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Equality of pressures for rational functions

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Abstract. We prove that for all rational functions f on the Riemann sphere and potential $-t \ln |f'|$, $t \geq 0$ all the notions of pressure introduced in Przytycki (*Proc. Amer. Math. Soc.* **351**(5) (1999), 2081–2099) coincide. In particular, we get a new simple proof of the equality between the hyperbolic Hausdorff dimension and the minimal exponent of conformal measure on a Julia set. We prove that these pressures are equal to the pressure defined with the use of periodic orbits under an assumption that there are not many periodic orbits with Lyapunov exponent close to 1 moving close together, in particular under the Topological Collet–Eckmann condition. In Appendix A, we discuss the case $t < 0$.

0. *Introduction*

The thermodynamical formalism, the study of various pressures and spectra, proved to be an important tool for solving problems in hyperbolic dynamics and fractal geometry. In this paper we are interested in a setup involving iterations of a rational function on the Riemann sphere. The book [**Z**] is a good introduction, see also [**PU**]. Consult Ruelle's monograph [**R1**] for a general picture. We discuss several definitions of pressure for the potential $-t \ln |f'|$, frequently applied in rational holomorphic dynamics. We prove that all of them coincide (and under an additional assumption are equal to the periodic orbits pressure).

Part of our motivation comes from the holomorphic dynamics itself. In this setup, the thermodynamic formalism was first applied by Bowen in [**B**], where it was proved that for quasi-Fuchsian groups the first zero of the pressure function is equal to the Hausdorff dimension of the limit set. This formula, together with a more detailed study of pressure, was spectacularly applied by Ruelle in [**R2**] to show that the Hausdorff dimension of the Julia set for hyperbolic rational functions f depends real analytically on f.

Additional motivation comes from perspective applications to complex analysis. The study of pressure has already produced interesting examples for problems outside dynamics, see, e.g., [**E, BS**]. The pressure for all rational functions and potentials $-t \ln|f'|$ describes the multifractal spectra of the measure of maximal entropy. For polynomials this measure coincides with the harmonic measure for the basin of attraction to infinity, so pressure for polynomial Julia sets describes the multifractal spectra of the harmonic measure, see [**MS1, MS2**]. It has been recently announced [**BJ**] that the extremal values of such spectra for all planar domains coincide with those for polynomial Julia sets, meaning that several problems in complex analysis can be reduced to evaluating the pressure for polynomials. One example of a 'dynamical' solution to an 'analytical' problem along such lines can be found in [**BMS**]. We expect future applications of thermodynamical formalism in this setup which stretch beyond purely 'dynamical' problems, see [**M**] for an introduction.

In what follows we will deal with a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree $d \geq 2$ on the Riemann sphere. We denote by Crit or Crit(f) the set of critical points, i.e. $f'(x) = 0$ for $x \in$ Crit, and J stands for the Julia set of f. Absolute values of derivatives and distances are considered with respect to the standard Riemann sphere metric. We study various definitions of pressure with potentials $-t \ln|f'|$, with t real. In the main body of the paper the proofs are given for $t \geq 0$, with the Appendix devoted to $t \leq 0$, where some definitions and phenomena are different. In our paper we discuss only dynamical definitions (as given in [**P2**]) but, as was mentioned previously, in this setup pressure can also be defined geometrically through multifractal spectra for the measure of maximal entropy—harmonic measure. For the discussion of this approach, consult [**M, MS1, MS2**].

Definition 0.1. (Tree pressure) For every $z \in \overline{\mathbb{C}}$, define (on the tree of all its preimages)

$$P_{\text{tree}}(z, t) = \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{f^n(x)=z} |(f^n)'(x)|^{-t}.$$

This limit does not depend on z for *typical* z, i.e. for z outside the zero Hausdorff dimension exceptional set $E = E' \cup O(\text{Crit})$, where $O(\text{Crit}) = \bigcup_{j=1}^{\infty} f^j(\text{Crit})$ and $E' = \bigcap_N \bigcup_{n>N} B(f^n(\text{Crit}), \exp(-\sqrt{n}))$, see [**P2**, Theorem 3.3]. The resulting value for typical z we call the *tree pressure* $P_{\text{tree}}(t)$.

Definition 0.2. (Hyperbolic pressure)

$$P_{\text{hyp}}(t) := \sup_X P(f|_X, -t \ln|f'|),$$

where the supremum is taken over all compact f-invariant (i.e. $f(X) \subset X$) isolated hyperbolic subsets of J. We call such sets *expanding repellers*, following Ruelle. *Isolated* means that there is a neighbourhood U of X such that $f^n(x) \in U$ for all $n \geq 0$ implies $x \in X$. *Hyperbolic* or *expanding* means that there is a constant $\lambda_X > 1$ such that, for all n large enough and all $x \in X$, we have $|(f^n)'(x)| \geq \lambda_X^n$.

$P(f|_X, -t \ln|f'|)$ denotes the standard topological pressure for the continuous mapping $f|_X$ and continuous real valued potential function $-t \ln|f'|$ on X, see, for example, [**W**].

Definition 0.3. (Hyperbolic variational pressure)

$$P_{\text{hyp var}}(t) := \sup_{\mu} \left\{ \mathrm{h}_\mu(f) - t \int \ln|f'|\, d\mu \right\},$$

where the supremum is taken over all ergodic f-invariant measures of positive Lyapunov exponent, i.e. $\chi_\mu = \chi_\mu(f) = \int \ln|f'|\, d\mu > 0$.

Definition 0.4. (Variational pressure)

$$P_{\text{var}}(t) := \sup_{\mu} \left\{ \mathrm{h}_\mu(f) - t \int \ln|f'|\, d\mu \right\},$$

where the supremum is taken over all ergodic f-invariant measures supported on J.

Note that the good properties of P_{var} rely on $\chi_\mu \geq 0$ for all such measures [**P1**].

Definition 0.5. [**DU**] Let V be a neighbourhood of $\text{Crit} \cap J$ in J. Define $K(V) := J \setminus \bigcup_{n \geq 0} f^{-n}(V)$ and observe that $f(K(V)) = K(V)$. As $K(V) \cap \text{Crit} = \emptyset$, we can consider the topological pressure $P(f|_{K(V)}, -t \ln|f'|)$ for the map $f|_{K(V)}$ and the real continuous function $-t \ln|f'|$ on the compact set $K(V)$. Define

$$P_{\text{DU}}(t) := \sup_V P(f|_{K(V)}, -t \ln|f'|),$$

where the supremum is taken over all V as before.

Definition 0.6. (Conformal pressure [**P2**]) Set $P_{\text{Conf}}(t) := \ln \lambda(t)$, where

$$\lambda(t) = \inf\{\lambda > 0 : \exists \mu,\ \text{a probability measure on } J \text{ with Jacobian } \lambda|f'|^t\}.$$

We say that $\varphi : J \to \mathbb{R}$, $\varphi \geq 0$ is the *Jacobian for $f|_J$ with respect to* μ if φ is μ-integrable and for every Borel set $E \in J$ on which f is injective $\mu(f(E)) = \int_E \phi\, d\mu$. We write $\varphi = \text{Jac}_\mu(f|_J)$. We call any probability measure μ on J with Jacobian of the form $\lambda|f'|^t$ a *conformal measure*.

(Later on, working with P_{DU}, we shall use these notions also for $f|_K$ where K is an arbitrary compact forward f-invariant subset of J.)

All these pressure functions are Lipschitz continuous, monotone decreasing and convex. These properties follow easily from the definitions (or have been explained in [**P2**]), except $P_{\text{Conf}}(t)$.

Completing the previous results, we obtain the following theorem.

THEOREM A. *For all rational functions the pressures* $P_{\text{tree}}(t)$, $P_{\text{hyp}}(t)$, $P_{\text{hypvar}}(t)$, $P_{\text{var}}(t)$, $P_{\text{DU}}(t)$, $P_{\text{Conf}}(t)$ *coincide for all* $t \geq 0$.

Lipschitz continuity, monotone decreasing and the convexity of $P_{\text{Conf}}(t)$ follow, of course, from Theorem A and the properties of other pressures.

For $t \geq 0$, we will denote by $P(t)$ the common value of these functions at t.

In [**P2**, Appendix 2], it was proved that the so called hyperbolic dimension of f, denoted by t_0, was the first zero of each of these functions and that these functions coincide for $0 \leq t \leq t_0$. This extends the mentioned result of Bowen to (non-hyperbolic) rational maps.

The missing inequalities for $t \geq t_0$ in the chain

$$P_{\text{tree}}(t) \geq P_{\text{hyp}}(t) \geq P_{\text{hypvar}}(t) \geq P_{\text{var}}(t) \geq P_{\text{DU}}(t) \geq P_{\text{Conf}}(t) \geq P_{\text{tree}}(t) \qquad (*)$$

were $P_{\text{hypvar}}(t) \geq P_{\text{var}}(t)$ and $P_{\text{DU}}(t) \geq P_{\text{Conf}}(t)$. Here we complete the proof of Theorem A by showing these inequalities, see §1.

Recently Urbański [**U**] proved these missing inequalities (and, hence, the conclusion of Theorem A) for critically non-recurrent maps without parabolic periodic orbits. Our proof in the general case is based on his approach and some ideas of [**PRS**]. Historically the progress has been made consecutively in [**DU, P1, P2, PRS, BMS, U**].

In §2 we give a simple direct proof of the equality $P_{\text{tree}}(t) = P_{\text{hyp}}(t)$, for $t \geq 0$. This yields a new direct proof that the Hausdorff dimension of all variants of conical limit sets defined in [**P2**] coincide with t_0 (Remark 2.3). In particular, together with Patterson Sullivan's construction of conformal measures (the version in [**P2**, Remark 2.6 and proof of Theorem A2.9, step 6]) giving $P_{\text{Conf}}(t) \leq P_{\text{tree}}(t)$, we obtain for the hyperbolic dimension $\text{HD}_{\text{hyp}}(J)$ an easy proof of the following theorem very useful in the studies of continuity of the hyperbolic dimension of Julia set $J(f)$ with respect to f.

THEOREM. [**DU, P1**] $\text{HD}_{\text{hyp}}(J)$ *is equal to the minimal exponent of conformal measure on* J.

0.1. *Pressure at periodic points.* We consider the following additional definition of pressure.

Definition 0.7. (Periodic orbits pressure) For $n \geq 1$, let Per_n be the set of points $p \in \overline{\mathbb{C}}$ satisfying $f^n(p) = p$. Define

$$P_{\text{Per}}(t) = \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{z \in \text{Per}_n(f) \cap J} |(f^n)'(z)|^{-t}.$$

This function of t is also Lipschitz continuous, monotone decreasing and convex, which easily follows from the definition.

It is reasonable to think that this definition of pressure coincides with the previous ones, for $t \geq 0$. However, we are only able to prove the following inequality.

