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# Equal-Time Commutation Relations for Heisenberg Fields in the Lee Model

M. SEETHARAMAN and V. SRINIVASAN*)

*Department of Theoretical Physics, University of Madras, Madras-600025*

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The self-consistent method of Umezawa is applied to a model with mass and coupling constant renormalizations. The equal-time commutation relation (E.T.C.R.) for Heisenberg fields is derived and not assumed. Light is shed on the interrelationship between the microscopic causality condition and the existence of a bound state of an earlier paper.

## § 1. Introduction

The self-consistent method proposed by Umezawa and developed by him and his collaborators[1)~4)] has been successfully applied to problems in both high energy physics[5)] and the area of many-body problems, particularly in superconductivity[6)] and ferromagnetism.[7)] In this paper we apply this method to the solvable Dirac-Lee model which, unlike the pair model considered in Ref. 3), has mass and coupling constant renormalizations. The equal-time commutation relations among Heisenberg fields are derived and not assumed. Further we show that the wave-function renormalization constant is determined from microcausality. This result sheds light on the question why in Ref. 3) the existence of the bound state was found to be closely connected with microcausality.

The self-consistent method exploits the duality between basic fields and observed particles concretely exhibited in the presence of interactions. The basic Heisenberg fields obey nonlinear field equations which reflect the laws of nature, while the physical particles are the ones that appear in observations. These physical fields obey free field equations and define the physical Fock space. The field equations for the Heisenberg fields are to be regarded as operator equations in this physical Fock space. The Heisenberg field is related to the physical field by means of a mapping known as the dynamical map.[4)] The set of physical fields is taken to form an irreducible operator ring. Therefore, any operator of the Fock space can be written as a sum of normal products of physical fields. The ability of this method to predict new particles rests on this postulate. The asymptotic condition may tell us that the physical fields are not complete. The completion of this set of fields then requires the existence of other particles at the level of observation and should be incorporated.[3),8)] The success of this method therefore rests on the proper choice of the set of physical fields.

*) C.S.I.R. Pool Officer.

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Another ingredient in the self-consistent method which assists us in writing down the dynamical map is that the invariance properties of the field equations obeyed by the Heisenberg fields cannot dissappear but should remain at every stage of the calculation if we want the theory to be internally consistent. However due to nonlinearity of the field equations it may happen that the transformation generating the invariance appears in a new form when written in terms of physical fields. The underlying philosophy here is that the basic invariance of the equations is sacrosanct but the symmetry can manifest in a different shape. This view is termed as the "dynamical rearrangement" of symmetries.[4),6),9)]

It should be noted here that since the expansion of the Heisenberg field is in terms of "in fields", the coefficients of the dynamical map are "retarded" in nature. Also we do not need the equal time commutation relations of the Heisenberg fields as in conventional quantum field theory. However for the Heisenberg fields we shall assume the condition of microcausality that two local operators on a space-like surface (anti-) commute with each other. We thus require that the equal-time commutation relation (E.T.C.R.) among Heisenberg operators are made up of only terms proportional to the $\delta$-function and its space derivatives of finite order:

$$[\psi_i(\boldsymbol{x},t),\psi_j(\boldsymbol{y},t)]_{\pm}=P_{ij}(\boldsymbol{\nabla})\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{I}$$

Here $P$ is a finite (matrix) polynomial in $\boldsymbol{\nabla}$, with no *a priori* conditions on its coefficients. As a matter of fact, when the coefficients of the dynamical map are determined, the structure of $P$ is also completely determined. Thus in the self-consistent method, unlike in conventional QFT, the E.T.C.R.'s are to be computed.

