

# Equations for secant varieties of Veronese and other varieties

J. M. Landsberg · Giorgio Ottaviani

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**Abstract** New classes of modules of equations for secant varieties of Veronese varieties are defined using representation theory and geometry. Some old modules of equations (catalecticant minors) are revisited to determine when they are sufficient to give scheme-theoretic defining equations. An algorithm to decompose a general ternary quintic as the sum of seven fifth powers is given as an illustration of our methods. Our new equations and results about them are put into a larger context by introducing vector bundle techniques for finding equations of secant varieties in general. We include a few homogeneous examples of this method.

**Keywords** Secant variety · Tensor rank · Veronese variety

**Mathematics Subject Classification (2000)** 14M12 · 14J60 · 15A21 · 15A69

## 1 Introduction

### 1.1 Statement of problem and main results

Let  $S^d\mathbb{C}^{n+1} = S^dV$  denote the space of homogeneous polynomials of degree  $d$  in  $n + 1$  variables, equivalently the space of symmetric  $d$ -way tensors over  $\mathbb{C}^{n+1}$ . It is an

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J. M. Landsberg  
Teitler Department of Mathematics, Texas A&M University,  
College Station, TX, USA  
e-mail: jml@math.tamu.edu

G. Ottaviani (✉)  
Dipartimento di Matematica “U. Dini”, Università di Firenze,  
viale Morgagni 67/A, Firenze, Italy  
e-mail: ottavian@math.unifi.it

important problem for complexity theory, signal processing, algebraic statistics, and many other areas (see e.g., [7, 11, 27, 41]) to find tests for the *border rank* of a given tensor. Geometrically, in the symmetric case, this amounts to finding set-theoretic defining equations for the *secant varieties* of the *Veronese variety*  $v_d(\mathbb{P}V) \subset \mathbb{P}W$ , the variety of rank one symmetric tensors.

For an algebraic variety  $X \subset \mathbb{P}W$ , the  $r$ -th secant variety  $\sigma_r(X)$  is defined by

$$\sigma_r(X) = \overline{\bigcup_{x_1, \dots, x_r \in X} \mathbb{P}\langle x_1, \dots, x_r \rangle} \subset \mathbb{P}W \tag{1}$$

where  $\langle x_1, \dots, x_r \rangle \subset W$  denotes the linear span of the points  $x_1, \dots, x_r$  and the overline denotes Zariski closure. When  $X = v_d(\mathbb{P}V)$ ,  $\sigma_r(X)$  is the Zariski closure of the set of polynomials that are the sum of  $r$   $d$ -th powers.

When  $d = 2$ ,  $S^2V$  may be thought of as the space of  $(n + 1) \times (n + 1)$  symmetric matrices via the inclusion  $S^2V \subset V \otimes V$  and the equations for  $\sigma_r(v_2(\mathbb{P}V))$  are just the size  $r + 1$  minors (these equations even generate the ideal). The first equations found for secant varieties of higher Veronese varieties were obtained by imitating this construction, considering the inclusions  $S^dV \subset S^aV \otimes S^{d-a}V$ , where  $1 \leq a \leq \lfloor \frac{d}{2} \rfloor$ : Given  $\phi \in S^dV$ , one considers the corresponding linear map  $\phi_{a,d-a} : S^aV^* \rightarrow S^{d-a}V$  and if  $\phi \in \sigma_r(v_d(\mathbb{P}V))$ , then  $\text{rank}(\phi_{a,d-a}) \leq r$ , see Sect. 2.2. Such equations are called *minors of symmetric flattenings* or *catalecticant minors* and date back at least to Sylvester who coined the term ‘‘catalecticant’’. See [20] for a history.

These equations are usually both too numerous and too few; that is, there are redundancies among them and even all of them usually will not give enough equations to define  $\sigma_r(v_d(\mathbb{P}V))$  set-theoretically.

In this paper we

- Describe a large class of new sets of equations for  $\sigma_r(v_d(\mathbb{P}V))$ , which we call *Young Flattenings*, that generalize the classical *Aronhold invariant*, see Proposition 4.1.1.
- Show a certain Young Flattening,  $YF_{d,n}$ , provides scheme-theoretic equations for a large class of cases where usual flattenings fail, see Theorem 1.2.3.
- Determine cases where flattenings are sufficient to give defining equations (more precisely, scheme-theoretic equations), Theorem 3.2.1. Theorem 3.2.1 is primarily a consequence of work of Iarrobino and Kanev [25] and Diesel [12].
- Put our results in a larger context by providing a uniform formulation of all known equations for secant varieties via vector bundle methods. We use this perspective to prove some of our results, including a key induction Lemma 6.2.1. The discussion of vector bundle methods is postponed to the latter part of the paper to make the results on symmetric border rank more accessible to readers outside of algebraic geometry.

Here is a chart summarizing what is known about equations of secant varieties of Veronese varieties:

Case	Equations	Cuts out	References
$\sigma_r(v_2(\mathbb{P}^r))$	Size $r + 1$ minors	Ideal	Classical
$\sigma_r(v_d(\mathbb{P}^1))$	Size $r + 1$ minors of any $\phi_{s,d-s}$	Ideal	Gundelfinger, [25]
$\sigma_2(v_d(\mathbb{P}^d))$	Size 3 minors of any $\phi_{1,d-1}$ and $\phi_{2,d-2}$	Ideal	[26]
$\sigma_3(v_3(\mathbb{P}^3))$	Aronhold + size 4 minors of $\phi_{1,2}$	Ideal	Proposition 2.3.1 Aronhold for $n=2$ [25]
$\sigma_3(v_d(\mathbb{P}^d)), d \geq 4$	Size 4 minors of $\phi_{2,2}$ and $\phi_{1,3}$	Scheme	Theorem 3.2.1 (1) [46] for $n = 2, d = 4$
$\sigma_4(v_d(\mathbb{P}^2))$	Size 5 minors of $\phi_{a,d-a}, a = \lfloor \frac{d}{2} \rfloor$	Scheme	Theorem 3.2.1 (2) [46] for $d = 4$
$\sigma_5(v_d(\mathbb{P}^2)), d \geq 6$ and $d = 4$	Size 6 minors of $\phi_{a,d-a}, a = \lfloor \frac{d}{2} \rfloor$	Scheme	Theorem 3.2.1 (3) Clebsch for $d = 4$ [25]
$\sigma_r(v_5(\mathbb{P}^2)), r \leq 5$	Size $2r + 2$ subPfaffians of $\phi_{31,31}$	Irred. comp.	Theorem 4.2.7
$\sigma_6(v_5(\mathbb{P}^2))$	Size 14 subPfaffians of $\phi_{31,31}$	Scheme	Theorem 4.2.7
$\sigma_6(v_d(\mathbb{P}^2)), d \geq 6$	Size 7 minors of $\phi_{a,d-a}, a = \lfloor \frac{d}{2} \rfloor$	Scheme	Theorem 3.2.1 (4)
$\sigma_7(v_6(\mathbb{P}^2))$	Symm.flat. + Young flat.	Irred.comp.	Theorem 4.2.9
$\sigma_8(v_6(\mathbb{P}^2))$	Symm.flat. + Young flat.	Irred.comp.	Theorem 4.2.9
$\sigma_9(v_6(\mathbb{P}^2))$	det $\phi_{3,3}$	Ideal	Classical
$\sigma_j(v_7(\mathbb{P}^2)), j \leq 10$	Size $2j + 2$ subPfaffians of $\phi_{41,41}$	Irred.comp.	Theorem 1.2.3
$\sigma_j(v_{2\delta}(\mathbb{P}^2)), j \leq \binom{\delta+1}{2}$	Rank $\phi_{a,d-a} = \min(j, \binom{a+2}{2}), 1 \leq a \leq \delta$ open and closed conditions	Scheme	[25, Theorem 4.1A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^2)), j \leq \binom{\delta+1}{2} + 1$	Rank $\phi_{a,d-a} = \min(j, \binom{a+2}{2}), 1 \leq a \leq \delta$ open and closed conditions	Scheme	[25, Theorem 4.5A]
$\sigma_j(v_{2\delta}(\mathbb{P}^n)), j \leq \binom{\delta+n-1}{n}$	Size $j + 1$ minors of $\phi_{\delta,\delta}$	Irred. comp.	[25, Theorem 4.10A]
$\sigma_j(v_{2\delta+1}(\mathbb{P}^n)), j \leq \binom{\delta+n}{n}$	Size $\binom{n}{a}j + 1$ minors of $Y_{d,n}, a = \lfloor n/2 \rfloor$ if $n = 2a, a$ odd, $\binom{n}{a}j + 2$ subpfaff. of $Y_{d,n}$	Irred. comp.	Theorem 1.2.3

### 1.2 Young flattenings

The simplest case of equations for secant varieties is for the space of rank at most  $r$  matrices of size  $p \times q$ , which is the zero set of the minors of size  $r + 1$ . Geometrically let  $A = \mathbb{C}^p, B = \mathbb{C}^q$  and let  $\text{Seg}(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$  denote the Segre variety of rank one matrices. Then the ideal of  $\sigma_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B))$  is generated by the space of minors of size  $r + 1$ , which is  $\wedge^{r+1} A^* \otimes \wedge^{r+1} B^*$ . Now if  $X \subset \mathbb{P}W$  is a variety and there is a linear injection  $W \rightarrow A \otimes B$  such that  $X \subset \sigma_p(\text{Seg}(\mathbb{P}A \times \mathbb{P}B))$ , then the minors of size  $pr + 1$  furnish equations for  $\sigma_r(X)$ . Flattenings are a special case of this method where  $W = S^d V, A = S^a V$  and  $B = S^{d-a} V$ .

When a group  $G$  acts linearly on  $W$  and  $X$  is invariant under the group action, then the equations of  $X$  and its secant varieties will be  $G$ -modules, and one looks for a  $G$ -module map  $W \rightarrow A \otimes B$ . Thus one looks for  $G$ -modules  $A, B$  such that  $W$  appears in the  $G$ -module decomposition of  $A \otimes B$ . We discuss this in detail for  $X = v_d(\mathbb{P}V)$  in Sect. 4 and in general in Sects. 10 and 11. For now we focus on a special class of Young flattenings that we describe in elementary language. We begin by reviewing the classical *Aronhold invariant*.

*Example 1.2.1* [The Aronhold invariant] The classical Aronhold invariant is the equation for the hypersurface  $\sigma_3(v_3(\mathbb{P}^2)) \subset \mathbb{P}^9$ . Map  $S^3 V \rightarrow (V \otimes \wedge^2 V) \otimes (V \otimes V^*)$ , by first embedding  $S^3 V \subset V \otimes V \otimes V$ , then tensoring with  $Id_V \in V \otimes V^*$ , and then skew-symmetrizing. Thus,

when  $n = 2$ ,  $\phi \in S^3V$  gives rise to an element of  $\mathbb{C}^9 \otimes \mathbb{C}^9$ . In bases, if we write

$$\begin{aligned} \phi = & \phi_{000}x_0^3 + \phi_{111}x_1^3 + \phi_{222}x_2^3 + 3\phi_{001}x_0^2x_1 + 3\phi_{011}x_0x_1^2 + 3\phi_{002}x_0^2x_2 \\ & + 3\phi_{022}x_0x_2^2 + 3\phi_{112}x_1^2x_2 + 3\phi_{122}x_1x_2^2 + 6\phi_{012}x_0x_1x_2, \end{aligned}$$

the corresponding matrix is:

$$\begin{bmatrix} & & & \phi_{002} & \phi_{012} & \phi_{022} & -\phi_{010} & -\phi_{011} & -\phi_{012} \\ & & & \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{011} & -\phi_{111} & -\phi_{112} \\ & & & \phi_{012} & \phi_{112} & \phi_{222} & -\phi_{012} & -\phi_{112} & -\phi_{122} \\ -\phi_{002} & -\phi_{012} & -\phi_{022} & & & & \phi_{000} & \phi_{001} & \phi_{002} \\ -\phi_{012} & -\phi_{112} & -\phi_{122} & & & & \phi_{001} & \phi_{011} & \phi_{012} \\ -\phi_{012} & -\phi_{112} & -\phi_{222} & & & & \phi_{001} & \phi_{011} & \phi_{022} \\ \phi_{010} & \phi_{011} & \phi_{012} & -\phi_{000} & -\phi_{001} & -\phi_{002} & & & \\ \phi_{011} & \phi_{111} & \phi_{112} & -\phi_{001} & -\phi_{011} & -\phi_{012} & & & \\ \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{001} & -\phi_{011} & -\phi_{022} & & & \end{bmatrix}.$$

All the principal Pfaffians of size 8 of the this matrix coincide, up to scale, with the classical *Aronhold invariant*. (Redundancy occurs here is because one should really work with the submodule  $S_{21}V \subset V \otimes \wedge^2 V \simeq V \otimes V^*$ , where the second identification uses a choice of volume form. The Pfaffian of the map  $S_{21} \rightarrow S_{21}$  is the desired equation.)

This construction, slightly different from the one in [40], shows how the Aronhold invariant is analogous to the invariant in  $S^9(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$  that was discovered by Strassen [48], (see also [39], and the paper [2] by Barth, all in different settings.)

Now consider the inclusion  $V \subset \wedge^k V^* \otimes \wedge^{k+1} V$ , given by  $v \in V$  maps to the map  $\omega \mapsto v \wedge \omega$ . In bases one obtains a matrix whose entries are the coefficients of  $v$  or zero. In the special case  $n + 1 = 2a + 1$  is odd and  $k = a$ , one obtains a square matrix  $K_n$ , which is skew-symmetric for odd  $a$  and symmetric for even  $a$ . For example, when  $n = 2$ , the matrix is

$$K_2 = \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}$$

and, when  $n = 4$ , the matrix is

$$K_4 = \begin{bmatrix} & & & & & & & & & & x_4 & -x_3 & x_2 \\ & & & & & & & & & & -x_4 & x_3 & -x_1 \\ & & & & & & & & & & x_4 & -x_2 & x_1 \\ & & & & & & & & & & -x_3 & x_2 & -x_1 \\ & & & & & & & & & & x_4 & -x_3 & x_0 \\ -x_4 & & & & & & & & & & -x_4 & x_2 & -x_0 \\ & & & & & & & & & & x_3 & -x_2 & x_0 \\ x_4 & & & & & & & & & & -x_4 & -x_1 & x_0 \\ -x_3 & & & & & & & & & & x_3 & x_1 & -x_0 \\ x_2 & -x_1 & & & & & & & & & x_2 & -x_1 & x_0 \end{bmatrix}.$$

Finally, consider the following generalization of both the Aronhold invariant and the  $K_n$ . Let  $a = \lfloor \frac{n}{2} \rfloor$  and let  $d = 2\delta + 1$ . Map  $S^dV \rightarrow (S^\delta V \otimes \wedge^a V^*) \otimes (S^\delta V \otimes \wedge^{a+1} V)$  by first performing the inclusion  $S^dV \rightarrow S^\delta V \otimes S^\delta V \otimes V$  and then using the last factor to obtain a map  $\wedge^a V \rightarrow \wedge^{a+1} V$ . We get:

$$YF_{d,n}(\phi) : S^\delta V^* \otimes \wedge^a V \rightarrow S^\delta V \otimes \wedge^{a+1} V. \tag{2}$$

If  $n + 1$  is odd, the matrix representing  $YF_{d,n}(\phi)$  is skew-symmetric, so we may take Pfaffians instead of minors.

