker \overline{\varphi})$, namely $\varphi = \varphi_{q_G(I)}$;*
(b) *$I \supseteq N(G)$ and $q_G(I) \cap \beta_G(G) = (0)$;*
(c) *$\mathscr{I} \supseteq \mathscr{N}(G)$ and $\mathscr{I} \cap \operatorname{coz} G = \{\varnothing\}$; that is, for all $g \in G, \infty(g) \in \mathscr{I}$ and $\operatorname{coz} g \notin \mathscr{I}$ if $g \neq 0$.*

To sum up, let $BI(G)$ be the l-group σ-ideals I satisfying 8.4, and let $\mathscr{B}\mathscr{I}(G)$ be the Boolean σ-ideals $\mathscr{I}$ satisfying 8.4.

8.5 Corollary. *$\mathscr{B}\mathscr{C}(G)/\sim$ is in one-to-one order-reversing correspondence with $BI(G)$, and $BI(G)$ is in one-to-one order-preserving correspondence with $\mathscr{B}\mathscr{I}(G)$.*

Finally, let $\mathscr{B}\mathscr{C}_\sigma(G)$ be the class of $\mathscr{B}_\sigma$-completions of G per Section 7, as a sub-quasi-ordered-set of $\mathscr{B}\mathscr{C}(G)$, so $\mathscr{B}\mathscr{C}_\sigma(G)/\sim$ has maximum μ_G. Let $BI_\sigma(G) = \{I \in BI(G) | I \supseteq Z(Y(G))\}$ and $\mathscr{B}I_\sigma(G) = \{\mathscr{I} \in \mathscr{B}\mathscr{I}(G) | \mathscr{I} \supseteq \mathscr{Z}(Y(G))\}$.

8.6 Corollary. *$\mathscr{B}\mathscr{C}_\sigma(G)/\sim$ is in one-to-one order-reversing correspondence with $BI_\sigma(G)$, and $BI_\sigma(G)$ is in one-to-one order preserving correspondence with $\mathscr{B}\mathscr{I}_\sigma(G)$.*

9. The $\mathscr{B}$-completion in which G is essentially and completely embedded

A *complete homomorphism* is one which preserves all existing suprema and infima, and a complete embedding is an embedding which is a complete homomorphism. Let us call a $\mathscr{B}$-completion $\varphi: G \to E$ for which φ is a complete embedding, a $\mathscr{B}_\infty$-completion. We now construct a $\mathscr{B}_\infty$-completion λG over which every complete homomorphism to a $\mathscr{B}$-object lifts uniquely to a complete homomorphism, and show that λG is the only $\mathscr{B}_\infty$-completion.

It is important to recognize at the outset that we are not dealing with an automatic extension of Section 7 to higher cardinals: λG is not any more complete than $\lambda G \in \mathscr{B}$ entails (that is, λG is conditionally and laterally σ-complete, but generally no more; in particular, not generally conditionally and laterally complete), nor do homomorphisms out of λG preserve any more existing suprema than $\lambda G \in \mathscr{B}$ entails (that is, countable sups are preserved, but generally no more).

9.1 Construction of λG. This proceeds just as with μG: let $G \in \mathscr{W}$, and let $\mathscr{M}(Y(G)) \equiv \{A \in \mathscr{B}(Y(G)) | A \text{ is meagre}\}$. This is a σ-ideal in the σ-field $\mathscr{B}(Y(G))$, and depends only on $Y(G)$. Let

$$M(Y(G)) \equiv \{f \in B(Y(G)) | \operatorname{coz} f \in \mathscr{M}(Y(G))\}$$

be the associated l-group σ-ideal of "null functions", and let

$$\lambda G \equiv B(Y(G))/M(Y(G)) \in \mathscr{W}$$

be the l-group quotient. There is an embedding of G into λG which can be described as follows. Since $N(G) \subseteq M(Y(G))$, there is the quotient $\overline{\lambda}_G: \beta G \twoheadrightarrow \lambda G$. Set $\lambda_G = \overline{\lambda}_G \beta_G: G \to \lambda G$; then $\overline{\lambda}_G$ is the extension of λ_G over βG provided by 5.1. And λ_G is one-to-one for the same reason that μ_G and β_G were: $\mathscr{M}(Y(G))$ consists of meagre sets.

By 6.2, $\lambda_G: G \to \lambda G$ is a $\mathscr{B}$-completion of G. (Indeed, $Z(Y(G)) \subseteq M(Y(G))$, so by 7.5, it is a $\mathscr{B}_\sigma$-completion.)

