Epigraphical splitting for solving constrained convex formulations of inverse problems with proximal tools

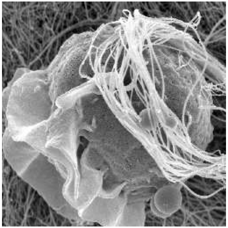
Nelly PUSTELNIK
Laboratoire de Physique, ENS Lyon – CNRS UMR 5672

Grenoble Optimization Day
November 5th, 2014

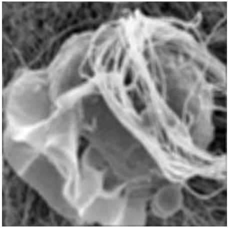
Collaboration

- Giovanni CHIERCHIA, Béatrice PESQUET-POPESCU
  LTCI, Télécom ParisTech/Institut Télécom– CNRS UMR 5141
- Jean-Christophe PESQUET
  LIGM, Université Paris-Est – CNRS UMR 8049

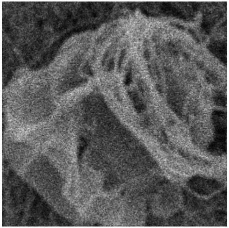
Motivation



$\overline{x} \in \mathbb{R}^{\overline{N}}$



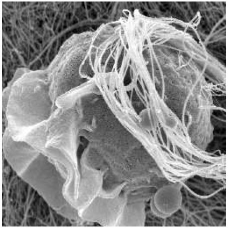
$A\overline{x} \in \mathbb{R}^{N}$



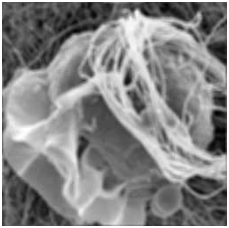
$z = \mathcal{P}_\alpha(A\overline{x})$

- $\overline{x}$: original image
- A : linear operator from $\mathbb{R}^{\overline{N}}$ to $\mathbb{R}^{N}$
- $\mathcal{P}_\alpha$: effect of noise where $\alpha > 0$ is the scaling parameter
- z : degraded image of size N

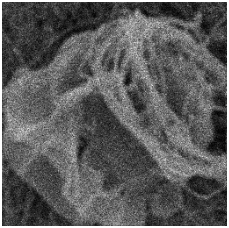
Motivation



$\overline{x} \in \mathbb{R}^{\overline{N}}$



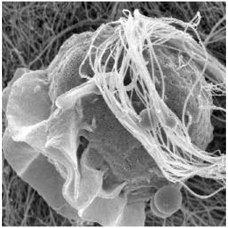
$A\overline{x} \in \mathbb{R}^{N}$



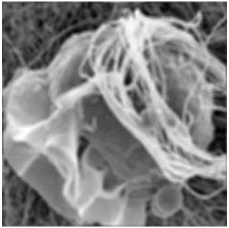
$z = \mathcal{P}_\alpha(A\overline{x})$

- $\overline{x}$: original image
  Assumption: sparse after some appropriate transform
- A : linear operator from $\mathbb{R}^{\overline{N}}$ to $\mathbb{R}^{N}$
- $\mathcal{P}_\alpha$: effect of noise where $\alpha > 0$ is the scaling parameter
- z : degraded image of size N

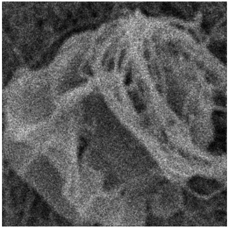
Motivation



$\overline{x} \in \mathbb{R}^{\overline{N}}$



$A\overline{x} \in \mathbb{R}^{N}$



$z = \mathcal{P}_\alpha(A\overline{x})$

- $\overline{x}$: original image

  Assumption: sparse after some appropriate transform
- A : linear operator from $\mathbb{R}^{\overline{N}}$ to $\mathbb{R}^{N}$
- $\mathcal{P}_\alpha$: effect of noise where $\alpha > 0$ is the scaling parameter
- z : degraded image of size N

  Objective: recover $\overline{x}$ from the observations z

Motivation

$$\widehat{x} \in \underset{x \in \mathbb{R}^{\overline{N}}}{\operatorname{Argmin}} \underbrace{g(Ax, z)}_{\text{Data fidelity term}} + \lambda \underbrace{f(x)}_{\text{Regularization term}}$$

$g(A\cdot, z) \in \Gamma_0(\mathbb{R}^{\overline{N}})$ $\quad$ $f \in \Gamma_0(\mathbb{R}^{\overline{N}})$

where $\lambda > 0$

Motivation : Existing works – Gaussian noise

Regularized approach	Constrained approach
$\min_{x \in \mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$	$\min_{\|Ax-z\|^2 \leq \eta} f(x)$
[Tikhonov, 1963]	[Combettes, Trussell, 1991]

Motivation : Existing works – Gaussian noise

Regularized approach	Constrained approach
$\min_{x\in\mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$	$\min_{\|Ax-z\|^2\leq\eta} f(x)$
[Tikhonov, 1963]	[Combettes, Trussell, 1991]

If $f = \|F\cdot\|^2$

Regularized approach	Constrained approach
→ Gradient-based methods	→ POCS [Trussell, Civanlar, 1984] → Subgradient projections [Luo, Combettes, 1999]

Motivation : Existing works – Gaussian noise

Regularized approach	**Constrained approach**
$$\min_{x\in\mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$$ [Tikhonov, 1963]	$$\min_{\|Ax-z\|^2\leq\eta} f(x)$$ [Combettes, Trussell, 1991]

If $f = \|F\cdot\|^2$

→ Gradient-based methods	→ POCS [Trussell, Civanlar, 1984] → Subgradient projections [Luo, Combettes, 1999]

If $f(x) = \sum_i |(Fx)^{(i)}|_1$

(where F is a wavelet transform, a frame)

→ Proximal methods [Combettes, Pesquet, 2011]	→ Proximal methods [Combettes, Pesquet, 2011]

Motivation : Existing works – Gaussian noise

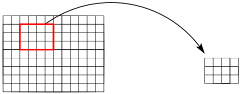
Regularized approach	**Constrained approach**
$$\min_{x\in\mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$$ [Tikhonov, 1963]	$$\min_{f(x)\leq\eta} \|Ax - z\|^2$$

Motivation : Existing works – Gaussian noise

Regularized approach	**Constrained approach**
$\min_{x\in\mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$	$\min_{f(x)\leq\eta} \|Ax - z\|^2$
[Tikhonov, 1963]	

If $f = \|\cdot\|_{1,p} = \sum_{b\in\mathbb{L}} \|B_b \cdot\|$ with $p \geq 1$

$\rightarrow$ block sparsity measure :
for every $b \in \mathbb{L} \subset \mathbb{K}$, B_b is a block selection transform.

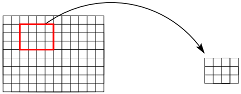


Motivation : Existing works – Gaussian noise

Regularized approach	**Constrained approach**
$$\min_{x\in\mathbb{R}^N} \|Ax - z\|^2 + \lambda f(x)$$ [Tikhonov, 1963]	$$\min_{f(x)\le\eta} \|Ax - z\|^2$$

If $f = \|\cdot\|_{1,p} = \sum_{b\in\mathbb{L}} \|B_b \cdot\|$ with $p \geq 1$

$\rightarrow$ block sparsity measure :
for every $b \in \mathbb{L} \subset \mathbb{K}$, B_b is a block selection transform.