## § 2. Description of model

We postulate the following set of equations[10)] for the Lee-Dirac model:

$$\left(\frac{\partial}{\partial t}+im_0\right)\psi(y)=-ig\int d^4x\alpha(x-y)\theta^\dagger(x)V(y), \tag{1}$$

$$\left(\frac{\partial}{\partial t}+iM_0\right)V(y)=-ig\int d^4x\alpha(x-y)\theta(\boldsymbol{x})\psi(y), \tag{2}$$

$$\left(\frac{\partial}{\partial t}+i\sqrt{\mu_0^2-\boldsymbol{\nabla}^2}\right)\theta(y)=-ig\int d^4x\alpha(y-x)\psi^\dagger(x)V(x). \tag{3}$$

Here $\alpha(x)$ is the cutoff function

$$\alpha(x)=\delta(t_x)\int\frac{d^3k}{\sqrt{(2\pi)^3}}\frac{\alpha(\omega_{\boldsymbol{k}})}{\sqrt{2\omega_{\boldsymbol{k}}}}e^{i|\boldsymbol{k}|\cdot|x|} \tag{4}$$

and

$$\omega_{\boldsymbol{k}}=\sqrt{\boldsymbol{k}^2+\mu_{\boldsymbol{k}}^2}.$$

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To effect the dynamical mapping we now choose a set of free fields ($\psi^{\text{in}}$, $V^{\text{in}}$, $\theta^{\text{in}}$) with Fourier expansions

$$\psi^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3kN_k^{\text{in}}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-m_kt)\}, \tag{5}$$

$$V^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3kV_k^{\text{in}}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-M_kt)\}, \tag{6}$$

$$\theta^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3k\theta_k^{\text{in}}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega_kt)\}, \tag{7}$$

and we shall assume that

$$[N_p^{\text{in}}, N_q^{\text{in}\dagger}]_+=[V_p^{\text{in}}, V_q^{\text{in}\dagger}]_+=[\theta_p^{\text{in}}, \theta_q^{\text{in}\dagger}]=\delta(\boldsymbol{p}-\boldsymbol{q}). \tag{8}$$

It may be noted that the masses occurring in Eqs. (5)~(7) are also to be determined from the dynamical map with the help of the field equations.

As pointed out in the Introduction we shall now expand $\psi$, $V$, $\theta$ in terms of normal products of $\psi^{\text{in}}$, $\theta^{\text{in}}$ and $V^{\text{in}}$ and then determine the coefficients of the expansion by using (8) and the equations of motion. To write the map we note that Eqs. (1)~(3) are invariant under the transformation

$$\psi\to\psi e^{i\phi},\qquad \theta\to\theta e^{i\phi},\qquad V\to Ve^{2i\phi}. \tag{9}$$

Considering a special solution where the transformation (9) is induced by

$$\begin{aligned}&\psi^{\text{in}}(x)\to\psi^{\text{in}}(x)e^{i\phi},\\&\theta^{\text{in}}(x)\to\theta^{\text{in}}(x)e^{i\phi},\\&V^{\text{in}}(x)\to V^{\text{in}}(x)e^{2i\phi},\end{aligned} \tag{10}$$

the dynamical map is written as*)

$$\begin{aligned}\psi(x)=\psi^{\text{in}}(x)&+\int d^3pd^3qd^3ra_p(\boldsymbol{q},\boldsymbol{r})N_{-p+q+r}^{\text{in}\dagger}N_q^{\text{in}}N_r^{\text{in}}\\&\times\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(m_r+m_q-m_{-p+q+r})t\}+\int d^3pd^3qd_p(\boldsymbol{q})\theta_q^{\text{in}\dagger}\\&\times V_{p+q}^{\text{in}}\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}+i(\omega_q-M_{p+q})t\}+\int d^3pd^3qd^3rc_p(\boldsymbol{q},\boldsymbol{r})\\&\times\theta_q^{\text{in}\dagger}\theta_{p+q-r}^{\text{in}}N_q^{\text{in}}\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(\omega_{p+q-r}+m_r-\omega_q)t\}+\cdots,\end{aligned} \tag{11}$$

$$\begin{aligned}V(x)=\sqrt{Z}V^{\text{in}}(x)&+\int d^3pd^3qh_p(\boldsymbol{q})N_{p-q}^{\text{in}}\theta_q^{\text{in}}\\&\times\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(\omega_q+m_{p-q})t\}+\cdots,\end{aligned} \tag{12}$$

*) There is a factor appearing in front of $V^{\text{in}}$ only. This is because in the field equations (1)~(3) a change in scale $V\to aV$, $\theta\to a\theta$, $\psi\to a\psi$ merely changes the coupling constant $g$ into $ag$, so we are free to choose the factors in front of the $\psi^{\text{in}}$, $\theta^{\text{in}}$ as unity. See also Ref. 3).