From 3.3, $G \in \mathscr{B}$ implies β_G is an isomorphism, and then λ_G is an isomorphism, and in particular, $M(Y(G)) = N(G)$ for $G \in \mathscr{B}$. We digress to note that we have now recovered the classical situation mentioned in 3.6: let Y be compact and basically disconnected; by 3.5 (2), each Baire set differs from a clopen set by a meager set, which is the essence of the Loomis-Sikorski Theorem (see [10]); also, $C(Y) = D(Y)^* = (\lambda D(Y))^* = B(Y)^*/M \cap B^*$, that is, each bounded Baire function differs from some continuous function only on a meagre set, which is Stone's Theorem [25].

We now do some ground-clearing needed to establish the properties of $\lambda_G: G \to \lambda G$. In particular, the completeness of λ_G is not particularly obvious, and for that, and other reasons, it seems better to focus on "essentiality".

9.2 Lemma. *Let $G \xrightarrow{\varphi} H \in \mathscr{W}$ be an embedding, with $Y(G) \xleftarrow{\tau} Y(H)$ the continuous map realizing φ. The following are equivalent (and define "φ is essential"):*

(a) *whenever $H \xrightarrow{\psi} K$, with K archimedean (or, with $\psi \in \mathscr{W}$) has $\psi\varphi$ one-to-one, then ψ is one-to-one;*

(b) *whenever I is an ideal of H with H/I archimedean has $I \cap \varphi(G) = (0)$, then $I = (0)$;*

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(c) *whenever* $0 < h \in H$, *there are* $0 < g \in G$ *and* n, *with* $\varphi(g) < nh$;

(d) τ *is "irreducible"; that is, whenever* U *is open nonvoid in* $Y(H)$, *there is open nonvoid* V *in* $Y(G)$, *with* $\tau^{-1}(V) \subseteq U$; *if* U *is dense, then* V *may be chosen dense.*

If G *is divisible* (c) *is equivalent to*

(e) *for all* $h \in Y^{+}$, $h = \bigvee\{\varphi(g) | \varphi(g) \leq h\}$.

Most of this well known, and it's all easy. Concerning (a), (b), (c), see [3] and [4], and also note that in (a), if the property holds for $\psi \in \mathscr{W}$, and we have $H \xrightarrow{\rho} K$ with $K \in$ Arch and $\rho\varphi$ is one-to-one, then $\psi\colon H \xrightarrow{\rho} K \twoheadrightarrow K/\rho(e_H)^{\perp} \in \mathscr{W}$ has $\psi\varphi$ one-to-one, and hence ψ is one-to-one, whence φ is one-to-one. For (d), see 4.1 of [12]. To prove the density statement, let U be dense. For each nonvoid open W in $Y(G)$, choose nonvoid open V_W in $Y(G)$ with $\tau^{-1}(V_W) \subseteq \tau^{-1}(W) \cap U$. Set $V = \bigcup_W V_W$.

The following is one of the essential features of λ_G.

9.3 THEOREM. *For any* $G \in \mathscr{W}$, *the embedding* $\lambda_G\colon G \to \lambda G$ *of* 9.1 *is essential.*

PROOF. Let $B = B(Y(G))$ and $M = M(Y(G))$. To satisfy 9.2(c), we are to show that for each $f \in B^{+}$ with $f \in M$, there are $g \in G$ and n with $0 < g + M \leq nf + M$ or $0 < (nf - g) + M$.

Given such an f, $\{f^{-1}[n, n+1) | n \in N\}$ is a cover of $Y(G)$ by Baire sets, and by the Baire Category Theorem, at least one of them, say $A = f^{-1}[n, n+1)$ is not meagre. We have $\chi_A \leq nf$. Now choose open U with $(A - U) \cup (U - A)$ meagre, by [13, page 58], and then choose $g \in G$ with $\varnothing \neq \operatorname{coz} g \subseteq U$, $0 \leq g \leq 1$. Then, $\operatorname{coz} g \subseteq A \cup (\operatorname{coz} g - A)$, and $\operatorname{coz} g - A \subseteq U - A$, which is meagre. Such g works in the first paragraph.

We now turn to complete homomorphisms and embeddings.

9.4 LEMMA. *Let* $G \xrightarrow{\varphi} H \in W$, *with* $Y(G) \xleftarrow{\tau} Y(G)$ *the continuous map realizing* φ. *Then* φ *is a complete homomorphism if and only if* U *dense open in* $Y(G)$ *implies* $\tau^{-1}(U)$ *dense (open) in* $Y(H)$ *(or, dually with nowhere dense closed sets); and this implies* $\tau^{-1}(\mathscr{M}(Y(G))) \subseteq \mathscr{M}(Y(H))$.

