Equational Systems and Free Constructions

(Extended Abstract)

Marcelo Fiore and Chung-Kil Hur*

Computer Laboratory, University of Cambridge

Abstract. The purpose of this paper is threefold: to present a general abstract, yet practical, notion of equational system; to investigate and develop a theory of free constructions for such equational systems; and to illustrate the use of equational systems as needed in modern applications, specifically to the theory of substitution in the presence of variable binding and to models of name-passing process calculi.

1 Introduction

The import of equational theories in theoretical computer science is by now well established. Traditional applications include the initial algebra approach to the semantics of computational languages and the specification of abstract data types pioneered by the ADJ group [11], and the abstract description of powerdomain constructions as free algebras of non-determinism advocated by Plotkin [13, 16] (see also [1]). While these developments essentially belong to the realm of universal algebra, more recent applications have had to be based on the more general categorical algebra. Examples include theories of abstract syntax with variable binding [6, 8], the algebraic treatment of computational effects [17, 18], and models of name-passing process calculi [5, 21].

In the above and most other applications of equational theories, the existence and construction of initial and/or free algebras, and consequently of monads, plays a central role; as so does the study of categories of algebras. These topics are investigated here in the context of *equational systems*, a very broad notion of equational theories. Examples of equational systems include enriched algebraic theories [14, 20], algebras for a monad, monoids in a monoidal category, *etc.* (see Section 3).

The original motivation for the development of the theory of equational systems arose from the need of a mathematical theory readily applicable to two further examples of equational systems: (*i*) Σ-monoids (see Section 6.1), which are needed for the initial algebra approach to the semantics of languages with variable binding and capture-avoiding simultaneous substitution [6]; and (*ii*) π-algebras (see Section 6.2), which provide algebraic models of the finitary π-calculus [21]. Indeed, these two examples respectively highlight two inadequacies of enriched algebraic theories in applications: (*i*) the explicit presentation of an enriched algebraic theory may be hard to give, as it is the case with

Σ-monoids; and (ii) models may require a theory based on more than one enrichment, as it is the case with π-algebras.

Further benefits of equational systems over enriched algebraic theories are that the theory can be developed for cocomplete, not necessarily locally presentable, categories (examples of which are the category of topological spaces, the category of directed-complete posets, and the category of complete semi-lattices), and that the concept of equational system is straightforwardly dualizable: a coequational system on a category is simply an equational system on the opposite category (thus, for instance, comonoids in a monoidal category are coalgebras for a coequational system). On the other hand, the price paid for all this generality is that the important connection between enriched algebraic theories and enriched Lawvere theories [19] is lost for equational systems.

An equational system $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ is defined as a parallel pair $L, R : F\text{-}\mathbf{Alg} \to D\text{-}\mathbf{Alg}$ of functors between categories of algebras over a base category $\mathscr{C}$. In this context, the endofunctor F on $\mathscr{C}$, which generalizes the notion of algebraic signature, is called a functorial signature; the functors L, R over $\mathscr{C}$, generalize the notion of equation, and are called functorial terms; the endofunctor D on $\mathscr{C}$ corresponds to the arity of the equation. The category of $\mathbb{S}$-algebras is the equalizer $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow F\text{-}\mathbf{Alg}$ of L, R. Thus, an $\mathbb{S}$-algebra is an F-algebra $(X, s : FX \to X)$ such that $L(X, s) = R(X, s)$ as D-algebras on X.

We have learnt during the course of this work that variations on the concept of equational system have already been considered in the literature. For instance, Fokkinga [7] introduces the more general concept of law between transformers, but only studies initial algebras for the laws that are equational systems; Cîrstea [3] introduces the concept of coequation between abstract cosignatures, which is equivalent to our notion of coequational system, and studies final coalgebras for them; Ghani, Lüth, De Marchi, and Power [10] introduce the concept of functorial coequational presentations, which is equivalent to our notion of coequational system on a locally presentable base category with an accessible functorial signature and an accessible arity endofunctor, and study cofree constructions for them.

Our theory of equational systems (and its dual), which we present in Sections 4 and 5, is more general and comprehensive than that of [7] and [3]; and we relate it to that of [10] in the Concluding Remarks (Section 7).

Free constructions for equational systems are investigated in Section 4. For an equational system $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$, the existence of free $\mathbb{S}$-algebras on objects in $\mathscr{C}$ is considered in two stages: (i) the construction of free F-algebras on objects in $\mathscr{C}$, and (ii) the construction of free $\mathbb{S}$-algebras over F-algebras. The former captures the construction of freely generated terms with operations from the functorial signature F; the latter that of quotienting F-algebras by the equation $L = R$ and congruence rules. We give two sufficient conditions for the existence of free $\mathbb{S}$-algebras on F-algebras. The first condition can be used to deduce the existence of free algebras for enriched algebraic theories, but it applies more generally. The second condition may be applied to functorial signatures and arities that are not accessible. The proofs of these results provide constructions of free algebras that may lead to explicit descriptions. As a concrete example of this situation, we observe that for the functorial signature Σ_λ of the λ-calculus, the initial Σ_λ-monoid satisfying β, η equations consists of λ-terms (up to α-equivalence) quotiented by the β, η equalities (see Section 6.1).