PROPOSITION B. *For* $t \geq 0$, *we have* $P_{\text{Per}}(t) \geq P(t)$.

The reverse inequality was established in [**BMS**] for polynomials without indifferent periodic orbits. In this paper we prove the reverse inequality for rational functions under the following hypothesis.

HYPOTHESIS H. *For every* $\delta > 0$ *and all n large enough, if for a set* $P \subset \text{Per}_n$ *for all* $p, q \in P$ *and all* $i : 0 \leq i < n$ $\text{dist}(f^i(p), f^i(q)) < \exp(-\delta n)$, *then* $\#P \leq \exp(\delta n)$.

THEOREM C. *Under Hypothesis H, we have* $P_{\text{Per}}(t) = P(t)$ *for every* $t \geq 0$.

Remark. It is reasonable to think that Hypothesis H holds for all rational maps. Of course it holds for Topological Collet–Eckmann Maps (TCEs), see [**PRS**] and the definition of UHP later. Its negation would imply the existence of exponentially many periodic orbits of period n of multiplier exponentially close to 1, for a sequence of ns tending to ∞ (see Proposition 3.10).

0.2. *Preliminaries.* We shall use the following definitions of TCEs, which are equivalent by [**PRS**].

- TCE. *Topological Collet–Eckmann condition.* There exist $M \geq 0$, $P \geq 1$ and $r > 0$ such that for every $x \in J(f)$ there exists a strictly increasing sequence of positive integers n_j, for $j = 1, 2, \dots$ such that $n_j \leq P \cdot j$ and for each j
$$\#\{i : 0 \leq i < n_j, \mathrm{Comp}_{f^i(x)}\, f^{-(n_j-i)} B(f^{n_j}(x), r) \cap \mathrm{Crit} \neq \emptyset\} \leq M,$$
where Comp_y means the connected component containing y (above $y = f^i(x)$).
- CE2(z_0). *Backward* or *second Collet–Eckmann condition at* $z_0 \in \overline{\mathbb{C}}$. There exist $\lambda_{\mathrm{CE2}} > 1$ and $C > 0$ such that, for every $n \geq 1$ and every $w \in f^{-n}(z_0)$,
$$|(f^n)'(w)| \geq C\lambda_{\mathrm{CE2}}^n.$$
(In this case z_0 is necessarily not in the forward orbit of a critical point.)
We write CE2(some z_0) if there exists $z_0 \in \overline{\mathbb{C}}$ such that CE2(z_0) holds.
- UHP. *Uniform hyperbolicity on repelling periodic orbits.* There exists $\lambda_{\mathrm{Per}} > 1$ such that every repelling periodic point $p \in J(f)$ of period $k \geq 1$ satisfies
$$|(f^k)'(p)| \geq \lambda_{\mathrm{Per}}^k.$$
- Lyapunov: *Lyapunov exponents of invariant measures are bounded away from zero.* There is a constant $\lambda_{\mathrm{Lyap}} > 1$ such that the Lyapunov exponent of any ergodic invariant probability measure μ supported on Julia set satisfies $\chi_\mu(f) := \int \ln|f'|\, d\mu \geq \ln \lambda_{\mathrm{Lyap}}$.
- Negative pressure: *Pressure for large t is negative*. For large values of t, the pressure function $P_{\mathrm{var}}(t)$ is negative.

Note that TCE implies immediately that $P_{\mathrm{var}}(t)$ is strictly decreasing with its 'slope' bounded away from 0 (use the Lyapunov condition).

TCE not satisfied implies immediately by negation of the negative pressure condition, that $P_{\mathrm{var}}(t) \equiv 0$ for all $t \geq t_0$.

We shall sometimes consider these two cases separately.

For a study of some classes of non-uniformly hyperbolic rational maps, larger than TCE, see [**GS**].

1. *The inequalities $P_{\mathrm{hypvar}}(t) \geq P_{\mathrm{var}}(t)$ and $P_{\mathrm{DU}}(t) \geq P_{\mathrm{Conf}}(t)$ in the chain (*)*

PROPOSITION 1.1. *$P_{\mathrm{var}}(t) = P_{\mathrm{hypvar}}(t)$ for all $t \geq 0$.*

Proof. In the TCE case, the equality holds since the measures under supremum in both definitions are the same (by the Lyapunov condition).

In the general case, for $0 \leq t \leq t_0$ the equality has been proved in [**P2**]. (Let us recall the argument. Suppose $t < t_0$. If $\mathrm{h}_\mu(f) - t\chi_\mu(f)$ is close to $P_{\mathrm{var}}(t)$ then it is positive; hence, due to $\chi_\mu(f) \geq 0$ (see [**P1**]) we get $h_\mu(f) > 0$. Hence, by Ruelle inequality,

$\chi_\mu(f) \geq h_\mu(f)/2 > 0$; hence, this measure is being taken in account also in the definition of $P_{\mathrm{hypvar}}(t)$.) For $t \geq t_0$ in the non-TCE case, both pressures are identically equal to 0 by the monotonicity and negation of the UHP condition. (One can support invariant measures μ on repelling periodic orbits with $\chi_\mu(f) \to 0$.) □

PROPOSITION 1.2. *$P_{\mathrm{DU}}(t) \geq P_{\mathrm{Conf}}(t)$ for all $t \geq 0$.*

Proof. We shall find a right conformal measure μ by repeating the construction from [**DU**]. We follow, in particular, the strategy and use some tricks from [**P2**] and [**U**].

For each critical point $c \in \mathrm{Crit} \cap J$, choose an arbitrary ergodic probability invariant measure μ_c supported on the ω-limit set $\omega(c)$ and choose a point $y_c \in \omega(c)$ such that, denoting $\chi_{\mu_c}(f)$ by χ_{μ_c},

$$\limsup_{n\to\infty}(\exp(-\chi_{\mu_c} n))|(f^n)'(y_c)| \geq 1. \tag{1.1}$$

(This is a standard fact, compare [**P1**, proof of Corollary A]. For the convenience of the reader we provide a proof of a general Lemma 1.3 implying this, at the end of the section.)

In fact, μ_c-a.e. y_c satisfies (1.1) and since by Poincaré's recurrence theorem μ_c-a.e. y is recurrent we can assume that y_c is recurrent.

For each $c \in \mathrm{Crit} \cap J$ and positive integer n, let

$$V_{c,n} := \bigcup_{k=0}^{\infty} f^{-k} B(y_c, 1/n)$$

and put $V_n = \bigcup_{c\in \mathrm{Crit}\, \cap J} V_{c,n}$.

Denote also $B_n := \bigcup_{c\in \mathrm{Crit}\, \cap J} B(y_c, 1/n)$.

From now on fix an arbitrary $t \geq 0$. Then there is a measure μ_n supported on $K(V_n)$, conformal for $f|_{K(V_n)}$, with Jacobian larger or equal to $\lambda_n|f'|^t$ (equality holds on $K(V(n)) \setminus \partial B_n$), such that

$$P_n(t) := P(f|_{K(V_n)}, -t \ln|f'|) = \ln \lambda_n.$$

Provided $P_n(t)$ can be approximated by $h_\mu(f|_{K(V_n)}) - t\chi_\mu(f|_{K(V_n)})$ with $\chi_\mu(f|_{K(V_n)}) > 0$, this holds by [**DU**, proof of Lemma 3.7]. In the remaining case $P_n(t) = 0$ and there exists an f-invariant ergodic probability measure μ on $K(V_n)$, such that $\chi_\mu(f|_{K(V_n)}) = 0$. Then by Birkhoff's ergodic theorem applied to the natural extension (inverse limit) of $f|_{K(V_n)}$, there exists a backward trajectory $x_n \in K(V_n)$, $f(x_n) = x_{n-1}$, $n = 0, 1, \ldots$, such that $\lim \frac{1}{n} \ln|(f^n)'(x_n)| = 0$. Therefore, Patterson–Sullivan's construction on $\cup f^{-k}(x_0)$ applied in [**DU**] gives μ_n with $\lambda_n \geq 1$. The opposite inequality, and the general $\lambda_n \leq \exp P_n(t)$ was proved in [**DU**, Lemmas 3.5, 3.6] (relying on [**MP**] and [**P3**, Lemma 3]).

Note that μ_n considered on the whole $J(f)$ (rather that $K(V_n)$) need not be even quasi-invariant, namely $\mu_n(V_n) = 0$ but usually $\mu_n(f(V_n)) > 0$. Nevertheless, the inequality

$$\mu_n(f(E)) \geq \int_E \lambda_n |f'|^t \, d\mu_n \tag{1.2}$$

still holds for every E on which f is injective, with the equality for E disjoint with $\mathrm{cl} B_n$.

Let, as in [**DU**], μ be a weak* limit of a subsequence $\{\mu_{n_j}\}_{j\geq 1}$ and $\lambda := \lim_{n\to\infty}\lambda_n \leq \exp P_{\rm DU}(t)$ (this limit exists since the sequence $P_n(t), n = 1, 2, \ldots$ is monotone increasing). This μ satisfies (1.2) with λ_n replaced by λ, with the equality for E disjoint with $\cup\{y_c\}$, since $\mathrm{cl} B_n \to \cup\{y_c\}$ in a Hausdorff distance.

We will prove that, in fact, μ is conformal with Jacobian $\lambda|f'|^t$.

To prove this, we only need to check that μ is conformal, with the Jacobian equal to $\lambda|f'|^t$ at each y_c or that $\mu(y_c) = 0$. To this end, it is sufficient to check that, for every f invariant ergodic probability measure ν on J, we have $\ln\lambda + t\chi_\nu(f) \geq 0$. Indeed, this inequality for each $\nu = \mu_c$ implies by (1.1) that

$$\limsup_{n\to\infty}\lambda^n|(f^n)'(y_c)|^t \geq \limsup_{n\to\infty}(\exp(-t\chi_{\mu_c}n))\ |(f^n)'(x)|^t \geq 1. \tag{1.3}$$

If y_c is not periodic then, being recurrent, it is not eventually periodic; hence, all the points $f^k(y_c)$ are pairwise distinct for $k = 0, 1, \ldots$. Hence, by (1.2) and (1.3), $\mu(y_c) > 0$ yields $\mu(J) \geq \sum_{k=0}^{\infty}\mu(\{f^k(y_c)\}) = \infty$, which contradicts the finiteness of μ.

Consider now the case y_c is periodic of period, say, k. First note that

$$\lambda^k|(f^k)'(y_c)|^t \geq 1. \tag{1.4}$$

Indeed, $\lambda^k|(f^k)'(y_c)|^t \leq \xi < 1$ leads to $\lambda^{mk}|(f^{mk})'(y_c)|^t \leq \xi^m$ for all positive integers m; hence, for all $0 \leq s < m$, $\lambda^{mk+s}|(f^{mk+s})'(y_c)|^t \leq \xi^m\lambda^s\sup|f'|^s$ converging to 0 as $mk + s \to \infty$. This contradicts (1.3).