(For "if and only if", use 4.1 and copy the proof of 4.2, giving up countability. If $E \in \mathscr{M}(Y(G))$, then $E \subseteq \bigcup_n F_n$ with the F_n nowhere dense closed, whence $\tau^{-1}(E) \subseteq \bigcup_n \tau^{-1}(F_n)$. So $\tau^{-1}(E) \in \mathscr{M}(Y(G))$.)

9.5 LEMMA. *An essential embedding in* $\mathscr{W}$ *is complete.*

This follows from 9.4 and 9.2(d). It is well known in more generality; see [3] and [4].

The following gives the other main features of λG.

9.6 THEOREM. *For each G, $\lambda_G: G \to \lambda G$ has these properties:*

(a) *λ_G is a complete embedding;*

(b) *if $E \in \mathscr{B}$, and $\varphi: G \to E$ is a complete homomorphism, then there is a complete homomorphism $\varphi': \lambda G \to E$ with $\varphi' \lambda_G = \varphi$;*

(c) *λ_G is epic; thus, the φ' in* (b) *is unique.*

PROOF. (a) This follows from 9.3 and 9.5.

(c) Now $\lambda_G = \bar{\lambda}_G \beta_G$ is the composition of two epics, and hence is epic.

(b) As in the proof of 7.2, we shall prove

9.7 LEMMA. *If $\varphi: G \to H$ is a complete homomorphism, there is a unique complete homomorphism $\varphi'': \lambda G \to \lambda H$ with $\varphi'' \lambda_G = \lambda_H \varphi$.*

Note that 9.7 implies 9.6(b), since $H \in \mathscr{B}$ implies λ_H is an isomorphism (9.1), and λ_H is complete by (a).

PROOF OF 9.7. As in 7.2 and 5.2, consider the diagram

$$\begin{tikzcd}
 & B(Y(G)) \arrow[dd, "B(\varphi)"] \arrow[dr] & \\
G \arrow[rr, "\lambda_G"] \arrow[dd, "\varphi"'] & & B(Y(G))/M(Y(G)) = \lambda G \arrow[dd, dashed, "\varphi'' = B(\varphi) + M(Y(H))"] \\
 & B(Y(H)) \arrow[dr] & \\
H \arrow[rr, "\lambda_H"'] & & B(Y(H))/M(Y(H)) = \lambda H
\end{tikzcd}$$

in which φ complete implies $\tau^{-1}(M(Y(G))) \subseteq M(Y(H))$ (by 9.4) which then implies that φ'', as defined, makes sense.

The verification that $\varphi'' \lambda_G = \lambda_H \varphi$ is similar to that of 5.3(d), and uniqueness of φ'' (even without knowing φ'' is complete) follows, since λ_G is epic.

We must show φ'' is complete. The "commuting square" $\varphi''\lambda_G = \lambda_H\varphi$, is realized by the commuting square of continuous maps

$$\begin{array}{ccc} Y(G) & \xleftarrow{\rho_G} & Y(\lambda G) \\ \tau\uparrow & & \uparrow\tau'' \\ Y(H) & \xleftarrow[\rho_H]{} & Y(\lambda G) \end{array}$$

By 9.4, we are to show that $\tau''^{-1}(U)$ is dense whenever U is dense open in $Y(\lambda G)$. Since λG is essential, 9.2(a) provides dense open V in $Y(G)$, with $\rho_G^{-1}(V) \subseteq U$. Since φ is complete, 9.4 shows $\tau^{-1}(V)$ is dense in $Y(H)$, and then $\rho_H^{-1}\tau^{-1}(V)$ is dense in $Y(\lambda H)$, by 9.4, since λ_H is complete. But $\rho_H^{-1}\tau^{-1}(V) = \tau''^{-1}(\rho_G^{-1}(V)) \subseteq \tau''^{-1}(U)$, so this last is dense.

This completes the proof of 9.6.

9.8 $\mathscr{W}$ WITH COMPLETE HOMOMORPHISMS. Let $\mathscr{W}_\infty$ denote the category whose objects are the $\mathscr{W}$-objects and whose morphisms are the $\mathscr{W}$-morphisms which are complete. Let $\mathscr{B}_\infty$ be the full subcategory of $\mathscr{W}_\infty$, whose objects are the $\mathscr{B}$-objects.