Monads and categories of algebras for equational systems are discussed in Section 5. In the vein of the above results, we give two sufficient conditions under which the monadicity and cocompleteness of categories of algebras follow. As an application, we observe that the category of π-algebras is monadic and cocomplete (see Section 6.2).

2 Algebraic Equational Theories

To set our work in context, we briefly review the classical concept of algebraic equational theory and some aspects of the surrounding theory.

An algebraic equational theory consists of a signature defining its operations and a set of equations describing the axioms that it should obey.

A signature Σ is given by a set of operators O together with a function $|-|\colon O \to \mathbb{N}$ giving an arity to each operator. The set of terms $T_\Sigma(V)$ on a set of variables V is built up from the variables and the operators of the signature Σ by the following grammar

$$t \in T_\Sigma(V) \ ::= \ v \ \mid \ o(t_1, \ldots, t_k)$$

where $v \in V$, o is an operator of arity k, and $t_i \in T_\Sigma(V)$ for all $i = 1, \ldots, k$.

An equation of arity V, written $V \vdash l = r$, for a signature Σ is a pair of terms $l, r \in T_\Sigma(V)$.

An algebraic equational theory $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ together with a set of equations E.

An algebra for a signature Σ is a pair $(X, [\![-]\!]_X)$ consisting of a carrier set X together with interpretation functions $[\![o]\!]_X : X^{|o|} \to X$ for each operator o in Σ. By structural induction, such an algebra induces interpretations $[\![t]\!]_X : X^V \to X$ of terms $t \in T_\Sigma(V)$ as follows:

$$[\![t]\!]_X = \begin{cases} X^V \xrightarrow{\pi_v} X & \text{, for } \ t = v \in V \\[2ex] X^V \xrightarrow{\langle [\![t_1]\!]_X, \ldots, [\![t_k]\!]_X \rangle} X^k \xrightarrow{[\![o]\!]_X} X & \text{, for } \ t = o(t_1, \ldots, t_k) \end{cases}$$

An algebra for the theory $\mathbb{T} = (\Sigma, E)$ is an algebra for the signature Σ that satisfies the constraints given by the equations in E, where a Σ-algebra $(X, [\![-]\!]_X)$ is said to satisfy the equation $V \vdash l = r$ whenever $[\![l]\!]_X \vec{x} = [\![r]\!]_X \vec{x}$ for all $\vec{x} \in X^V$.

An homomorphism of $\mathbb{T}$-algebras from $(X, [\![-]\!]_X)$ to $(Y, [\![-]\!]_Y)$ is a function $h : X \to Y$ between their carrier sets that commutes with the interpretation of each operator; that is, such that $h([\![o]\!]_X(x_1, \ldots, x_k)) = [\![o]\!]_Y\big(h(x_1), \ldots, h(x_k)\big)$. Algebras and homomorphisms form the category $\mathbb{T}$-**Alg**.

The existence of free algebras for algebraic theories is one of the most significant properties that they enjoy. For an algebraic theory $\mathbb{T} = (\Sigma, E)$, the free algebra over a set X has as carrier the set $T_\Sigma(X)/_{\sim_E}$ of equivalence classes of terms on X under the equivalence relation $\sim_E$ defined by setting $t \sim_E t'$ iff t is provably equal to t' by the equations given in E and the congruence rules. The interpretation of each operator on $T_\Sigma(X)/_{\sim_E}$ is given syntactically: $[\![o]\!]([t_1]_{\sim_E}, \ldots, [t_k]_{\sim_E}) = [o(t_1, \ldots, t_k)]_{\sim_E}$. This construction gives rise to a left adjoint $F_\mathbb{T}$ to the forgetful functor $U_\mathbb{T} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$. Moreover, the adjunction is monadic: $\mathbb{T}$-**Alg** is equivalent to the category of algebras for the associated monad on **Set**.

We recall the notion of algebra for an endofunctor and how it generalizes that of algebra for a signature.

An algebra for an endofunctor F on a category $\mathscr{C}$ is a pair (X, s) of a carrier object X in $\mathscr{C}$ together with a structure algebra map $s : FX \to X$. A homomorphism of F-algebras from (X, s) to (Y, t) is a map $h : X \to Y$ in $\mathscr{C}$ such that $h \cdot s = t \cdot Fh$. F-algebras and homomorphisms form the category F-**Alg**, and the forgetful functor $U_F : F\text{-}\mathbf{Alg} \to \mathscr{C}$ maps an F-algebra (X, s) to its carrier object X.

As it is well-known, every signature can be turned into an endofunctor on **Set** preserving its algebras. Indeed, for a signature Σ, one defines the corresponding endofunctor as $F_\Sigma(X) = \coprod_{o \in \Sigma} X^{|o|}$, so that Σ-**Alg** and F_Σ-**Alg** are isomorphic. In this view, we will henceforth take endofunctors as a general abstract notion of signature.

Definition 2.1 (*Functorial signature*). *A functorial signature on a category is an endofunctor on it.*

3 Equational Systems

We motivate and subsequently present an abstract notion of equation for functorial signatures, leading to the concept of equational system. Free constructions for equational systems are considered in the following section.