Observe that, using (1.2) for μ and λ,

$$\mu(y_c) = \mu(f^k(y_c)) \geq \mathrm{Jac}_\mu(f^k)(y_c)\mu(y_c) \geq \lambda^k|(f^k)'(y_c)|^t\mu(y_c),$$

hence, using also (1.4), $\mathrm{Jac}_\mu(f^k)(y_c) = \lambda^k|(f^k)'(y_c)|^t = 1$.

Due to this,

$$\mathrm{Jac}_\mu(f^k)(y_c) = \prod_{j=0}^{k-1}\mathrm{Jac}_\mu(f)(f^j(y_c))$$

and

$$|(f^n)'(y_c)| = \prod_{j=0}^{k-1}|f'(f^j(y_c))|,$$

the inequalities $\mathrm{Jac}_{\mu_n}(f^j(y_c)) \geq \lambda|f'(f^j(y_c))|^t$ are, in fact, equalities. In particular, μ is conformal with Jacobian $\lambda|f'|^t$ at y_c.

To prove $\ln\lambda + t\chi_\nu(f) \geq 0$, assume first that $\chi_\nu(f) > 0$. Then there are infinitely many periodic orbits $O(p)$ with $\chi_{\mu_{O(p)}}$ arbitrarily close to χ_ν, where $\chi_{\mu_{O(p)}}$ is the probability measure equidistributed on $O(p)$ (use Katok–Pesin Theory, cf. [**P2**]). All $O(p)$ except at most one do not contain y_c; hence, they are in $K(V_n)$ for n large enough. Therefore,

$$\ln\lambda \geq \lim_{n_j\to\infty}P_{n_j}(t) \geq h_{\mu_{O(p)}}(f) - t\chi_{\mu_{O(p)}} \geq -t\chi_\nu(f) - \varepsilon$$

for $\varepsilon > 0$ arbitrarily small. If $\chi_\nu(f) = 0$ we are in the non-TCE case where by non-UHP there are infinitely many repelling periodic orbits $O(p)$ with $(1/n(p))\ln|(f^{n(p)})'|$ arbitrarily close to 1, where $n(p)$ is a period of p. Therefore, as before, taking the infimum over all $O(p)$ except the finite number of them containing points y_c,

$$\ln\lambda \geq -\inf_p t\chi_{\mu_{O(p)}} \geq 0 = -t\chi_\nu(f). \qquad \square$$

LEMMA 1.3. *For every probability space $(X, \mathcal{F}, \mu)$, every ergodic endomorphism $f : X \to X$ preserving the measure μ, and every μ-integrable real function $\phi : X \to \mathbb{R}$,*

$$\limsup_{n\to\infty} \sum_{j=0}^{n-1} \left(\phi(f^j(y)) - \int_X \phi \, d\mu \right) \geq 0, \quad \textit{for a.e. } y.$$

Proof. This lemma follows easily from the Birkhoff ergodic theorem. For example, let $C < 0$, $n_0 > 0$ and suppose there exists E of positive measure μ such that for all $x \in E$ and $n \geq n_0$ we have

$$\sum_{j=0}^{n-1} \left(\phi(f^j(y)) - \int_X \phi \, d\mu \right) \leq C < 0.$$

Then by Birkhoff's ergodic theorem applied to the indicator function of E to estimate from below the number of consecutive hits of E by the trajectory of y, considering only every n_0th hit, for a.e. $y \in E$, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(\phi(f^j(y)) - \int_X \phi \, d\mu \right) \leq \mu(E)C/n_0 < 0.$$

So, again by Birkhoff's ergodic theorem applied this time to ϕ and by the ergodicity of f, we get $\int_X \phi \, d\mu - \int_X \phi \, d\mu < 0$, a contradiction. □

2. *The equality $P_{\text{tree}}(t) = P_{\text{hyp}}(t)$, for $t \geq 0$*

Two of the most elementary notions of pressure are P_{tree} and P_{hyp}. Theorem A asserts that these functions coincide for $t \geq 0$. In this section we give a simple and nice proof of the inequality $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$, for $t \geq 0$. The proof relies on the same idea as the proof of the implication UHP $\Rightarrow$ CE2(z_0) in [**PRS**]: given a backward branch from z_0, we capture a periodic orbit (this procedure has a name: *Bowen's specification* or *shadowing*). Here we shall capture a large hyperbolic set.

Note that the proof of $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$ provided in §1 via the chain of inequalities (*) has been quite complicated and used the results of [**P2**].

PROPOSITION 2.1. *For $t \geq 0$, we have $P_{\text{tree}}(t) = P_{\text{hyp}}(t)$.*

Proof. We start with a direct proof of the inequality $P_{\text{tree}}(t) \geq P_{\text{hyp}}(t)$ (in [**P2**] the proof was also indirect). It follows immediately from $f^{-n}(\{z_0\}) \supset f^{-n}(\{z_0\}) \cap X$ for every f-invariant isolated hyperbolic set $X \subset J$, $z_0 \in X$ and from the fact that for expanding repellers X and Hölder $\varphi : X \to \mathbb{R}$ all notions of pressure $P(f|_X, \varphi)$ coincide [**W**] and [**PU**, Ch. 3]. Note that we need z_0 typical (see Definition 0.1): we can achieve it by replacing $z_0 \in X$ by a point z_0' close to z_0 not necessarily in X (for example, when X is just a repelling periodic orbit). Any backward trajectory of z_0 in X is close to a backward trajectory of z_0' being attracted to X, thus contributing to the sum under the logarithm in P_{tree} the same summand, up to a bounded factor.

To prove $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$, we will follow the proof of Lemma 3.1 in [**PRS**]. First, we find a typical $z_0 \in J$ in an expanding repeller $X \subset J$ such that

$$W := \mathrm{Comp}_{z_0} f^{-l} B(f^l(z_0), 2\delta) \subset B(z_0, \varepsilon n^{-\alpha}),$$

f^l is univalent on W, $l = l(n) := [\beta \ln n]$ and

$$B(z_0, n^{-\alpha}) \cap \bigcup_{j=1}^{2n} f^j(\mathrm{Crit}) = \emptyset$$

for constants $\alpha, \beta > 0$ depending on X, arbitrary $\varepsilon > 0$ and all n large enough.

By the Koebe distortion lemma for ε small enough, for every $1 \le j \le 2n$ and $z_j \in f^{-j}(z_0)$, we have

$$\mathrm{Comp}_{z_j} f^{-j} B(z_0, \varepsilon n^{-\alpha}) \subset B(z_j, \delta).$$

Let $m = m(\delta)$ be such that $f^m(B(y, \delta/2)) \supset J$ for every $y \in J$. Then, putting $y = f^l(z_0)$, for every $z_n \in f^{-n}(z_0)$, we find a component W_{z_n} of $f^{-m}(\mathrm{Comp}_{z_n} f^{-n}(B(z_0, \varepsilon n^{-\alpha})))$ $\subset B(f^l(z_0, \frac{3}{2}\delta))$ on which f^{m+n} is univalent (provided $m \le n$).

Therefore, f^{m+n+l} is univalent from $W'_{z_n} := \mathrm{Comp}(f^{-(m+n+l)}(B(f^l(z_0), 2\delta))) \subset W_{z_n}$ onto $B(f^l(z_0), 2\delta)$. The mapping

$$F = f^{m+n+l} : \bigcup_{z_n \in f^{-n}(z_0)} W'_{z_n} \to B(f^l(z_0), 2\delta)$$

has no critical points; hence, $Z := \bigcap_{k=0}^{\infty} F^{-k}(B(f^l(z_0), 2\delta))$ is an isolated expanding F-invariant (Cantor) subset of J.

For each z_n denote the point in $f^{-m}(z_n) \cap W'_{z_n}$ by z'_n. We obtain for a constant $C > 0$ resulting from distortion and $L = \sup |f'|$,

$$\begin{aligned} P(F|_Z, -t \ln |F'|) &\ge \ln \Bigg(C \sum_{z_n \in f^{-n}(z_0)} |(f^{m+n+l})'(z'_n)|^{-t} \Bigg) \\ &\ge \ln \Bigg(C \sum_{z_n \in f^{-n}(z_0)} |(f^n)'(z_n)|^{-t} L^{-t(m+l)} \Bigg). \end{aligned} \tag{2.1}$$

Hence, on the expanding f-invariant set $Z' := \bigcup_{j=0}^{m+n+l-1} f^j(Z)$, we obtain

$$\begin{aligned} P(f|_{Z'}, -t \ln |f'|) &\ge \frac{1}{m+n+l} P(F, -t \ln |F'|) \\ &\ge \frac{1}{m+n+l} \Bigg(\ln C - t(m+l) \ln L + \ln \sum_{z_n \in f^{-n}(z_0)} |(f^n)'(z_n)|^{-t} \Bigg). \end{aligned}$$

Passing with n to ∞ and using notation in Definition 0.1 we obtain

$$P(f|_{Z'}, -t \ln |f'|) \ge P_{\mathrm{tree}}(z_0, t) = P_{\mathrm{tree}}(t). \qquad \square$$

Remark 2.2. Note that the proof of the inequality $P_{\mathrm{hypvar}}(t) \le P_{\mathrm{hyp}}(t)$ for all $t \ge 0$ provided in [**P2**] relying on Pesin–Katok theory (cf. [**PU**]) uses, in fact, the same idea as in the proof of Proposition 2.1: capturing of a large hyperbolic set using good backward branches.

Remark 2.3. In [**P2**] several notions of a conical limit set have been provided. The one (formally not defined in [**P2**]) smaller than all others is

$$\Lambda_{\mathrm{hyp}} := \bigcup \{X : X \subset J,\ f\text{-invariant isolated hyperbolic}\}.$$

The largest is

$$\Lambda_{\mathrm{ML1}} := \{x \in J : (\exists \eta, M > 0, n_j \to \infty), \text{ such that each } f^{n_j} \text{ has degree bounded by } M \text{ on } \mathrm{Comp}_x\, f^{-n_j}(B(f^{n_j}(x), \eta))\}.$$

For the hyperbolic Hausdorff dimension of J defined by

$$\mathrm{HD}_{\mathrm{hyp}}(J) := \sup\{\mathrm{HD}(X) : X \subset J\ f\text{-invariant isolated hyperbolic}\}$$

the inequality $\mathrm{HD}_{\mathrm{hyp}}(J) \le \mathrm{HD}(\Lambda_{\mathrm{hyp}})$ is obvious. The inequality $\mathrm{HD}(\Lambda_{\mathrm{ML1}}) \le \alpha(f)$, where $\alpha(f)$ denotes the smallest exponent α of a conformal measure, i.e. the probability measure on J with Jacobian $|f'|^\alpha$, easily follows from distortion estimates, see [**P2**, Proposition A3.7].