For a decomposable  $w^d \in S^d V$ , the map is

$$\alpha^\delta \otimes v_1 \wedge \cdots \wedge v_a \mapsto (\alpha(w))^\delta w^\delta \otimes w \wedge v_1 \wedge \cdots \wedge v_a.$$

In bases, one obtains a matrix in block form, where the blocks correspond to the entries of  $K_n$  and the matrices in the blocks are the square catalecticants  $\pm \left(\frac{\partial \phi}{\partial x_i}\right)_{\delta,\delta}$  in the place of  $\pm x_i$ .

Let

$$YF_{d,n}^r := \left\{ \phi \in S^d V \mid \text{rank}(YF_{d,n}(\phi)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} r \right\}.$$

Two interesting cases are  $YF_{3,2}^3 = \sigma_3(v_3(\mathbb{P}^2))$ , which defines the quartic Aronhold invariant, and  $YF_{3,4}^7 = \sigma_7(v_3(\mathbb{P}^4))$  which defines the invariant of degree 15 considered in [40].

*Remark 1.2.2* Just as with the Aronhold invariant above, there will be redundancies among the minors and Pfaffians of  $YF_{d,n}(\phi)$ . See Sect. 4 for a description without redundancies.

**Theorem 1.2.3** *Let  $n \geq 2$ , let  $a = \lfloor \frac{n}{2} \rfloor$ , let  $V = \mathbb{C}^{n+1}$ , and let  $d = 2\delta + 1$ .*

*If  $r \leq \binom{\delta+n}{n}$  then  $\sigma_r(v_d(\mathbb{P}^n))$  is an irreducible component of  $YF_{d,n}^r$ , the variety given by the size  $\binom{n}{a}r + 1$  minors of  $YF_{d,n}$ .*

*In the case  $n = 2a$  with odd  $a$ ,  $YF_{d,n}$  is skew-symmetric (for any  $d$ ) and one may instead take the size  $\binom{n}{a}r + 2$  sub-pfaffians of  $YF_{d,n}$ .*

*In the case  $n = 2a$  with even  $a$ ,  $YF_{d,n}$  is symmetric.*

The bounds given in Theorem 1.2.3 for  $n = 2$  are sharp (see Proposition 4.2.3).

### 1.3 Vector bundle methods

As mentioned above, the main method for finding equations of secant varieties for  $X \subset \mathbb{P}V$  is to find a linear embedding  $V \subset A \otimes B$ , where  $A, B$  are vector spaces, such that  $X \subset \sigma_q(\text{Seg}(\mathbb{P}A \times \mathbb{P}B))$ , where  $\text{Seg}(\mathbb{P}A \times \mathbb{P}B)$  denotes the Segre variety of rank one elements. What follows is a technique to find such inclusions using vector bundles.

Let  $E$  be a vector bundle on  $X$  of rank  $e$ , write  $L = \mathcal{O}_X(1)$ , so  $V = H^0(X, L)^*$ . Let  $v \in V$  and consider the linear map

$$A_v^E: H^0(E) \rightarrow H^0(E^* \otimes L)^* \tag{3}$$

induced by the natural map

$$A: H^0(E) \otimes H^0(L)^* \rightarrow H^0(E^* \otimes L)^*$$

where  $A_v^E(s) = A(s \otimes v)$ . In examples  $E$  will be chosen so that both  $H^0(E)$  and  $H^0(E^* \otimes L)$  are non-zero, otherwise our construction is vacuous. A key observation (Proposition 5.1.1) is that *the size  $(re + 1)$  minors of  $A_v^E$  give equations for  $\sigma_r(X)$ .*

### 1.4 Overview

In Sect. 2 we establish notation and collect standard facts that we will need later. In Sect. 3, we first review work of Iarrobino and Kanev [25] and Diesel [12] and then show how their results imply several new cases where  $\sigma_r(v_d(\mathbb{P}V))$  is cut out scheme-theoretically by flattenings (Theorem 3.2.1). In Sect. 4 we discuss Young Flattenings for Veronese varieties.

The possible inclusions  $S^d V \rightarrow S_\pi V \otimes S_\mu V$  follow easily from the Pieri formula, however, which of these are useful is still not understood. In Sect. 4.2 we make a detailed study of the  $n = 2$  case. The above-mentioned Proposition 5.1.1 is proved in Sect. 5, where we also describe simplifications when  $(E, L)$  is a symmetric or skew-symmetric pair. We also give a sufficient criterion for  $\sigma_r(X)$  to be an irreducible component of the equations given by the  $(re + 1)$  minors of  $A_v^E$  (Theorem 5.4.3). In Sect. 6 we prove a downward induction lemma (Lemma 6.2.1). We prove Theorem 1.2.3 in Sect. 7, which includes Corollary 7.0.10 on linear systems of hypersurfaces, which may be of interest in its own right. To get explicit models for the maps  $A_v^E$  it is sometimes useful to factor  $E$ , as described in Sect. 8. In Sect. 9 we explain how to use equations to obtain decompositions of polynomials into sums of powers, illustrating with an algorithm to decompose a general ternary quintic as the sum of seven fifth powers. We conclude, in Sects. 10, 11 with a few brief examples of the construction for homogeneous varieties beyond Veronese varieties.

## 2 Background

### 2.1 Notation

We work exclusively over the complex numbers.  $V, W$  will generally denote (finite dimensional) complex vector spaces. The dual space of  $V$  is denoted by  $V^*$ . The projective space of lines through the origin of  $V$  is denoted by  $\mathbb{P}V$ . If  $A \subset W$  is a subspace,  $A^\perp \subset W^*$  is its annihilator, the space of  $f \in W^*$  such that  $f(a) = 0 \forall a \in A$ .

For a partition  $\pi = (p_1, \dots, p_r)$  of  $d$ , we write  $|\pi| = d$  and  $\ell(\pi) = r$ . If  $V$  is a vector space,  $S_\pi V$  denotes the irreducible  $GL(V)$ -module determined by  $\pi$  (assuming  $\dim V \geq \ell(\pi)$ ). In particular  $S^d W = S_{(d)} W$  and  $\wedge^a W = S_{1^a} W$  are respectively the  $d$ -th symmetric power and the  $a$ -th exterior power of  $W$ .  $S^d W = S_{(d)} W$  is also the space of homogeneous polynomials of degree  $d$  on  $W^*$ . Given  $\phi \in S^d W$ ,  $\text{Zeros}(\phi) \subset \mathbb{P}W^*$  denotes its zero set.

For a subset  $Z \subseteq \mathbb{P}W$ ,  $\hat{Z} \subseteq W \setminus 0$  denotes the affine cone over  $Z$ . For a projective variety  $X \subset \mathbb{P}W$ ,  $I(X) \subset \text{Sym}(W^*)$  denotes its ideal and  $I_X$  its ideal sheaf of (regular) functions vanishing at  $X$ . For a smooth point  $z \in X$ ,  $\hat{T}_z X \subset V$  is the affine tangent space and  $\hat{N}_z^* X = (\hat{T}_z X)^\perp \subset V^*$  the affine conormal space.

We make the standard identification of a vector bundle with the corresponding locally free sheaf. For a sheaf  $E$  on  $X$ ,  $H^i(E)$  is the  $i$ -th cohomology space of  $E$ . In particular  $H^0(E)$  is the space of global sections of  $E$ , and  $H^0(I_Z \otimes E)$  is the space of global sections of  $E$  which vanish on a subset  $Z \subset X$ . According to this notation  $H^0(\mathbb{P}V, \mathcal{O}(1)) = V^*$ .

If  $G/P$  is a rational homogeneous variety and  $E \rightarrow G/P$  is an irreducible homogeneous vector bundle, we write  $E = E_\mu$  where  $\mu$  is the highest weight of the irreducible  $P$ -module inducing  $E$ . We use the conventions of [4] regarding roots and weights of simple Lie algebras.

### 2.2 Flattenings

Given  $\phi \in S^d V$ , write  $\phi_{a,d-a} \in S^a V \otimes S^{d-a} V$  for the  $(a, d - a)$ -polarization of  $\phi$ . We often consider  $\phi_{a,d-a}$  as a linear map  $\phi_{a,d-a} : S^a V^* \rightarrow S^{d-a} V$ . This notation is compatible with the more general one of Young flattenings that we will introduce in Sect. 4.

If  $[\phi] \in v_d(\mathbb{P}V)$ , then for all  $1 \leq a \leq d - 1$ ,  $\text{rank}(\phi_{a,d-a}) = 1$ , and the  $2 \times 2$  minors of  $\phi_{a,d-a}$  generate the ideal of  $v_d(\mathbb{P}V)$  for any  $1 \leq a \leq d - 1$ . If  $[\phi] \in \sigma_r(v_d(\mathbb{P}V))$ , then  $\text{rank}(\phi_{a,d-a}) \leq r$ , so, the  $(r + 1) \times (r + 1)$  minors of  $\phi_{a,d-a}$  furnish equations for  $\sigma_r(v_d(\mathbb{P}V))$ , i.e.,

$$\wedge^{r+1}(S^a V^*) \otimes \wedge^{r+1}(S^{d-a} V^*) \subset I_{r+1}(\sigma_r(v_d(\mathbb{P}V))).$$

Since  $I_r(\sigma_r(v_d(\mathbb{P}V))) = 0$ , these modules, obtained by *symmetric flattenings*, also called *catalecticant homomorphisms*, are among the modules generating the ideal of  $\sigma_r(v_d(\mathbb{P}V))$ . Geometrically the symmetric flattenings are the equations for the varieties

$$\text{Rank}_{a,d-a}^r(S^d V) := \sigma_r(\text{Seg}(\mathbb{P}S^a V \times \mathbb{P}S^{d-a} V)) \cap \mathbb{P}S^d V.$$

*Remark 2.2.1* The equations of  $\sigma_r(v_2(\mathbb{P}W)) \subset \mathbb{P}S^2 W$  are those of  $\sigma_r(\text{Seg}(\mathbb{P}W \times \mathbb{P}W))$  restricted to  $\mathbb{P}S^2 W$ . Since  $S^{2p} V \subset S^2(S^p V)$ , when  $d = 2a$  we may also describe the symmetric flattenings as the equations for  $\sigma_r(v_2(\mathbb{P}S^a V)) \cap \mathbb{P}S^d V$ .

### 2.3 Inheritance

The purpose of this subsection is to explain why it is only necessary to consider the “primitive” cases of  $\sigma_r(v_d(\mathbb{P}^n))$  for  $n \leq r - 1$ .

Let

$$\begin{aligned} \text{Sub}_r(S^d V) &:= \mathbb{P}\{\phi \in S^d V \mid \exists V' \subset V, \dim V' = r, \phi \in S^d V'\} \\ &= \{[\phi] \in \mathbb{P}S^d V \mid \text{Zeros}(\phi) \subset \mathbb{P}V^* \text{ is a cone over a linear space of codimension } r\} \end{aligned}$$

denote the *subspace variety*. The ideal of  $\text{Sub}_r(S^d V)$  is generated in degree  $r + 1$  by all modules  $S_\pi V^* \subset S^{r+1}(S^d V^*)$  where  $\ell(\pi) > r + 1$ , see [49, §7.2]. These modules may be realized explicitly as the  $(r + 1) \times (r + 1)$  minors of  $\phi_{1,d-1}$ . Note in particular that  $\sigma_r(v_d(\mathbb{P}V)) \subset \text{Sub}_r(S^d V)$  and that equality holds for  $d \leq 2$  or  $r = 1$ . Hence, the equations of  $\text{Sub}_r(S^d V)$  appear among the equations of  $\sigma_r(v_d(\mathbb{P}V))$ .

Let  $X \subset \mathbb{P}W$  be a  $G$ -variety for some group  $G \subset GL(W)$ . We recall that a module  $M \subset \text{Sym}(W^*)$  defines  $X$  *set-theoretically* if  $\text{Zeros}(M) = X$  as a set, that it defines  $X$  *scheme-theoretically* if there exists a  $\delta$  such that the ideal generated by  $M$  equals the ideal of  $X$  in all degrees greater than  $\delta$ , and that  $M$  defines  $X$  *ideal theoretically* if the ideal generated by  $M$  equals the ideal of  $X$ .

**Proposition 2.3.1 (Symmetric Inheritance)** *Let  $V$  be a vector space of dimension greater than  $r$ .*

*Let  $M \subset \text{Sym}((\mathbb{C}^r)^*)$  be a module and let  $U = [\wedge^{r+1} V^* \otimes \wedge^{r+1}(S^{d-1} V^*)] \cap S^{r+1}(S^d V^*) \subset S^{r+1}(S^d V^*)$  be the module generating the ideal of  $\text{Sub}_r(S^d V)$  given by the  $(r + 1) \times (r + 1)$ -minors of the flattening  $\phi \mapsto \phi_{1,d-1}$ .*

*If  $M$  defines  $\sigma_r(v_d(\mathbb{P}^{r-1}))$  set-theoretically, respectively scheme-theoretically, resp. ideal theoretically, let  $\tilde{M} \subset \text{Sym}(V^*)$  be the module induced by  $M$ , then  $\tilde{M} + U$  defines  $\sigma_r(v_d(\mathbb{P}V))$  set-theoretically, resp. scheme-theoretically, resp. ideal-theoretically.*

See [32, Chapter 8] for a proof.

### 2.4 Results related to degree

Sometimes it is possible to conclude global information from local equations if one has information about degrees.

We need the following result about excess intersection.

**Theorem 2.4.1** *Let  $Z \subset \mathbb{P}^n$  be a variety of codimension  $e$  and  $L \subset \mathbb{P}^n$  a linear subspace of codimension  $f$ . Assume that  $Z \cap L$  has an irreducible component  $Y$  of codimension  $f + e \leq n$  such that  $\deg Y = \deg Z$ . Then  $Z \cap L = Y$ .*

*Proof* This is an application of the refined Bézout theorem of [16, Thm. 12.3]. □

The degrees of  $\sigma_r(v_d(\mathbb{P}^2))$  for certain small values of  $r$  and  $d$  were computed by Ellingsrud and Stromme in [15], see also [25, Rem. 7.20].