Statement 9.6 says that $\mathscr{B}_\infty$ is monoreflective in $\mathscr{W}_\infty$, with $\lambda_G \colon G \to \lambda G$ being the reflection of G (that is, the operator $W_\infty \xrightarrow{\lambda} \mathscr{B}_\infty$ is, in fact, a functor, which is left-adjoint to the inclusion $\mathscr{W}_\infty \hookleftarrow \mathscr{B}_\infty$).

Reflections are unique, as discussed in Section 5. This means that if $\varphi\colon G \to E$ is a $\mathscr{B}_\infty$-reflection of G, then there is a $\mathscr{B}_\infty$-isomorphism $\psi\colon \lambda G \to E$ with $\psi\lambda_G = \varphi$; and in particular, $(\varphi, E) \in \mathscr{BE}(G)$ and $(\varphi, E) \sim (\lambda_G, \lambda G)$ (Condition 9.10(d) below is a strengthening of this unicity.)

One is led to compare $\mathscr{W}_\infty \overset{\lambda}{\rightleftarrows} \mathscr{B}_\infty$ with $\mathscr{W}_\sigma \overset{\mu}{\rightleftarrows} \mathscr{B}_\sigma$ (7.3). For the latter, the fact that every $\mathscr{W}$-homomorphism out of a $\mathscr{B}$-object is a σ-homomorphism (4.5) had the following consequences: $\mathscr{B}_\sigma = \mathscr{B}$; $\beta G = \mu G$ if and only if each $\mathscr{W}$-homomorphism out of G is a σ-homomorphism; and for $G \xrightarrow{\varphi} H \in \mathscr{W}_\sigma$ with $H \in \mathscr{B}$, φ is $\mathscr{W}$-epic if and only if φ is $\mathscr{W}_\sigma$-epic.

However, not every $\mathscr{W}$-homomorphism out of a $\mathscr{B}$-object is complete (see (a) below): so $\mathscr{B}_\infty \neq \mathscr{B}$; $\beta_G = \lambda G$ only if each $\mathscr{W}$-homomorphism out of G is complete, but not conversely ((a) below); for $G \xrightarrow{\varphi} H \in \mathscr{W}_\infty$ with $H \in \mathscr{B}$, φ $\mathscr{W}$-epic implies φ is $\mathscr{W}_\infty$-epic (of course), but not conversely, even with H conditionally and laterally complete (see 9.12 below).

(a) Let X be an uncountable set, and let $Y = X \cup \{p\}$, with neighborhoods of p having countable complement in X, and each $x \in X$ is isolated. Let $G = C(Y)$. Now Y is basically disconnected, and thus so is $\beta Y = Y(G)$, and $G = D(\beta Y)$, so $G \in \mathscr{B}$. (These are proved in [1].) Thus $\beta G = \lambda G$.

Define $G \xrightarrow{\varphi} \mathbf{R} \in \mathscr{W}$ by $\varphi(g) = g(p)$. Then φ is realized by the inclusion $Y(G) = \beta Y \xleftarrow{\tau} \{p\} = Y(\mathbf{R})$. Evidently, $\tau^{-1}(X) = \varnothing$, whence φ is not complete by 9.4. (This expresses the fact that $\bigvee\{\chi_x | x \in X\} = 1_G$, where χ_x is the characteristic function of $\{x\}$, while in $\mathbf{R}, \bigvee\{\varphi(\chi_x) | x \in X\} = \bigvee\{0\} = 0$.)

(b) Let $G = C(\mathbf{N}), \mathbf{N}$ the discrete natural numbers (that is, $G = \mathbf{R}^{\mathbf{N}}$). Every $\mathscr{W}$-homomorphism out of G is complete: give $G \xrightarrow{\varphi} H \in \mathscr{W}$, realized by $Y(G) = \beta\mathbf{N} \xleftarrow{\tau} Y(H)$, we have $\mathbf{N} = g^{-1}(\mathbf{R})$ for various $g \in G$, whence $\tau^{-1}(\mathbf{N}) = \varphi(g)^{-1}(\mathbf{R})$ is dense. But, any open dense U contains $\mathbf{N}$, whence $\tau^{-1}(U)$ is dense, whence φ is complete by 9.4.

Many interesting questions about $\mathscr{W}_\infty$ naturally occur to one. But this doesn't seem to have to do with the real topic of this paper, epicompletions in $\mathscr{W}$, so, except for some remarks in 9.11 below, we leave the subject for now.

We turn to the remarkable uniqueness properties of λG.