Also $\alpha(f) \le t_0(P_{\mathrm{tree}})$, the first zero of $P_{\mathrm{tree}}(t)$ easily follows from a Patterson–Sullivan type of construction, see [**P2**, the construction following Remark 2.6].

The equality $t_0(P_{\mathrm{hyp}}) = \mathrm{HD}_{\mathrm{hyp}}(J)$, where $t_0(P_{\mathrm{hyp}})$ denotes the first zero of $P_{\mathrm{hyp}}(t)$, follows immediately from the definitions and the equality of the zero of pressure and Hausdorff dimension on hyperbolic sets.

Summarizing, in the chain

$$t_0(P_{\mathrm{tree}}) \le t_0(P_{\mathrm{hyp}}) = \mathrm{HD}_{\mathrm{hyp}}(J) \le \mathrm{HD}(\Lambda_{\mathrm{hyp}}) \le \mathrm{HD}(\Lambda_{\mathrm{ML1}}) \le \alpha(f) \le t_0(P_{\mathrm{tree}})$$

all the inequalities have had easy proofs except the first one. Proposition 2.1 fills this gap. It gives a new proof of the equality of the Hausdorff dimensions of all the conical limit sets to $\mathrm{HD}_{\mathrm{hyp}}(J)$, omitting delicate considerations with the use of $P_{\mathrm{DU}}(t)$ and $P_{\mathrm{Conf}}(t)$.

Remark 2.4. Restricting our interests to conformal pressure and measures, we get $P_{\mathrm{Conf}} \le P_{\mathrm{tree}}$ using Patterson–Sullivan's construction, and $P_{\mathrm{tree}} \le P_{\mathrm{hyp}}$ by Proposition 2.1. These inequalities, together with the easy $P_{\mathrm{hyp}} \le P_{\mathrm{Conf}}$, give a simple proof of

$$P_{\mathrm{hyp}}(t) = P_{\mathrm{Conf}}(t) \quad \text{for all } t \ge 0.$$

Hence a simple proof of the theorem $\mathrm{HD}_{\mathrm{hyp}}(J) = \alpha(f)$, [**DU, P1**], given in the Introduction.

3. *Pressure on periodic orbits*

The proof of the inequality $P_{\mathrm{Per}}(t) \ge P_{\mathrm{tree}}(t)$ for $t \ge 0$ (Proposition B) is a repetition of [**PRS**, Lemma 3.1] (shadowing by periodic orbits) and is, in fact, contained in the proof of Proposition 2.1. It also follows directly from Proposition 2.1, since for f on isolated hyperbolic sets where the mapping is topologically transitive, pressures for all Hölder functions coincide by classical theory.

The main goal of this section is to prove Theorem C, by proving the reverse inequality under Hypothesis H.

We start with a sequence of standard lemmas.

LEMMA 3.0. *Given a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for any family of holomorphic branches g_j of f^{-n_j} on $B(x_j, r)$ (provided they exist), where $x_j \in J$, $n_j \to \infty$ and for any family of conformal parametrizations by the unit disc $h_j : B(0,1) \to B(x_j, r)$ with $h_j(0) = x_j$, the family $g_j \circ h_j$ is normal, with all limit functions being constant.*

This lemma is often called Fatou's argument (it is implicitly present in Fatou's papers). For Mañé's more difficult version, in the presence of critical points, see, for example, [**Ma**] or [**P4**, Lemma 1.1] (for a more detailed proof).

Proof. Normality follows from the omission by $g_j(B(x_j, r))$ of at least three points by $\deg f > 1$ and Montel's theorem. If $G = \lim_{k\to\infty} g_{j_k} \circ h_{j_k} \neq$ Const then $G(\mathbb{D})$ contains a closed disc $\mathrm{cl}B(G(0), \delta)$. Then, by Hurwitz theorem, for all k large enough, $g_{j_k} \circ h_{j_k}(\mathbb{D})$ contains $\mathrm{cl}B(G(0), \delta)$ and $f^{n_{j_k}}|_{B(G(0),\delta)} \subset B(x_{j_k}, r)$. Therefore, by Montel's theorem, the family $f^{n_{j_k}}|_{B(G(0),\delta)}$ is normal, which contradicts the fact that $G(0) = \lim_{k\to\infty} g_{j_k}(x_{j_k}) \in J$. □

LEMMA 3.1. *Given a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ there exists a $C_1 > 0$ such that for every positive integer k and $z \in \mathrm{Per}_k(f) \cap J$, $\mathrm{dist}(z, \mathrm{Crit}) \geq \exp(-C_1 k)$ holds.*

LEMMA 3.2. *A $C_2 > 0$ exists such that for every positive integer k and $z \in \mathrm{Per}_k(f) \cap J$ for every $\delta > 0$*

$$\#\{j : 0 \leq j < k,\ B(f^j(z), \exp -\delta k) \cap \mathrm{Crit} \neq \emptyset\} \leq C_2/\delta.$$

Lemmas 3.1 and 3.2 immediately follow from [**P1**, Lemma 1].

LEMMA 3.3. (Telescope, [**P3, PRS**]) *For every $\lambda > 1$, there exist $a > 0$ and $C_3 > 0$ such that for every $z \in J, k \geq 0$ and $r > 0$, if $\mathrm{diam}\,\mathrm{Comp}_{f^j(z)} f^{-(k-j)} B(f^k(z), r) \leq a$ for every $0 \leq j < k$, then*

$$\mathrm{diam}\,\mathrm{Comp}_z\, f^{-k} B(f^k(z), r)/r \leq C_3 \lambda^k |(f^k)'(z)|^{-1}.$$

LEMMA 3.4. (Distortion) *For every $M \geq 0$, there exists a $C(M) \geq 1$ such that for every $\eta > 0$ small enough, $k \geq 0$ and $z \in J$ if*

$$\#\{j : 0 \leq j < k,\ \mathrm{Comp}_{f^j(z)} f^{-(k-j)} B(f^k(z), 2\eta) \cap \mathrm{Crit} \neq \emptyset\} \leq M,$$

then for every $w \in \mathrm{Comp}_z\, f^{-k} B(f^k(z), \eta)$

$$|(f^k)'(w)| \leq C(M) \frac{\eta}{\mathrm{diam}\,\mathrm{Comp}_z\, f^{-k} B(f^k(z), \eta)}.$$

LEMMA 3.5. (Koebe) *If, in Lemma 3.4, we assume $M = 0$ then, there exists a $K > 1$ such that for every $w_1, w_2 \in \mathrm{Comp}_z\, f^{-k} B(f^k(z), \eta)$*

$$|(f^k)'(w_1)|/|(f^k)'(w_2)| \leq K.$$

Definition 3.6. Denote $PC^n := \bigcup_{i=1}^n f^i(\mathrm{Crit})$. Fix n and arbitrary $x_0 \in J \setminus PC^n$ and $r > 0$. For every backward trajectory of x_0, namely a sequence of points $(x_i, i = 0, 1, \ldots, n)$ such that $f(x_i) = x_{i-1}$ do the following procedure. Take the smallest $k = k_1 \geq 0$ such that $\mathrm{Comp}_{x_{k_1}} f^{-k_1} B(x_0, r)$ contains a critical point. Next let k_2 be the smallest $k > k_1$ such that $\mathrm{Comp}_{x_{k_2}} f^{-(k_2-k_1)} B(x_1, r)$ contains a critical point. Etc. until $k = n$. Let the largest $k_j \leq n$ for the sequence (x_i) be denoted by $k((x_i))$ and the set $\{y : y = x_{k((x_i))}$ for a backward trajectory $(x_i)\}$ by $N(x_0) = N(x_0, n, r)$.

LEMMA 3.7. See [**PRS**, the displayed estimate in Lemma A.2] *For every* $\varepsilon > 0$ *for all* $r_0 > 0$ *small enough and* n *sufficiently large, for every* $x_0 \in J$, $\#N(x_0, n, r_0) \leq \exp(\varepsilon n)$ *holds.*

Proof of Theorem C. $P_{\mathrm{Per}}(t) \leq P_{\mathrm{tree}}(t)$ *for* $t \geq 0$.

Step 1: *Regular and singular periodic orbits.* Fix $\varepsilon > 0$ and $r > 0$ small enough so that the conclusion of Lemma 3.7 holds with $r_0 = 2r$.

Definition 3.8. We say that a periodic orbit O of period $n \geq 1$ is *regular* if there exists $p \in O$ such that f^n is injective on $\mathrm{Comp}_p\, f^{-n}(B(p, r))$. If O is not regular, then we say that O is *singular*.

(Caution. Given p, this definition depends on n. Indeed, a parabolic periodic orbit of period n is regular if r is small enough but for period kn with k large enough this orbit is singular.)

We denote by Per_n^r and Per_n^s the set of points $p \in \mathrm{Per}_n \cap J$ whose periodic orbit is regular and singular, respectively. We shall prove that, for $t \geq 0$,

$$P^r_{\mathrm{Per}}(t) = \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{p \in \mathrm{Per}_n^r} |(f^n)'(p)|^{-t} \leq P_{\mathrm{tree}}(t),$$

$$P^s_{\mathrm{Per}}(t) = \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{p \in \mathrm{Per}_n^s} |(f^n)'(p)|^{-t} \leq (\exp\varepsilon) P_{\mathrm{tree}}(t).$$

This implies that $P_{\mathrm{Per}}(t) \leq (\exp\varepsilon) P_{\mathrm{tree}}(t)$ and since $\varepsilon > 0$ can be chosen arbitrarily small, we obtain $P_{\mathrm{Per}}(t) \leq P_{\mathrm{tree}}(t)$.

The first inequality will be proven in Step 3. In Step 4 we shall bound the number of singular periodic orbits of period n: this will be the only part where Hypothesis H will be used. In Step 5 we shall complete the proof of the second inequality by bounding $|(f^n)'(p)|^{-t}$, for $p \in \mathrm{Per}_n^s$.

Step 2: Take an arbitrary constant $\delta : 0 < \delta < 1/2$ and a positive integer n. Set $m := n[1/\delta]$, where $[\cdot]$ is the integer part function. Let $X(n, \delta) \subset J$ be an $(\exp(-\delta n))$-dense set in J (i.e. $\bigcup_{x \in X(n,\delta)} B(x, \exp(-\delta n)) \supset J$), consisting of points x satisfying $\mathrm{dist}(x, PC^m) \geq \exp(-2n\delta)$ (see Definition 3.6). It is easy to see that there is a constant $C > 0$ such that $\#X(n, \delta) \leq C \exp(2n\delta)$.