$\rightarrow$ Proximal methods [Combettes, Pesquet, 2011]	$\rightarrow$ Inner iterations, **?** [Van Den Berg,Friedlander,2008]

Motivation : Existing works – Poisson noise

Regularized approach	**Constrained approach**
$\min_{x\in\mathcal{H}} D_{KL}(Tx,z)+\lambda f(x)$	$\min_{D_{KL}(Tx,z)\leq\eta} f(x)$

Motivation : Existing works – Poisson noise

Regularized approach

$$\min_{x\in\mathcal{H}} D_{KL}(Tx,z)+\lambda f(x)$$

If $f=\|F\cdot\|^2$

→ Cross-Entropy minimization
[Byrne, 1993]

→ Barrier function optimization
[Chouzenoux *et al.*, 2011]

Constrained approach

$$\min_{D_{KL}(Tx,z)\leq\eta} f(x)$$

→ **?**

Motivation : Existing works – Poisson noise

Regularized approach	**Constrained approach**
$\min_{x\in\mathcal{H}} D_{KL}(Tx,z)+\lambda f(x)$	$\min_{D_{KL}(Tx,z)\le\eta} f(x)$

If $f=\|F\cdot\|^2$

→ Cross-Entropy minimization [Byrne, 1993] | → **?**

→ Barrier function optimization [Chouzenoux *et al.*, 2011]

If $f(x)=\sum_i |(Fx)^{(i)}|_1$

(where F can denote a gradient filter, a wavelet transform, a frame)

→ Proximal methods [Combettes, Pesquet, 2011] | → **?**

Problem

$$\widehat{x} \in \operatorname*{Argmin}_{x \in \mathbb{R}^{\overline{N}}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} h_1(H_1 x) \leq \eta_1, \\ \vdots \\ h_S(H_S x) \leq \eta_S, \end{cases}$$

- $(\forall s \in \{1, \ldots, S\})$, $H_r \colon \mathbb{R}^{\overline{N}} \to \mathbb{R}^{M_s}$ is a linear operator,
- $(\forall s \in \{1, \ldots, S\})$, $h_s \in \Gamma_0(\mathbb{R}^{M_s})$,
- $(\forall r \in \{1, \ldots, R\})$, $T_r \colon \mathbb{R}^{\overline{N}} \to \mathbb{R}^{N_r}$ is a linear operator,
- $(\forall r \in \{1, \ldots, R\})$, $g_r \in \Gamma_0(\mathbb{R}^{N_r})$.

$\Rightarrow$ Any closed convex subset C_s of $\mathbb{R}^{M_s}$ can be expressed in this way by setting $\eta_s = 0$, $L = 1$ and $h_s = d_{C_s} = \| \cdot - P_{C_s} \|$

Problem

$$\widehat{x} \in \operatorname*{Argmin}_{x \in \mathbb{R}^{\overline{N}}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- $(\forall s \in \{1,\ldots,S\})$, $H_r \colon \mathbb{R}^{\overline{N}} \to \mathbb{R}^{M_s}$ is a bounded linear operator,
- $(\forall s \in \{1,\ldots,S\})$, C_s is a nonempty closed convex subset of $\mathbb{R}^{M_s}$,
- $(\forall r \in \{1,\ldots,R\})$, $T_r \colon \mathbb{R}^{\overline{N}} \to \mathbb{R}^{N_r}$ is a bounded linear operator,
- $(\forall r \in \{1,\ldots,R\})$, $g_r \in \Gamma_0(\mathbb{R}^{N_r})$.

Problem

$$\widehat{x} \in \operatorname*{Argmin}_{x \in \mathbb{R}^{\overline{N}}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- Forward-Backward [Combettes,Wajs,2005]
  $\rightarrow \min_x g_1(T_1 x) + g_2(x)$ with g_1 gradient Lipschitz function

Problem

$$\widehat{x} \in \operatorname*{Argmin}_{x \in \mathbb{R}^{\overline{N}}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- **Forward-Backward** [Combettes,Wajs,2005]
  $\rightarrow \min_x g_1(T_1 x) + g_2(x)$ with g_1 gradient Lipschitz function
- **Douglas-Rachford** [Combettes,Pesquet,2007]
  $\rightarrow \min_x g_1(x) + g_2(x)$

Problem

$$\widehat{x} \in \underset{x\in\mathbb{R}^{\overline{N}}}{\operatorname{Argmin}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- **Forward-Backward** [Combettes,Wajs,2005]
  $\rightarrow \min_x g_1(T_1 x) + g_2(x)$ with g_1 gradient Lipschitz function
- **Douglas-Rachford** [Combettes,Pesquet,2007]
  $\rightarrow \min_x g_1(x) + g_2(x)$
- **PPXA** [Combettes,Pesquet,2008]
  $\rightarrow \min_x \sum_{r=1}^{R} g_r(x) + \sum_{s=1}^{S} \iota_{C_s}(x)$

Problem

$$\widehat{x} \in \operatorname*{Argmin}_{x \in \mathbb{R}^{\overline{N}}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- PPXA + [Pesquet,Pustelnik,2012] / ADMM [Setzer,Steidl,Teuber,2009]
  $\rightarrow$ $\min_x \sum_{r=1}^{R} g_r(T_r x) + \sum_{s=1}^{S} \iota_{C_s}(H_s x)$
  $\rightarrow \sum_{r=1}^{R} T_r^* T_r + \sum_{s=1}^{S} H_s^* H_s$ invertible

Problem

$$\widehat{x} \in \underset{x\in\mathbb{R}^{\overline{N}}}{\operatorname{Argmin}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} H_1 x \in C_1, \\ \vdots \\ H_S x \in C_S, \end{cases}$$

- PPXA + [Pesquet,Pustelnik,2012] / ADMM [Setzer,Steidl,Teuber,2009]
  $\rightarrow \min_x \sum_{r=1}^{R} g_r(T_r x) + \sum_{s=1}^{S} \iota_{C_s}(H_s x)$
  $\rightarrow \sum_{r=1}^{R} T_r^* T_r + \sum_{s=1}^{S} H_s^* H_s$ invertible

- M+SFBF [Briceño-Arias,Combettes,2011]
  M+LFBF [Combettes,Pesquet,2012] and others [Vũ,2013][Condat,2013]
  $\rightarrow \min_x \sum_{r=1}^{R} g_r(T_r x) + \sum_{s=1}^{S} \iota_{C_s}(H_s x)$

Problem

For $n = 0, 1, \ldots$

$x^{[n]} = \sum_{r=1}^{R} \omega_r u_r^{[n]} + \sum_{s=1}^{S} \omega_s \overline{u}_s^{[n]}$ ⟵ Under technical assumptions, $(x^{[n]})_{n\in\mathbb{N}}$ generated by M+SFBF [Combettes,Briceño-Arias,2011] converges to $\hat{x}$

For $r = 1, \ldots, R$

$w_{1,r}^{[n]} = u_r^{[n]} - \gamma_\ell T_r^* v_r^{[n]}$

$w_{2,r}^{[n]} = v_r^{[n]} + \gamma_\ell T_r u_r^{[n]}$

For $s = 1, \ldots, S$

$\overline{w}_{1,s}^{[n]} = \overline{u}_s^{[n]} - \gamma_n H_s^* \overline{v}_s^{[n]}$

$\overline{w}_{2,s}^{[n]} = \overline{u}_s^{[n]} + \gamma_n H_s \overline{u}_s^{[n]}$

$p_1^{[n]} = \sum_{r=1}^{R} \omega_r w_{1,r}^{[n]} + \sum_{s=1}^{S} \omega_s \overline{w}_{1,s}^{[n]}$

For $r = 1, \ldots, R$

$$p_{2,r}^{[n]} = w_{2,r}^{[n]} - \frac{\gamma_n}{\omega_r} \operatorname{prox}_{\frac{\omega_r}{\gamma_n} g_r}\left(\frac{\omega_r}{\gamma_n} w_{2,r}^{[n]}\right)$$ ⟵ Proximity operator computation