**Proposition 2.4.2** [15]

- (1)  $\deg(\sigma_3(v_4(\mathbb{P}^2))) = 112.$
- (2)  $\deg(\sigma_4(v_4(\mathbb{P}^2))) = 35.$
- (3)  $\deg(\sigma_6(v_5(\mathbb{P}^2))) = 140.$
- (4)  $\deg(\sigma_6(v_6(\mathbb{P}^2))) = 28, 314.$

**Proposition 2.4.3** (see, e.g., [31, Cor. 3.2]) *The minimal possible degree of a module in  $I(\sigma_r(v_d(\mathbb{P}^n)))$  is  $r + 1.$*

We recall the following classical formulas, due to C. Segre. For a modern reference see [23].

**Proposition 2.4.4**

$$\text{codim } \sigma_r(v_2(\mathbb{P}^n)) = \binom{n-r+2}{2} \quad \deg \sigma_r(v_2(\mathbb{P}^n)) = \prod_{i=0}^{n-r} \frac{\binom{n+1+i}{n-r-i+1}}{\binom{2i+1}{i}}$$

$$\text{codim } \sigma_r(G(2, n+1)) = \binom{n-2r+1}{2} \quad \deg \sigma_r(G(2, n+1)) = \frac{1}{2^{n-2r}} \prod_{i=0}^{n-2r-1} \frac{\binom{n+1+i}{n-2r-i}}{\binom{2i+1}{i}}.$$

2.5 Conormal spaces

The method used to prove a module of equations locally defines  $\sigma_r(v_d(\mathbb{P}^n))$  will be to show that the conormal space at a smooth point of the zero set of the module equals the conormal space to  $\sigma_r(v_d(\mathbb{P}^n))$  at that point.

Let  $A, B$  be vector spaces and let  $\text{Seg}(\mathbb{P}A \times \mathbb{P}B) \subset \mathbb{P}(A \otimes B)$  denote the Segre variety, so if  $[x] \in \text{Seg}(\mathbb{P}A \times \mathbb{P}B)$ , then  $x = a \otimes b$  for some  $a \in A$  and  $b \in B$ . One has the affine tangent space  $\hat{T}_{[x]} \text{Seg}(\mathbb{P}A \times \mathbb{P}B) = a \otimes B + A \otimes b \subset A \otimes B$ , and the affine conormal space  $\hat{N}_{[x]}^* \text{Seg}(\mathbb{P}A \times \mathbb{P}B) = a^\perp \otimes b^\perp = \ker(x) \otimes \text{Image}(x)^\perp \subset A^* \otimes B^*$ , where in the latter description we think of  $a \otimes b : A^* \rightarrow B$  as a linear map. Terracini’s lemma implies that if  $[z] \in \sigma_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B))$  is of rank  $r$ , then

$$\hat{N}_{[z]}^* \sigma_r(\text{Seg}(\mathbb{P}A \times \mathbb{P}B)) = \ker(z) \otimes \text{Image}(z)^\perp. \tag{4}$$

In particular, letting  $\text{Rank}_{(a,d-a)}^r(S^d V) \subset \mathbb{P}(S^d V)$  denote the zero set of the size  $(r + 1)$ -minors of the flattenings  $S^a V^* \rightarrow S^{d-a} V$ , one has

**Proposition 2.5.1** *Let  $[\phi] \in \text{Rank}_{(a,d-a)}^r(S^d V)$  be a sufficiently general point, then*

$$\hat{N}_{[\phi]}^* \text{Rank}_{(a,d-a)}^r(S^d V) = \ker(\phi_{a,d-a}) \circ \text{Image}(\phi_{a,d-a})^\perp \subset S^d V^*.$$

Now let  $[y^d] \in v_d(\mathbb{P}V)$ , then

$$\begin{aligned} \hat{N}_{[y^d]}^* v_d(\mathbb{P}V) &= \{P \in S^d V^* \mid P(y) = 0, dP_y = 0\} \\ &= S^{d-2} V^* \circ S^2 y^\perp \\ &= \{P \in S^d V^* \mid \text{Zeros}(P) \text{ is singular at } [y]\}. \end{aligned}$$

Applying Terracini’s lemma yields:



**Proposition 2.5.2** *Let  $[\phi] = [y_1^d + \dots + y_r^d] \in \sigma_r(v_d(\mathbb{P}V))$ . Then*

$$\begin{aligned} \hat{N}_{[\phi]}^* \sigma_r(v_d(\mathbb{P}V)) &\subseteq (S^{d-2}V^* \circ S^2y_1^\perp) \cap \dots \cap (S^{d-2}V^* \circ S^2y_r^\perp) \\ &= \{P \in S^dV^* \mid \text{Zeros}(P) \text{ is singular at } [y_1], \dots, [y_r]\} \end{aligned}$$

and equality holds if  $\phi$  is sufficiently general.

### 3 Symmetric flattenings (catalecticant minors)

#### 3.1 Review of known results

The following result dates back to the work of Sylvester and Gundelfinger.

**Theorem 3.1.1** (see, e.g., [25, Thm. 1.56]) *The ideal of  $\sigma_r(v_d(\mathbb{P}^1))$  is generated in degree  $r + 1$  by the size  $r + 1$  minors of  $\phi_{u,d-u}$  for any  $r \leq u \leq d - r$ , i.e., by any of the modules  $\wedge^{r+1} S^u \mathbb{C}^2 \otimes \wedge^{r+1} S^{d-u} \mathbb{C}^2$ . The variety  $\sigma_r(v_d(\mathbb{P}^1))$  is projectively normal and arithmetically Cohen-Macaulay, its singular locus is  $\sigma_{r-1}(v_d(\mathbb{P}^1))$ , and its degree is  $\binom{d-r+1}{r}$ .*

**Corollary 3.1.2** (Kanev, [26]) *The ideal of  $\sigma_2(v_d(\mathbb{P}^n))$  is generated in degree 3 by the 3 by 3 minors of the  $(1, d - 1)$  and  $(2, d - 2)$  flattenings.*

*Proof* Apply Proposition 2.3.1 to Theorem 3.1.1 in the case  $r = 2$ . □

**Remark 3.1.3** Raicu [43] recently proved that in Corollary 3.1.2 it is possible to replace the  $(2, d - 2)$  flattening with any  $(i, d - i)$ -flattening such that  $2 \leq i \leq d - 2$ .

**Theorem 3.1.4** (see [25, Thms. 4.5A, 4.10A]) *Let  $n \geq 3$ , and let  $V = \mathbb{C}^{n+1}$ . Let  $\delta = \lfloor \frac{d}{2} \rfloor$ ,  $\delta' = \lceil \frac{d}{2} \rceil$ . If  $r \leq \binom{\delta-1+n}{n}$  then  $\sigma_r(v_d(\mathbb{P}V))$  is an irreducible component of  $\sigma_r(\text{Seg}(\mathbb{P}S^\delta V \times \mathbb{P}S^{\delta'} V)) \cap \mathbb{P}(S^d V)$ . In other words,  $\sigma_r(v_d(\mathbb{P}^n))$  is an irreducible component of the size  $(r + 1)$  minors of the  $(\delta, \delta')$ -flattening.*

The bound obtained in this theorem is the best we know of for even degree, apart from some cases of small degree listed below. Theorem 1.2.3 is an improvement of this bound in the case of odd degree.

When  $\dim V = 3$ , then a Zariski open subset of  $\sigma_r(v_d(\mathbb{P}V))$  can be characterized by open and closed conditions, that is, with the same assumptions of Theorem 3.1.4: If the rank of all  $(a, d - a)$ -flattenings computed at  $\phi$  is equal to  $\min\left(r, \binom{a+2}{2}\right)$ , then  $\phi \in \sigma_r(v_d(\mathbb{P}V))$  and the general element of  $\sigma_r(v_d(\mathbb{P}V))$  can be described in this way (see [25, Thm. 4.1A]).

**Remark 3.1.5** The statement of Theorem 3.1.4 cannot be improved, in the sense that the zero locus of the flattenings usually is reducible. Consider the case  $n = 2, d = 8, r = 10$ , in this case the  $10 \times 10$  flattenings of  $\phi_{4,4}$  define a variety with at least two irreducible components, one of them is  $\sigma_{10}(v_8(\mathbb{P}^2))$  (see [25, Ex. 7.11]).

**Proposition 3.1.6** *Assume  $\sigma_r(v_d(\mathbb{P}^n))$  is not defective and  $r < \frac{1}{n+1} \binom{n+d}{d}$  so that  $\sigma_r(v_d(\mathbb{P}^n))$  is not the ambient space. Let  $\delta = \lfloor \frac{d}{2} \rfloor$ ,  $\delta' = \lceil \frac{d}{2} \rceil$ . There are non-trivial equations from flattenings iff  $r < \binom{\delta+n}{\delta}$ . (The defective cases where  $r$  is allowed to be larger are understood as well.)*

*Proof* The right-hand side is the size of the maximal minors of  $\phi_{\delta,\delta'}$ , which give equations for  $\sigma_{\binom{\delta+n}{\delta}-1}(v_d(\mathbb{P}^n))$ . □

For example, when  $n = 3$  and  $d = 2\delta$ ,  $\sigma_r(v_d(\mathbb{P}^3))$  is an irreducible component of the zero set of the flattenings for  $r \leq \binom{\delta+2}{3}$ , the flattenings give some equations up to  $r \leq \binom{\delta+3}{3}$ , and there are non-trivial equations up to  $r \leq \frac{1}{4} \binom{2\delta+3}{3}$ .

We will have need of more than one symmetric flattening, so we make the following definitions, following [25], but modifying the notation:

**Definition 3.1.7** Fix a sequence  $\vec{r} := (r_1, \dots, r_{\lfloor \frac{d}{2} \rfloor})$  and let

$$\begin{aligned} \text{SFlat}_{\vec{r}}(S^d W) &:= \{ \phi \in S^d W \mid \text{rank}(\phi_{j,d-j}) \leq r_j, j = 1, \dots, \lfloor \frac{d}{2} \rfloor \} \\ &= \left[ \bigcap_{j=1}^{\lfloor \frac{d}{2} \rfloor} \sigma_{r_j}(\text{Seg}(\mathbb{P}S^j W \times \mathbb{P}S^{d-j} W)) \right] \cap S^d W \end{aligned}$$

Call a sequence  $\vec{r}$  *admissible* if there exists  $\phi \in S^d W$  such that  $\text{rank}(\phi_{j,d-j}) = r_j$  for all  $j = 1, \dots, \lfloor \frac{d}{2} \rfloor$ . It is sufficient to consider  $\vec{r}$  that are admissible because if  $\vec{r}$  is non-admissible, the zero set of  $\text{SFlat}_{\vec{r}}$  will be contained in the union of admissible  $\text{SFlat}$ 's associated with smaller  $\vec{r}$ 's in the natural partial order. Note that  $\text{SFlat}_{\vec{r}}(S^d W) \subseteq \text{Sub}_{r_1}(S^d W)$ .

Even if  $\vec{r}$  is admissible, it still can be the case that  $\text{SFlat}_{\vec{r}}(S^d W)$  is reducible. For example, when  $\dim W = 3$  and  $d \geq 6$ , the zero set of the size 5 minors of the  $(2, d - 2)$ -flattening has two irreducible components, one of them is  $\sigma_4(v_d(\mathbb{P}^2))$  and the other has dimension  $d + 6$  [25, Ex. 3.6].

To remedy this, let  $\vec{r}$  be admissible, and consider

$$\text{SFlat}_{\vec{r}}^0(S^d W) := \left\{ \phi \in S^d W \mid \text{rank}(\phi_{j,d-j}) = r_j, j = 1, \dots, \lfloor \frac{d}{2} \rfloor \right\}$$

and let

$$\text{Gor}(\vec{r}) := \overline{\text{SFlat}_{\vec{r}}^0(S^d W)}.$$

*Remark 3.1.8* In the commutative algebra literature (e.g., [12,25]), ‘‘Gor’’ is short for Gorenstein, see [25, Def. 1.11] for a history.

Unfortunately, defining equations for  $\text{Gor}(\vec{r})$  are not known. One can test for membership of  $\text{SFlat}_{\vec{r}}^0(S^d W)$  by checking the required vanishing and non-vanishing of minors.

**Theorem 3.1.9** [12, Thm 1.1] *If  $\dim W = 3$ , and  $\vec{r}$  is admissible, then  $\text{Gor}(\vec{r})$  is irreducible.*

Theorem 3.1.9 combined with Theorem 3.1.4 allows one to extend the set of secant varieties of Veronese varieties defined by flattenings.

### 3.2 Consequences of Theorems 3.1.9 and 3.1.4

**Theorem 3.2.1** *The following varieties are defined scheme-theoretically by minors of flattenings:*

- (1) *Let  $d \geq 4$ . The variety  $\sigma_3(v_d(\mathbb{P}^n))$  is defined scheme-theoretically by the  $4 \times 4$  minors of the  $(1, d - 1)$  and  $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$  flattenings.*

- (2) For  $d \geq 4$  the variety  $\sigma_4(v_d(\mathbb{P}^2))$  is defined scheme-theoretically by the  $5 \times 5$  minors of the  $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$  flattenings.
- (3) For  $d \geq 6$  the variety  $\sigma_5(v_d(\mathbb{P}^2))$  is defined scheme-theoretically by the  $6 \times 6$  minors of the  $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$  flattenings.
- (4) Let  $d \geq 6$ . The variety  $\sigma_6(v_d(\mathbb{P}^2))$  is defined scheme-theoretically by the  $7 \times 7$  minors of the  $(\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor)$  flattenings.

*Remark 3.2.2* In the recent preprint [6] it is proved that the variety  $\sigma_r(v_d(\mathbb{P}^n))$  is defined set-theoretically by the  $(r + 1) \times (r + 1)$  minors of the  $(i, d - i)$  flattenings for  $n \leq 3, 2r \leq d$  and  $r \leq i \leq d - r$ . Schreyer proved (2) in the case  $d = 4$ , [46, Thm. 2.3].

By [47, Thm 4.2], when  $n \leq 2$ ,  $\text{rank}(\phi_{s,d-s})$  is non-decreasing in  $s$  for  $1 \leq s \leq \lfloor \frac{d}{2} \rfloor$ . We will use this fact often in this section.

To prove each case, we simply show the scheme defined by the flattenings in the hypotheses coincides with the scheme of some  $\text{Gor}(\vec{r})$  with  $\vec{r}$  admissible and then Theorem 3.1.4 combined with Theorem 3.1.9 implies the result. The first step is to determine which  $\vec{r}$  are admissible.

**Lemma 3.2.3** *The only admissible sequences  $\vec{r}$  with  $r_1 = 3$  and  $r_i \leq 5$  are*

- (1)  $(3, \dots, 3)$  for  $d \geq 3$ ,
- (2)  $(3, 4, 4, \dots, 4)$  for  $d \geq 4$ ,
- (3)  $(3, 5, 5, \dots, 5)$  for  $d \geq 4$ ,
- (4)  $(3, 4, 5, \dots, 5)$  for  $d \geq 6$ .