Let $t \in T_\Sigma(V)$ be a term on a set of variables V for a signature Σ. Recall from the previous section that for every Σ-algebra $(X, [\![-]\!]_X)$, the term t gives an interpretation function $[\![t]\!]_X : X^V \to X$. Thus, the term t determines a function $\widetilde{t}$ assigning to a Σ-algebra $(X, [\![-]\!]_X)$ the D-algebra $(X, [\![t]\!]_X)$, for D the endofunctor $(-)^V$ on **Set**. Note that the function $\widetilde{t}$ does not only preserves carrier objects but, furthermore, by the uniformity of the interpretation of terms, that a Σ-homomorphism $(X, [\![-]\!]_X) \to (Y, [\![-]\!]_Y)$ is also a D-homomorphism $(X, [\![t]\!]_X) \to (Y, [\![t]\!]_Y)$. In other words, the function $\widetilde{t}$ extends to a functor $\Sigma\text{-}\mathbf{Alg} \to D\text{-}\mathbf{Alg}$ over **Set**, *i.e.* a functor preserving carrier objects and homomorphisms. These considerations lead us to define abstract notions of term and equations as follows.

Definition 3.1 (*Functorial terms and equations*). *A functorial term T of arity D for a functorial signature F on a category $\mathscr{C}$, consists of an endofunctor D on $\mathscr{C}$ and a functor $T : F\text{-}\mathbf{Alg} \to D\text{-}\mathbf{Alg}$ over $\mathscr{C}$, that is, a functor such that*

$U_D \cdot T = U_F$. *A functorial equation is given by a pair of functorial terms of the same arity.*

We are now ready to define equational systems, our abstract notion of equational theory.

Definition 3.2 (*Equational systems*). *An equational system*

$$\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$$

is given by a category $\mathscr{C}$ *and a functorial equation* $L = R$ *of arity* D *for a functorial signature* F.

We have restricted attention to equational systems subject to a single equation. The consideration of multi-equational systems $(\mathscr{C} \rhd F \vdash \{L_i = R_i : D_i\}_{i \in I})$ subject to a set of equations in what follows is left to the interested reader. We remark however that our development is typically without loss of generality; as, whenever $\mathscr{C}$ has I-indexed coproducts, a multi-equational system as above can be expressed as the equational system $(\mathscr{C} \rhd F \vdash [L_i]_{i \in I} = [R_i]_{i \in I} : \coprod_{i \in I} D_i)$ with a single equation.

Recall that an equation $l = r$ in an algebraic theory is interpreted as the constraint that the interpretation functions associated with the terms l and r coincide. Hence, for an equational system $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$, it is natural to say that an F-algebra A satisfies the functorial equation $L = R$ whenever $L(A) = R(A)$, and consequently define the category of algebras for the equational system as the full subcategory of F-**Alg** consisting of the F-algebras that satisfy the functorial equation $L = R$. Equivalently, we introduce the following definition.

Definition 3.3. *For an equational system* $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$, *the category* $\mathbb{S}$-**Alg** *of* $\mathbb{S}$-*algebras is the equalizer of* $L, R : F\text{-}\mathbf{Alg} \to D\text{-}\mathbf{Alg}$ (*in the large category of locally small categories over* $\mathscr{C}$).

Examples of equational systems together with their induced categories of algebras follow.

1. The equational system $\mathbb{S}_\mathbb{T}$ associated to the algebraic theory $\mathbb{T} = (\Sigma, E)$ is given by $(\mathbf{Set} \rhd F_\mathbb{T} \vdash L_\mathbb{T} = R_\mathbb{T} : D_\mathbb{T})$, with $F_\mathbb{T} X = \coprod_{o \in \Sigma} X^{|o|}$, $D_\mathbb{T} X = \coprod_{(V \vdash l = r) \in E} X^V$, and
$$L_\mathbb{T}(X, [\![-]\!]_X) = \big(X, \big[\, [\![l]\!]_X \,\big]_{(l=r) \in E}\big) \ ,$$
$$R_\mathbb{T}(X, [\![-]\!]_X) = \big(X, \big[\, [\![r]\!]_X \,\big]_{(l=r) \in E}\big) \ .$$
It follows that $\mathbb{T}$-**Alg** is isomorphic to $\mathbb{S}_\mathbb{T}$-**Alg**.
2. More generally, consider an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$ on a locally finitely presentable category $\mathscr{C}$ enriched over a suitable category $\mathscr{V}$, see [14]. Recall that this is given by functors $B, E : |\mathscr{C}_{\mathrm{fp}}| \to \mathscr{C}_0$ and a pair of morphisms $\sigma, \tau : FE \to FB$ between the free finitary monads FB and FE on $\mathscr{C}$ respectively arising from B and E. The equational system $\mathbb{S}_\mathbb{T}$ associated to such an enriched algebraic theory $\mathbb{T}$ is given by $(\mathscr{C}_0 \rhd (GB)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0 : (GE)_0)$, where GB and GE are the free finitary endofunctors on $\mathscr{C}$ respectively arising from B and E, and where $\overline{\sigma}$ and $\overline{\tau}$ are

respectively the functors corresponding to σ and τ by the bijection between morphisms $FE \to FB$ and functors $GB\text{-}\mathbf{Alg} \cong \mathscr{C}^{FB} \to \mathscr{C}^{FE} \cong GE\text{-}\mathbf{Alg}$ over $\mathscr{C}$. It follows that $(\mathbb{T}\text{-}\mathbf{Alg})_0$ is isomorphic to $\mathbb{S}_{\mathbb{T}}\text{-}\mathbf{Alg}$.