$q_{1,r}^{[n]} = p_1^{[n]} - \gamma_n (T_r^* p_{2,r}^{[n]})$

$q_{2,r}^{[n]} = p_{2,r}^{[n]} + \gamma_n (T_r p_1^{[n]})$

Update $u_1^{[n+1]}$ and $v_1^{[n+1]}$

For $s = 1, \ldots, S$

$$\overline{p}_{2,s}^{[n]} = \overline{w}_{2,r}^{[n]} - \frac{\gamma_n}{\omega_s} P_{C_s}\left(\frac{\omega_s}{\gamma_n} \overline{w}_{2,r}^{[n]}\right)$$ ⟵ Projection computation

$\overline{q}_{1,s}^{[n]} = \overline{p}_1^{[n]} - \gamma_n (H_s^* \overline{p}_{2,s}^{[n]})$

$\overline{q}_{2,s}^{[n]} = \overline{p}_{2,s}^{[n]} + \gamma_n (H_s \overline{p}_1^{[n]})$

Update $\overline{u}_1^{[n+1]}$ and $\overline{v}_1^{[n+1]}$

Proximity operator

Definition [Moreau,1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of f at point $u \in \mathcal{H}$ is the unique point denoted by $\mathrm{prox}_f u$ such that

$$(\forall u \in \mathcal{H}) \qquad \mathrm{prox}_f u = \arg\min_{v \in \mathcal{H}} f(v) + \frac{1}{2}\|u - v\|^2$$

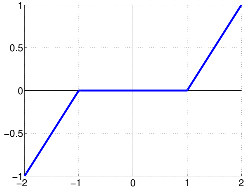
Proximity operator

Definition [Moreau,1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of f at point $u \in \mathcal{H}$ is the unique point denoted by $\mathrm{prox}_f u$ such that

$$(\forall u \in \mathcal{H}) \qquad \mathrm{prox}_f u = \arg\min_{v \in \mathcal{H}} f(v) + \frac{1}{2}\|u - v\|^2$$

Examples: closed form

- $\mathrm{prox}_{\chi\|\cdot\|_1}$: soft-thresholding with a fixed threshold $\chi > 0$



Proximity operator

Definition [Moreau,1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of f at point $u \in \mathcal{H}$ is the unique point denoted by $\mathrm{prox}_f u$ such that

$$(\forall u \in \mathcal{H}) \qquad \mathrm{prox}_f u = \arg\min_{v \in \mathcal{H}} f(v) + \frac{1}{2}\|u - v\|^2$$

Examples: closed form

- $\mathrm{prox}_{\chi\|\cdot\|_1}$: soft-thresholding with a fixed threshold $\chi > 0$
- $\mathrm{prox}_{\|\cdot\|_{1,2}}$ [Peyré,Fadili,2011].
- $\mathrm{prox}_{D_{KL}}$ [Chaux,Combettes,Pesquet,Wajs,2005].
- $\mathrm{prox}_{\iota_C} = P_C$: projection onto the convex set C.

Proximity operator

Definition [Moreau,1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of f at point $u \in \mathcal{H}$ is the unique point denoted by $\mathrm{prox}_f u$ such that

$$(\forall u \in \mathcal{H}) \qquad \mathrm{prox}_f u = \arg\min_{v \in \mathcal{H}} f(v) + \frac{1}{2}\|u - v\|^2$$

Examples: closed form

- $\mathrm{prox}_{\chi\|\cdot\|_1}$: soft-thresholding with a fixed threshold $\chi > 0$
- $\mathrm{prox}_{\|\cdot\|_{1,2}}$ [Peyré,Fadili,2011].
- $\mathrm{prox}_{D_{KL}}$ [Chaux,Combettes,Pesquet,Wajs,2005].

- $\mathrm{prox}_{\iota_C} = P_C$: projection onto the convex set C.
  - $\rightarrow$ range constraint: hypercube projection,
  - $\rightarrow$ closed half-space: half-space projection,

Proximity operator

Definition [Moreau,1965] Let $f \in \Gamma_0(\mathcal{H})$ where $\mathcal{H}$ denotes a real Hilbert space. The proximity operator of f at point $u \in \mathcal{H}$ is the unique point denoted by $\mathrm{prox}_f u$ such that

$$(\forall u \in \mathcal{H}) \qquad \mathrm{prox}_f u = \arg\min_{v \in \mathcal{H}} f(v) + \frac{1}{2}\|u - v\|^2$$

Examples: NO closed form

- $\mathrm{prox}_{\iota_C} = P_C$: projection onto the convex set C.

  → C models a $\ell_{1,p}$-ball constraint: iterative procedure for projection [Quattoni,Carreras,Collins,Darrell,2007] [Van Den Berg,Friedlander,2008].

  → constraint associated with the Kullback-Leibler divergence

  → constraint associated with the logistic cost function

Solution

- **Assumption**: separable function

  For every $y = [\underbrace{(\mathrm{y}^{(1)})^\top}_{\text{size } M^{(1)}}, \ldots, \underbrace{(\mathrm{y}^{(L)})^\top}_{\text{size } M^{(L)}}]^\top \in \mathbb{R}^M$,

$$y \in C \quad \Leftrightarrow \quad h(y) \leq \eta \quad \Leftrightarrow \quad \sum_{\ell=1}^{L} h^{(\ell)}(y^{(\ell)}) \leq \eta.$$

Solution

- **Assumption**: separable function

  For every $y = [\underbrace{(\mathrm{y}^{(1)})^\top}_{\text{size } M^{(1)}}, \ldots, \underbrace{(\mathrm{y}^{(L)})^\top}_{\text{size } M^{(L)}}]^\top \in \mathbb{R}^M$,

$$y \in C \quad \Leftrightarrow \quad h(y) \leq \eta \quad \Leftrightarrow \quad \sum_{\ell=1}^{L} h^{(\ell)}(y^{(\ell)}) \leq \eta.$$

- **Solution**: splitting the constraint into simpler constraints by introducing the auxiliary vector $\zeta = \left(\zeta^{(\ell)}\right)_{1 \leq \ell \leq L} \in \mathbb{R}^L$,

$$y \in C \quad \Leftrightarrow \quad \begin{cases} \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta, \\ (\forall \ell \in \{1, \ldots, L\}) \qquad h^{(\ell)}(\mathrm{y}^{(\ell)}) \leq \zeta^{(\ell)}. \end{cases}$$

Solution

- **Assumption**: separable function

  For every $y = [\underbrace{(\mathrm{y}^{(1)})^\top}_{\text{size } M^{(1)}}, \ldots, \underbrace{(\mathrm{y}^{(L)})^\top}_{\text{size } M^{(L)}}]^\top \in \mathbb{R}^M$,

$$y \in C \quad \Leftrightarrow \quad h(y) \leq \eta \quad \Leftrightarrow \quad \sum_{\ell=1}^{L} h^{(\ell)}(y^{(\ell)}) \leq \eta.$$

- **Solution**: splitting the constraint into simpler constraints by introducing the auxiliary vector $\zeta = \left(\zeta^{(\ell)}\right)_{1 \leq \ell \leq L} \in \mathbb{R}^L$,

$$y \in C \quad \Leftrightarrow \quad \begin{cases} \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta, \\ (\forall \ell \in \{1, \ldots, L\}) \qquad (\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} h^{(\ell)}. \end{cases}$$

Solution

$$y \in C \Leftrightarrow \begin{cases} \zeta \in V \\ (y, \zeta) \in E \end{cases}$$

- V denotes a closed half-space such that:

$$V = \left\{ \zeta \in \mathbb{R}^L \mid 1_L^\top \zeta \leq \eta \right\}$$

- E is the closed convex set associated to the epigraphical constraint:

$$E = \left\{ (y, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\}) \ (\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} h^{(\ell)} \right\}$$

Solution

$$y \in C \Leftrightarrow \begin{cases} \zeta \in V \\ (y, \zeta) \in E \end{cases}$$

- V denotes a closed half-space such that:

$$V = \left\{ \zeta \in \mathbb{R}^L \mid 1_L^\top \zeta \leq \eta \right\}$$

$\rightarrow$ P_V has a closed form: projection onto an half-space.