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$$\theta(x)=\theta^{\text{in}}(x)+\int d^3p d^3q d^3r f_{\boldsymbol{p}}(\boldsymbol{q},\boldsymbol{r})\theta^{\text{in}\dagger}_{\boldsymbol{r}+\boldsymbol{q}-\boldsymbol{p}}\theta_{\boldsymbol{q}}{}^{\text{in}}\theta_{\boldsymbol{r}}{}^{\text{in}}$$
$$\times\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(\omega_{\boldsymbol{q}}+\omega_{\boldsymbol{r}}-\omega_{\boldsymbol{r}+\boldsymbol{q}-\boldsymbol{p}})t\}+\int d^3p d^3q d^3r$$
$$\times g_{\boldsymbol{p}}(\boldsymbol{q},\boldsymbol{r})N_{\boldsymbol{q}}{}^{\text{in}\dagger}N^{\text{in}}_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r}}\theta_{\boldsymbol{r}}{}^{\text{in}}\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(m_{\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r}}+\omega_{\boldsymbol{r}}-m_{\boldsymbol{q}})t\}$$
$$+\int d^3p d^3q K_{\boldsymbol{p}}(\boldsymbol{q})N_{\boldsymbol{q}}{}^{\text{in}\dagger}V^{\text{in}}_{\boldsymbol{p}+\boldsymbol{q}}\exp\{i\boldsymbol{p}\cdot\boldsymbol{x}-i(M_{\boldsymbol{p}+\boldsymbol{q}}-m_{\boldsymbol{q}})t\}+\cdots. \tag{13}$$

Here the dots stand for higher products.

a) *Mass determination*

Let us feed (11), (12) and (13) into the matrix element $\langle 0|$ Eq. (1) $|N_{\boldsymbol{l}}{}^{\text{in}}\rangle$. This gives $(\partial/\partial t+im_0)\langle 0|\psi^{\text{in}}(x)|N_{\boldsymbol{l}}{}^{\text{in}}\rangle=0$, which tells us that $m=m_0=m_{\boldsymbol{l}}$. Similarly $\langle 0|$ Eq. (3) $|\theta_{\boldsymbol{l}}{}^{\text{in}}\rangle$ shows that $\mu=\mu_0=\mu_{\boldsymbol{l}}$. By considering the matrix elements $\langle 0|$ Eq. (2) $|V_{\boldsymbol{l}}{}^{\text{in}}\rangle$ and $\langle\theta_{\boldsymbol{s}}{}^{\text{in}}|$ Eq. (1) $|V_{\boldsymbol{l}}{}^{\text{in}}\rangle$, one easily finds that $M_{\boldsymbol{l}}$ is given by

$$M_{\boldsymbol{l}}=M_0+g^2\int d^3k\frac{\alpha^2(\omega_{\boldsymbol{k}})}{2\omega_{\boldsymbol{k}}}\frac{1}{M_{\boldsymbol{l}}-\omega_{\boldsymbol{k}}-m}. \tag{14}$$

It is clear that $M_{\boldsymbol{l}}$ is independent of $\boldsymbol{l}$ and hence we shall drop its subscript.

b) *Coefficients of the map*

The coefficients of the map are determined by considering the matrix elements of Eqs. (1)～(3) between the states $|0\rangle$, $|\theta^{\text{in}}\rangle$, $|N^{\text{in}}\rangle$, $|V^{\text{in}}\rangle$ and $|\theta^{\text{in}}N^{\text{in}}\rangle$. The derivation proceeds parallel to that given in Ref. 3) and so we present below only the final results giving the explicit forms of the coefficients which we need in the next subsection:

$$a_{\boldsymbol{p}}(\boldsymbol{q},\boldsymbol{r})=0,\qquad f_{\boldsymbol{p}}(\boldsymbol{q},\boldsymbol{r})=0,$$
$$c_{\boldsymbol{u}+\boldsymbol{t}-\boldsymbol{s}}(\boldsymbol{s},\boldsymbol{t})=\frac{1}{(2\pi)^{3/2}}\left(\frac{\omega_{\boldsymbol{u}}}{\omega_{\boldsymbol{s}}}\right)^{1/2}\frac{\alpha(\omega_{\boldsymbol{s}})}{\alpha(\omega_{\boldsymbol{u}})}\frac{f(\omega_{\boldsymbol{u}})}{\omega_{\boldsymbol{u}}-\omega_{\boldsymbol{s}}+i\epsilon},$$
$$d_{\boldsymbol{l}-\boldsymbol{s}}(\boldsymbol{s})=\frac{\sqrt{Z}}{(2\pi)^{3/2}}\frac{g\alpha(\omega_{\boldsymbol{s}})}{(2\omega_{\boldsymbol{s}})^{1/2}(M-\omega_{\boldsymbol{s}}-m)},$$
$$g_{\boldsymbol{u}+\boldsymbol{t}-\boldsymbol{s}}(\boldsymbol{s},\boldsymbol{t})=\frac{1}{(2\pi)^{3/2}}\left(\frac{\omega_{\boldsymbol{t}}}{\omega_{\boldsymbol{t}+\boldsymbol{u}-\boldsymbol{s}}}\right)^{1/2}\frac{\alpha(\omega_{\boldsymbol{t}+\boldsymbol{u}-\boldsymbol{s}})}{\alpha(\omega_{\boldsymbol{t}})}\frac{f(\omega_{\boldsymbol{t}})}{\omega_{\boldsymbol{t}}-\omega_{\boldsymbol{t}+\boldsymbol{u}-\boldsymbol{s}}+i\epsilon},$$
$$h_{\boldsymbol{t}+\boldsymbol{u}}(\boldsymbol{u})=\frac{1}{(2\pi)^{3/2}}\frac{f(\omega_{\boldsymbol{u}})(2\omega_{\boldsymbol{u}})^{1/2}}{g\alpha(\omega_{\boldsymbol{u}})}, \tag{15}$$
$$K_{\boldsymbol{l}-\boldsymbol{s}}(\boldsymbol{s})=\frac{\sqrt{Z}}{(2\pi)^{3/2}}\frac{g\alpha(\omega_{\boldsymbol{l}-\boldsymbol{s}})}{(\omega_{\boldsymbol{l}-\boldsymbol{s}})^{1/2}}\frac{1}{M-m-\omega_{\boldsymbol{l}-\boldsymbol{s}}}.$$

Here $g'$, $f(\omega_{\boldsymbol{u}})$ and $I(\omega_{\boldsymbol{u}})$ are defined by

$$g'=g'(\omega_{\boldsymbol{u}})=\frac{g^2}{\omega_{\boldsymbol{u}}+m-M_0},\qquad f(\omega_{\boldsymbol{u}})=\frac{g^2\alpha^2(\omega_{\boldsymbol{u}})}{2\omega_{\boldsymbol{u}}(\omega_{\boldsymbol{u}}+m-M_0)}\frac{1}{1-g'I(\omega_{\boldsymbol{u}})},$$

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$$I(\omega_u)=\int d^3k \frac{\alpha^2(\omega_k)}{2\omega_k(\omega_u-\omega_k+i\epsilon)} \tag{16}$$

and the $i\epsilon$ factor in the denominators is due to the retarted nature of the coefficients of the dynamical map.

c) *Microcausality and Z*

We have thus determined all the coefficients of the expansion in terms of $Z$, to obtain which the microscopic causality condition will be used now. For this, consider

$$\langle N_{l+k}^{\text{in}}|[\psi^\dagger(x),\theta(y)]_{t_x=t_y}|\theta_k{}^{\text{in}}\rangle .$$

On evaluating this matrix element it is found that the microscopic causality condition (I) demands that

$$\left\{\frac{1}{(2\pi)^{3/2}}[g_p(l+k,k)+c^*_{p+l}(k,l+k)]+\int d^3r g_p(l+k,r)c^*_{p+l}(k,p+l+k-r)\right\}$$
$$+K_p(l+k)d^*_{p+l}(k)=\text{a finite polynomial in } p. \tag{17}$$