Fix a typical point $z_0 \in \overline{\mathbb{C}}$ as in Definition 0.1 of P_{tree}. By geometric Lemma 3.1 of [**P2**], there is a constant $C' > 0$ such that for every $j : n \leq j \leq m$ the quasi-hyperbolic distance in $U_j := \overline{\mathbb{C}} \setminus PC^j$ between z_0 and a point x_0 outside $B(PC^j, \exp(-2n\delta))$ is bounded by

$$C'\sqrt{j}\sqrt{\ln 1/\min\{\mathrm{dist}(z_0, PC^j), \mathrm{dist}(x_0, PC^j)\}} \leq C'\sqrt{j}\sqrt{2n\delta} = l_{n,j}.$$

In other terms, this means that there exists a sequence of discs $B_1 = B(q_1, r_1), \ldots, B_k = B(q_k, r_k)$ for some $k \leq l_{n,j}$, such that for every $j = 1, \ldots, k$ each $2B_j := B(q_j, 2r_j)$ is disjoint from W, $z_1 \in B_1$, $z_2 \in B_k$, and $\bigcup_{j=1}^k B_j$ is connected.

(For more discussion and further references, see [**P2**]. See also [**HH**] and [**PRS**].)

Consider a continuous path $\gamma \subset \bigcup_{j=1}^k B_j$ without self-intersections joining z_0 and x_0.

Given $x_j \in f^{-j}(x_0)$ let $z_j \in f^{-j}(z_0)$ be the point such that x_j and z_j are the endpoints of a connected component of $f^{-j}(\gamma)$. Note that this yields a one-to-one correspondence between points in $f^{-j}(x_0)$ and points in $f^{-j}(z_0)$.

Then for n large enough the Koebe Distortion Lemma (Lemma 3.5) applied to all B_j implies

$$|(f^j)'(x_j)| \geq K^{-l_{n,j}} |(f^j)'(z_j)|.$$

Step 3: *Pressure on regular periodic orbits.* Let $O \subset \mathrm{Per}_n^r$ be a regular periodic orbit and let $p_0 \in O$ such that f^n is injective on $\mathrm{Comp}_{p_0} f^{-n}(B(p, r))$, see Definition 3.8. Provided n is large enough that $r/12 \geq \exp(-\delta n)$, we can choose $x_0 \in X(n, \delta) \cap B(p_0, r/12)$. Consider the (unique) backward orbit $(x_i, i = 0, \ldots, n)$ such that $x_j \in \mathrm{Comp}_{f^{n-j}(p_0)} f^{-j}(B(p_0, r/12))$. Moreover, let z_n be the nth preimage of z_0 associated to x_n, as in Step 2.

Applying the Koebe distortion lemma (Lemma 3.5) to f^n at p_0 and x_n, we obtain, for n large enough,

$$|(f^n)'(p_0)| \geq K^{-1}|(f^n)'(x_n)| \geq K^{-1}K^{-l_{n,n}}|(f^n)'(z_n)| = K^{-1-C'n\sqrt{2\delta}}|(f^n)'(z_n)|.$$

However, for n large enough,

$$\mathrm{Comp}_{x_n} f^{-n}B(x_0, r/2) \subset \mathrm{Comp}_{p_0} f^{-n}B(p_0, 2r/3) \subset B(p_0, r/4) \subset B(x_0, r/3),$$

where the middle inclusion holds for large n, as f^n is injective in $\mathrm{Comp}_{p_0} f^{-n}(B(p_0, r))$, by Lemma 3.0.

By the Schwarz lemma, it follows that the map $f^{-n} : B(x_0, r/2) \to B(x_0, r/3)$ has only one fixed point. Therefore, the backward orbit $(x_i, i = 0, 1, \ldots, n)$ cannot be associated to a regular periodic orbit, other than O.

Thus, for a given $z_n \in f^{-n}(z_0)$, there is at most $\#X(n, \delta) \leq C\exp(2n\delta)$ regular orbits associated, at most one for each choice of $x_0 \in X(n, \delta)$. We conclude that

$$\sum_{p \in \mathrm{Per}_n^r} |(f^n)'(p)|^{-t} \leq Cn(\exp(2n\delta))K^{t(1+C'n\sqrt{2\delta})} \sum_{z_n \in f^{-n}(z_0)} |(f^n)'(z_n)|^{-t}.$$

Since $\delta > 0$ can be chosen arbitrarily small, we conclude that $P_{\mathrm{Per}}^r(t) \leq P_{\mathrm{tree}}(t)$.

Step 4: *Upper bound on the number of singular periodic orbits.* Consider a periodic orbit $O \subset \mathrm{Per}_n \cap J$. Since O is non-attracting, we can choose $p_0 \in O$ such that for $k \geq 0$ we have $|(f^k)'(p_k)| \geq 1$, where $p_k \in O$ is determined by $f^k(p_k) = p_0$. For $\rho > 0$ and $i \geq 0$, we denote $W_i(p_0, \rho) = \mathrm{Comp}_{p_i} f^{-i}(B(p_0, \rho))$.

Choose $x_0 = x_0(O) \in X(n, \delta)$ such that $p_0 \in B(x_0, \exp(-n\delta))$. Assign to each singular periodic orbit O a backward trajectory $(x_i, i = 0, 1, \ldots, n)$ of $x_0(O)$, such that $x_i \in W_i(p_0, \exp(-n\delta))$.

By Lemma 3.3 applied to $\lambda := \exp(\delta/3)$, if n is large enough we can prove by induction in i that

$$\mathrm{diam}\, W_i(p_0, \exp(-n\delta)) \leq \exp(-n\delta/2),$$

for all $0 \leq i \leq n$. In particular, we have $\mathrm{dist}(p_i, x_i) \leq \exp(-n\delta/2)$, for $i = 0, \ldots, n$. Therefore, Hypothesis H implies that the number of singular periodic orbits $O \subset \mathrm{Per}_n^s$ with the same backward trajectory $(x_0,\dots,x_n)$ assigned, is bounded by $\exp(n\delta/2)$. (We have not used yet the assumption the orbits are singular.)

Step 4.1: Fix $k = k((x_i))$ given by Lemma 3.7. Denote by X_{k,x_k} the set of all backward trajectories $(x_i, i = 0,1,\dots,n)$ assigned to singular periodic orbits $O \subset \mathrm{Per}^s_n$, given k and x_k. We shall prove that $\# X_{k,x_k} \le k(2d-2)$, where d is the degree of f.

Write $W'_i := \mathrm{Comp}_{x_{k+i}} f^{-i} B(x_k, 2r)$ for $i = 0,1,\dots,n-k$. By the definition of k (no hitting of critical points going backward to W'_{n-k} along trajectories with this k) the sets W'_{n-k} are in one-to-one correspondence with $(x_0,\dots,x_n) \in X_{k,x_k}$, and they are pairwise disjoint as different components of one $f^{-(n-k)}(W')$. For each $i : 0 < i \le k$ at most $\#\,\mathrm{Crit} \le 2d-2$ pairwise disjoint sets $f^{-i}(W'_{n-k})$ capture critical points, so summing over i we see that the number of such W'_{n-k} is bounded by $k(2d-2)$. Observe now that $\mathrm{dist}(p_k, x_k) \le \exp(-n\delta/2)$ implies $B(x_k, 2r) \supset B(p_k, r)$. Pulling this inclusion back we obtain $W'_i \supset W_i(p_k, r)$, as $x_{k+i} \in W'_i$ and $x_{k+i} \in f^{-i}_{p_{k+i}}(W_k(p_0, \exp(-n\delta))) \subset W_i(p_k, r)$ for all $i = 0,\dots,n-k$. In particular, $W_i(p_k, r)$ does not capture critical points for $i = 1,\dots,n-k$.

Therefore, $W_i(p_k, r)$ captures a critical point for some $i : n-k < i \le n$ as O is singular; hence, also $f^{-i}(W'_{n-k})$ contains this critical point.

We conclude that $\# X_{k,x_k} \le k(2d-2)$.

Step 4.2: Considering that $x_0 \in X(n,\delta)$, the estimate in Part 4.1 and the definition of $N(x_0) = N(x_0, n, 2r)$ (Definition 3.6), we obtain, by Lemma 3.7, that the number of backward trajectories $(x_0,\dots,x_n)$ associated to singular periodic orbits is bounded by

$$\# X(n,\delta)\cdot N(x_0)\cdot n(2d-2) \le C(\exp n(2\delta+\varepsilon))n(2d-2).$$

Since the number of singular periodic orbits assigned to a given backward trajectory is at most $\exp(n\delta/2)$, we conclude that, for n big,

$$\#\,\mathrm{Per}^s_n \le n^2(2d-2)C\exp(n(\delta/2+2\delta+\varepsilon)) \le \exp(n(3\delta+\varepsilon)). \tag{3.1}$$

Step 5: *Bound of the multiplier of a singular periodic orbit.* Consider a singular periodic orbit O. As in Step 4, we can choose $p_0 \in O$ such that for $k \ge 0$ we have $|(f^k)'(p_k)| \ge 1$, where $p_k \in O$ is determined by $f^k(p_k) = p_0$.

Recall that $m := n[1/\delta]$. Consider $W_i := W_i(p_0, 2\exp(-\delta n))$ and $\frac{1}{2}W_i := W_i(p_0, \exp(-\delta n))$ for $i = 0,1,\dots,m$, as defined in Part 4.

Set $\lambda = \exp(n\delta/3m) \le \exp(\delta^2/2)$ and consider a as in Lemma 3.3. By increasing n if necessary we may assume that $\exp(-n\delta/2) \le a$. So we can prove by induction in i that $\mathrm{diam}\, W_i \le \exp(-n\delta/2)$, for $i = 0,\dots,m$. So, by Lemma 3.2, the number of integers $i : 0 < i \le m$ such that W_i contains a critical point, is bounded by $M = C_2(m/n)(2/\delta) \le C_2 3/\delta^2$. Note that M is independent of n.

Choose $x_0 \in X(n,\delta)$ in $B(p_0, \exp(-n\delta))$ and a backward trajectory $(x_i, i = 0,1,\dots,m)$ such that $x_i \in \frac{1}{2}W_i$. For $0 \le i \le m$ let $z_i \in f^{-i}(z_0)$ be chosen as in Part 2, for x_i.

Fix an arbitrary $\kappa : 0 < \kappa < 1/2$ and set $m_1 = n[1/\delta^\kappa]$. Then there are two cases.

Step 5.1: Case 1. $\mathrm{diam}\, \frac{1}{2}W_{m_1} \ge \lambda^{-m}\exp(-C_1 n)$. By Lemma 3.4, we have

$$|(f^{m_1})'(x_{m_1})| \le C(M)\frac{\exp(-\delta n)}{\mathrm{diam}\,\frac{1}{2}W_{m_1}} \le C(M)\exp(C_1 n) \le \exp(2C_1\delta^\kappa m_1).$$

Thus, letting $\delta_1 = 2C_1\delta^\kappa + C'\sqrt{\delta}\ln K$, we have

$$|(f^{m_1})'(z_{m_1})| \leq |(f^{m_1})'(x_{m_1})| K^{C'\sqrt{m_1}\sqrt{2n\delta}} \leq \exp(\delta_1 m_1).$$

Observe that $\delta_1 \to 0$ as $\delta \to 0$.