- E is the closed convex set associated to the epigraphical constraint:

$$E = \left\{ (y, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\}) \ (\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} h^{(\ell)} \right\}$$

Solution

$$y \in C \Leftrightarrow \begin{cases} \zeta \in V \\ (y, \zeta) \in E \end{cases}$$

- V denotes a closed half-space such that:

$$V = \left\{ \zeta \in \mathbb{R}^L \mid 1_L^\top \zeta \leq \eta \right\}$$

$\rightarrow$ P_V has a closed form: projection onto an half-space.

- E is the closed convex set associated to the epigraphical constraint:

$$E = \left\{ (y, \zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1, \ldots, L\})\ (\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} h^{(\ell)} \right\}$$

$\rightarrow$ P_E has a closed form for specific choice of $h^{(\ell)}$.

Solution

- Euclidean norm functions defined as:

$$\boxed{\big(\forall \ell \in \{1,\ldots,L\}\big)\big(\forall \mathrm{y}^{(\ell)} \in \mathbb{R}^{M^{(\ell)}}\big) \qquad h^{(\ell)}(\mathrm{y}^{(\ell)}) = \tau^{(\ell)}\|\mathrm{y}^{(\ell)}\|}$$

where $\tau^{(\ell)} \in \,]0,+\infty[$.

Solution

- Euclidean norm functions defined as:

$$\big(\forall \ell \in \{1,\dots,L\}\big)\big(\forall \mathrm{y}^{(\ell)} \in \mathbb{R}^{M^{(\ell)}}\big) \qquad h^{(\ell)}(\mathrm{y}^{(\ell)}) = \tau^{(\ell)} \|\mathrm{y}^{(\ell)}\|$$

where $\tau^{(\ell)} \in \,]0,+\infty[$.

- **Epigraphic projection**: for every $(\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) \in \mathbb{R}^{M^{(\ell)}} \times \mathbb{R}$

$$P_{\operatorname{epi} h^{(\ell)}}(\mathrm{y}^{(\ell)}, \zeta^{(\ell)}) = \begin{cases} (\mathrm{y}^{(\ell)}, \zeta^{(\ell)}), & \text{if } \|\mathrm{y}^{(\ell)}\| < \frac{\zeta^{(\ell)}}{\tau^{(\ell)}}, \\ (0,0), & \text{if } \|\mathrm{y}^{(\ell)}\| < -\tau^{(\ell)}\zeta^{(\ell)}, \\ \alpha^{(\ell)} \left(\mathrm{y}^{(\ell)}, \tau^{(\ell)}\|\mathrm{y}^{(\ell)}\|\right), & \text{otherwise,} \end{cases}$$

where $\alpha^{(\ell)} = \dfrac{1}{1+(\tau^{(\ell)})^2}\left(1 + \dfrac{\tau^{(\ell)}\zeta^{(\ell)}}{\|\mathrm{y}^{(\ell)}\|}\right)$.

Solution

- Infinity norms defined as:

$$\Big(\forall \ell \in \{1,\ldots,L\}\Big)\Big(\forall \mathrm{y}^{(\ell)} = (\mathrm{y}^{(\ell,m)})_{1\le m\le M^{(\ell)}} \in \mathbb{R}^{M^{(\ell)}}\Big)$$

$$\boxed{h^{(\ell)}(\mathrm{y}^{(\ell)}) = \max\left\{ \frac{|\mathrm{y}^{(\ell,m)}|}{\tau^{(\ell,m)}} \mid 1 \le m \le M^{(\ell)} \right\}}$$

where $(\tau^{(\ell,m)})_{1\le m\le M^{(\ell)}} \in \,]0,+\infty[^{M^{(\ell)}}$.

Solution

- **Epigraphic projection**:
  - $(\nu^{(\ell,m)})_{1\le m\le M^{(\ell)}}$: sequence of reals by sorting $(|y^{(\ell,m)}|/\tau^{(\ell,m)})_{1\le m\le M^{(\ell)}}$ in ascending order ($\nu^{(\ell,0)} = -\infty$ and $\nu^{(\ell,M^{(\ell)}+1)} = +\infty$).
  - $\overline{m}$ is the unique integer in $\{1,\ldots,M^{(\ell)}+1\}$ such that

$$\nu^{(\ell,\overline{m}-1)} < \frac{\zeta^{(\ell)} + \sum_{m=\overline{m}}^{M^{(\ell)}} \nu^{(\ell,m)}(\tau^{(\ell,m)})^2}{1+\sum_{m=\overline{m}}^{M^{(\ell)}}(\tau^{(\ell,m)})^2} \le \nu^{(\ell,\overline{m})}.$$

  - $(p^{(\ell)},\theta^{(\ell)}) = P_{\text{epi}\, h^{(\ell)}}(y^{(\ell)},\zeta^{(\ell)})$ with $p^{(\ell)} = (p^{(\ell,m)})_{1\le m\le M^{(\ell)}}$,

where

$$p^{(\ell,m)} = \begin{cases} y^{(\ell,m)}, & \text{if } |y^{(\ell,m)}| \le \tau^{(\ell,m)}\theta^{(\ell)}, \\ \tau^{(\ell,m)}\theta^{(\ell)}, & \text{if } y^{(\ell,m)} > \tau^{(\ell,m)}\theta^{(\ell)}, \\ -\tau^{(\ell,m)}\theta^{(\ell)}, & \text{if } y^{(\ell,m)} < -\tau^{(\ell,m)}\theta^{(\ell)}, \end{cases}$$

and

$$\theta^{(\ell)} = \frac{\max\left(\zeta^{(\ell)} + \sum_{m=\overline{m}}^{M^{(\ell)}} \nu^{(\ell,m)}(\tau^{(\ell,m)})^2, 0\right)}{1+\sum_{m=\overline{m}}^{M^{(\ell)}}(\tau^{(\ell,m)})^2}.$$

RGB image restoration with missing samples

Original
$\underline{x}$

Degraded
$z = A\overline{x} + w$

- Original (multicomponent) image: $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_R) \in (\mathbb{R}^M)^R$
- Linear operator: $A = (A_{j,i})_{1 \leq j \leq S, 1 \leq i \leq R}$, with $A_{j,i} \in \mathbb{R}^{K \times M}$
- Zero-mean white Gaussian noise: $w \in (\mathbb{R}^K)^S$
- Degraded image: $z = (z_1, \ldots, z_S) \in (\mathbb{R}^K)^S$

RGB image restoration with missing samples

$$\widehat{x} \in \underset{x\in(\mathbb{R}^M)^R}{\text{Argmin}} \; \underbrace{\|Ax - z\|_2^2}_{\text{Data fidelity term}} + \lambda \underbrace{g(x)}_{\text{Regularization term}}$$

- Component-wise Total Variation *(CC-TV)*
  [Blomgren 1998] [Zach 2007]

- Structure Tensor TV *(ST-TV)*
  $\rightarrow \ell_p$ matrix-norm regularization
  [Di Zenzo 1986] [Sapiro 1996] [Weickert 1999] [Tschumperlé 2001] [Bresson 2008] [Duval 2009][Goldluecke 2012]