3. The definition of Eilenberg-Moore algebras for a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts can be directly encoded as the equational system $\mathbb{S}_{\mathbf{T}} = (\mathscr{C} \rhd T \vdash L = R : D)$ with $D(X) = X + T^2X$ and

$$L(X, s) = (\, X, [\, s \cdot \eta_X \;,\; s \cdot \mu_X \,]\,) \;,$$
$$R(X, s) = (\, X, [\; \mathrm{id}_X \;,\; s \cdot Ts \,]\,) \;.$$

It follows that $\mathbb{S}_{\mathbf{T}}\text{-}\mathbf{Alg}$ is isomorphic to the category $\mathscr{C}^{\mathbf{T}}$ of Eilenberg-Moore algebras for $\mathbf{T}$.

4. The definition of monoid in a monoidal category $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ with binary coproducts yields the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})} = (\mathscr{C} \rhd F \vdash L = R : D)$ with $F(X) = (X \otimes X) + I$, $D(X) = \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I)$, and

$$L(X, [m, e]) = (\, X, [\; m \cdot (m \otimes \mathrm{id}_X) \;,\; \lambda_X \;,\; \rho_X \;]\,) \;,$$
$$R(X, [m, e]) = (\, X, [\; m \cdot (\mathrm{id}_X \otimes m) \cdot \alpha_{X,X,X} \;,\; m \cdot (e \otimes \mathrm{id}_X) \;,\; m \cdot (\mathrm{id}_X \otimes e) \;]\,) \;.$$

It follows that $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}\text{-}\mathbf{Alg}$ is isomorphic to the category of monoids and monoid homomorphisms in $\mathscr{C}$.

4 Free Constructions for Equational Systems

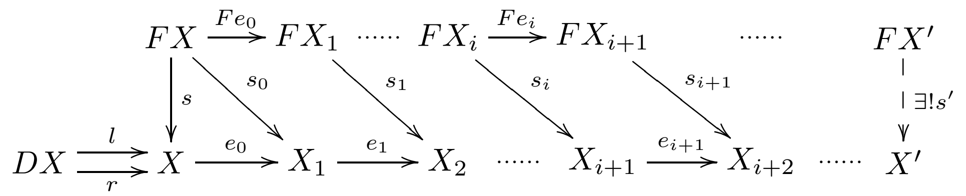
We investigate sufficient conditions for the existence of free algebras for equational systems; that is, for the existence of a left adjoint to the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$, for $\mathbb{S}$ an equational system. Since, by definition, the forgetful functor $U_{\mathbb{S}}$ decomposes as $\mathbb{S}\text{-}\mathbf{Alg} \overset{J_{\mathbb{S}}}{\hookrightarrow} F\text{-}\mathbf{Alg} \overset{U_F}{\longrightarrow} \mathscr{C}$, we will concentrate on obtaining a left adjoint to the embedding $J_{\mathbb{S}}$. Conditions for the existence of a left adjoint to U_F have already been studied in the literature (see *e.g.* [2]).

Theorem 4.1. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system. If $\mathscr{C}$ is cocomplete, and F and D preserve colimits of α-chains for some infinite limit ordinal α, then the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow F\text{-}\mathbf{Alg}$ has a left adjoint.*

Theorem 4.2. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system. If $\mathscr{C}$ is well-copowered and cocomplete, and F preserves epimorphisms, then the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow F\text{-}\mathbf{Alg}$ has a left adjoint.*

Corollary 4.1. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system. If $\mathscr{C}$ is cocomplete, F preserves epimorphisms and colimits of ω-chains, and D preserves epimorphisms, then the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow F\text{-}\mathbf{Alg}$ has a left adjoint. Furthermore the free algebra functor is constructed in ω steps.*

These results are proved by performing an iterative, possibly transfinite, construction that associates a free $\mathbb{S}$-algebra to every F-algebra. The cocompleteness of the base category allows one to perform the construction, whilst the other conditions guarantee that the process will eventually stop. We present the construction in the simplest case, *viz.* that of Corollary 4.1. To this end, for an F-algebra (X, s), let $L(X, s) = (X, l : DX \to X)$ and $R(X, s) = (X, r : DX \to X)$, and consider the following diagram



(1)

where e_0 is a coequalizer of l, r and where (e_{i+1}, s_{i+1}) is a pushout of (s_i, Fe_i) for all $i \geq 0$. Further, let X' be a colimit of the ω-chain $\langle e_i \rangle$, so that FX' is a colimit of the ω-chain $\langle Fe_i \rangle$, and define the algebra map s' to be the unique mediating morphism between them. It follows that (X', s') is a free $\mathbb{S}$-algebra on the F-algebra (X, s).

The intuition behind the construction is that of first quotienting the carrier object by the equation $L = R$, and then by congruence rules as much as needed. If free algebras are constructed in ω steps, then, roughly speaking, they arise by quotienting a finite number of times.

Finally, we remark that in the presence of binary coproducts the problem of finding free algebras reduces to that of finding initial algebras.