Since $|(f^n)'(p_0)| \geq 1$, we have

$$|(f^n)'(p_0)|^{-t/n} \leq (\exp(\delta_1 t))|(f^{m_1})'(z_{m_1})|^{-t/m_1}.$$

Step 5.2: Case 2. $\operatorname{diam}\frac{1}{2}W_{m_1} < \lambda^{-m}\exp -C_1 n$. Put $m_2 := m_1[1/\delta^\kappa]$. Considering that $|(f^k)'(p_k)| \geq 1$, for $k \geq 0$, Lemma 3.3 implies that for

$$W_i'' := \operatorname{Comp}_{p_i} f^{-(i-m_1)} B(p_{m_1}, 2\lambda^{-m}\exp(-C_1 n)),$$

we have

$$\operatorname{diam} W_i'' < \exp(-C_1 n) \quad \text{for all } m_1 \leq i \leq m_2.$$

Hence, Lemma 3.1 implies $W_i'' \cap \mathrm{Crit} = \emptyset$ and we can use the Koebe distortion lemma (Lemma 3.5) to obtain

$$|(f^{m_2-m_1})'(x_{m_2})| \leq K|(f^{m_2-m_1})'(p_{m_2})|.$$

Since $|(f^{m_1})'(x_{m_1})| \leq L^{m_1}$, where $L := \sup|f'|$, we obtain, writing $\lambda_{p_0} := |(f^n)'(p)|^{1/n} \geq 1$,

$$|(f^{m_2})'(x_{m_2})| \leq K(\lambda_{p_0})^{m_2-m_1}L^{m_1} \leq K\lambda_{p_0}^{m_2}\exp m_2(2\delta^\kappa \ln L).$$

Letting $\delta_2 = 2\delta^\kappa \ln L + C'\sqrt{\delta}\ln K$, we have

$$|(f^{m_2})'(z_{m_2})| \leq |(f^{m_2})'(x_{m_2})| K^{C'\sqrt{m_2}\sqrt{2n\delta}} \leq K\lambda_{p_0}^{m_2}(\exp\delta_2 m_2).$$

Observe that $\delta_2 \to 0$ as $\delta \to 0$.

Equivalently, we have

$$|(f^n)'(p_0)|^{-t/n} \leq (K^{t/m_2}\exp(\delta_2 t))|(f^{m_2})'(z_{m_2})|^{-t/m_2}.$$

Step 5.3: Let $p_0 \in \mathrm{Per}_n^s$ with minimizing $|(f^n)'(p)|$ and put $m_0 = m_1$ or m_2 and $\delta_0 = \delta_1$ or δ_2 depending on whether we are in Case 1 or 2. Using estimate (3.1) and the estimates in 4.1 and 4.2, we have

$$\begin{aligned}\Big(\sum_{p\in\mathrm{Per}_n^s}|(f^n)'(p)|^{-t}\Big)^{1/n} &\leq \exp(3\delta+\varepsilon)|(f^n)'(p_0)|^{-t/n} \\ &\leq \exp(3\delta+\varepsilon)(\mathrm{Const}(\exp\delta_0 t)|(f^{m_0})'(z_{m_0})|)^{-t/m_0}.\end{aligned}$$

Letting $\delta \to 0$ we obtain $P_{\mathrm{Per}}^s(t) \leq (\exp\varepsilon)P_{\mathrm{tree}}(t)$, as desired. □

Remark 3.9. The strategy for the proof of Theorem C is similar to [**BMS**]. Regular and singular periodic orbits have been introduced in [**BMS**], under the names: *good* and *bad*. The estimate of the number of bad orbits by $\exp\varepsilon n$, in the absence of indifferent cycles, has been achieved in [**BMS**] with the help of Markov *puzzle structure*. Hypothesis H, and even a stronger property, followed from an estimate for the degree of appropriate polynomial-like maps $f^n : B \to f^n(B)$ on thickened puzzle pieces B.

The following proposition explains the meaning of Hypothesis H in more detail.

PROPOSITION 3.10. *If Hypothesis H does not hold then there exist* $0 < \delta < \Delta$, $C > 0$ *and a sequence of ns such that for each n from the sequence there exists a* $P \subset \mathrm{Per}_n$ *with* $\#P \geq \exp(\delta n)$ *and there exist* $x \in J$ *and* $\Delta_n : \delta < \Delta_n \leq \Delta$ *satisfying the following conditions.*

(1) *For each* $i : 0 \leq i \leq n$, $f^i(P) \subset B(f^i(x), \exp(-\Delta_n n))$.

(2) f^n *is injective on the disc* $B' = B(x, \exp(-(\Delta_n - \delta)n))$*; in particular, there are no critical points for* f^n *in* B' *and the distortion of* f^n *on* $B := B(x, \exp(-\Delta_n n))$ *(i.e. K from Lemma 3.5 for* $w_1, w_2 \in B$*) is bounded by* $1 + C\exp(-\delta n)$.

(3) *Neither* $f^n(B')$ *contains the closure of* B' *nor is it contained in* B'.

(4) *For each* $p \in P$, $|(f^n)'(p) - 1| \leq C\exp(-\delta n)$.

Proof. The existence of a constant δ, a sequence of ns and $P = P(n)$, $x = x(n)$ to satisfy Property (1) (with $\Delta_n = \delta$) follows immediately from the negation of Hypothesis H. Since these δ, P, x are not our final ones, we use for them the notation δ^0, P^0, x^0.

Fix an arbitrary n from this sequence. We can assume that all $p \in P^0$ satisfy $|(f^k)'(p_k)| \geq 1$ for all $k = 0, \ldots, n$ with p_k in the periodic trajectory of p and $f^k(p_k) = p$ by choosing in each periodic orbit of $p \in P^0$ an appropriate point p_0 as in the proof of Theorem C, Step 4, and choosing those p_0s which lie in <u>one</u> $B(f^i(x^0), \exp(-\delta^0 n))$. This gives a new set $P^1 \subset \bigcup_{j=0}^{n-1} f^j(P^0)$ with $\#P^1 \geq (\exp(\delta^0 n))/n^2$ for the right choice of i. Therefore, for δ^1 arbitrarily close to δ^0, we get, provided n is large enough, $\#P^1 \geq \exp(\delta^1 n)$. We replace x^0 by a point $x^1 \in P^1$. Property (1), still holds with $P^1 \subset B(x^1, 2\exp(-\delta^0 n))$.

Now we choose, by induction, the components $U_k = \mathrm{Comp}\ f^{-k}(B(x^1, 2\exp(-n\delta^0)))$ so that $f(U_k) = U_{k-1}$, $2U_k := \mathrm{Comp}\ f^{-k}(B(x^1, 4\exp(-n\delta^0))) \supset U_k$ for $k = 0, 1, \ldots, n$ and choose the sets $P_k \subset P^1$ so that the following holds:

(a) $\operatorname{diam} 2U_k \leq \exp(-\frac{1}{2}\delta^0 n)$;

(b) $\#\{i : 0 \leq i < n, 2U_i \cap \mathrm{Crit} \neq \emptyset\} \leq \mathrm{Const}/\delta^0$; and

(c) $\#(P_n \cap U_n) \geq \mathrm{Const}\exp(\delta^1 n)$.

We achieve this as follows. Having chosen U_k, take as U_{k+1} the component of $f^{-1}(U_k)$ containing the largest portion of $f^{n-k-1}(P_k)$ and set $P_{k+1} = f^{k+1}(f^{n-k-1}(P_k) \cap U_{k+1})$.

If there is only one component of $f^{-1}(U_k)$ intersecting $f^{n-k-1}(P_k)$, then $P_{k+1} = P_k$. If more, at most $d = \deg f$, then $\#P_{k+1} \geq d^{-1}\#P_k$.

Note that if the latter case happens, then there are two distinct components, of $f^{-1}(U_k)$ intersecting $B(f^{n-k-1}(x^1), 2\exp(-\delta^0 n))$. Denote these components by A_1, A_2. Observe that $\operatorname{diam} A_i \leq \exp(-\frac{1}{2}\delta^0 n)$ for $i = 1, 2$. This follows from Lemma 3.3, where the role of z is played by a point $p \in A_i \cap f^{n-k-1}(P_k)$ (cf. the proof of Theorem C). Therefore, there exist $y_1 \in A_1, y_2 \in A_2$ such that $f(y_1) = f(y_2)$ and $\mathrm{dist}(y_1, y_2) \leq 2\exp(-\frac{1}{2}\delta^0 n) + 4\exp(-\delta^0 n) := t_n$.

This is possible only if $B(f^{n-k-1}(x^1), d \cdot t_n) \cap \mathrm{Crit} \neq \emptyset$; therefore, by Lemma 3.2, only for at most a Const/δ^0 number of times (independent of n). This yields property (c). Property (a) follows from Lemma 3.3 and property (b) from Lemma 3.2.

(The choice of x^1 has been made to ease the notation, to replace f^i and appropriate pre-images by the periodic orbit of x^1. It can happen that $x^1 \notin P_k$ for some $k > 0$; therefore, $x^1 \notin P_n$.)

Set $P^2 := P_n$ in the previous construction. By Lemma 3.1 for every $p \in P^2$ and $0 \le i \le n$, we have $\mathrm{dist}(f^i(p), \mathrm{Crit}) \ge \exp(-C_1 n)$. Hence, by $\mathrm{Crit}(f^n) = \bigcup_{i=0}^{n-1} f^{-i}(\mathrm{Crit})$, we have

$$\mathrm{dist}(p, \mathrm{Crit}(f^n)) \ge (\exp(-C_1 n))(\sup |f'|)^{-(n-1)} := \tau_n.$$

Note also that by (b) we have $\deg f^n|_{2U_n} \le d^{\mathrm{Const}/\delta^0}$; hence, $\#(\mathrm{Crit}(f^n) \cap 2U_n) \le (\mathrm{Const}/\delta^0) d^{\mathrm{Const}/\delta^0} := d_1$, not depending on n.

For each $c \in \mathrm{Crit}(f^n) \cap 2U_n$, we cover $U_n \setminus B(c, \tau_n)$ by a family $\mathcal{D}(c)$ of discs $B(y_j, a(\mathrm{dist}(y_j, c)))$ such that $\#(\mathcal{D}(c)) = \mathrm{Const}\, 2\pi (\ln \mathrm{diam}\, U_n - \ln \tau_n)/a^2$. This covering comes from the partition of $U_n \setminus B(c, \tau_n)$ in the logarithmic coordinates, namely the partition of the rectangle $[\ln \tau_n, \ln \mathrm{diam}\, U_n] \times [0, 2\pi]$, into equal squares of side $\mathrm{Const}\, a$. We set here $a := \exp -(\delta^1/5d_1)n$.