9.9 Lemma. *Let $\varphi: G \to E$ be a $\mathscr{B}$-completion of G. Then φ is essential if and only if (φ, E) is minimal in $\mathscr{B}\mathscr{C}(G)$ (as per Section 8).*

Proof. Let φ be essential, and suppose $(\varphi, E) \geq (\delta, D)$ is expressed by $E \xrightarrow{\psi} D$ with $\delta = \psi\varphi$. Then δ is one-to-one, so ψ is one-to-one. Since ψ is a surjection, $(\varphi, E) \sim (\delta, D)$.

Suppose $E \xrightarrow{\psi} H \in \mathscr{W}$ has $\psi\varphi$ one-to-one. Let $E \xrightarrow{\psi'} \psi(H)$ be the "range restriction of ψ." Then $(\psi'\varphi, \psi(H)) \in \mathscr{B}\mathscr{C}(G)$ (as in Section 8), and $(\varphi, E) \geq (\psi'\varphi, \psi(H))$. If (φ, E) is minimal, then $(\varphi, E) \sim (\psi'\varphi, \psi(H))$, which says ψ' is an isomorphism. Thus ψ is one-to-one.

9.10 Corollary. *Let $\varphi: G \to E$ be a $\mathscr{B}$-completion of G. The following are equivalent:*

(a) *(φ, E) is minimal in $\mathscr{B}\mathscr{C}(G)$;*
(b) *φ is essential;*
(c) *φ is complete;*
(d) *$(\varphi, E) \sim (\lambda_G, \lambda G)$.*

Proof. From 9.9, (a) implies (b), and from 9.5, (b) implies (c).

(c) implies (d). (We did not show this in the discussion of 9.8.) If φ is complete, there is $\varphi': \lambda G \to E$ with $\varphi'\lambda_G = \varphi$, by 9.6(b). Since λ_G is essential (9.3), φ' is one-to-one. Since $\varphi'(\lambda G) \in \mathscr{B}$, and φ is $\mathscr{B}$-minimal, φ' is onto E. (d) implies (a). By 9.3, λ_G is essential; now apply 9.9.

9.11 Is λG least? We just showed that $(\lambda_G, \lambda G)$ is the essentially unique minimal element of $\mathscr{B}\mathscr{C}(G)$, so we address the obvious question of whether or not it is the least element. The rough answer is, sometimes (9.13 and 9.14, below), but usually not.

By Section 8, the equivalence classes of elements of $\mathscr{B}\mathscr{C}(G)$ correspond one-to-one with σ-ideals $\mathscr{I}$ in the Baire field $\mathscr{B}(Y(G))$ with $\mathscr{N}(G) \subseteq \mathscr{I}$ and $\operatorname{coz} G \cap \mathscr{I} = \{\varnothing\}$; and an element is not above λG if and only if its ideal $\mathscr{I}$ is not contained in $\mathscr{M}(Y(G))$.

For simplicity, we specialize to $G = C(Y)$ with Y compact, whence $\mathscr{N}(C(Y)) = \{\varnothing\}$.

(a) $\lambda C(Y)$ is not least in $\mathscr{B}\mathscr{C}(C(Y))$ if there is a nonmeagre Baire set P in Y which contains no nonvoid cozero set.

(b) Suppose Y contains a nonvoid cozero set U, U having a dense set E which is countable and consists of non-isolated G_δ-points. Then $P = U - E$ has the properties in (a).

(c) In $Y = [0, 1]$, P being the set of irrational points shows $\lambda C[0, 1]$ is not least in $\mathscr{B}\mathscr{C}(C[0, 1])$.

Proof. (a) If $\mathscr{I}$ is the ideal associated with a $\mathscr{B}$-completion not above $\lambda C(Y)$, then one chooses nonmeagre $P \in \mathscr{I}$. Conversely, given P, $\mathscr{I} = \{A | A \subseteq P\}$ is a σ-ideal associated with a $\mathscr{B}$-completion not above $\lambda C(Y)$.

(b) The points of E are nowhere dense zero-sets (see [8]), so $U - E$ is Baire, co-meagre in U, and hence not meagre. Any cozero set contained in U hits E, so $U - E$ contains no nonvoid cozero set.

(c) $P = [0, 1] - E$, where E is the set of rational points.

A related question, not answered by the above construction, is whether $(\lambda_G, \lambda G)$ is always least in $\mathscr{B}\mathscr{C}_\sigma(G)$? We don't know the answer. The question is the l-group version of an old question of Sikorski about "σ-extensions" of Boolean algebras; see 10.5 and 10.6(e) below.