RGB image restoration with missing samples

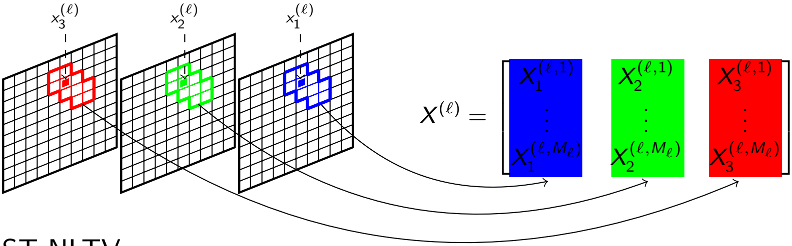
$$\widehat{x} \in \underset{x \in C \subset (\mathbb{R}^M)^R}{\operatorname{Argmin}} \|Ax - z\|_2^2 \quad \text{subj. to} \quad g(x) \leq \eta$$

- Constrained approach
- Regularization by ST Non-Local TV *(ST-NLTV)*
  $\rightarrow$ NLTV better preserves texture, details and fine structures
  $\rightarrow$ ST better reveals features not visible in single components

RGB image restoration with missing samples

- Non-Local gradient at point $\ell \in \{1, \dots, M\}$

$$X^{(\ell)} = \left(\omega_{\ell,n}\,(x_i^{(\ell)} - x_i^{(n)})\right)_{n\in\mathcal{N}_\ell,\,1\le i\le R} \in \mathbb{R}^{M_\ell\times R}$$



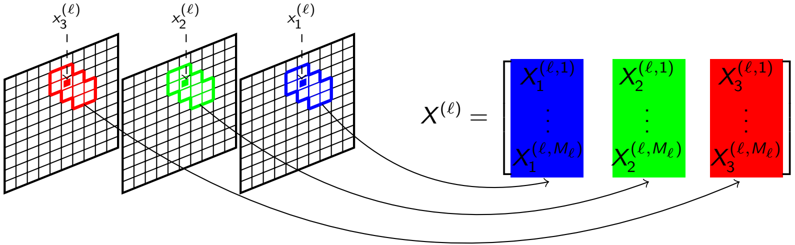
- ST-NLTV

$$g(x) = \sum_{\ell=1}^{M} \tau_\ell \,\|X^{(\ell)}\|_p \quad \Leftrightarrow \quad g(x) = \sum_{\ell=1}^{M} \tau_\ell \left(\sum_{m=1}^{\min\{M_\ell,R\}} \left(\sigma_{X^{(\ell)}}^{(m)}\right)^p \right)^{1/p}$$

RGB image restoration with missing samples

- Non-Local gradient at point $\ell \in \{1, \dots, M\}$

$$X^{(\ell)} = \left(\omega_{\ell,n}\,(x_i^{(\ell)} - x_i^{(n)})\right)_{n\in\mathcal{N}_\ell,\ 1\leq i\leq R} \in \mathbb{R}^{M_\ell \times R}$$



- Special case: ST-TV
  - $\mathcal{N}_\ell \rightarrow$ horizontal/vertical neighbours
  - $\omega_{\ell,n} \equiv 1$

RGB image restoration with missing samples

$$\widehat{x} \in \operatorname*{Argmin}_{x \in C} \|Ax - z\|_2^2 \quad \text{subject to} \quad Fx \in D$$

RGB image restoration with missing samples

$$\widehat{x} \in \underset{x \in C}{\mathrm{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad Fx \in D$$

⋮

$$(\widehat{x}, \widehat{\zeta}) \in \underset{(x,\zeta) \in C \times V}{\mathrm{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E$$

RGB image restoration with missing samples

$$\widehat{x} \in \underset{x \in C}{\operatorname{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad Fx \in D$$

⋮

$$(\widehat{x}, \widehat{\zeta}) \in \underset{(x,\zeta) \in C \times V}{\operatorname{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E$$

- Collection of epigraphs

$$E = \left\{(X, \zeta) \;\middle|\; (X^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} \|\cdot\|_p \quad (\forall \ell \in \{1, \ldots, M\})\right\}$$

RGB image restoration with missing samples

$$\widehat{x} \in \underset{x \in C}{\operatorname{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad Fx \in D$$

$$\vdots$$

$$(\widehat{x}, \widehat{\zeta}) \in \underset{(x,\zeta) \in C \times V}{\operatorname{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E$$

- Collection of epigraphs

$$E = \left\{(X, \zeta) \;\middle|\; (X^{(\ell)}, \zeta^{(\ell)}) \in \operatorname{epi} \|\cdot\|_p \quad (\forall \ell \in \{1, \ldots, M\})\right\}$$

- Closed half-space

$$V = \left\{\zeta \in \mathbb{R}^M \mid 1_M^\top \zeta \leq \eta\right\}$$

with $1_M = (1, \ldots, 1)^\top \in \mathbb{R}^M$

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  $\rightarrow$ can we extend these results to matrix norms ?

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$ [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  $\rightarrow$ can we extend these results to matrix norms ?
- S.V.D. : $X^{(\ell)} = U^{(\ell)}\ \mathrm{Diag}(s^{(\ell)})\ {V^{(\ell)}}^\top$

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  $\rightarrow$ can we extend these results to matrix norms ?
- S.V.D. : $X^{(\ell)} = U^{(\ell)}\ \mathrm{Diag}(s^{(\ell)})\ {V^{(\ell)}}^\top$
- Proximity operator of spectral functions [Lewis 1995]

$$\mathrm{prox}_{\|\cdot\|_p}(X^{(\ell)}) = U^{(\ell)}\ \mathrm{Diag}(\mathrm{prox}_{\|\cdot\|_p}(s^{(\ell)}))\ {V^{(\ell)}}^\top$$

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with p $\in \{1, 2, +\infty\}$
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  $\rightarrow$ can we extend these results to matrix norms ?
- S.V.D. : $X^{(\ell)} = U^{(\ell)}\ \text{Diag}(s^{(\ell)})\ {V^{(\ell)}}^\top$
- Proximity operator of spectral functions [Lewis 1995]

$$\text{prox}_{\|\cdot\|_p}(X^{(\ell)}) = U^{(\ell)}\ \text{Diag}(\text{prox}_{\|\cdot\|_p}(s^{(\ell)}))\ {V^{(\ell)}}^\top$$

Epigraphical projection

1. $P_{\text{epi}\,\|\cdot\|_p}(X^{(\ell)}, \zeta^{(\ell)}) = \left(U^{(\ell)}\ \text{Diag}(t^{(\ell)})\ {V^{(\ell)}}^\top,\ \theta^{(\ell)}\right)$

RGB image restoration with missing samples

- $P_{\text{epi}\,\|\cdot\|_p}$ exists for vectorial norms with $p \in \{1, 2, +\infty\}$
  [Pang 2003] [Pock 2010] [Ding 2012] [Chierchia 2012]
  $\rightarrow$ can we extend these results to matrix norms ?
- S.V.D. : $X^{(\ell)} = U^{(\ell)}\ \text{Diag}(s^{(\ell)})\ {V^{(\ell)}}^\top$
- Proximity operator of spectral functions [Lewis 1995]

$$\text{prox}_{\|\cdot\|_p}(X^{(\ell)}) = U^{(\ell)}\ \text{Diag}(\text{prox}_{\|\cdot\|_p}(s^{(\ell)}))\ {V^{(\ell)}}^\top$$