Next consider the covering $\mathcal{D} := \bigvee_{c \in \mathrm{Crit}(f^n) \cap 2U_n} \mathcal{D}(c)$. We have $\#\mathcal{D} \le (\#\mathcal{D}(c))^{d_1} \le \mathrm{Const}(n/a^2)^{d_1} \le \exp(\frac{1}{2}\delta^1 n)$. Thus, we can choose $D \in \mathcal{D}$ such that

$$\#(D \cap P^2) \ge \#P^2/\#\mathcal{D} \ge \mathrm{Const} \exp(\delta^1 n)/\exp(\tfrac{1}{2}\delta^1 n) = \mathrm{Const} \exp(\tfrac{1}{2}\delta^1 n).$$

For our final x choose an arbitrary point in D. Then f^n is injective on $B(x, \mathrm{dist}(x, \mathrm{Crit}(f^n)))$.

We set our final $P := D \cap P^2$, $\delta := \delta^1/6d_1$ and $\Delta := C_1 + \ln \sup |f'| + 2\delta$ (so it can happen that $\Delta \gg \delta$) and $\Delta_n := \delta - 1/n \ln \mathrm{dist}(D, \mathrm{Crit}(f^n) \cap 2U_n)$.

Property (1) for $i = 0$ and property (2) follow from our definitions. The bound $1 + C\exp(-\delta n)$ for distortion follows from standard Koebe's distortion estimate.

Property (3) holds since otherwise either B' contains only one repelling fixed point for f^n due to the absence of critical points, by the Schwarz Lemma or B' contains an attracting fixed point for f^n.

Property (4) follows from property (3) for B in place of B' and from Koebe's estimate.

Property (1) for $i > 0$ follows from $\mathrm{diam}\, f^i(D) \le \frac{3}{2} \mathrm{diam}\, D$, resulting from $|(f^{n-i})'(f^i(p))| \ge 1$ for any $p \in P$ (by definition of P) and from property (3) and the bound on distortion in property (2). □

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A. *Appendix. Pressure for $t < 0$*

A.1. *Comparison with standard pressure.* First recall the standard definition of pressure for a continuous mapping $f : X \to X$ for a compact metric space X and a function $\phi : X \to \mathbb{R} \cup \{-\infty\}$, see **[W]** and **[K]** where ϕ taking values $-\infty$ has been considered.

Definition A.1.

$$P(f,\phi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\ln\sup_{S_{n,\varepsilon}}\sum_{x\in S_{n,\varepsilon}}\exp\sum_{k=0}^{n-1}\phi\circ f^k(x)$$

where $S_{n,\varepsilon}$ denote (n,ε)-separated sets, i.e. for any two $x\neq y$ belonging to $S_{n,\varepsilon}$ we have $\max_{k=0,1,\dots,n-1}\{\rho(f^k(x), f^k(y))\}\geq\varepsilon$ for the spherical metric ρ.

We use it for $\phi=-t\ln|f'|$ and $X=J$, Julia set, obtaining

$$P(f|_J,\phi) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\ln\sup_{S_{n,\varepsilon}}\sum_{x\in S_{n,\varepsilon}}|(f^n)'(x)|^{-t}.$$

Note that for $t>0$ in the presence of critical points in J where $\phi=\infty$, the notion is not interesting because it is always equal to ∞.

THEOREM A.2. *For all $t<0$, we have $P(f|_J,-t\ln|f'|)=P_{\text{var}}(t)$.*

This theorem follows immediately from the following version of the Variational Principle, see [**K**, Theorem 4.4.11].

THEOREM A.3. (Variational principle) *Let $f: X\to X$ and $\phi: X\to\mathbb{R}\cup\{-\infty\}$ be as above, with ϕ upper semicontinuous. Then*

$$P(f,\phi)=\sup_\mu\left\{h_\mu(f)+\int\phi\,d\mu\right\}$$

where the supremum is taken over all ergodic f-invariant probability measures on X.

For $t<0$, part of the assertion of Theorem A, see the Introduction, holds:

THEOREM A.4. *For all $t<0$,*

$$P_{\text{tree}}(t)=P_{\text{hyp}}(t)=P_{\text{hypvar}}(t)=P_{\text{var}}(t).$$

Proof. The proof of $P_{\text{hypvar}}(t)=P_{\text{var}}(t)$ is the same as for $0\leq t\leq t_0$, see the Proof of Proposition 1.1 and [**P2**]. Namely $P_{\text{var}}(t)>0$ implies $h_\mu(f)-t\chi_\mu(f)>0$ for μ such that this expression is close to $\sup_\mu$. This yields either $\chi_\mu(f)>0$ or $h_\mu(f)>0$ which also implies $\chi_\mu(f)>0$.

(Note that unlike in the $t>0$ case, we do not need to rely on $\chi_\mu(f)\geq 0$ [**P1**]. Note also that for $t<0$ the possibility $h_\mu(f)=0$ can happen. Consider, for example, $f(z)=z^2-2$ and $-t$ large. Then μ supported on the repelling fixed point $z=2$ is the only measure where $h_\mu(f)-t\chi_\mu(f)$ is close (and actually equal) to the supremum.)

Also the proof of $P_{\text{hyp}}(t)=P_{\text{hypvar}}(t)$ holds for $t<0$ as well as for $t\geq 0$, see Remark 2.2 and [**P2**].

The proof of $P_{\text{tree}}(t)=P_{\text{tree}}(z,t)\geq P_{\text{hyp}}(t)$ for typical z is immediate, see the beginning of the proof of Proposition 2.1. The opposite inequality follows from $P_{\text{hyp}}(t)=P(f|_J,-t\ln|f'|)$ following from Theorem A.2 and the already proven $P_{\text{hyp}}(t)=P_{\text{var}}(t)$ and from

$$P_{\text{tree}}(z,t)\leq P(f|_J,-t\ln|f'|) \tag{A.1}$$

for every $z \in J$. The latter inequality can be proved similarly to this inequality with $-t \ln |f'|$ replaced by ϕ real-valued, for $z \in J$, see [**P3**, Lemma 4]. (The main idea, the existence of large subtrees of well-separated branches, comes from [**MP**].) □

Remark. The inequality (A.1) holds, in fact, for all $z \in \overline{\mathbb{C}}$. The proof uses the geometric Lemma 3.1 of [**P2**] and in the case where z is superattracting, the proof uses the fact that there is no contribution to the pressure along its periodic orbit, since for negative t we have $|(f^n)'(z)|^{-t} = 0$. We do not provide details since we are interested only in $z \in J$.

We denote all the pressures in Theorems A.2, A.4 for $t < 0$ by $P(t)$.

It is not interesting to consider $P_{\mathrm{DU}}(t)$ since one can consider pressure as in Definition A.1 for $\phi = -t \ln |f'|$ directly on J.

A.2. *Conformal pressure.* To find conformal measures to study $P_{\mathrm{Conf}}(t)$, one can use Perron–Frobenius operator $\mathcal{L} : C(X) \to C(X)$,

$$\mathcal{L}(h)(x) = \sum_{y \in f^{-1}(x)} |f'(y)|^{-t} h(y).$$

The conjugate operator $\mathcal{L}^*$ acts on measures and we find a probability eigenmeasure μ, i.e. such that $\mathcal{L}^*(\mu) = \lambda\mu$ for an eigenvalue $\lambda > 0$, see [**W**]. It follows from the definition that for every Borel set $E \subset J$ on which f is injective,

$$\lambda\mu(E) = \int_{f(E)} |f'((f|_E)^{-1}(x))|^{-t} \, d\mu(x). \tag{A.2}$$

This property is equivalent to being an eigenmeasure for $\mathcal{L}^*$. We call it *backward conformal* with Jacobian $\lambda^{-1}|f'|^{-t}$.

Note that since f' is equal to zero at critical points, μ need not be forward quasi-invariant; we cannot rewrite this property to $\mu(f(E)) = \int \lambda |f'|^t \, d\mu$ as in Definition 0.6 in the case μ has an atom at a critical value, because at critical points $|f'|^t$ is infinite. This can happen indeed: consider $f(z) = z^2 - 2$ and μ supported at $\{-2, 2\}$ and $\lambda = 4^{-t}$.

We gather part of this discussion in the following proposition.

PROPOSITION A.5. *For $t < 0$ any probability measure μ on J is conformal with Jacobian $\lambda|f'|^t$ iff it is backward conformal with Jacobian $\lambda^{-1}|f'|^{-t}$ and has no atoms at critical values.*

Proof. For μ conformal and any $c \in \mathrm{Crit}(f)$, $\mu(c) > 0$ would imply $\mu(f(c)) = \int_{\{c\}} \lambda |f'(c)|^t \delta\mu = \infty$, which is impossible, and $\mu(c) = 0$ implies $\mu(f(c)) = \int_{\{c\}} \lambda |f'(c)|^t \delta\mu = 0$. The integral conditions for conformality and backward conformality become equivalent. □

In the case of $t < 0$, we replace in Definition 0.6 of P_{Conf}, $\inf \lambda$ by $\sup \lambda$ and conformal by backward conformal.

Definition A.6. (Backward conformal pressure for $t < 0$) $\bar{P}_{\mathrm{BConf}}(t) = \ln \lambda(t)$, where

$$\lambda(t) := \sup\{\lambda > 0 : \exists\mu \text{ backward conformal on } J \text{ with Jacobian } \lambda^{-1}|f'|^{-t}\}.$$

Now we can complete Theorem A.4.

THEOREM A.7. *For $t < 0$ we have $P_{\text{tree}}(t) = \bar{P}_{\text{BConf}}(t)$.*

Proof. By definition, we obtain for every backward conformal measure μ with Jacobian $\lambda^{-1}|f'|^{-t}$ and every n

$$\int \sum_{f^n(x)=z} |(f^n)'(x)|^{-t}\, d\mu(z) = \int \mathcal{L}^n(\mathbb{1})\, d\mu = \lambda^n, \tag{A.3}$$

hence,

$$\lambda^n \le \sup_{z\in J} \sum_{f^n(x)=z} |(f^n)'(x)|^{-t} \le \text{Const} \exp n(P(f|_J, -t\ln|f'|) + \delta) \tag{A.4}$$

for an arbitrary $\delta > 0$, by [**P3**, Lemma 4] (allowing the value $-\infty$ does not change the proof).

We conclude with $\ln\lambda \le P(f|_J, -t\ln|f'|)$.

To prove the opposite inequality, we find an appropriate backward conformal measure by the Patterson–Sullivan method. We just repeat the construction from [**P2**, Remark 2.6] for $z_0 \in J(f)$ regular non-periodic.