Lemma 3.2.3 is proved below. We first show how it implies the results above, by showing in each case a general  $\phi$  satisfying the hypotheses of the theorem must be in the appropriate  $\text{SFlat}_{\vec{r}}^0$  with  $\vec{r}$  admissible. Since  $\sigma_r(v_d(\mathbb{P}^V))$  is irreducible, one concludes.

*Proof of (1)* We may assume that  $\text{rank}(\phi_{1,d-1}) = 3$ ; otherwise, we are in the case of two variables. By inheritance, it is sufficient to prove the result for  $n = 2$ . In this case, if  $\text{rank}(\phi_{a,d-a}) \leq 3$  for  $a = \lfloor \frac{d}{2} \rfloor$ , then  $\text{rank}(\phi_{a,d-a}) \leq 3$  for all  $a$  such that  $2 \leq a \leq d - 2$ , and  $\vec{r} = (3, \dots, 3)$  is admissible. □

*Proof of (2)* We may assume that  $\text{rank}\phi_{1,d-1} = 3$ . Let  $d = 4$ . If  $\text{rank}(\phi_{2,d-2}) \leq 4$  then by Lemma 3.2.3 the only possible  $\vec{r}$  are  $(3, 3)$  and  $(3, 4)$ . The first case corresponds to  $\sigma_3(v_4(\mathbb{P}^2)) \subset \sigma_4(v_4(\mathbb{P}^2))$  and the second case is as desired. The case  $d \geq 5$  is analogous. □

*Proof of (3)* If  $\phi \notin \sigma_4(v_d(\mathbb{P}^W))$ , then  $\phi \in \text{Gor}(3, 5, \dots, 5) = \sigma_5(v_d(\mathbb{P}^2))$  or  $\phi \in \text{Gor}(3, 4, 5, \dots, 5)$ .

It remains to show  $\text{Gor}(3, 4, 5, \dots, 5) \subset \sigma_5(v_d(\mathbb{P}^2))$ . To prove this we generalize the argument of [25, Ex. 5.72].

Let  $\phi \in \text{SFlat}_{3,4,5,\dots,5}^0(S^d W)$  and consider the kernel of  $\phi_{2,d-2}$  as a 2-dimensional space of plane conics. If the base locus is a zero-dimensional scheme of length four, then  $\phi \in \sigma_4(v_d(\mathbb{P}^2))$ , and  $\text{rank}(\phi_{3,d-3}) \leq 4$  which contradicts  $\phi \in \text{SFlat}_{3,4,5,\dots,5}^0(S^d W)$ . Thus the base locus has dimension one, and in convenient coordinates, the kernel of  $\phi_{2,d-2}$  is  $\langle xy, xz \rangle$ . It follows that  $\phi = x^d + \psi(y, z)$  and from  $\text{rank}(\phi_{a,d-a}) = 5$  it follows  $\text{rank}(\psi_{a,d-a}) = 4$ . Since  $\psi$  has two variables, this implies  $\psi \in \sigma_4(v_d(\mathbb{P}^2))$  and  $\phi \in \sigma_5(v_d(\mathbb{P}^2))$  as required. □

*Proof of (4)* If the sequence of ranks of  $\phi_{a,d-a}$  is  $\vec{r} = (1, 3, 6, \dots, 6, 3, 1)$  then  $\phi \in \text{Gor}(\vec{r}) = \sigma_6(v_d(\mathbb{P}^n))$ .

Otherwise  $\text{rank}(\phi_{2,d-2}) \leq 5$ . In this case, by an extension of Lemma 3.2.3, there are just two other possibilities for  $\vec{r}$ , namely  $\vec{r}_1 = (1, 3, 5, 6, \dots, 6, 5, 3, 1)$  and  $\vec{r}_2 = (1, 3, 4, 5, 6, \dots, 6, 5, 4, 3, 1)$  for  $d \geq 8$ .

Consider  $\phi \in \vec{r}_2$ . As above, we may assume that  $\ker(\phi_{2,d-2}) = \langle xy, xz \rangle$ . It follows that  $\phi = x^d + \psi(y, z)$ . If  $d \leq 9$ , then  $\psi \in \sigma_5(v_d(\mathbb{P}^2))$ . If  $d \geq 10$ ,  $\text{rank}(\psi_{5,d-5}) = 5$  because  $\text{rank}(\phi_{5,d-5}) = 6$ . Since  $\psi$  has two variables,  $\psi \in \sigma_5(v_d(\mathbb{P}^2))$ . Thus  $\phi \in \sigma_6(v_d(\mathbb{P}^2))$  as desired.

Consider  $\phi \in \vec{r}_1$ . The cases  $d = 6$  and  $d \geq 8$  respectively follow from [25, Ex. 5.72] and [25, Thm. 5.71 (i)]. Here is a uniform argument for all  $d \geq 6$ : There is a conic  $C$  in the kernel of  $\phi_{2,d-2}$ . The four-dimensional space of cubics which is the kernel of  $\phi_{3,d-3}$  is generated by  $C \cdot L$  where  $L$  is any line and a cubic  $F$ . Then the 6 points of  $C \cap F$  are the base locus, by [25, Cor 5.69], so  $\phi \in \sigma_6(v_d(\mathbb{P}^2))$ . □

Let  $\dim W = 3$ . We will need the following facts to prove Lemma 3.2.3:

- (1) For  $\phi \in S^d W$ , consider the ideal  $\phi^\perp$  generated in degrees  $\leq d$ , by  $\ker(\phi_{a,d-a}) \subset S^a W^*$ ,  $1 \leq a \leq d$ , and the ring  $A_\phi := \text{Sym}(W^*)/\phi^\perp$ . Note that the values of the Hilbert function of  $A_\phi$ ,  $H_{A_\phi}(j)$ , are  $T(\vec{r}) := (1, r_1, \dots, r_{\lfloor \frac{d}{2} \rfloor}, r_{\lfloor \frac{d}{2} \rfloor}, \dots, r_1, 1, 0, \dots, 0)$ , where recall that  $r_j = \text{rank}(\phi_{j,d-j})$ .
- (2) [5, Thm. 2.1] The ring  $A_\phi$  has a minimal free resolution of the form

$$0 \longrightarrow \text{Sym}(W^*)(-d-3) \xrightarrow{g^*} \bigoplus_{i=1}^u \text{Sym}(W^*)(-d+3-q_i) \xrightarrow{h} \bigoplus_{i=1}^u \text{Sym}(W^*)(-q_i) \xrightarrow{g} \text{Sym}(W^*)(0) \rightarrow A_\phi \rightarrow 0 \tag{5}$$

where the  $q_j$  are non-decreasing. Here  $\text{Sym}(W^*)(j)$  denotes  $\text{Sym}(W^*)$  with the labeling of degrees shifted by  $j$ , so  $\dim(\text{Sym}(W^*)(j))_d = \binom{d+j+2}{2}$ .

- (3) Moreover [12, Thm 1.1] there is a unique resolution with the properties  $q_1 \leq \frac{d+3}{2}$ ,  $q_i + q_{u-i+2} = d + 2$ ,  $2 \leq i \leq \frac{u+1}{2}$  having  $T(\vec{r})$  as the values of its Hilbert function.
- (4) Recall that  $H_{A_\phi}(j)$  is also the alternating sum of the dimensions of the degree  $j$  term in (5) (forgetting the last term). Thus the  $q_j$  determine the  $r_i$ .
- (5) [12, Thm. 3.3] Letting  $j_0$  be the smallest  $j$  such that  $\phi_{j,d-j}$  is not injective, then  $u = 2j_0 + 1$  in the resolution above.

*Remark 3.2.4* Although we do not need these facts from [5] here, we note that above:  $h$  is skew-symmetric,  $g$  is defined by the principal sub-Pfaffians of  $h$ ,  $A_\phi$  is an Artinian Gorenstein graded  $\mathbb{C}$ -algebra and any Artinian Gorenstein graded  $\mathbb{C}$ -algebra is isomorphic to  $A_\phi$  for a polynomial  $\phi$  uniquely determined up to constants. The ring  $A_\phi$  is called the *apolar ring* of  $\phi$ , the resolution above is called *saturated*.

*Proof of Lemma 3.2.3* Let  $\vec{r}$  be a sequence as in the assumptions. Since  $\dim S^2\mathbb{C}^3 = 6 > 5$ ,  $u = 5$  by Fact 3.2(5). Consider the unique resolution satisfying Fact 3.2(3) having  $T(\vec{r})$  as Hilbert function. The number of generators in degree 2 is the number of the  $q_i$  equal to 2. We must have  $q_1 = 2$ , otherwise, by computing the Hilbert function,  $H(A_\phi)(2) = 6$  contrary to assumption. The maximal number of generators in degree 2 is three; otherwise, we would have  $2 + 2 = q_4 + q_{5-4+2} = d + 2$ , a contradiction.

If there are three generators in degree 2, then  $q_i = (2, 2, 2, d, d)$ , and computing the Hilbert function via the Euler characteristic, we are in case (1) of the Lemma.

If there are two generators in degree 2, then there are the two possibilities  $q_i = (2, 2, 3, n - 1, n)$  (case(2)) or  $q_i = (2, 2, 4, n - 2, n)$  (case (4)). Note that  $q_i = (2, 2, 5, n - 3, n)$  for  $n \geq 8$  is impossible because otherwise  $\vec{r} = (1, 3, 4, 5, 6, \dots)$  contrary to assumption. By similar arguments, one shows that there are no other possibilities.

If there is one generator in degree 2, then  $q_i = (2, 3, 3, n - 1, n - 1)$  (case (3)). □

**Proposition 3.2.5** *Let  $\dim V = 3$  and  $p \geq 2$ . If the variety  $\sigma_r(v_{2p}(\mathbb{P}V))$  is an irreducible component of  $\sigma_r(v_2(\mathbb{P}S^pV)) \cap \mathbb{P}S^{2p}V$  then  $r \leq \frac{1}{2}p(p + 1)$  or  $(p, r) = (2, 4)$  or  $(3, 9)$ .*

*Proof* Recall that for  $(p, r) \neq (2, 5)$

$$\text{codim}_{\sigma_r(v_{2p}(\mathbb{P}V))} = \binom{2p + 2}{2} - 3r \tag{6}$$

and from Proposition 2.4.4 every irreducible component  $X$  of  $\sigma_r(v_2(\mathbb{P}S^pV)) \cap \mathbb{P}S^{2p}V$  satisfies

$$\text{codim}X \leq \frac{1}{2} \left[ \binom{p + 2}{2} - r \right] \left[ \binom{p + 2}{2} - r + 1 \right] \tag{7}$$

The result follows by solving the inequality. □

For the case  $(p, r) = (2, 4)$  see Theorem 3.2.1(2) and for the case  $(p, r) = (3, 9)$  see Theorem 4.2.9.

Corollary 3.2.6  $\text{deg } \sigma_{\binom{p+1}{2}}(v_{2p}(\mathbb{P}^2)) \leq \prod_{i=0}^p \frac{\binom{(p+2)(p+1)/2+i}{p-i+1}}{\binom{2i+1}{i}}.$

*If equality holds then  $\sigma_{\binom{p+1}{2}}(v_{2p}(\mathbb{P}^2))$  is given scheme-theoretically by the  $\left(\frac{1}{2}\right)$   $(p^2 + p + 2)$  minors of the  $(p, p)$ -flattening.*

The values of the right-hand side of the inequality for  $p = 1, \dots, 4$  are respectively 4, 112, 28314, 81662152.

*Proof* The right-hand side is  $\text{deg } \sigma_{\binom{p+1}{2}}(v_2(\mathbb{P}^{p(p+3)/2}))$  by Proposition 2.4.4. If  $\phi \in \sigma_{\binom{p+1}{2}}(v_{2p}(\mathbb{P}^2))$  then  $\text{rank}(\phi_{p,p}) \leq \binom{p+1}{2}$ . This means  $\sigma_{\binom{p+1}{2}}(v_{2p}(\mathbb{P}^2))$  is contained in a linear section of  $\sigma_{\binom{p+1}{2}}(v_2(\mathbb{P}^{p(p+3)/2}))$ , and by Theorem 3.1.4 it is a irreducible component of this linear section. The result follows by the refined Bezout Theorem 2.4.1. □

Equality holds in Corollary 3.2.6 for the cases  $p = 1, 2, 3$  by Proposition 2.4.2. The case  $p = 1$  corresponds to the quadratic Veronese surface, and the cases  $p = 2, 3$  will be considered respectively in Theorem 3.2.1 (1) and Theorem 4.2.8. For  $p \geq 4$  these numbers are out of the range of the results in [15] (the points are too few to be fixed points of a torus action), and we do not know if equality holds.

### 4 Young flattenings for Veronese varieties

#### 4.1 Preliminaries

In what follows we fix  $n + 1 = \dim V$  and endow  $V$  with a volume form and thus identify (as  $SL(V)$ -modules)  $S_{(p_1, \dots, p_{n+1})}V$  with  $S_{(p_1 - p_{n+1}, p_2 - p_{n+1}, \dots, p_n - p_{n+1}, 0)}V$ . We will say  $(p_1 - p_n, p_2 - p_n, \dots, p_{n-1} - p_n, 0)$  is the *reduced partition* associated with  $(p_1, \dots, p_n)$ .

The *Pieri formula* states that  $S_\pi V^* \subset S_\nu V^* \otimes S^d V^*$  iff the Young diagram of  $\pi$  is obtained by adding  $d$  boxes to the Young diagram of  $\nu$ , with no two boxes added to the same column. Moreover, if this occurs, the multiplicity of  $S_\pi V^*$  in  $S_\nu V^* \otimes S^d V^*$  is one.

We record the basic observation that the dual  $SL(V)$ -module to  $S_\pi V$  is obtained by considering the complement to  $\pi$  in the  $\ell(\pi) \times \dim V$  rectangle and rotating it to give a Young diagram whose associated partition we denote  $\pi^*$ .

Say  $S_\pi V^* \subset S_\nu V \otimes S^d V^*$  and consider the map  $S^d V \rightarrow S_\pi V \otimes S_\nu V^*$ . Let  $S_\mu V = S_\nu V^*$  where  $\mu$  is the reduced partition with this property. We obtain an inclusion  $S^d V \rightarrow S_\pi V \otimes S_\mu V$ .

Given  $\phi \in S^d V$ , let  $\phi_{\pi,\mu} \in S_\pi V \otimes S_\mu V$  denote the corresponding element. If  $S_\mu V = S_\nu V^*$  as an  $SL(V)$ -module, we will also write  $\phi_{\pi,\nu^*} = \phi_{\pi,\mu}$  when we consider it as a linear map  $S_\nu V \rightarrow S_\pi V$ .

The following proposition is an immediate consequence of the subadditivity for ranks of linear maps.