By substituting the explicit forms of the coefficients obtained in the previous subsection, we can simplify the l.h.s. of (17). For instance, the terms inside the curly bracket in (17) can be reduced to the form

$$A(p,k)(\omega_p-\omega_k)I_1 \tag{18}$$

with

$$A(p,k)=\frac{\alpha(\omega_k)\alpha(\omega_p)}{(2\pi)^3(4\omega_p\omega_k)^{1/2}}\frac{1}{(\omega_k-\omega_p+i\epsilon)}$$

and

$$I_1=\int_C\frac{g'}{1-g'I(\omega)}\frac{d\omega}{(\omega-\omega_p)(\omega-\omega_k)}.$$

The contour $C$ is the cut-plane, cut along the positive real axis from $\omega=\mu$ to $\omega=\infty$. The contour encloses a pole at $\omega=(M-m)$ as $I_1-g'(M-m)=0$. Evaluation of the residue at this pole $I_1$ is easily obtained. Using this value of $I_1$ in (18), we see that the l.h.s. of (17) takes the form

$$K_p(l+k)d^*_{p+l}(k)+\frac{A(p,k)(\omega_p-\omega_k)}{(M-m-\omega_p)(M-m-\omega_k)}$$
$$\times g^2\left[1+g^2\int\frac{d^3k}{2\omega_k}\frac{\alpha^2(\omega_k)}{(M-m-\omega_k)^2}\right]^{-1}.$$

Finally substituting the expressions for $K_p(l+k)$ and $d_p{}^*(l+k)$ in this equation, we find that the microcausality condition (Eq. (17)) will be satisfied if

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$$Z=\left[1+g^2\int\frac{d^3k}{2\omega_{\boldsymbol{k}}}\frac{\alpha^2(\omega_{\boldsymbol{k}})}{(M-m-\omega_{\boldsymbol{k}})^2}\right]^{-1}.$$

Thus we have obtained all the coefficients of the map in the lowest sector. If we now compute the equal-time commutators in the subspace under consideration, the result is found to be the same as the usual canonical equal-time commutators.

Equations (11) $\sim$ (13) show that

$$H=m\int d^3kN_{\boldsymbol{k}}^{\text{in}\dagger}N_{\boldsymbol{k}}^{\text{in}}+\int d^3k\omega_{\boldsymbol{k}}\theta_{\boldsymbol{k}}^{\text{in}\dagger}\theta_{\boldsymbol{k}}^{\text{in}}+M\int d^3kV_{\boldsymbol{k}}^{\text{in}\dagger}V_{\boldsymbol{k}}^{\text{in}}, \qquad (19)$$

which acts as the Hamiltonian*)

$$i\frac{\partial\psi(x)}{\partial t}=[\psi(x),H],$$

$$i\frac{\partial\theta(x)}{\partial t}=[\theta(x),H],$$

$$i\frac{\partial V(x)}{\partial t}=[V(x),H]. \qquad (20)$$

## § 3. Discussion

We have thus demonstrated that the self-consistent method can be carried out with satisfactory results in a solvable model with mass and coupling constant renormalization. In this model $Z$ was determined by microcausality. In Ref. 3) it was found that in the pair model, microscopic causality leads to the existence of a bound state. In the model under consideration if we invert the mapping (this is possible now since the E.T.C.R.'s of $\psi$, $V$ and $\theta$ have been calculated) for $V^{\text{in}}$, $\psi^{\text{in}}$ and $\theta^{\text{in}}$ in terms of $V$, $\psi$, $\theta$ and consider the limit $Z\to 0$, $M_0\to\infty$, $g\to\infty$ such that $g_r=\sqrt{Z}g$ is finite, we find that $V^{\text{in}}$ is just the bound state found in Ref. 3), and the expansion for $V^{\text{in}}$ is identical with the expansion of the bound state. This is perhaps the reason why we have found the existence of the bound state closely connected with microcausality.

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*) The conventional interaction Hamiltonian when written in terms of the physical fields, is in the form of a free Hamiltonian. See also Ref. 6).

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