At this point we abandon our prohibition on speaking about epics and epicompletions, and permit ourselves to use Section 1 and a few abstract features of epics.

9.12 The essential closure. In [4], Conrad has shown that the archimedean l-groups G which admit no proper essential extension in Arch ($\equiv$ essentially closed) are exactly of the form $D(Y)$, Y compact and extremally disconnected, and that to each $G \in$ Arch there is an essentially closed εG and various essential embeddings $\varphi: G \to \varepsilon G$ such that whenever $G \xrightarrow{\psi} H$ is an essential embedding in Arch, there is an essential embedding $H \xrightarrow{\alpha} \varepsilon G$ (so $G \xrightarrow{\alpha\psi} \varepsilon G$ is one of the φ's).

When $G \in \mathscr{W}$, we may construe $\varepsilon G = D(Y)$ to be in W by choosing the constant function 1 as weak unit. Then (as is clear from [4]) there is exactly

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one essential $\varepsilon_G\colon G \to \varepsilon G$ in $\mathscr{W}$, and whenever $G \xrightarrow{\psi} H \in \mathscr{W}$ is essential, there is $H \xrightarrow{\alpha} \varepsilon G \in \mathscr{W}$ which is essential, with $\alpha\psi = \varepsilon_G$.

We thus have $G \le \lambda G \le \varepsilon G$ over G (suppressing some notation); this can also be inferred immediately from 9.6, since ε_G is complete (by 9.5). We deal briefly with several obvious questions.

(a) $\lambda G = \varepsilon G$ if and only if ε_G is $\mathscr{W}$-epic.

(b) If $Y(G)$ is metrizable, or admits no uncountable family of pairwise disjoint open sets, then (a) holds.

(c) Let $G = C(Y)$, where Y is the one-point compactification of uncountable discrete X. Then $\lambda C(Y)$ is example 9.8(a), while $\varepsilon C(Y) = C(X)$ $(= R^X)$. So ε_G is not $\mathscr{W}$-epic.

(Statement (a) is obvious. In one form or another, (b) is known; see [5], for example; (c) is easy.)

(d) $\lambda G = \bigcap\{H | G \le H \le \varepsilon G, H \in \mathscr{B}\}$.

(e) λG is the maximum $\mathscr{W}$-epic extension of G within εG (alluded to in the proof of 1.2).

(We show statement (d) as follows. The intersection in $\mathscr{W}$ of $\mathscr{B}$-objects is in $\mathscr{B}$, as a formal consequence of reflectivity of $\mathscr{B}$ in $\mathscr{W}$; see [16]. So the construction here is a $\mathscr{B}$-completion within εG, thus essential, and 9.9 applies. Statement (e) follows from 9.9 and 1.1.)

(f) $\varepsilon_G\colon G \to \varepsilon G$ is $\mathscr{W}_\infty$-epic.

(g) *Conjecture*: G is $\mathscr{W}_\infty$-epicomplete if and only if $G = \varepsilon G$; and $\varepsilon_G\colon G \to \varepsilon G$ is the functorial $\mathscr{W}_\infty$-epicompletion of G.

(Let dG be the divisible closure of G in εG. then, for all $h \in \varepsilon G^+, h = \bigvee\{g \in dG | g \le h\}$ by 9.2(e), and (f) follows.)

9.13 Corollary. *Let $G \in \mathscr{W}$. The following are equivalent*:

(a) *$\beta_G\colon G \to \beta G$ is essential (or complete)*;

(b) *$\beta G = \lambda G$, that is, $N(G) = M(Y(G))$*;

(c) *There is exactly one $\mathscr{B}$-completion, or epicompletion, of G (up to equivalence in $\mathscr{B}\mathscr{C}(G)$)*;

(d) *Every epic embedding of G is essential*;

(e) *Every epic embedding of G is complete.*

Proof. That (a) implies (b) follows from 9.9.

(b) implies (c). A partially ordered set with a minimal maximum has one element.

(c) implies (d). If $G\varphi \le H$ is epic, then so is $\lambda_H\varphi\colon G \to \lambda H$. By (c), $\lambda_H\varphi$ "is" λ_G, and hence is essential, so φ is essential.

That (d) implies (e) follows from 9.5.

That (e) implies (a) is obvious.

9.14 **Corollary.** (a) *If* $\beta G = \varepsilon G$, *then* G *satisfies* 9.13.

(b) *If* Y *is a compact space with no nonvoid nowhere dense zero sets* (Y *is "almost-P" as per* [5]) *then* $C(Y)$ *satisfies* 9.13.