Proposition 4.1. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system on a category $\mathscr{C}$ with binary coproducts. An $\mathbb{S}$-algebra is free over $A \in \mathscr{C}$ iff it is an initial $\mathbb{S}^A$-algebra for $\mathbb{S}^A = (\mathscr{C} \rhd (A + F) \vdash L \cdot U^A = R \cdot U^A : D)$ where U^A denotes the forgetful functor $(A + F)$-$\mathbf{Alg} \to F$-$\mathbf{Alg}$.*

5 Categories of Algebras for Equational Systems

We consider monads and categories of algebras for equational systems, and give some basic applications of our results.

Theorem 5.1. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system with $\mathscr{C}$ cocomplete.*

1. *If F and D preserve colimits of α-chains for some infinite limit ordinal α, then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-$\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}$-$\mathbf{Alg}$ is cocomplete.*
2. *If $\mathscr{C}$ is well-copowered, F preserves epimorphisms, and the forgetful functor $U_{\mathbb{S}} : \mathbb{S}$-$\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, then $U_{\mathbb{S}}$ is monadic and $\mathbb{S}$-$\mathbf{Alg}$ is cocomplete.*

Proposition 5.1. *Let $\mathbb{S} = (\mathscr{C} \rhd F \vdash L = R : D)$ be an equational system. If the functors F and D preserve $\mathbb{I}$-indexed colimits for a small category $\mathbb{I}$ and $U_{\mathbb{S}}$ has a left adjoint, then the induced monad on $\mathscr{C}$ also preserves $\mathbb{I}$-indexed colimits.*

We revisit the examples of equational systems given in Section 3 in the light of the above results.

1. For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathbf{Set} \rhd F_{\mathbb{T}} \vdash L_{\mathbb{T}} = R_{\mathbb{T}} : D_{\mathbb{T}})$ representing an algebraic theory $\mathbb{T}$, the category $\mathbb{S}_{\mathbb{T}}$-$\mathbf{Alg}$ is monadic over $\mathbf{Set}$ and cocomplete

by Theorem 5.1(1); as **Set** is cocomplete and $F_{\mathbb{T}}$ and $D_{\mathbb{T}}$ preserve colimits of ω-chains.

2. For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathscr{C}_0 \rhd (GB)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0 : (GE)_0)$ representing an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$, the category $\mathbb{S}_{\mathbb{T}}$-**Alg** is monadic over $\mathscr{C}_0$ and cocomplete by Theorem 5.1(1); as $\mathscr{C}_0$ is locally finitely presentable and thus cocomplete, and $(GB)_0$ and $(GE)_0$ are finitary and thus preserve colimits of ω-chains. Furthermore, the monad arising from the monadicity of $\mathbb{S}_{\mathbb{T}}$-**Alg** is finitary by Proposition 5.1 as so are the functors $(GB)_0$ and $(GE)_0$.
3. One may apply Theorem 5.1(1) to the equational system $\mathbb{S}_{\mathbf{T}}$ representing a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts as follows. If $\mathscr{C}$ is cocomplete and T preserves colimits of ω-chains, then $\mathbb{S}_{\mathbf{T}}$-**Alg** is monadic over $\mathscr{C}$ and cocomplete.
   As another example, consider the powerset monad $\mathbf{P} = (P, \{-\}, \cup)$ on **Set**. Since **Set** is cocomplete and well-copowered, and the powerset functor P preserves epimorphisms, by Theorem 4.2, the embedding $\mathbf{Set}^{\mathbf{P}} \hookrightarrow P$-**Alg** has a left adjoint. We also see that the forgetful functor $\mathbf{Set}^{\mathbf{P}} \to \mathbf{Set}$ has a left adjoint from the fact that $\mathbf{P}$ is a monad. Therefore, $\mathbf{Set}^{\mathbf{P}}$, which is isomorphic to the category of complete semi-lattices, is cocomplete (by Theorem 5.1(2)).
4. To the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$ of monoids in a monoidal category $\mathscr{C}$ with binary coproducts, we can apply Theorem 5.1(1) as follows. If $\mathscr{C}$ is cocomplete and the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves colimits of ω-chains, then $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** is monadic over $\mathscr{C}$ and cocomplete.

6 Two Applications

We consider applications of equational systems to the theory of abstract syntax supporting variable binding and substitution [6], and to algebraic models of the π-calculus [21].

6.1 Σ-monoids

Following [6], we introduce the concept of Σ-monoid, for a functorial signature Σ with a pointed strength, and consider it from the point of view of equational systems. The theory of equational systems is then used to provide an explicit description of free Σ-monoids. We then show that, for Σ_λ the functorial signature of the lambda calculus, the β, η identities are straightforwardly expressible as functorial equations. The theory of equational systems is further used to relate the arising algebraic models by adjunctions.