Epigraphical projection

1. $P_{\text{epi}\,\|\cdot\|_p}(X^{(\ell)}, \zeta^{(\ell)}) = \left(U^{(\ell)}\ \text{Diag}(t^{(\ell)})\ {V^{(\ell)}}^\top,\ \theta^{(\ell)}\right)$
2. $(t^{(\ell)},\ \theta^{(\ell)}) = P_{\text{epi}\,\|\cdot\|_p}(s^{(\ell)}, \zeta^{(\ell)})$

RGB image restoration with missing samples

$$(\widehat{x}, \widehat{\zeta}) \in \underset{(x,\zeta)\in C\times W}{\text{Argmin}} \|Ax - z\|_2^2 \quad \text{subject to} \quad (Fx, \zeta) \in E$$

- Degradation: 3×3 uniform blur, 90% of decimation, AWGN with $\alpha = 10$
- Color space: RGB
  → pixels of z have missing colors
  → impossible to work into YCbCr, CIELab, ...
- Dynamics range constraint: $x_i^{(\ell)} \in [0, 255]$
- Weights $\omega_{\ell,n}$ estimated as in [Foi 2012]
- Choice of η based on image characteristics

RGB image restoration with missing samples


Original

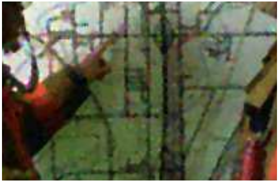

Noisy


Zoom

RGB image restoration with missing samples

ℓ_1-CC-TV
16.15 dB

ℓ_2-CC-TV
16.32 dB



ℓ_∞-CC-TV
16.05 dB

RGB image restoration with missing samples

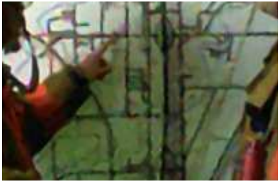
ℓ_1-ST-TV
17.08 dB

ℓ_2-ST-TV
16.84 dB

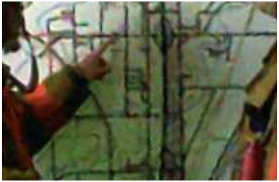
ℓ_∞-ST-TV
16.43 dB

RGB image restoration with missing samples

ℓ_1-CC-NLTV
16.87 dB



ℓ_2-CC-NLTV
17.20 dB



ℓ_∞-CC-NLTV
17.22 dB

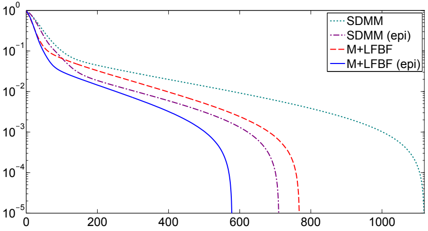
RGB image restoration with missing samples

ℓ_1-**ST-NLTV**
18.20 dB

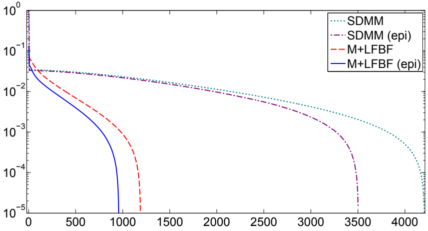
ℓ_2-ST-NLTV
17.46 dB

ℓ_∞-ST-NLTV
16.67

RGB image restoration with missing samples



ℓ_1-ST-TV

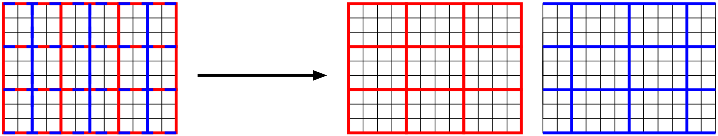


ℓ_1-ST-NLTV

Poisson based restoration

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{b\in\mathbb{L}} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ g(Ax, z) \leq \eta. \end{cases}$$

- For computational reasons, it will be assumed that there exists a partition of $\mathbb{L}$ in S subsets $(\mathbb{L}_s)_{1\leq s\leq S}$ such that $\sum_{b\in\mathbb{L}} \|B_b \cdot\| = \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b \cdot\|$ (i.e. grouped into S sets of non-overlapping blocks).



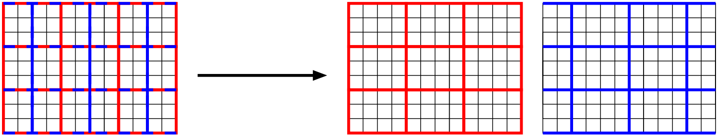
$(S = 2)$

- *Particular case : $S = 1$, $\mathbb{L} = \mathbb{L}_1 = \mathbb{K}$ and, for every $b \in \mathbb{L}$, B_b selects one element (i.e. one pixel) $\to$* **the classical ℓ^1-norm is obtained**.

Poisson based restoration

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ g(Ax, z) \le \eta. \end{cases}$$

- For computational reasons, it will be assumed that there exists a partition of $\mathbb{L}$ in S subsets $(\mathbb{L}_s)_{1\le s\le S}$ such that $\sum_{b\in\mathbb{L}} \|B_b \cdot\| = \sum_{s=1}^{S}\sum_{b\in\mathbb{L}_s} \|B_b \cdot\|$ (i.e. grouped into S sets of non-overlapping blocks).



$(S = 2)$

- *<u>Particular case</u> : $S = 1$, $\mathbb{L} = \mathbb{L}_1 = \mathbb{K}$ and, for every $b \in \mathbb{L}$, B_b selects one element (i.e. one pixel) $\rightarrow$* **the classical ℓ^1-norm is obtained**.

Poisson based restoration

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ g(Ax, z) \le \eta. \end{cases}$$

that is equivalent to

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ Ax \in D \end{cases}$$

with $D = \{u \in \mathbb{R}^K \mid g(u,z) \le \eta\} = \mathrm{lev}_{\le\eta}\, g(\cdot, z)$.

Poisson based restoration

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ g(Ax, z) \leq \eta. \end{cases}$$

that is equivalent to

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{s=1}^{S} \sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ Ax \in D \end{cases}$$

with $D = \{u \in \mathbb{R}^K \mid g(u, z) \leq \eta\} = \text{lev}_{\leq\eta}\, g(\cdot, z)$.

- **Projection onto** D
  - $\rightarrow$ Closed form if $g(\cdot, z) = \|\cdot - z\|^2$ [Rockafellar, 1969].
  - $\rightarrow$ NO closed form in a general context .

Poisson based restoration

Explicit form of the projection operator associated with :

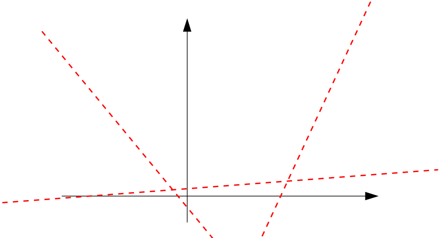
$$h_\ell(\mathrm{v}^\ell) = \max\{\mathrm{v}^{(\ell,j)} + \eta^{(\ell,j)} \mid 1 \le j \le M^{(\ell)}\}$$

where

→ $\mathrm{v}^{(\ell)} = (\mathrm{v}^{(\ell,1)}, \ldots, \mathrm{v}^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}}$

→ $\ell \in \{1, \ldots, L\}$ and $(\eta^{(\ell,1)}, \ldots, \eta^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}}$

Example for $L = 1$ and $M^{(1)} = 3$:



Poisson based restoration

Explicit form of the projection operator associated with :