(c) $Y = \beta N - N$ *satisfies* (b).

(d) *The converse of* (a) *fails, with the* Y *of* 9.8(a).

Proof. (a) β_G is essential.

(b) Since Y is compact, $N(C(Y)) = \{\varnothing\}$. In general, every nonvoid Baire set contains a nonvoid zero set (by an induction on the Baire classes), so Y almost-P implies $\mathscr{M}(Y) = \{\varnothing\}$. Thus $\beta C(X) = \lambda C(Y)$.

(c) The fact is known for $\beta N - N$ [8], and clear for 9.8(a).

(d) See 9.8(a).

10. Miscellania

We present an assortment of observations. First, we give the best cardinal bound on epic extensions (which improves the result in [1]).

10.1 **Theorem.** *Let* $G \in \mathscr{W}$.

(a) *If* $G \leq H$ *is* $\mathscr{W}$*-epic, then* $|H| \leq |G|^{\aleph_0}$.

(b) *Any* $\mathscr{W}$*-epicompletion of* G *has cardinality* $|G|^{\aleph_0}$.

Proof. (a) Each cozero set in $Y(G)$ is a countable union of sets $\operatorname{coz} g$ ($g \in G$), as in the proof of 4.1, so $|\operatorname{coz} Y(g)| \leq |G|^{\aleph_0}$. By transfinite induction on the Baire classification (see the proof of 5.3), it follows that $|\mathscr{B}(Y(G))| \leq |G|^{\aleph_0}$. Then, since each $f \in B(Y(G))$ can be sequentially approximated by rational-valued step functions, $|B(Y(G))| \leq |G|^{\aleph_0}$ follows. Then βG is a quotient of $B(Y(G))$, so $|\beta G| \leq |G|^{\aleph_0}$.

If $G \leq H$ is $\mathscr{W}$-epic, then, by Sections 1 and 6, βH is a quotient of βG, so $|H| \leq |\beta H| \leq |\beta G| \leq |G|^{\aleph_0}$.

(b) For any X, $|C(X)|^{\aleph_0} = |C(X)|$, by [18], and $|D(X)| = |C(X)|$ so $|D(X)|^{\aleph_0} = |D(X)|$. If $G \leq H$ is an epicompletion of G, it is a $\mathscr{B}$-completion (Section 1), so H is of the form $D(X)$ and $|H| = |H|^{\aleph_0}$. Then, with (a), we have $|H| \leq |G|^{\aleph_0} \leq |H|^{\aleph_0} = |H|$.

10.2 **Corollary.** *Let* $G \leq H$ *be Arch-epic. Then* $|H| \leq |G|^{\aleph_0}$.

Proof. If $G \leq H$ is Arch-epic, then by [1], for all $u \in G^+, G/u^{\perp_H}$ is $\mathscr{W}$-epic, and $\bigcap\{u^{\perp_H} | u \in G^+\} = (0)$. Thus, H embeds in $\prod\{H/u^{\perp_H} | u \in G^+\}$, each $|H/u^{\perp_H}| \leq |G/u^{\perp_G}|^{\aleph_0} \leq |G|^{\aleph_0}$, and so $|H| \leq |G| \cdot |G|^{\aleph_0} = |G|^{\aleph_0}$.

10.3 Remark. While [1] and [2] treat Arch as well as $\mathscr{W}$, 10.2 above is our first and essentially last encounter with Arch in this paper. We hope to return to Arch in later work, and to treat at least the analogue of the operator β (which exists, by [2]). The difficulty with Arch is, briefly, the lack of a canonical Yosida representation.

10.4 Basically disconnected covers of compacta. Let X be compact, and consider a $\mathscr{B}$-completion $\varphi\colon C(X) \to E$. Then φ is realized by a continuous map $X \overset{\tau(\varphi)}{\twoheadleftarrow} Y(E)$, $Y(E)$ is basically disconnected, and $\tau(\varphi)$ has some special properties. We call such a pair $(\tau(\varphi), Y(E))$ *a basically disconnected cover* of X. The particular one $(\tau(\lambda_{(X)}), Y(\lambda C(X)))$ is the topic of [26].

We are trying to develop this subject as a piece of topology, and shall report on it later.

10.5 $\mathscr{W}$ versus Boolean algebras. Let $\mathscr{A}$ be a Boolean algebra, $S(\mathscr{A})$ the Stone space, and $G(\mathscr{A})$ the $\mathscr{W}$-object in $C(S(\mathscr{A}))$ of functions of the form $\sum_{i=1}^{n} r_i\chi_{A_i}$ ($n \in N$; A_i a clopen set in $S(\mathscr{A})$, that is, $A_i \in \mathscr{A}$; χ_{A_i}, the characteristic function; $r_i \in R$). Then $Y(G(\mathscr{A})) = S(\mathscr{A})$, by Section 2.