Let Σ be a functorial signature on a monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$. A pointed strength for Σ is a natural transformation

$$\mathsf{st}_{X,(Y,y:I\to Y)} : \Sigma(X) \otimes Y \stackrel{.}{\to} \Sigma(X \otimes Y)$$

between functors $\mathscr{C} \times (I/\mathscr{C}) \to \mathscr{C}$ satisfying coherence conditions similar to those of strength [15]:

$$\rho_{\Sigma A} = \Sigma(\rho_A) \cdot \mathsf{st}_{A,(I,\mathrm{id}_I)} \colon \Sigma(A) \otimes I \to \Sigma A \ ,$$

$$\mathsf{st}_{A,(B\otimes C,(b\otimes c)\cdot\rho_I{}^{-1})} \cdot \alpha_{\Sigma A,B,C}$$
$$= \Sigma(\alpha_{A,B,C}) \cdot \mathsf{st}_{A\otimes B,(C,c)} \cdot (\mathsf{st}_{A,(B,b)} \otimes \mathrm{id}_C) \colon (\Sigma(A) \otimes B) \otimes C \to \Sigma(A \otimes (B \otimes C))$$

for all $A \in \mathscr{C}$ and $(B, b : I \to B), (C, c : I \to C) \in I/\mathscr{C}$.

For a functorial signature Σ with a pointed strength st on a monoidal category $\mathscr{C}$, the category of *Σ-monoids* Σ-**Mon**($\mathscr{C}$) has objects given by quadruples (X, s, m, e) where (X, s) is a Σ-algebra and (X, m, e) is a monoid in $\mathscr{C}$ satisfying the following compatibility law

$$m \cdot (s \otimes \mathrm{id}_X) = s \cdot \Sigma(m) \cdot \mathsf{st}_{X,(X,e)} : \Sigma(X) \otimes X \to X \ ;$$

morphisms are maps of $\mathscr{C}$ which are both Σ-algebra and monoid homomorphisms.

For $\mathscr{C}$ with binary coproducts, the equational system $\mathbb{M}_\Sigma$ of Σ-monoids is defined as $(\mathscr{C} \rhd F_\Sigma \vdash L_\Sigma = R_\Sigma : D_\Sigma)$, with $F_\Sigma X = \Sigma(X) + (X \otimes X) + I$, $D_\Sigma X = \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I) + (\Sigma(X) \otimes X)$, and

$$L_\Sigma(\, X, [s, m, e]\,) = (\, X, [\quad m \cdot (m \otimes \mathrm{id}_X) \quad , \quad \lambda_X \quad , \quad \rho_X \quad , \quad m \cdot (s \otimes \mathrm{id}_X) \quad]\,)$$
$$R_\Sigma(\, X, [s, m, e]\,) = (\, X, [\, m \cdot (\mathrm{id}_X \otimes m) \cdot \alpha_{X,X,X} \ , m \cdot (e \otimes \mathrm{id}_X) \ , m \cdot (\mathrm{id}_X \otimes e) \ , \ s \cdot \Sigma(m) \cdot \mathsf{st}_{X,(X,e)} \,]\,) \,.$$

The functoriality of L_Σ and R_Σ follows from the naturality of α, λ, ρ, and st. The isomorphism of $\mathbb{M}_\Sigma$-**Alg** and Σ-**Mon**($\mathscr{C}$) follows trivially from their definitions.

Consequently, one can apply the theory of equational systems developed in this paper to the algebra of Σ-monoids. For instance, by Theorem 4.1, if $\mathscr{C}$ is cocomplete, and the functorial signature Σ and the tensor product $\otimes$ preserve colimits of ω-chains, then there exists a free Σ-monoid over every object in $\mathscr{C}$. While this only shows the existence of free Σ-monoids, when the monoidal structure is closed, we can go further and give an explicit description of free Σ-monoids using the fact that in this case the initial Σ-monoid exists if so does the initial $(I + \Sigma)$-algebra $\mu X.\, I + \Sigma X$, and has carrier object $\mu X.\, I + \Sigma X$ equipped with an appropriate Σ-monoid structure, see [6]. Indeed, by Proposition 4.1, a free Σ-monoid over $A \in \mathscr{C}$ is an initial $\mathbb{M}^A_\Sigma$-algebra for the equational system $\mathbb{M}^A_\Sigma = (\mathscr{C} \rhd (A + F_\Sigma) \vdash L_\Sigma \cdot U_A = R_\Sigma \cdot U_A : D_\Sigma)$, where U_A denotes the forgetful functor $(A + F_\Sigma)$-**Alg** $\to F_\Sigma$-**Alg**. Furthermore, one can readily establish the isomorphism $\mathbb{M}^A_\Sigma$-**Alg** $\cong \mathbb{M}_{(A\otimes-)+\Sigma}$-**Alg**, where the pointed strength $\mathsf{st}'_{X,(Y,y)}$ for $(A \otimes -) + \Sigma(-)$ is given by the composite

$$\big((A \otimes X) + \Sigma(X)\big) \otimes Y \cong \big((A \otimes X) \otimes Y\big) + \Sigma(X) \otimes Y \xrightarrow{\alpha_{A,X,Y} + \mathsf{st}_{X,(Y,y)}} \big(A \otimes (X \otimes Y)\big) + \Sigma(X \otimes Y) \ .$$

Thus, we have the following result.