**Proposition 4.1.1** *Rank conditions on  $\phi_{\pi,\mu}$  provide equations for the secant varieties of  $v_d(\mathbb{P}V)$  as follows: Let  $[x^d] \in v_d(\mathbb{P}V)$ . Say  $\text{rank}(x^d_{\pi,\mu}) = t$ . If  $[\phi] \in \sigma_r(v_d(\mathbb{P}V))$ , then  $\text{rank}(\phi_{\pi,\mu}) \leq rt$ . Thus if  $r + 1 \leq \min\{\dim S_\pi V, \dim S_\mu V\}$ , the  $(rt + 1) \times (rt + 1)$  minors of  $\phi_{\pi,\mu}$  provide equations for  $\sigma_r(v_d(\mathbb{P}V))$ , i.e.,*

$$\wedge^{rt+1}(S_\pi V^*) \otimes \wedge^{rt+1}(S_\nu V^*) \subset I_{rt+1}(\sigma_r(v_d(\mathbb{P}V))).$$

*Remark 4.1.2* From the inclusion  $S^d V \subset V \otimes S^a V \otimes S^{d-a-1} V$  we obtain  $v_d(\mathbb{P}V) \subset \text{Seg}(\mathbb{P}V \times \mathbb{P}S^a V \times \mathbb{P}S^{d-a-1} V)$ . Strassen’s equations for  $\sigma_{n+s}(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}))$  give equations for secant varieties of Veronese varieties via this three-way flattening. The Aronhold equation comes from the Strassen equations for  $\sigma_3(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ .

So far we have not obtained any new equations using three-way symmetric flattenings and Young flattenings. We mention three-way symmetric flattenings because they may be useful in future investigations, especially when further modules of equations for secant varieties of triple Segre products are found.

Let

$$\text{YFlat}^t_{\pi,\mu}(S^d V) := \overline{\{\phi \in S^d V \mid \text{rank}(\phi_{\pi,\mu}) \leq t\}} = \sigma_t(\text{Seg}(\mathbb{P}S_\pi V \times \mathbb{P}S_\mu V)) \cap \mathbb{P}S^d V.$$

*Remark 4.1.3* Recall  $YF^r_{d,n}$  from Sect. 1.2 whose description had redundancies. We can now give an irredundant description of its defining equations:

$$YF^r_{d,n} = \text{YFlat}^{\binom{n}{1 \frac{r}{2} 1}}_{((\delta+1)^a, \delta^{n-a}), (\delta+1, 1^a)}(S^d V).$$

This is a consequence of Schur’s Lemma, because the module  $S_{(\delta+1, 1^a)} V$  is the only one appearing in both sides of  $S^d V \otimes S^\delta V^* \otimes \wedge^a V \rightarrow S^\delta V \otimes \wedge^{a+1} V$

Note that if  $S_\pi V \simeq S_\mu V$  as  $SL(V)$ -modules and the map is symmetric, then

$$\text{YFlat}^t_{\pi,\mu}(S^d V) = \sigma_t(v_2(\mathbb{P}S_\pi V)) \cap \mathbb{P}S^d V,$$

and if it is skew-symmetric, then

$$\text{YFlat}^t_{\pi,\mu}(S^d V) = \sigma_t(G(2, S_\pi V)) \cap \mathbb{P}S^d V.$$

4.2 The surface case,  $\sigma_r(v_d(\mathbb{P}^2))$

In this subsection fix  $\dim V = 3$  and a volume form  $\Omega$  on  $V$ . From the general formula for  $\dim S_\pi V$  (see, e.g., [17, p. 78]), we record the special case:

$$\dim S_{a,b}\mathbb{C}^3 = \frac{1}{2}(a + 2)(b + 1)(a - b + 1). \tag{8}$$

**Lemma 4.2.1** *Let  $a \geq b$ . Write  $d = \alpha + \beta + \gamma$  with  $\alpha \leq b$ ,  $\beta \leq a - b$  so  $S_{(a+\gamma-\alpha, b+\beta-\alpha)}V \subset S_{a,b}V \otimes S^dV$ . For  $\phi \in S^dV$ , consider the induced map*

$$\phi_{(a,b),(a+\gamma-\alpha, b+\beta-\alpha)} : S_{a,b}V^* \rightarrow S_{(a+\gamma-\alpha, b+\beta-\alpha)}V. \tag{9}$$

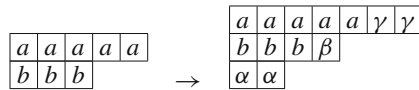
Let  $x \in V$ , then

$$\text{rank}((x^d)_{(a,b),(a+\gamma-\alpha, b+\beta-\alpha)}) = \frac{1}{2}(b - \alpha + 1)(a - b - \beta + 1)(a + \beta - \alpha + 2) =: R. \tag{10}$$

Thus in this situation  $\wedge^{pR+1}(S_{ab}V) \otimes \wedge^{pR+1}(S_{a+\gamma-\alpha, b+\beta-\alpha}V)$  gives non-trivial degree  $pR + 1$  equations for  $\sigma_p(v_d(\mathbb{P}^2))$ .

*Remark 4.2.2* Note that the right-hand sides of Eqs. (8) and (10) are the same when  $\alpha = \beta = 0$ . To get useful equations one wants  $R$  small with respect to  $\dim S_{a,b}\mathbb{C}^3$ .

*Proof* In the following picture we label the first row containing  $a$  boxes with  $a$  and so on.



Assume we have chosen a weight basis  $x_1, x_2, x_3$  of  $V$  and  $x = x_3$  is a vector of lowest weight. Consider the image of a weight basis of  $S_{a,b}V$  under  $(x_3^d)_{(a,b),(a+\gamma-\alpha, b+\beta-\alpha)}$ . Namely consider all semi-standard fillings of the Young diagram corresponding to  $(a, b)$  and count how many do not map to zero. By construction, the images of all the vectors that do not map to zero are linearly independent, so this count indeed gives the dimension of the image.

In order to have a vector not in the kernel, the first  $\alpha$  boxes of the first row must be filled with 1's and the first  $\alpha$  boxes of the second row must be filled with 2's.

Consider the next  $(b - \alpha, b - \alpha)$  subdiagram. Let  $\mathbb{C}_{ij}^2$  denote the span of  $x_i, x_j$  and  $\mathbb{C}_i^1$  the span of  $x_i$ . The boxes here can be filled with any semi-standard filling using 1's 2's and 3's, but the freedom to fill the rest will depend on the nature of the filling, so split  $S_{b-\alpha, b-\alpha}\mathbb{C}^3$  into two parts, the first part where the entry in the last box in the first row is 1, which has  $\dim S_{b-\alpha}\mathbb{C}_{23}^2$  fillings and second where the entry in the last box in the first row is 2, which has  $(\dim S_{b-\alpha, b-\alpha}\mathbb{C}^3 - \dim S_{b-\alpha}\mathbb{C}_{23}^2)$  fillings.

In the first case, the next  $\beta$ -entries are free to be any semi-standard row filled with 1's and 2's, of which there are  $\dim S_\beta\mathbb{C}_{12}^2$  in number, but we split this further into two sub-cases, depending on whether the last entry is a 1 (of which there is one  $(= \dim S_\beta\mathbb{C}_1^1)$  such) or 2 (of which there are  $(\dim S_\beta\mathbb{C}_{12}^2 - \dim S_\beta\mathbb{C}_1^1)$  such). In the first sub-case the last  $a - (b + \beta)$  entries admit  $\dim S_{a-(b+\beta)}\mathbb{C}^3$  fillings, and in the second sub-case there are  $\dim S_{a-(b+\beta)}\mathbb{C}_{23}^2$  such. Putting these together, the total number of fillings for the various paths corresponding to the first part is

$$\begin{aligned} & \dim S_{b-\alpha}\mathbb{C}_{12}^2 [ (\dim S_\beta\mathbb{C}_1^1) (\dim S_{a-b-\beta}\mathbb{C}^3) + (\dim S_\beta\mathbb{C}_{12}^2 - \dim S_\beta\mathbb{C}_1^1) (\dim S_{a-b-\beta}\mathbb{C}_{23}^2) ] \\ & = (b - \alpha + 1) [ (1) \binom{a - b - \beta + 2}{2} + (\beta - 1)(a - b - \beta + 1) ]. \end{aligned}$$

For the second part, the next  $\beta$  boxes in the first row must be filled with 2's (giving  $1 = \dim S_\beta \mathbb{C}_2^1$ ), and the last  $a - (b + \beta)$  boxes can be filled with 2's or 3's semi-standardly; that is, there are  $\dim S_{a-b-\beta} \mathbb{C}_{23}^2$ 's worth. So the contribution of the second part is

$$\begin{aligned}
 & (\dim S_{b-\alpha, b-\alpha} \mathbb{C}^3 - \dim S_{b-\alpha} \mathbb{C}_{12}^2)(\dim S_\beta \mathbb{C}_2^1)(\dim S_{a-b-\beta} \mathbb{C}_{23}^2) \\
 &= \left( \binom{b-\alpha+2}{b-\alpha} - (b-\alpha+1) \right) (1)(a-b-\beta+1).
 \end{aligned}$$

Adding up gives the result. □

We are particularly interested in cases where  $(a, b) = (a + \gamma - \alpha, b + \beta - \alpha)$ . In this case

$$\alpha = \gamma = \frac{1}{3}(d + 2b - a) \tag{11}$$

$$\beta = \frac{1}{3}(d - 4b + 2a). \tag{12}$$

Plugging into the conclusion of Lemma 4.2.1, the rank of the image of a  $d$ -th power in this situation is

$$\frac{1}{9}(a + b - d + 3)^2(a - b + 1).$$

To keep this small, it is convenient to take  $d = a + b$  so the rank is  $a - b + 1$ . One can then fix this number and let  $a, b$  grow to study series of cases.

If (10) has rank one when  $d = 2p$ , we just recover the usual symmetric flattenings as  $S_{(p,p)}V = S_p V^*$ . We consider the next two cases in the theorems below,  $(a, b) = (p + 1, p)$  when  $d = 2p + 1$  and  $(a, b) = (p + 2, p)$  when  $d = 2p + 2$ . Recall that  $(p + q, p)^* = (p + q, q)$  in the notation of Sect. 4.1.1.

Let  $d = 2p + 1$ . The skew analog of Proposition 3.2.5 is the following proposition, which shows that the bound in the assumption of Theorem 1.2.3 is sharp.

**Proposition 4.2.3** *Let  $\dim V = 3$ . If the variety  $\sigma_r(v_{2p+1}(\mathbb{P}V))$  is an irreducible component of  $\text{YFlat}_{(p+1,p),(p+1,p)}^r(S^{2p+1}V)$ , then  $r \leq \binom{p+2}{2}$ .*

*Proof* Recall from Proposition 2.4.4

$$\text{codim}_{\sigma_r(v_{2p+1}(\mathbb{P}V))} = \binom{2p+3}{2} - 3r \tag{13}$$

$$\begin{aligned}
 \text{codim}_{\sigma_r(G(2, S_{p+1,p}V) \cap \mathbb{P}S^{2p+1}V)} &\leq \frac{1}{2} [(p+1)(p+3) - 2r] \\
 &\quad \times [(p+1)(p+3) - 2r - 1] \tag{14}
 \end{aligned}$$

The inequality is a consequence of  $\text{codim}_{\sigma_r(v_{2p+1}(\mathbb{P}V))} \leq \text{codim}_{\sigma_r(G(2, S_{p+1,p}V) \cap \mathbb{P}S^{2p+1}V)}$ . □

Here is a pictorial description when  $p = 2$  of  $\phi_{31,31}: S_{32}V \rightarrow S_{31}V$  in terms of Young diagrams:





Corollary 4.2.4  $\deg \sigma_{\binom{p+2}{2}}(v_{2p+1}(\mathbb{P}^2)) \leq \frac{1}{2^p} \prod_{i=0}^{p-1} \frac{\binom{(p+3)(p+1)+i}{p-i}}{\binom{2i+1}{i}}$ .

If equality holds, then  $\sigma_{\binom{p+2}{2}}(v_{2p+1}(\mathbb{P}^2))$  is given scheme-theoretically by the size  $(p + 2)(p + 1) + 2$  sub-Pfaffians of  $\phi_{(p+1,p),(p+1,p)}$ .

The values of the right-hand side for  $p = 1, \dots, 4$  are respectively 4, 140, 65780, 563178924.

*Proof* The right-hand side is  $\deg \sigma_{\binom{p+2}{2}}(G(\mathbb{C}^2, S_{(p+1,p)}V))$  by Proposition 2.4.4. Now  $\sigma_{\binom{p+1}{2}}(v_{2p+1}(\mathbb{P}^2))$  is contained in a linear section of  $\sigma_{\binom{p+2}{2}}(G(\mathbb{C}^2, S_{(p+1,p)}V))$ , and by Theorem 1.2.3 it is a irreducible component of this linear section. The result follows by the refined Bezout theorem 2.4.1.  $\square$

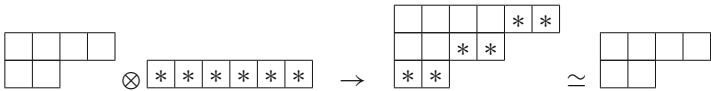
In Corollary 4.2.4 equality holds in the cases  $p = 1, 2$  by Proposition 2.4.2. The case  $p = 1$  is just the Aronhold case, and the case  $p = 2$  will be considered in Theorem 4.2.7 (2). For  $p \geq 3$  these numbers are out of the range of [15], and we do not know if equality holds.

Note that the usual symmetric flattenings only give equations for  $\sigma_{k-1}(v_{2p+1}(\mathbb{P}^2))$  for  $k \leq \frac{1}{2}(p^2 + 3p + 2)$ .

Now let  $d = 2p + 2$  be even, requiring  $\pi = \mu$ , the smallest possible  $\text{rank}((x^d)_{\pi,\mu})$  is three, which we obtain with  $\phi_{(p+2,p),(p+2,p)}$ .