Let us recall the construction. We define $\mu := \lim \mu_\lambda$, where

$$\mu_\lambda = \sum_{n\ge 0} \sum_{f^n(x)=z_0} D_x \cdot b_n \lambda^{-n} |(f^n)'(x)|^{-t}/B_\lambda,$$

where D_x is the Dirac measure, $\lambda \searrow \exp P(f|_J, -t\ln|f'|)$ and $b_n/b_{n+1} \to 1$ such that the series for λ replaced by $\exp P(f|_J, -t\ln|f'|)$ is divergent and B_λ is the normalizing factor, making the measure of $J(f)$ equal to one.

Clearly μ is backward conformal with Jacobian $(\exp - P(f|_J, -t\ln|f'|))|f'|^{-t}$. □

Remark A.8. We cannot replace in Theorem A.7 $\bar{P}_{\text{BConf}}(t)$ defined with $\sup\lambda$ by $\underline{P}_{\text{BConf}}(t)$ defined with $\inf\lambda$, since it can happen that $\ln\lambda < P_{\text{tree}}(t)$.

Consider again, for example, $f(z) = z^2 - 2$ and negative t. Then $P_{\text{tree}}(t) > P(0) = \ln 2$, whereas for μ supported on $\{-2, 2\}$, we have $\ln\lambda = -t\ln 4$ close to 0 for t close to 0.

$\bar{P}_{\text{BConf}}(t) > \underline{P}_{\text{BConf}}(t)$ can happen only if there is a backward conformal measure which is not conformal. This will follow from Proposition A.11.

Denote by X the set of all points $z_0 \in J$ whose every backward trajectory $z_1, z_2, \dots$ contains a critical point. The existence of a backward conformal measure not being conformal is equivalent to $X \ne \emptyset$. The proof is easy, use Proposition A.5.

In the example $f(z) = z^2 - 2$, for $-1 < t < 0$, we have $\overline{P}_{\text{BConf}}(t) = (-t+1)\ln 2 > \underline{P}_{\text{BConf}}(t) = -t\ln 4$. For $t < -1$, we have $\overline{P}_{\text{BConf}}(t) = -t\ln 4$ attained on μ above and no conformal measure exists.

The parameter $t = -1$ is called the 'phase transition' parameter. The phenomena related to it from the point of view of equilibrium measures and the non-differentiability of $P(t)$ at -1 have been comprehensively described in [**MS0**] and [**MS1**].

Recall one more equality for $\sigma(\mathcal{L})$ being the spectral radius of Perron–Frobenius operator $\mathcal{L}$, following, in fact, from (A.4) and Theorems A.2, A.4.

THEOREM A.9. [**MS1, P3**] $\log \sigma(\mathcal{L}) = P(t)$.

We end §A.2 with the proof that for conformal measures all $\ln \lambda$s coincide with $P(t)$ (remember, however, that the set of conformal measures can be empty). We start with the following lemma.

LEMMA A.10. *The inequality* $P_{\text{tree}}(z,t) \geq P(t)$ *for* $t < 0$ *holds for all* $z \in \overline{\mathbb{C}} \setminus X$.

Proof. For every $z \in \overline{\mathbb{C}} \setminus X$ we choose a backward trajectory $z_k \in f^{-k}(z)$ disjoint with $\mathrm{Crit}(f)$, converging to a hyperbolic repeller $Z \subset J$. Then, for k large, clearly

$$\begin{aligned} P(f|_Z, -t\ln|f'|) \leq P_{\text{tree}}(z_k,t) &= \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{f^n(x)=z_k} |(f^n)'(x)|^{-t} \\ &= \limsup_{n\to\infty} \frac{1}{n} \ln \sum_{f^n(x)=z_k} |(f^{n+k})'(x)|^{-t} |(f^k)'(z_k)|^t \leq P_{\text{tree}}(z,t), \end{aligned}$$

since $|(f^k)'(z_k)|^t \leq 1$ for k large enough. Choosing Z such that $P(f|_Z, -t\ln|f'|)$ is arbitrarily close to $P(t)$, we get the desired inequality. □

PROPOSITION A.11. *For every conformal measure* μ *with Jacobian* $\lambda|f'|^t$, $t < 0$, *we have* $\ln\lambda = P(t)$.

Proof. $\ln\lambda \leq P_{\text{tree}}(t)$ follows from Theorem A7. To prove the opposite inequality, note that the Tchebyshev inequality and (A.3) yield, for every $\delta > 0$,

$$\mu\left\{ z \in J : \sum_{f^n(x)=z} |(f^n)'(x)|^{-t} \geq \lambda^n \exp(\delta n) \right\} \leq \exp(-\delta n).$$

Hence, by the Borel–Cantelli lemma and letting $\delta \to 0$, we get $P_{\text{tree}}(z,t) \leq \ln\lambda$ for all z in a full μ-measure set B. By Proposition A.5, repeating the proof by induction along critical orbits, we see that μ has no atoms in $\bigcup_{i=1}^{\infty} f^i(\mathrm{Crit}) \supset X$. So $B \setminus X \neq \emptyset$ and for $z \in B \setminus X$ we have, by Lemma A.10, that $P(t) = P_{\text{tree}}(z,t) \leq \ln\lambda$. □

A.3. $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$ *revisited.* Here we give another direct proof of $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$ trying to repeat the proof of Proposition 2.1: that proof must, however, be slightly modified. We should be especially careful in the presence of eventually periodic critical points in J.

Let $O_1, \dots, O_I$ be periodic orbits in J contained in $\bigcup_{j=0}^{\infty} f^j(\mathrm{Crit})$. Denote $\lambda_j := |(f^{k_j})'(p_j)|^{1/k_j}$ for $p_j \in O_j$ of period k_j, for $j = 1, \dots, I$.

If there exists j such that $-t\ln\lambda_j \geq P_{\text{tree}}(t)$, then we choose as a hyperbolic set $X = O_j$ which proves $P_{\text{tree}}(t) \leq P_{\text{hyp}}(t)$. So we can assume that all $-t\ln\lambda_j$ are less than a constant $P_0 < P_{\text{tree}}(t)$.

Let $Y = \bigcup_{j=1}^{m} f^j(\mathrm{Crit})$, for $m = m(\delta)$ and δ as in the proof of Proposition 2.1. For every $k = 0, \dots, n$, define for a constant $\xi > 0$ and for z_0 as in Proposition 2.1

$$\begin{aligned} X_k := \{ z_n \in f^{-n}(z_0) : \mathrm{dist}(f^s(z_n), Y) \geq \xi \text{ for } s = n-k \text{ and } \mathrm{dist}(f^s(z_n), Y) \\ < \xi \text{ for all } s = 0, \dots, n-k-1 \}. \end{aligned}$$

Consider also $k = -1$, where the condition for $s = n - k$ should be omitted.

Note that if ξ is small enough, then $n - k > m_0 := \#\mathrm{Crit}(f) \cdot m$ implies for $z_n \in X_k$ that there exists j such that

$$\mathrm{dist}(f^s(z_n), O_j) < \xi \quad \text{for all } s = m_0 + 1, \ldots, n - k - 1. \tag{A.5}$$

(This is the only possibility, if $f^s(z_n)$ follows Y.) Denote

$$R_k := \sum_{x \in f^{-n}(z_0) \cap X_k} |(f^k)'(f^{n-k}(x))|^{-t}$$

and

$$S_k = \sum_{x \in f^{-k}(z_0)} |(f^k)'(x)|^{-t}.$$

Observe that, due to (A.5) for ξ small enough, there exists a bound A (independent of n) such that for every k and $z_k \in f^{-k}(z_0)$, we have

$$\#\{z_n \in X_k : f^{n-k}(z_n) = z_k\} \le A.$$

We have, for $L = \sup |f'|$,

$$\begin{aligned} S_n &= \sum_{k=-1,\ldots,n} \sum_{f^n(x)=z_0, x \in X_k} |(f^n)'(x)|^{-t} \\ &\le L^{-tm_0} \Bigg(\sum_{k=n-m_0,\ldots,n} R_k + C \sum_{k=-1,\ldots,n-m_0-1} (\exp(n-k-m_0)P_0) R_k \Bigg). \end{aligned}$$

Here we have $\exp(n - k - m_0)P_0$ due to (A.5). The constant C is necessary in case only a piece of O_j is followed by $f^s(z_n)$.

By the definition of $P_{\mathrm{tree}}(t)$ for an arbitrary constant $a > 0$, there is $k(a)$ such that for all $k \ge k(a)$, we have $S_k \le \exp k(P_{\mathrm{tree}}(t) + a)$; hence, $R_k \le A \exp k(P_{\mathrm{tree}}(t) + a)$. Assume also that n satisfies $S_n \ge \exp n P_1$ for $P_1 = (P_0 + P_{\mathrm{tree}}(t))/2$. Fix finally $a = P_{\mathrm{tree}}(t) - P_0$.

Suppose that, for all $k : \frac{1}{5}n \le k \le n$, we have $R_k \le \exp(kP_0)$. Then we obtain

$$\begin{aligned} \exp(nP_1) \le CL^{-tm_0} \Bigg(&\mathrm{Const} + \sum_{k(a) \le k \le n/5} A \exp((n-k-m_0)P_0) \exp k(P_{\mathrm{tree}}(t) + a)) \\ &+ \sum_{n/5 < k \le n-m_0-1} \exp((n-k-m_0)P_0) \exp(kP_0) \\ &+ (m_0+1) \sum_{n-m_0 \le k \le n} \exp(kP_0) \Bigg) \\ \le CL^{-tm_0} &\left(\mathrm{Const} + \frac{n}{5} A \exp n \left(P_1 - \frac{a}{10} \right) + \frac{4n}{5} \exp(nP_0) \right) \end{aligned}$$

which, however, is obviously false for n large enough.

Therefore, there exists $k : n/5 < k \le n$ such that $R_k \ge \exp(kP_0)$. For all $z_k \in f^{n-k}(X_k)$, we find z'_k as in the proof of Proposition 2.1 and construct invariant sets Z, Z', with n replaced by k.

Recall that for all $x_k \in f^{n-k}(X_k)$, we have $\mathrm{dist}(x_k, Y) \geq \xi$. Hence, there exists $\xi' > 0$ such that $|f'(f^s(z_k'))| \geq \xi'$ for all $s : 0 \leq s \leq m$. We can also assume that $|f'(x)| \geq \xi'$ for $x \in B(X, 2\delta)$.

We can repeat now and modify calculation (2.1).

$$\begin{aligned}
P(F|_Z, -t\ln|F'|) &\geq \ln\left(C \sum_{z_k \in f^{n-k}(X_k)} |(f^{m+k+l})'(z_k')|^{-t}\right) \\
&\geq \ln C \sum_{z_k \in f^{n-k}(X_k)} |(f^k)'(z_k)|^{-t} (\xi')^{-t(m+l)} \\
&\geq \ln C - t(m+l)\ln \xi' + \ln A^{-1} + kP_0.
\end{aligned}$$

Passing to Z' we finish the proof as in Proposition 2.1. □

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