Proposition 6.1. *For $\mathscr{C}$ a monoidal closed category with binary coproducts, the free Σ-monoid on $A \in \mathscr{C}$ exists if so does the initial $\big(I + (A\otimes -) + \Sigma(-)\big)$-algebra $\mu X.\, I + A \otimes X + \Sigma X$, and has carrier object $\mu X.\, I + A \otimes X + \Sigma X$ equipped with an appropriate Σ-monoid structure.*

As a concrete example, we now consider the λ-calculus. A λ-model [6] is a Σ_λ-monoid for the functorial signature $\Sigma_\lambda X = X^V + X^2$ with a suitable pointed strength on the presheaf category $\mathbf{Set}^{\mathbb{F}}$, where $\mathbb{F}$ is the (essentially small) category of finite sets and functions, equipped with the substitution monoidal structure $(\bullet, V)$. The operations of a Σ_λ-monoid $(X, [\mathsf{abs}, \mathsf{app}, \mathsf{sub}, \mathsf{var}] : X^V + X^2 + (X \bullet X) + V \to X)$ provide interpretations of λ-abstraction ($\mathsf{abs} : X^V \to X$), application ($\mathsf{app} : X^2 \to X$), capture-avoiding simultaneous substitution ($\mathsf{sub} : X \bullet X \to X$), and variables ($\mathsf{var} : V \to X$). The initial λ-model has carrier object $\mu X. V + X^V + X^2$, and provides an abstract notion of syntax for the λ-calculus. A syntactic description of free Σ_λ-monoids has been considered by Hamana in [12].

The β, η identities for a λ-model on X are expressed by the following equations in the internal language

$$(\beta) \quad f : X^V, x : X \vdash \mathsf{app}(\mathsf{abs}(f), x) = \mathsf{sub}\big(f\langle x\rangle\big) : X$$
$$(\eta) \quad x : X \vdash \mathsf{abs}\big(\lambda v : V.\, \mathsf{app}(x, \mathsf{var}\, v)\big) = x : X$$

where the map $-\langle=\rangle : X^V \times X \to X \bullet X$ embeds $X^V \times X$ into $X \bullet X$. These internal equations provide functorial equations on λ-models, and yield a further equational system $\mathbb{M}_{\Sigma_\lambda/\beta,\eta}$. From two applications of Corollary 4.1, we obtain the following adjoint situations:

$$\mathbb{M}_{\Sigma_\lambda/\beta,\eta}\text{-}\mathbf{Alg} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbb{M}_{\Sigma_\lambda}\text{-}\mathbf{Alg} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \big(\Sigma_\lambda(-) + (- \bullet -) + V\big)\text{-}\mathbf{Alg} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Set}^{\mathbb{F}}$$

Further, by examining the construction (1) for the free $\mathbb{M}_{\Sigma_\lambda/\beta,\eta}$-algebra on the initial $\mathbb{M}_{\Sigma_\lambda}$-algebra, one sees that the presheaf of (α-equivalence classes of) λ-terms is first quotiented by the β, η identities, and then by the congruence rules for the operations abs, app, and sub as much as needed. Thus, the initial $\mathbb{M}_{\Sigma_\lambda/\beta,\eta}$-algebra is the presheaf of β, η-equivalence classes of λ-terms.

6.2 Pi-calculus Algebras

We briefly discuss π-algebras, an algebraic model of the finitary π-calculus introduced by Stark in [21], as algebras for an equational system. The existence of free models is deduced from the theory of equational systems.

We need consider the presheaf category $\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the (essentially small) category of finite sets and injections, with the symmetric monoidal closed structure $(1, \otimes, \multimap)$ induced by the symmetric monoidal structure $(\emptyset, \uplus)$ on $\mathbb{I}$ by Day's construction [4].

A π-algebra is an object $A \in \mathbf{Set}^{\mathbb{I}}$ together with operations $\mathsf{choice} : A^2 \to A$, $\mathsf{nil} : 1 \to A$, $\mathsf{out} : N \times N \times A \to A$, $\mathsf{in} : N \times A^N \to A$, $\mathsf{tau} : A \to A$, and $\mathsf{new} : (N \multimap A) \to A$ satisfying the equations of [21, Sections 3.1–3.3 and 3.5]. These algebras, and their homomorphisms, form the category $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}})$.

The equational theory for π-algebras is expressed entirely in the internal language of $\mathbf{Set}^{\mathbb{I}}$ (see also [5]). For example, the equation establishing the inactivity of a process that inputs on a restricted channel is given by

$$p : (A^N)^N \vdash \mathsf{new}\big(\nu(\lambda x : N.\, \mathsf{in}(x, p\, x))\big) = \mathsf{nil} : A$$

where $\nu : A^N \to (N \multimap A)$ is the composite

$$A^N \xrightarrow{{up_A}^{up_N}} (N \multimap A)^{N \multimap N} \xrightarrow{\mathrm{id}^{e_N}} (N \multimap A)^1 \xrightarrow{\cong} (N \multimap A)$$

for up_X and e_X respectively the monoidal transposes of

$$X \otimes N \xrightarrow{id_X \otimes !} X \otimes 1 \xrightarrow{\cong} X \quad \text{and} \quad 1 \otimes X \xrightarrow{\cong} X \quad .$$

All these internal equations yield functorial equations, and induce an equational system $\mathbb{S}_\pi$.

Since every endofunctor of $\mathbb{S}_\pi$ is finitary, the following result follows from Theorem 5.1(1).