**Proposition 4.2.5** *Let  $d = 2p + 2$ . The Young flattening  $\phi_{(p+2,2),(p+2,2)} \in S_{p+2,2}V \otimes S_{p+2,2}V$  is symmetric. It is of rank three for  $\phi \in v_d(\mathbb{P}^2)$  and gives degree  $3(k + 1)$  equations for  $\sigma_r(v_{2p+2}(\mathbb{P}^2))$  for  $r \leq \frac{1}{2}(p^2 + 5p + 4) - 1$ . A convenient model for the equations is given in the proof.*

A pictorial description when  $p = 2$  is as follows:



*Proof* Let  $\tilde{\Omega} \in \wedge^3 V$  be dual to the volume form  $\Omega$ . To prove the symmetry, for  $\phi = x^{2p+2}$ , consider the map,

$$M_{x^{2p+2}} : S^p V^* \otimes S^2(\wedge^2 V^*) \rightarrow S^p V \otimes S^2(\wedge^2 V)$$

$$\alpha_1 \cdots \alpha_p \otimes (\gamma_1 \wedge \delta_1) \circ (\gamma_2 \wedge \delta_2) \mapsto \alpha_1(x) \cdots \alpha_p(x) x^p \tilde{\Omega}(x \lrcorner \gamma_1 \wedge \delta_1) \circ \tilde{\Omega}(x \lrcorner \gamma_2 \wedge \delta_2)$$

and define  $M_\phi$  for arbitrary  $\phi \in S^{2p+2}V$  by linearity and polarization. If we take bases of  $S^2 V \otimes S^2(\wedge^2 V)$  as above, with indices  $((i_1, \dots, i_p), (kl), (k'l'))$ , most of the matrix of  $M_{e_1^{2p+2}}$  is zero. The upper-right hand  $6 \times 6$  block, where  $(i_1, \dots, i_p) = (1, \dots, 1)$  in both rows and columns and the order on the other indices

$$((12), (12)), ((13), (13)), ((12), (13)), ((12), (23)), ((13), (23)), ((23), (23)),$$

is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

showing the symmetry. Now

$$\begin{aligned} (S^p V \otimes S^2(\wedge^2 V))^{\otimes 2} &= (S^p V \otimes (S_{22} V \oplus S_{1111} V))^{\otimes 2} \\ &= S_{p+2,2} V \oplus \text{stuff} \end{aligned}$$

where all the terms in stuff have partitions with at least three parts. On the other hand, from the nature of the image we conclude that it is just the first factor and  $M_\phi \in S^2(S_{p+2,2} V)$ .  $\square$

Note that the usual symmetric flattenings give non-trivial equations for  $\sigma_{k-1}(v_{2p+2}(\mathbb{P}^2))$  for  $k \leq \frac{1}{2}(p^2 + 5p + 6)$ , a larger range than in Proposition 4.2.5. However, we show (Theorem 4.2.9) that the symmetric flattenings alone are not enough to cut out  $\sigma_7(v_6(\mathbb{P}^2))$ , but with the  $((p + 1, 2), (p + 1, 2))$ -Young flattening they are.

Here is a more general Young flattening:

**Proposition 4.2.6** *Let  $d = p + 4q - 1$ . The Young flattening*

$$\phi_{(p+2q,2q-1),(p+2q,2q-1)} \in S_{(p+2q,2q-1)} V \otimes S_{(p+2q,2q-1)} V,$$

is skew-symmetric if  $p$  is even and symmetric if  $p$  is odd.

Since it has rank  $p$  if  $\phi \in v_d(\mathbb{P}^2)$ , if  $p$  is even (resp. odd), the size  $kp + 2$  sub-Pfaffians (resp. size  $kp + 1$  minors) of  $\phi_{(p+2q,2q-1),(p+2q,2q-1)}$  give degree  $\frac{kp}{2} + 1$  (resp.  $kp + 1$ ) equations for  $\sigma_k(v_{p+4q-1}(\mathbb{P}^2))$  for

$$k \leq \frac{q(p + 2q + 2)(p + 2)}{p}.$$

*Proof* Consider  $M_\phi : S^{p-1} V^* \otimes S^q(\wedge^2 V^*) \rightarrow S^p V \otimes S^q(\wedge^2 V)$  given for  $\phi = x^{p+4q-1}$  by

$$\alpha_1 \dots \alpha_{p-1} \otimes \beta_1 \wedge \gamma_1 \dots \beta_q \wedge \gamma_q \mapsto \prod_j (\alpha_j(x)) x^{p-1} \otimes \tilde{\Omega}(x \lrcorner \beta_1 \wedge \gamma_1) \dots \tilde{\Omega}(x \lrcorner \beta_q \wedge \gamma_q)$$

and argue as above.  $\square$

Here the usual flattenings give degree  $k$  equations for  $\sigma_{k-1}(v_d(\mathbb{P}^2))$  in the generally larger range  $k \leq \frac{1}{8}(p + 4q + 2)(p + 4q)$ .

Recall that we have already determined the ideals of  $\sigma_r(v_d(\mathbb{P}^2))$  for  $d \leq 4$ , (Theorem 3.2.1 and the chart in the introduction) so we next consider the case  $d = 5$ .

*Case  $d = 5$ :* The symmetric flattening given by the size  $(k + 1)$  minors of  $\phi_{2,3}$  define  $\sigma_k(v_5(\mathbb{P}^2))$  up to  $k = 4$  by Theorem 3.1.4 (3). Note that the size 6 minors define a subvariety of codimension 5, strictly containing  $\sigma_5(v_5(\mathbb{P}^2))$ , which has codimension 6. So, in this case, the bound provided by Theorem 3.1.4 is sharp.

**Theorem 4.2.7** (1)  $\sigma_k(v_5(\mathbb{P}^2))$  for  $k \leq 5$  is an irreducible component of  $\text{YFlat}_{31,31}^{2k}(S^5 \mathbb{C}^3)$ , the variety given by the principal size  $2k + 2$  Pfaffians of the  $[(31), (31)]$ -Young flattenings.

(2) the principal size 14 Pfaffians of the  $[(31), (31)]$ -Young flattenings are scheme-theoretic defining equations for  $\sigma_6(v_5(\mathbb{P}^2))$ , i.e., as schemes,  $\sigma_6(v_5(\mathbb{P}^2)) = \text{YFlat}_{31,31}^{12}(S^5 \mathbb{C}^3)$ .

(3)  $\sigma_7(v_5(\mathbb{P}^2))$  is the ambient space.

*Proof* (1) is a consequence of Theorem 1.2.3.

To prove the second assertion, by the Segre formula (Proposition 2.4.4) the subvariety of size 15 skew-symmetric matrices of rank  $\leq 12$  has codimension 3 and degree 140. By Proposition 2.4.2,  $\sigma_6(v_5(\mathbb{P}^2))$  has codimension 3 and degree 140. We conclude as in the proof of Corollary 3.2.6.  $\square$

*Case  $d = 6$ :* In the case  $\sigma_k(v_6(\mathbb{P}^2))$ , the symmetric flattening given by the  $(k + 1)$  minors of  $\phi_{2,4}$  defines  $\sigma_k(v_6(\mathbb{P}^2))$  as an irreducible component up to  $k = 4$  (in the case  $k = 4$  S. Diesel [25, Ex. 3.6] showed that there are two irreducible components) and the symmetric flattening given by the  $(k + 1)$  minors of  $\phi_{3,3}$  defines  $\sigma_k(v_6(\mathbb{P}^2))$  as an irreducible component up to  $k = 6$ .

The following is a special case of the Theorem 3.2.1(4), and we include this second proof because it is very short.

**Theorem 4.2.8** *As schemes,  $\sigma_6(v_6(\mathbb{P}^2)) = \text{Rank}_{3,3}^6(S^6\mathbb{C}^3)$ , i.e., the size 7 minors of  $\phi_{3,3}$  cut out  $\sigma_6(v_6(\mathbb{P}^2))$  scheme-theoretically.*

*Proof* By the Segre formula (Proposition 2.4.4) the subvariety of symmetric  $10 \times 10$  matrices of rank  $\leq 6$  has codimension 10 and degree 28, 314. By Proposition 2.4.2,  $\sigma_6(v_6(\mathbb{P}^2))$  has codimension 10 and degree 28, 314. We conclude as in the proof of Corollary 3.2.6.  $\square$

The size 8 minors of  $\phi_{3,3}$  define a subvariety of codimension 6, strictly containing  $\sigma_7(v_6(\mathbb{P}^2))$ , which has codimension 7. In the same way, the size 9 minors of  $\phi_{3,3}$  define a subvariety of codimension 3, strictly containing  $\sigma_8(v_6(\mathbb{P}^2))$ , which has codimension 4. Below we construct equations in terms of Young flattenings.

$\det(\phi_{42,42})$  is a polynomial of degree 27, which is not the power of a lower degree polynomial. This can be proved by cutting with a random projective line and using Macaulay2. The two-variable polynomial obtained is not the power of a lower degree polynomial.

If  $\phi$  is decomposable then  $\text{rank}(\phi_{42,42}) = 3$  by Lemma 4.2.1, so that when  $\phi \in \sigma_k(v_6(\mathbb{P}^2))$ , then  $\text{rank}(\phi_{42,42}) \leq 3k$ .

- Theorem 4.2.9** (1)  $\sigma_7(v_6(\mathbb{P}^2))$  is an irreducible component of  $\text{Rank}_{3,3}^7(S^6\mathbb{C}^3) \cap Y\text{Flat}_{42,42}^{22}(S^6\mathbb{C}^3)$ , i.e., of the variety defined by the size 8 minors of the symmetric flattening  $\phi_{3,3}$  and by the size 22 minors of the  $[(42), (42)]$ -Young flattenings.  
 (2)  $\sigma_8(v_6(\mathbb{P}^2))$  is an irreducible component of  $\text{Rank}_{3,3}^8(S^6\mathbb{C}^3) \cap Y\text{Flat}_{42,42}^{24}(S^6\mathbb{C}^3)$ , i.e., of the variety defined by the size 9 minors of the symmetric flattening  $\phi_{3,3}$  and by the size 25 minors of the  $[(42), (42)]$ -Young flattenings.  
 (3)  $\sigma_9(v_6(\mathbb{P}^2))$  is the hypersurface of degree 10 defined by  $\det(\phi_{3,3})$ .

*Proof* To prove (2), we picked a polynomial  $\phi$  which is the sum of 8 random fifth powers of linear forms and a submatrix of  $\phi_{42,42}$  of order 24 which is invertible. The matrix representing  $\phi_{42,42}$  can be constructed explicitly by the package PieriMaps of Macaulay2 [22, 45].

The affine tangent space at  $\phi$  of  $\text{Rank}_{3,3}^8(S^6\mathbb{C}^3)$  has codimension 3 in  $S^6\mathbb{C}^3$ . In order to compute a tangent space, differentiate, as usual, each line of the matrix, substitute  $\phi$  in the other lines, computes the determinant and then sums over the lines. It is enough to pick one minor of order 25 of  $\phi_{42,42}$  containing the invertible ones of order 24. The tangent space of this minor at  $\phi$  is not contained in the subspace of codimension 3, yielding the desired subspace of codimension 4.

(1) can be proved in the same way. (3) is well known.  $\square$

*Remark 4.2.10*  $\sigma_7(v_6(\mathbb{P}^2))$  is the first example where the known equations are not of minimal possible degree.

*Case  $d = 7$ :* In the case of  $\sigma_k(v_7(\mathbb{P}^2))$ , the symmetric flattening given by the size  $(k + 1)$  minors of  $\phi_{3,4}$  define  $\sigma_k(v_7(\mathbb{P}^2))$  as an irreducible component up to  $k = 8$ .

**Theorem 4.2.11** (1) For  $k \leq 10$   $\sigma_k(v_7(\mathbb{P}^2))$  is an irreducible component of  $Y\text{Flat}_{41,41}^k(S^7\mathbb{C}^3)$ , which is defined by the size  $(2k + 2)$  subpfaffians of  $\phi_{41,41}$ .

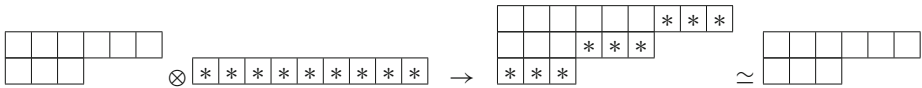
(2)  $\sigma_{11}(v_7(\mathbb{P}^2))$  has codimension 3, and it is contained in the hypersurface  $YFlat_{41,41}^{22}(S^7\mathbb{C}^3)$  of degree 12 defined by  $Pf(\phi_{41,41})$ .

*Proof* (1) follows from Theorem 1.2.3. (2) is obvious. □

*Remark 4.2.12* We emphasize that  $\sigma_{11}(v_7(\mathbb{P}^2))$  is the first case where we do not know, even conjecturally, further equations.

*Case  $d = 8$ :* It is possible that  $\sigma_k(v_8(\mathbb{P}^2))$  is an irreducible component of  $\text{Rank}_{4,4}^{k+1}(S^8\mathbb{C}^3)$  up to  $k = 12$ , but the bound given in Theorem 3.1.4 is just  $k \leq 10$ . The size 14 minors of  $\phi_{4,4}$  define a subvariety of codimension 4, which strictly contains  $\sigma_{13}(v_8(\mathbb{P}^2))$  which has codimension 6. Other equations of degree 40 for  $\sigma_{13}(v_8(\mathbb{P}^2))$  are given by the size 40 minors of  $\phi_{53,52}$ . It is possible that minors of  $\phi_{(53),(52)}$  could be used to get a collection of set-theoretic equations for  $\sigma_{12}(v_8(\mathbb{P}^2))$  and  $\sigma_{11}(v_8(\mathbb{P}^2))$ . In the same way,  $\det \phi_{4,4}$  is just an equation of degree 15 for  $\sigma_{14}(v_8(\mathbb{P}^2))$  which has codimension 3.

*Case  $d = 9$ :* It is possible that  $\sigma_k(v_9(\mathbb{P}^2))$  is an irreducible component of  $\text{Rank}_{4,5}^{k+1}(S^9\mathbb{C}^3)$  up to  $k = 13$ , but the bound given in Theorem 3.1.4 is just  $k \leq 11$ .  $\text{Rank}_{4,5}^{14}(S^9\mathbb{C}^3)$  strictly contains  $\sigma_{14}(v_9(\mathbb{P}^2))$  which has codimension 13. The principal Pfaffians of order 30 of  $\phi_{54,51}$  give further equations for  $\sigma_{14}(v_9(\mathbb{P}^2))$ . In the same way the principal Pfaffians of order 32 (resp. 34) of  $\phi_{54,51}$  give some equations for  $\sigma_{15}(v_9(\mathbb{P}^2))$  (resp.  $\sigma_{16}(v_9(\mathbb{P}^2))$ ). Another equation for  $\sigma_{15}(v_9(\mathbb{P}^2))$  is the Pfaffian of the skew-symmetric morphism  $\phi_{63,63}: S_{6,3}V \rightarrow S_{6,3}V$  which can be pictorially described in terms of Young diagrams, by



We do not know any equations for  $\sigma_k(v_9(\mathbb{P}^2))$  when  $k = 17, 18$ . The  $k = 18$  case is particularly interesting because the corresponding secant variety is a hypersurface.

### 5 Construction of equations from vector bundles

#### 5.1 Main observation

Write  $e := \text{rank}(E)$  and consider the *determinantal varieties* or *rank varieties* defined by the minors of  $A_v^E: H^0(E) \rightarrow H^0(E^* \otimes L)^*$  (defined by equation (3)),

$$\begin{aligned} \text{Rank}_k(E) &:= \mathbb{P}\{v \in V \mid \text{rank}(A_v^E) \leq ek\} \\ &= \sigma_{ek}(\text{Seg}(\mathbb{P}H^0(E)^* \times \mathbb{P}H^0(E^* \otimes L)^*)) \cap \mathbb{P}V. \end{aligned}$$

**Proposition 5.1.1** *Let  $X \subset \mathbb{P}V = \mathbb{P}H^0(L)^*$  be a variety, and  $E$  a rank  $e$  vector bundle on  $X$ . Then*

$$\sigma_r(X) \subseteq \text{Rank}_r(E)$$

*i.e., the size  $(re + 1)$  minors of  $A_v^E$  give equations for  $\sigma_r(X)$ .*

When  $E$  is understood, we will write  $A_v$  for  $A_v^E$ .