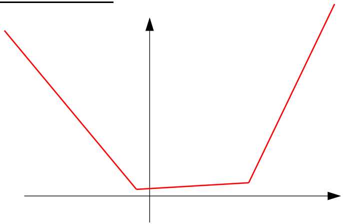
$$h_\ell(\mathrm{v}^\ell) = \max\{\mathrm{v}^{(\ell,j)} + \eta^{(\ell,j)} \mid 1 \leq j \leq M^{(\ell)}\}$$

where

→ $\mathrm{v}^{(\ell)} = (\mathrm{v}^{(\ell,1)}, \ldots, \mathrm{v}^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}}$

→ $\ell \in \{1, \ldots, L\}$ and $(\eta^{(\ell,1)}, \ldots, \eta^{(\ell,M^{(\ell)})})^\top \in \mathbb{R}^{M^{(\ell)}}$

Example for $L = 1$ and $M^{(1)} = 3$:

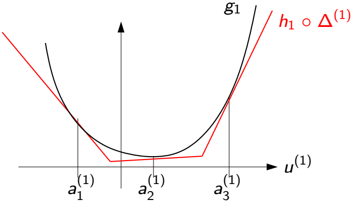


Poisson based restoration

$$g(u,z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \simeq \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)})$$

- $h_\ell(\mathrm{v}^{(\ell)}) = \max\{\mathrm{v}^{(\ell,j)} + \eta^{(\ell,j)} \mid 1 \leq j \leq M^{(\ell)}\}$,
- $\eta^{(\ell,j)} = g_\ell(a_j^{(\ell)}, z^{(\ell)}) - \delta_j^{(\ell)} a_j^{(\ell)}$,
- $\delta_j^{(\ell)} \in \mathbb{R}$ is any subgradient of $g_r(\cdot, z_\ell)$ at $a_j^{(\ell)}$,
- $\Delta^{(\ell)} = [\delta_1^{(\ell)}, \ldots, \delta_{M^{(\ell)}}^{(\ell)}]^\top$.

$\rightarrow$ The approximation can be as close as desired by choosing $M^{(\ell)}$ large enough.

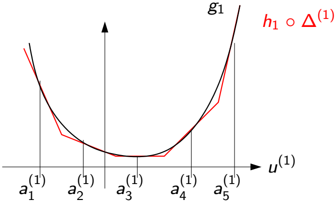


Poisson based restoration

$$g(u,z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \simeq \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)})$$

- $h_\ell(\mathrm{v}^{(\ell)}) = \max\{\mathrm{v}^{(\ell,j)} + \eta^{(\ell,j)} \mid 1 \le j \le M^{(\ell)}\}$,
- $\eta^{(\ell,j)} = g_\ell(a_j^{(\ell)}, z^{(\ell)}) - \delta_j^{(\ell)} a_j^{(\ell)}$,
- $\delta_j^{(\ell)} \in \mathbb{R}$ is any subgradient of $g_r(\cdot, z_\ell)$ at $a_j^{(\ell)}$,
- $\Delta^{(\ell)} = [\delta_1^{(\ell)}, \dots, \delta_{M^{(\ell)}}^{(\ell)}]^\top$.

→ The approximation can be as close as desired by choosing $M^{(\ell)}$ large enough.



Poisson based restoration

$$\underset{x\in\mathbb{R}^{\overline{N}}}{\text{minimize}} \sum_{s=1}^{S}\sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} x \in C \\ Ax \in D \end{cases}$$

$\Rightarrow$ Approximated criterion :

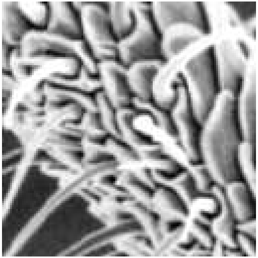
$$\underset{(x,\zeta)\in\mathbb{R}^{\overline{N}}\times\mathbb{R}^L}{\text{minimize}} \sum_{s=1}^{S}\sum_{b\in\mathbb{L}_s} \|B_b F x\| \quad \text{subject to} \quad \begin{cases} (x,\zeta) \in C \times V \\ \Delta A x \in E \end{cases}$$

where

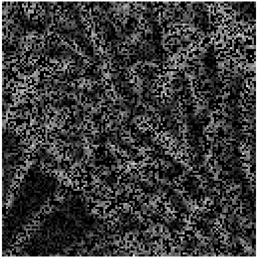
- $D = \{u \in \mathbb{R}^L \mid g(u,z) \leq \eta\}$,
- $V = \{\zeta \in \mathbb{R}^L \mid 1_L^\top \zeta \leq \eta\}$,
- $E = \{(\mathrm{v},\zeta) \in \mathbb{R}^M \times \mathbb{R}^L \mid (\forall \ell \in \{1,\ldots,L\})\ (\mathrm{v}^{(\ell)},\zeta^{(\ell)}) \in \operatorname{epi} h_\ell\}$,
- For every $u \in \mathbb{R}^L$, $g(u,z) = \sum_{\ell=1}^{L} g_\ell(u^{(\ell)}, z^{(\ell)}) \simeq \sum_{\ell=1}^{L} h_\ell(\Delta^{(\ell)} u^{(\ell)})$.

Poisson based restoration

- Electron microscopy image of size $\overline{N} = 128 \times 128$,
- T denotes a randomly decimated blur : uniform blur of size 3×3 and approximately 60% of missing data, that leads to $L = 9834$,
- Poisson noise with scaling parameter 0.5.



Original

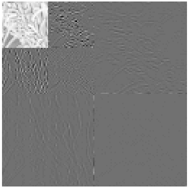


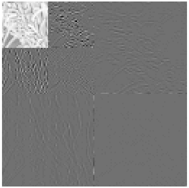
Degraded

Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,

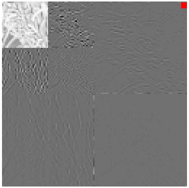


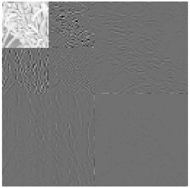


Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,

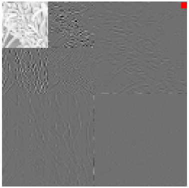


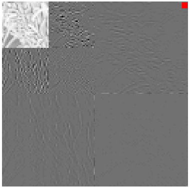


Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0,255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,
  - Block_PrimalDual : Blocks gathering primal and dual DTT coefficients,

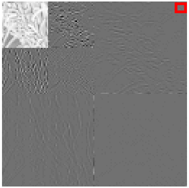


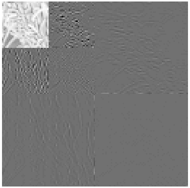


Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,
  - Block_PrimalDual : Blocks gathering primal and dual DTT coefficients,
  - Block_4Pixel_overlap : spatially overlapping blocks of size 2×2 are employed for each tree (primal or dual) separately.

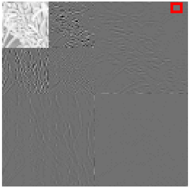


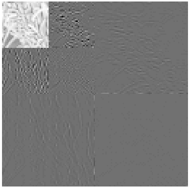


Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,
  - Block_PrimalDual : Blocks gathering primal and dual DTT coefficients,
  - Block_4Pixel_overlap : spatially overlapping blocks of size 2×2 are employed for each tree (primal or dual) separately.

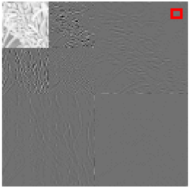


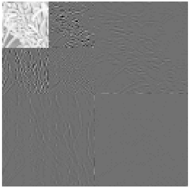


Poisson based restoration

Choice of the criterion :

- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,
  - Block_PrimalDual : Blocks gathering primal and dual DTT coefficients,
  - Block_4Pixel_overlap : spatially overlapping blocks of size 2×2 are employed for each tree (primal or dual) separately.