We construct a Boolean σ-algebra $\beta\mathscr{A}$ and a Boolean embedding $\beta_{\mathscr{A}}\colon \mathscr{A} \to \beta\mathscr{A}$ defined by $\beta\mathscr{A} = \operatorname{clop} Y(\beta G(\mathscr{A}))$ where $\operatorname{clop} Y$ is the algebra of clopen subsets of Y), and $\beta_{\mathscr{A}}$ is the Stone dual of the map $S(\mathscr{A}) = Y(G(\mathscr{A})) \overset{\tau}{\twoheadleftarrow} Y(\beta G(\mathscr{A}))$ which realizes $\beta_{G(\mathscr{A})}\colon G(\mathscr{A}) \to \beta G(\mathscr{A})$.

This, it is not hard to see, is the free σ-algebra over Boolean algebras generated by $\mathscr{A}$, first constructed in [20] and [29]. (This is not the free σ-algebra over sets treated in [22].)

In exactly the same way, using our μ and λ, we construct $\mu_{\mathscr{A}}\colon \mathscr{A} \to \mu\mathscr{A}$ and $\lambda_{\mathscr{A}}\colon \mathscr{A} \to \lambda\mathscr{A}$ which are, *respectively, the maximal and minimal "σ-extensions"* treated in [21, Section 36].

Our method of discussing β, μ, λ (topological "pseudo-duality") is much akin to the treatments in [29] and [21]. See also the discussion of "m-extensions" in [22], which proceeds differently.

It may be possible to argue from the Boolean theory to a limited version of the $\mathscr{W}$-theory, but there is a touchy difference: for $G \in \mathscr{B}$, every $\mathscr{W}$-homomorphism out of G is a σ-homomorphism (so sometimes $\beta H = \mu H$); no infinite Boolean algebra has the analogous property, (so $\beta\mathscr{A} \neq \mu\mathscr{A}$ for infinite $\mathscr{A}$). Further, this paper is about epicompletions.

10.6 Problems. We collect a list of issues encountered above which we have passed over without understanding completely.

(a) 7.4 (b) implies that $\beta G = \mu G$ if and only if each $\mathscr{W}$-homomorphism out of G is a σ-homomorphism. What are these G's? Corollary 4.4 contributes to that. Perhaps more interesting is the same question in Arch, where the answer should look more like algebra.

(b) We have seen in Section 9 that if each $\mathscr{W}$-homomorphism out of G is complete, then $\beta G = \lambda G$, but not conversely. What are each of these G's? The former class looks particularly interesting, and one asks about that property in Arch.

(c) In $\mathscr{W}_\infty$ (see 9.8), what are the epics and the epicomplete objects? likewise, for "Arch$_\infty$". This is, of course, related to (b). (The corresponding questions for $\mathscr{W}_\sigma$ have been answered in Section 7 above, and we feel that complete understanding of "Arch$_\sigma$" is not far off.)

(d) There are four operators in $\mathscr{W}$ involved in this paper: β, μ, λ, ε. Questions (a) and (b) above are two examples of $4!/2!2! = 6$ questions. The other four questions seem to be interesting as well. Several of these questions seem to be non-trivial questions about the Baire fields of compact spaces. We would not be surprised if some of the answers depend on the axioms of set theory.

(e) Is λG always the smallest element of $\mathscr{BC}_\sigma(G)$? This the analogue for $\mathscr{W}$ of an old question of Sikorski about Boolean algebras [21].

(f) Is there some sense in which $B(X)$ is a "canonical" $\mathscr{B}$-completion of $C(X)$?

(g) The cardinal number $\aleph_0$ is everywhere in the theory of $\mathscr{B}$-completions. In Section 9, we succumbed to the temptation to "replace $\aleph_0$ by ∞". We have not succumbed to the temptation to "replace $\aleph_0$ by m" (for various reasons, not least of which is that this paper is long enough already). However, the various questions in that vein undeniably exist and are probably interesting. In considering these, one should compare the theory of m-extensions of Boolean algebras in [22] with the theory of σ-extensions in [21]. The former is more complicated exactly because not every m-algebra is m-representable. The analogue of that in $\mathscr{W}$ is that, upon simple-minded replacement of $\aleph_0$ by m in 3.3, (a) does not imply (c).

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