Proposition 6.2. *The category of* π*-algebras* $\mathcal{PI}(\mathbf{Set}^{\mathbb{I}}) \cong \mathbb{S}_\pi$-$\mathbf{Alg}$ *is cocomplete and monadic over* $\mathbf{Set}^{\mathbb{I}}$*.*

The above discussion also applies more generally, to axiomatic settings as in [5] and, in particular, to π-algebras over the Schanuel topos, $\omega\mathbf{Cpo}^{\mathbb{I}}$, *etc.*

7 Concluding Remarks

Our theoretical development also includes the organization of equational systems over a base category into a category. The consideration of colimits, in particular coequalizers, of equational systems led us to introduce the more general concept of *iterated equational system*, for which the whole theory of equational systems generalizes. As an additional result, we have that the category of iterated equational systems over a cocomplete base category is itself cocomplete. This, together with the fact that it embeds the category of accessible monads on the base category as a full subcategory which is closed under colimits, proves that the category of accessible monads on a cocomplete category is also cocomplete. Details will appear elsewhere.

Our theory of equational systems dualizes to one for *coequational systems.* Besides this being of interest in its own right, we note that the proof of the dual of Theorem 4.2, together with the construction of cofree coalgebras for endofunctors by terminal sequences of Worrell [22], gives a construction of cofree coalgebras for coequational systems on a locally presentable base category with an accessible functorial signature that preserves monomorphisms. This result is a variation of a main result of the theory developed by Ghani, Lüth, De Marchi, and Power in [10] (see *e.g.* their Lemmas 5.8 and 5.14); which is there proved by means of the theory of accessible categories without assuming the preservation of monomorphisms but assuming an accessible arity endofunctor.

Ghani and Lüth [9] give an abstract presentation of term rewriting via coinserters in the context of algebraic theories on the category of preorders. In this vein, we have developed a theory of free constructions for *inequational systems* in a preorder-enriched setting, and we are considering applications to higher-order rewriting.

Acknowledgements. We are grateful to Sam Staton for discussions.

References

1. S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, pages 1–168. Oxford University Press, 1994.
2. M. Barr and C. Wells. *Toposes, Triples and Theories*. Springer-Verlag, 1985.
3. C. Cîrstea. An algebra-coalgebra framework for system specification. In *Proc. 3rd International Workshop on Coalgebraic Methods in Computer Science*, Elsevier ENTCS 33, pages 80–110, 2000.
4. B. Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, Springer LNM 137, pages 1–38, 1970.
5. M. Fiore, E. Moggi, and D. Sangiorgi. A fully abstract model for the π-calculus. *Information and Computation*, 179(1):76–117, 2002.
6. M. Fiore, G. Plotkin, and D. Turi. Abstract syntax and variable binding. In *Proc. 14th IEEE Symp. Logic in Computer Science*, pages 193–202. IEEE Press, 1999.
7. M. Fokkinga. Datatype laws without signatures. *Mathematical Structures in Computer Science*, 6(1):1–32, 1996.
8. M. J. Gabbay and A. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13:341–363, 2001.
9. N. Ghani and C. Lüth. Rewriting via coinserters. *Nordic Journal of Computing*, 10(4):290–312, 2003.
10. N. Ghani, C. Lüth, F. De Marchi, and A. J. Power. Dualising initial algebras. *Mathematical Structures in Computer Science*, 13(2):349–370, 2003.
11. J. Goguen, J. Thatcher, and E. Wagner. An initial algebra approach to the specification, correctness and implementation of abstract data types. In R. Yeh, editor, *Current Trends in Programming Methodology: Software Specification and Design*, volume IV, chapter 5, pages 80–149. Prentice Hall, 1978.
12. M. Hamana. Free Σ-monoids: A higher-order syntax with metavariables. In *Proc. 2nd Asian Symp. Programming Languages and Systems*, Springer LNCS 3202, pages 348–363, 2004.
13. M. Hennessy and G. Plotkin. Full abstraction for a simple parallel programming language. In *Mathematical Foundations of Computer Science*, Springer LNCS 74, pages 108–120, 1979.
14. G. M. Kelly and A. J. Power. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra*, 89:163–179, 1993.
15. A. Kock. Strong functors and monoidal monads. *Archiv der Mathematik*, 23, 1972.
16. G. Plotkin. Domains. Pisa Notes on Domain Theory, 1983.
17. G. Plotkin and A. J. Power. Algebraic operations and generic effects. *Applied Categorical Structures*, 11(1):69–94, 2003.
18. G. Plotkin and A. J. Power. Computational effects and operations: An overview. In *Proc. Workshop on Domains VI*, Elsevier ENTCS 73, pages 149–163, 2004.
19. A. J. Power. Enriched Lawvere theories. *Theory and Applications of Categories*, 6:83–93, 1999.
20. E. Robinson. Variations on algebra: Monadicity and generalisations of equational theories. *Formal Aspects of Computing*, 13(3–5):308–326, 2002.
21. I. Stark. Free-algebra models for the π-calculus. In *Proc. Foundations of Software Science and Computation Structures*, Springer LNCS 3441, pages 155–169, 2005.
22. J. Worrell. Terminal sequences for accessible endofunctors. In *Proc. 2nd International Workshop on Coalgebraic Methods in Computer Science*, Elsevier ENTCS 19, 1999.