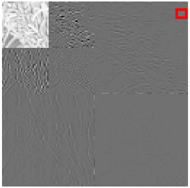


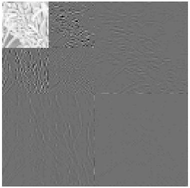


Poisson based restoration

Choice of the criterion :

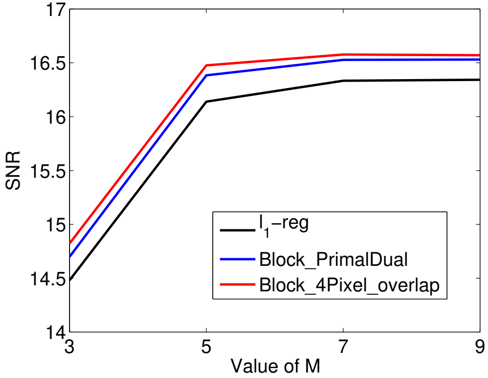
- Data fidelity : approximation of the Poisson likelihood,
  - Influence of $M \equiv M^{(\ell)}$,
- $C = [0, 255]^{\overline{N}}$,
- F : Dual-Tree Transform (DTT) – symmlet 6, 2 levels,
- Blocks :
  - ℓ_1-reg : Classical ℓ_1 cost function,
  - Block_PrimalDual : Blocks gathering primal and dual DTT coefficients,
  - Block_4Pixel_overlap : spatially overlapping blocks of size 2×2 are employed for each tree (primal or dual) separately.





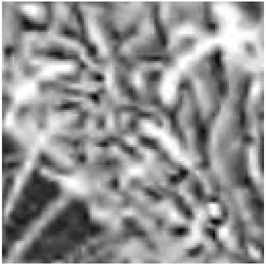
Poisson based restoration

- Impact of M and of the regularization term.

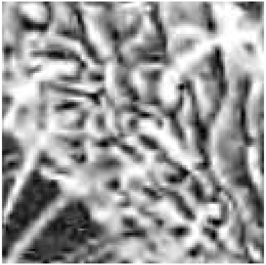


Poisson based restoration

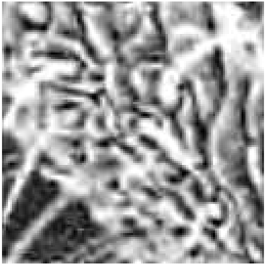
- $M = 7$,
- Impact of the regularization term.



ℓ_1-reg
SNR = 16.3 dB



Block_PrimalDual
SNR = 16.5 dB



Block_4Pixel_overlap
SNR = 16.6 dB

Conclusions

$$\operatorname*{Argmin}_{x} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} \sum_{\ell=1}^{L} h_1^{(\ell)}\big((H_1 x)^{(\ell)}\big) \leq \eta_1 \\ H_2 x \in C_2 \\ \vdots \\ H_S x \in C_S \end{cases}$$

$\vdots$

$$\operatorname*{Argmin}_{x,\zeta} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} (\forall \ell \in \{1,\ldots,L\}) \;\; h_1^{(\ell)}\big((H_1 x)^{(\ell)}\big) \leq \zeta^{(\ell)} \\ \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta_1 \\ H_2 x \in C_2 \\ \vdots \\ H_S x \in C_S \end{cases}$$

Conclusions

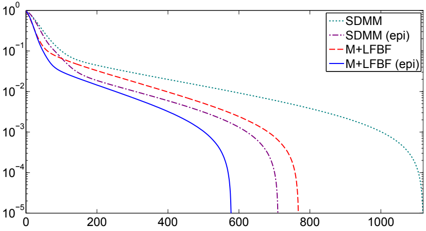
$$\underset{x}{\text{Argmin}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} \sum_{\ell=1}^{L} h_1^{(\ell)}\left((H_1 x)^{(\ell)}\right) \leq \eta_1 \\ H_2 x \in C_2 \\ \vdots \\ H_S x \in C_S \end{cases}$$

$$\vdots$$

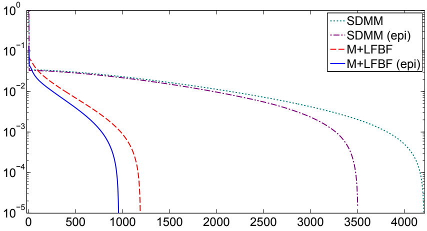
$$\underset{x,\zeta}{\text{Argmin}} \sum_{r=1}^{R} g_r(T_r x) \quad \text{s.t.} \quad \begin{cases} (\forall \ell \in \{1,\ldots,L\}) \;\; h_1^{(\ell)}\left((H_1 x)^{(\ell)}\right) \leq \zeta^{(\ell)} \\ \sum_{\ell=1}^{L} \zeta^{(\ell)} \leq \eta_1 \\ H_2 x \in C_2 \\ \vdots \\ H_S x \in C_S \end{cases}$$

$\rightarrow P_{\text{epi}\, h_1^{(\ell)}}$: closed form when $h_1^{(\ell)}$ models a Euclidean or infinity norm.

Conclusions

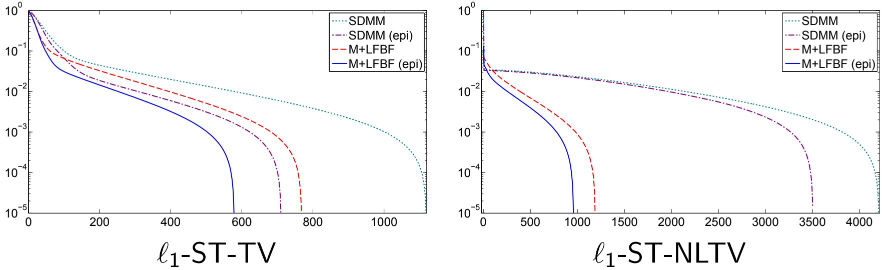


ℓ_1-ST-TV



ℓ_1-ST-NLTV

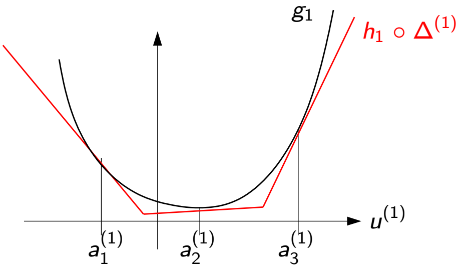
Conclusions



$\rightarrow$ Faster than direct methods

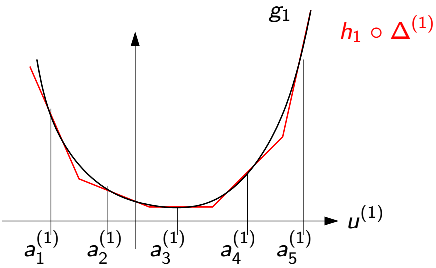
[Quattoni,Carreras,Collins,Darrell,2007] [Van Den Berg,Friedlander,2008] .

Conclusions



$\rightarrow$ Links with bundle methods ?

Conclusions



$\rightarrow$ Links with bundle methods ?

References

- G. Chierchia, N. Pustelnik, J.-C. Pesquet, B. Pesquet-Popescu, "*Epigraphical splitting for solving constrained convex formulations of inverse problems with proximal tools*," **Signal, Image and Video Processing**, to appear, 2014.

- G. Cherchia, N. Pustelnik, B. Pesquet-Popescu, J.-C. Pesquet, "*A Non-Local Structure Tensor Based Approach for Multicomponent Image Recovery Problems*," **IEEE Trans. Image processing**, to appear 2014

- G. Cherchia, N. Pustelnik, J.-C. Pesquet, and B. Pesquet-Popescu, "*Epigraphic proximal projection for sparse Multiclass SVM*," **IEEE ICASSP**, Florence, Italy, May 4-9, 2014