*Proof* By (3), if  $x = [v] \in X$ , then  $H^0(I_x \otimes E) \subseteq \ker A_v$ . The subspace  $H^0(I_x \otimes E) \subseteq H^0(E)$  has codimension at most  $e$  in  $H^0(E)$ ; hence, the same is true for the subspace  $\ker A_x$

and it follows  $\text{rank}(A_x) \leq e$ . If  $v \in \hat{\sigma}_r(X)$  is a general point, then it may be expressed as  $v = \sum_{i=1}^r x_i$ , with  $x_i \in \hat{X}$ . Hence

$$\text{rank}(A_v) = \text{rank} \left( \sum_{i=1}^r A_{x_i} \right) \leq \sum_{i=1}^r \text{rank}(A_{x_i}) \leq re$$

as claimed. Since the inequality is a closed condition, it holds for all  $v \in \hat{\sigma}_r(X)$ . □

*Example 5.1.2* Let  $X = v_d(\mathbb{P}W)$ ,  $L = \mathcal{O}(d)$ ,  $E = \mathcal{O}(a)$  so  $E^* \otimes L = \mathcal{O}(d - a)$ , then  $\text{Rank}_r(E)$  is the *catalecticant variety* of symmetric flattenings in  $S^a W \otimes S^{d-a} W$  of rank at most  $r$ .

*Example 5.1.3* Let  $E$  be a homogeneous bundle on  $\mathbb{P}^n$  with  $H^0(E)^* = S_\mu V$ ,  $H^0(E^* \otimes L)^* = S_\pi V$ . Then  $A_v$  corresponds to  $\phi_{\pi, \mu}$  of Sect. 4.

The generalization of symmetric flattenings to Young flattenings for Veronese varieties is a representation-theoretic version of the generalization from line bundles to higher rank vector bundles.

*Example 5.1.4* A general source of examples is given by curves obtained as determinantal loci. This topic is studied in detail in [14]. In [21], A. Ginzsky considers the secant varieties  $\sigma_k(C)$  to smooth curves  $C$  in their bicanonical embedding. With our notations this corresponds to the symmetric pair  $(E, L) = (K_C, K_C^2)$ . He proves [21, Thm. 2.1] that  $\sigma_k(C) = \text{Rank}_k(K_C)$  if  $k < \text{Cliff}(C)$  and  $\sigma_k(C) \subsetneq \text{Rank}_k(K_C)$  for larger  $k$ . Here  $\text{Cliff}(C)$  denotes the Clifford index of  $C$ .

### 5.2 The construction in the symmetric and skew-symmetric cases

We say that  $(E, L)$  is a *symmetric pair* if the isomorphism  $E \xrightarrow{\alpha} E^* \otimes L$  is symmetric, that is the transpose isomorphism  $E \otimes L^* \xrightarrow{\alpha^t} E^*$ , after tensoring by  $L$  and multiplying the map by  $1_L$  equals  $\alpha$ , i.e.,  $\alpha = \alpha^t \otimes 1_L$ . In this case  $S^2 E$  contains  $L$  as a direct summand, the morphism  $A_v$  is symmetric, and  $\text{Rank}_k(E)$  is defined by the  $(ke + 1)$ -minors of  $A_v$ .

Similarly,  $(E, L)$  is a *skew-symmetric pair* if  $\alpha = -\alpha^t \otimes 1_L$ . In this case  $e$  is even,  $\wedge^2 E$  contains  $L$  as a direct summand, the morphism  $A_v$  is skew-symmetric, and  $\text{Rank}_k(E)$  is defined by the size  $(ke + 2)$  subpfaffians of  $A_v$ , which are equations of degree  $\frac{ke}{2} + 1$ .

*Example 5.2.1* Let  $X = \mathbb{P}^2 \times \mathbb{P}^n$  embedded by  $L = \mathcal{O}(1, 2)$ . The equations for  $\sigma_k(X)$  recently considered in [8], where they have been called *exterior flattenings*, fit in this setting. Call  $p_1, p_2$  the two projections. Let  $Q$  be the tautological quotient bundle on  $\mathbb{P}^2$  and let  $E = p_1^* Q \otimes p_2^* \mathcal{O}(1)$ , then  $(E, L)$  is a skew-symmetric pair which gives rise to the Eq. (2) in Theorem 1.1 of [8], while the Eq. (1) are obtained with  $E = p_2^* \mathcal{O}(1)$ .

### 5.3 The conormal space

The results reviewed in Sect. 2.5, restated in the language of vector bundles, say the affine conormal space of  $\text{Rank}_k(E)$  at  $[v] \in \text{Rank}_k(E)_{\text{smooth}}$  is the image of the map

$$\ker A_v \otimes \text{Im } A_v^\perp \rightarrow H^0(L) = V^*.$$

If  $(E, L)$  is a symmetric (resp. skew-symmetric) pair, there is a symmetric (resp. skew-symmetric) isomorphism  $\ker A_v \simeq \text{Im } A_v^\perp$  and the conormal space of  $\text{Rank}_k(E)$  at  $v$  is given by the image of the map  $S^2(\ker A_v) \rightarrow H^0(L)$ , (resp.  $\wedge^2(\ker A_v) \rightarrow H^0(L)$ ).

5.4 A sufficient criterion for  $\sigma_k(X)$  to be an irreducible component of  $\text{Rank}_k(E)$

Let  $v \in \hat{X}$ , in the proof of Proposition 5.1.1, we saw  $H^0(I_v \otimes E) \subseteq \ker A_v$ . In the same way,  $H^0(I_v \otimes E^* \otimes L) \subseteq \text{Im } A_v^\perp$ , by taking transpose. Equality holds if  $E$  is spanned at  $x = [v]$ . This is generalized by the following Proposition.

**Proposition 5.4.1** *Let  $v = \sum_{i=1}^k x_i \in V$ , with  $[x_i] \in X$ , and let  $Z = \{[x_1], \dots, [x_k]\}$ . Then*

$$\begin{aligned} H^0(I_Z \otimes E) &\subseteq \ker A_v \\ H^0(I_Z \otimes E^* \otimes L) &\subseteq \text{Im } A_v^\perp. \end{aligned}$$

*The first inclusion is an equality if  $H^0(E^* \otimes L) \rightarrow H^0(E^* \otimes L|_Z)$  is surjective. The second inclusion is an equality if  $H^0(E) \rightarrow H^0(E|_Z)$  is surjective.*

*Remark 5.4.2* In [3], a line bundle  $F \rightarrow X$  is defined to be  $k$ -spanned if  $H^0(F)$  surjects onto  $H^0(F|_Z)$  for all  $Z = \{[x_1], \dots, [x_k]\}$ . In that paper and in subsequent work, they study which line bundles have this property. In particular  $k$ -spanned-ness of  $X \subset \mathbb{P}H^0(F)^*$  implies that for all  $k$ -tuples of points  $([x_1], \dots, [x_k])$  on  $X$ , writing  $Z = \{[x_1], \dots, [x_k]\}$ , then  $\langle Z \rangle = \mathbb{P}^{k-1}$ , as for example occurs with Veronese varieties  $v_d(\mathbb{P}V)$  when  $k \leq d + 1$ .

*Proof* If  $E^* \otimes L$  is spanned at  $x = [w]$  then we claim  $H^0(I_x \otimes E) = \ker A_w$ . To see this, work over an open set where  $E, L$  are trivialisable and take trivialisations. There are  $t_j \in H^0(E^* \otimes L)$  such that in a basis  $e_i$  of the  $\mathbb{C}^e$  which we identify with the fibers of  $E$  on this open subset,  $\langle e_i, t_j(x) \rangle = \delta_{ij}$ . Take  $s \in \ker(A_w)$ . By assumption  $\langle s(x), t_j(x) \rangle = 0$  for every  $j$ , hence, writing  $s = \sum s_i e_i$ ,  $s_j(x) = 0$  for every  $j$ , i.e.,  $s \in H^0(I_x \otimes E)$ .

Since

$$H^0(I_Z \otimes E) = \cap_{i=1}^k H^0(I_{x_i} \otimes E) \subseteq \cap_{i=1}^k \ker A_{x_i} \subseteq \ker A_v,$$

the inclusion for the kernel follows. To see the equality assertion, if  $H^0(E^* \otimes L) \rightarrow H^0(E^* \otimes L|_Z)$  is surjective, for every  $j = 1, \dots, k$  we can choose  $t_{h,j} \in H^0(E^* \otimes L)$ , for  $h = 1, \dots, e$ , such that  $t_{h,j}(x_i) = 0$  for  $i \neq j$ ,  $\forall h$  and  $t_{h,j}$  span the fiber of  $E^* \otimes L$  at  $x_j$ . It follows that if  $s \in \ker(A_v)$  then  $t_{h,j}(x_j) \cdot s(x_j) = 0$  for every  $h, j$  which implies  $s(x_j) = 0$ , that is  $s \in H^0(I_Z \otimes E)$ . The dual statement is similar. □

The following theorem gives a useful criterion to find local equations of secant varieties.

**Theorem 5.4.3** *Let  $v = \sum_{i=1}^r x_i \in V$  and let  $Z = \{[x_1], \dots, [x_r]\}$ , where  $[x_j] \in X$ . If*

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^* \otimes L) \longrightarrow H^0(I_{Z^2} \otimes L)$$

*is surjective, then  $\sigma_r(X)$  is an irreducible component of  $\text{Rank}_r(E)$ .*

*If  $(E, L)$  is a symmetric, resp. skew-symmetric pair, and*

$$\begin{aligned} &S^2(H^0(I_Z \otimes E)) \rightarrow H^0(I_{Z^2} \otimes L) \\ \text{resp. } &\wedge^2(H^0(I_Z \otimes E)) \rightarrow H^0(I_{Z^2} \otimes L) \end{aligned}$$

*is surjective, then  $\sigma_r(X)$  is an irreducible component of  $\text{Rank}_r(E)$ .*

*Proof* Write  $v = x_1 + \dots + x_r$  for a smooth point of  $\hat{\sigma}_r(X)$ . Recall that by Terracini’s Lemma,  $\mathbb{P}\hat{N}_{[v]}^* \sigma_r(X)$  is the space of hyperplanes  $H \in \mathbb{P}V^*$  such that  $H \cap \sigma_r(X)$  is singular at the  $[x_i]$ , i.e.,  $\hat{N}_{[v]}^* \sigma_r(X) = H^0(I_{Z^2} \otimes L)$ . Consider the commutative diagram

$$\begin{array}{ccc}
 H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^* \otimes L) & \longrightarrow & H^0(I_{Z^2} \otimes L) \\
 \downarrow i & & \downarrow i \\
 \ker A_v \otimes \operatorname{Im} A_v^\perp & \longrightarrow & H^0(L)
 \end{array}$$

The surjectivity of the map in the first row implies that the rank of map in the second row is at least  $\dim H^0(I_{Z^2} \otimes L)$ . By Proposition 5.4.1,  $\ker A_v \otimes \operatorname{Im} A_v^\perp \rightarrow H^0(I_{Z^2} \otimes L)$  is surjective, so that the conormal spaces of  $\sigma_k(X)$  and of  $\operatorname{Rank}_k(E)$  coincide at  $v$ , proving the general case. The symmetric and skew-symmetric cases are analogous.  $\square$

## 6 The induction Lemma

### 6.1 Weak defectivity

The notion of weak defectivity, which dates back to Terracini, was applied by Ciliberto and Chiantini in [10] to show many cases of tensors and symmetric tensors admitted a unique decomposition as a sum of rank one tensors. We review here it as it is used to prove the promised induction Lemma 6.2.1.

**Definition 6.1.1** [10, Def. 1.2] A projective variety  $X \subset \mathbb{P}^V$  is *k-weakly defective* if the general hyperplane tangent in  $k$  general points of  $X$  exists and is tangent along a variety of positive dimension.

**Notational warning:** what we call *k-weakly defective* is called  $(k - 1)$ -weakly defective in [10]. We shifted the index in order to uniformize to our notion of *k-defectivity*: by Terracini’s lemma, *k-defective* varieties (i.e., those where  $\dim \sigma_k(X)$  is less than the expected dimension) are also *k-weakly defective*.

The Veronese varieties which are weakly defective have been classified.

**Theorem 6.1.2 (Chiantini-Ciliberto-Mella-Ballico)** [1, 10, 36] *The k-weakly defective varieties  $v_d(\mathbb{P}^n)$  are the triples  $(k, d, n)$ :*

- (i) *the k-defective varieties, namely  $(k, 2, n)$ ,  $k = 2, \dots, \binom{n+2}{2}$ ,  $(5, 4, 2)$ ,  $(9, 4, 3)$ ,  $(14, 4, 4)$ ,  $(7, 3, 4)$ ,*  
*and*
- (ii)  $(9, 6, 2)$ ,  $(8, 4, 3)$ .

Let  $L$  be an ample line bundle on a variety  $X$ . Recall that the *discriminant variety* of a subspace  $V \subseteq H^0(L)$  is given by the elements of  $\mathbb{P}V$  whose zero sets (as sections of  $L$ ) are singular outside the base locus of common zeros of elements of  $V$ . When  $V$  gives an embedding of  $X$ , then the discriminant variety coincides with the dual variety of  $X \subset \mathbb{P}V^*$ .

**Proposition 6.1.3** *Assume that  $X \subset \mathbb{P}V = \mathbb{P}H^0(L)^*$  is not k-weakly defective and that  $\sigma_k(X)$  is a proper subvariety of  $\mathbb{P}V$ . Then for a general  $Z'$  of length  $k' < k$ , the discriminant subvariety in  $\mathbb{P}H^0(I_{Z^2} \otimes L)$ , given by the hyperplane sections having an additional singular point outside  $Z'$ , is not contained in any hyperplane.*

*Proof* If  $X$  is not *k-weakly defective*, then it is not  $k'$ -weakly defective for any  $k' \leq k$ . Hence we may assume  $k' = k - 1$ . Consider the projection  $\pi$  of  $X$  centered at the span of  $k'$  general tangent spaces at  $X$ . The image  $\pi(X)$  is not 1-weakly defective, [10, Prop 3.6] which means that the Gauss map of  $\pi(X)$  is non-degenerate [10, Rem. 3.1 (ii)]. Consider the dual variety

of  $\pi(X)$ , which is contained in the discriminant, see [34, p. 810]. If the dual variety of  $\pi(X)$  was contained in a hyperplane, then  $\pi(X)$  would be a cone (see, e.g., [13, Prop. 1.1]) and thus be 1-weakly defective, which is a contradiction. Hence the linear span of the discriminant variety is the ambient space.  $\square$

6.2 If  $X$  is not  $k$ -weakly defective and the criterion of Sect. 5.4 works for  $k$ , it works for  $k' \leq k$

**Lemma 6.2.1** *Let  $X \subset \mathbb{P}V = \mathbb{P}H^0(L)^*$  be a variety and  $E \rightarrow X$  be a vector bundle on  $X$ . Assume that*

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^* \otimes L) \longrightarrow H^0(I_{Z^2} \otimes L)$$

*is surjective for the general  $Z$  of length  $k$ , that  $X$  is not  $k$ -weakly defective, and that  $\sigma_k(X)$  is a proper subvariety of  $\mathbb{P}V$ .*

*Then*

$$H^0(I_{Z'} \otimes E) \otimes H^0(I_{Z'} \otimes E^* \otimes L) \longrightarrow H^0(I_{Z'^2} \otimes L)$$

*is surjective for general  $Z'$  of length  $k' \leq k$ .*

*The analogous results hold in the symmetric and skew-symmetric cases.*

*Proof* It is enough to prove the case when  $k' = k - 1$ . Let  $s \in H^0(I_{Z'}^2 \otimes L)$ . By the assumption and Proposition 6.1.3 there are sections  $s_1, \dots, s_t$ , with  $t$  at most  $\dim H^0(I_{Z'}^2 \otimes L)$ , with singular points respectively  $p_1, \dots, p_t$  outside  $Z'$ , such that  $s = \sum_{i=1}^t s_i$ . Let  $Z_i = Z' \cup \{p_i\}$  for  $i = 1, \dots, t$ . We may assume that the  $p_i$  are in general linear position. By assumption  $s_i$  is in the image of  $H^0(I_{Z_i} \otimes E) \otimes H^0(I_{Z_i} \otimes E^* \otimes L) \longrightarrow H^0(I_{Z_i^2} \otimes L)$ . So all  $s_i$  come from  $H^0(I_{Z'} \otimes E) \otimes H^0(I_{Z'} \otimes E^* \otimes L)$ . The symmetric and skew-symmetric cases are analogous.  $\square$

*Example 6.2.2 (Examples where downward induction fails)* Theorem 6.1.2 (ii) furnishes cases where the hypotheses of Lemma 6.2.1 are not satisfied. Let  $k = 9$  and  $L = \mathcal{O}_{\mathbb{P}^2}(6)$ . Here 9 general singular points impose independent conditions on sextics, but for  $k' = 8$ , all the sextics singular at eight general points have an additional singular point, namely the ninth point given by the intersection of all cubics through the eight points, so it is not general. Indeed, in this case  $\sigma_9(v_6(\mathbb{P}^2))$  is the catalecticant hypersurface of degree 10 given by the determinant of  $\phi_{3,3} \in S^3\mathbb{C}^3 \otimes S^3\mathbb{C}^3$ , but the  $9 \times 9$  minors of  $\phi_{3,3}$  define a variety of dimension 24 which strictly contains  $\sigma_8(v_6(\mathbb{P}^2))$ , which has dimension 23.

Similarly, the  $9 \times 9$  minors of  $\phi_{2,2} \in S^2\mathbb{C}^4 \otimes S^2\mathbb{C}^4$  define  $\sigma_8(v_4(\mathbb{P}^3))$ , but the  $8 \times 8$  minors of  $\phi_{2,2}$  define a variety of dimension 28 which strictly contains  $\sigma_7(v_4(\mathbb{P}^3))$ , which has dimension 27.

### 7 Proof of Theorem 1.2.3

Recall  $a = \lfloor \frac{n}{2} \rfloor$ ,  $d = 2\delta + 1$ ,  $V = \mathbb{C}^{n+1}$ . By Lemma 6.2.1, it is sufficient to prove the case  $t = \binom{\delta+n}{n}$ . Here  $E = \wedge^{n-a} Q(\delta)$  where  $Q \rightarrow \mathbb{P}V$  is the tautological quotient bundle.

By Theorem 5.4.3 it is sufficient to prove the map

$$H^0(I_Z \otimes \wedge^a Q(\delta)) \otimes H^0(I_Z \otimes \wedge^{n-a} Q(\delta)) \rightarrow H^0(I_Z^2(2\delta + 1)) \tag{15}$$



is surjective. In the case  $n = 2a$ ,  $a$  odd we have to prove that the map

$$\wedge^2 H^0(I_Z \otimes \wedge^a Q(\delta)) \rightarrow H^0(I_Z^2(2\delta + 1)) \tag{16}$$

is surjective. The arguments are similar.

We need the following lemma, whose proof is given below:

**Lemma 7.0.3** *Let  $Z$  be a set of  $\binom{\delta+n}{n}$  points in  $\mathbb{P}^n$  obtained as the intersection of  $\delta + n$  general hyperplanes  $H_1, \dots, H_{\delta+n}$  ( $\delta \geq 1$ ).*

*Let  $h_i \in V^*$  be an equation for  $H_i$ . A basis of the space of polynomials of degree  $2\delta + 1$  which are singular on  $Z$  is given by  $P_{I,J} := (\prod_{i \in I} h_i) \cdot (\prod_{j \in J} h_j)$  where  $I, J \subseteq \{1, \dots, \delta+n\}$  are multi-indices (without repetitions) satisfying  $|I| = \delta + 1$ ,  $|J| = \delta$ , and  $|I \cap J| \leq \delta - 1$  (that is  $J$  cannot be contained in  $I$ ).*

We prove that the map (15) is surjective for  $t = \binom{\delta+n}{n}$  by degeneration to the case when the  $\binom{\delta+n}{n}$  points are the vertices of a configuration given by the union of  $\delta + n$  general hyperplanes given by linear forms  $h_1, \dots, h_{\delta+n} \in V^*$ .

First consider the case  $n$  is even. For any multi-index  $I \subset \{1, \dots, \delta+n\}$  with  $|I| = \delta$  write  $q_I = \prod_{k \in I} h_k \in H^0(\mathcal{O}(\delta))$  and for any multi-index  $H$ , of length  $a + 1$ , let  $s_H$  be the section of  $\wedge^a Q$  represented by the linear subspace of dimension  $a - 1$  given by  $h_H = \{h_i = 0, i \in H\}$ . Indeed the linear subspace is a decomposable element of  $\wedge^a V = H^0(\wedge^a Q)$ . The section  $s_H$  is represented by the Plücker coordinates of  $h_H$ .

The section  $q_I s_H \in H^0(\wedge^a Q(\delta))$  vanishes on the reducible variety consisting of the linear subspace  $h_H$  of dimension  $a - 1$  and the degree  $\delta$  hypersurface  $q_I = 0$ . In particular, if  $I \cap H = \emptyset$ , then  $q_I s_H \in H^0(I_Z \otimes \wedge^a Q(\delta))$ .

Let  $I = I_0 \cup \{u\}$  with  $|I_0| = \delta$ , consider  $|J| = \delta$  such that  $u \notin J$  and  $|I \cap J| \leq \delta - 1$ . It is possible to choose  $K_1, K_2$  such that  $|K_1| = |K_2| = a$  and such that  $K_1 \cap K_2 = K_1 \cap I = K_2 \cap J = \emptyset$ .

We get  $q_{I_0 s_{\{u\} \cup K_1}} \wedge q_{J s_{\{u\} \cup K_2}}$  is the degree  $2\delta + 1$  hypersurface  $q_{I_0} q_J h_u = q_I q_J$ . By Lemma 7.0.3 these hypersurfaces generate  $H^0(I_Z^2(2\delta + 1))$ . Hence  $W_Z$  is surjective and the result is proved in the case  $n$  is even.

When  $n$  is odd, the morphism  $A_\phi$  is represented by a rectangular matrix, and it induces a map  $B: H^0(\wedge^a Q(\delta)) \otimes H^0(\wedge^{n-a} Q(\delta)) \rightarrow H^0(\mathcal{O}(2\delta + 1))$ .

For any multi-index  $H$ , of length  $n - a + 1$ , let  $s_H$  be the section of  $\wedge^a Q$  represented by the linear subspace of dimension  $a - 1$  given by  $h_H = \{h_i = 0, i \in H\}$ . Indeed the linear subspace is a decomposable element of  $\wedge^a V = H^0(\wedge^a Q)$ . The section  $s_H$  is represented by the Plücker coordinates of  $h_H$ .

The section  $q_I s_H \in H^0(\wedge^a Q(\delta))$  vanishes on the reducible variety consisting of the linear subspace  $h_H$  of dimension  $a - 1$  and the degree  $\delta$  hypersurface  $q_I$ . In particular, if  $I \cap H = \emptyset$ , then  $q_I s_H \in H^0(I_Z \otimes \wedge^a Q(\delta))$ .

Let  $I = I_0 \cup \{u\}$  with  $|I_0| = \delta$ , consider  $|J| = \delta$  such that  $u \notin J$  and  $|I \cap J| \leq \delta - 1$ . The modification to the above proof is that it is possible to choose  $K_1, K_2$  such that  $|K_1| = n - a = a + 1$ ,  $|K_2| = a$  and such that  $K_1 \cap K_2 = K_1 \cap I = K_2 \cap J = \emptyset$ .

Then  $q_{I_0 s_{\{u\} \cup K_1}} \in H^0(\wedge^{a+1} Q(\delta))$ ,  $q_{J s_{\{u\} \cup K_2}} \in H^0(\wedge^a Q(\delta))$ , and  $q_{I_0 s_{\{u\} \cup K_1}} \wedge q_{J s_{\{u\} \cup K_2}}$  is the degree  $2\delta + 1$  hypersurface  $q_{I_0} q_J h_u = q_I q_J$ . By Lemma 7.0.3 these hypersurfaces generate  $H^0(I_Z^2(2\delta + 1))$ . Hence, the map (15) is surjective, and the result is proved. □

Proof of Lemma 7.0.3

Let  $Z$  be a set of  $\binom{\delta+n}{n}$  points in  $\mathbb{P}^n = \mathbb{P}V$  obtained as the intersection of  $\delta + n$  general hyperplanes  $H_1, \dots, H_{\delta+n}$  ( $\delta \geq 1$ ), and  $h_i \in V^*$  is an equation for  $H_i$ .

We begin by discussing properties of the hypersurfaces through  $Z$ , which may be of independent interest.

- Proposition 7.0.4** (i) For  $d \geq \delta$ , the products  $\prod_{i \in I} h_i$  for every  $I \subseteq \{1, \dots, n + \delta\}$  such that  $|I| = d$  are independent in  $S^d V^*$ .  
 (ii) For  $d \leq \delta$ , the products  $\prod_{i \in I} h_i$  for every  $I \subseteq \{1, \dots, n + \delta\}$  such that  $|I| = d$  span  $S^d V^*$ .

*Proof* Write  $h_I = \prod_{i \in I} h_i$ . Consider a linear combination  $\sum_I a_I h_I = 0$ . Let  $d \geq \delta$ . If  $d \geq n + \delta$  the statement is vacuous, so assume  $d < n + \delta$ . For each  $I_0$  such that  $|I_0| = d$  there is a point  $P_0 \in V$  such that  $h_i(P_0) = 0$  for  $i \notin I_0$  and  $h_i(P_0) \neq 0$  for  $i \in I_0$ . The equality  $\sum_I a_I h_I(P_0) = 0$  implies  $a_{I_0} = 0$ , proving (i). For  $d = \delta$ , (ii) follows as well by counting dimensions, as  $\dim S^\delta V^* = \binom{\delta+n}{n}$ . For  $d \leq \delta - 1$  pick  $f \in S^d V^*$ . By the case already proved, there exist coefficients  $a_I, b_J$  such that

$$f \prod_{i=1}^{\delta-d} h_i = \sum_I a_I h_I + \sum_J b_J h_J \tag{17}$$

where the first sum is over all  $I$  with  $|I| = \delta$  such that  $\{1, \dots, \delta - d\} \subseteq I$  and the second sum over all  $J$  with  $|J| = \delta$  such that  $\{1, \dots, \delta - d\} \not\subseteq J$ .

For every  $J_0$  such that  $|J_0| = \delta$  and  $\{1, \dots, \delta - d\} \not\subseteq J_0$  there is a (unique) point  $[P_0]$  such that  $h_i(P_0) = 0$  for  $i \notin J_0$  and  $h_i(P_0) \neq 0$  for  $i \in J_0$ . In particular  $\prod_{i=1}^{\delta-d} h_i(P_0) = 0$  and substituting  $P_0$  into (17) shows  $b_{J_0} = 0$ , for each  $J_0$ . Thus

$$f \prod_{i=1}^{\delta-d} h_i = \sum_I a_I h_I$$

and dividing both sides by  $\prod_{i=1}^{\delta-d} h_i$  proves (ii). □

Recall that the subspace of  $S^d V^*$  of polynomials which pass through  $p$  points and are singular through  $q$  points has codimension  $\leq \min\left(\binom{n+d}{n}, p + q(n + 1)\right)$ . When equality holds, one says that the conditions imposed by the points are independent and that the subspace has the expected codimension.

Let  $I_Z$  denote the ideal sheaf of  $Z$  and  $I_Z(d) = I_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d)$ . These sheaves have cohomology groups  $H^p(I_Z(d))$ , where in particular  $H^0(I_Z(d)) = \{P \in S^d V^* \mid Z \subseteq \text{Zeros}(P)\}$ .

**Proposition 7.0.5** Let  $Z$  be a set of  $\binom{n+\delta}{n}$  points in  $\mathbb{P}^n$  obtained as above. Then

- (i)  $H^0(I_Z(d)) = 0$  if  $d \leq \delta$ .
- (ii)  $H^0(I_Z(\delta + 1))$  has dimension  $\binom{\delta+n}{\delta+1}$  and it is generated by the products  $h_I$  with  $|I| = \delta + 1$ .
- (iii) If  $d \geq \delta + 1$  then  $H^0(I_Z(d))$  is generated in degree  $\delta + 1$ , that is the natural morphism  $H^0(I_Z(\delta + 1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(d - \delta - 1)) \rightarrow H^0(I_Z(d))$  is surjective.

*Proof* Consider the exact sequence of sheaves

$$0 \rightarrow I_Z(d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathcal{O}_Z(d) \rightarrow 0.$$

Since  $Z$  is finite,  $\dim H^0(\mathcal{O}_Z(d)) = \binom{\delta+n}{n} = \text{deg } Z$  for every  $d$ .

The space  $H^0(\mathcal{O}_{\mathbb{P}^n}(\delta))$  has dimension  $\binom{\delta+n}{n}$  and by Proposition 7.0.4, for every  $I$  with  $|I| = \delta$  the element  $h_I$  vanishes on all the points of  $Z$  with just one exception, given by  $\bigcap_{j \notin I} H_j$ . Hence the restriction map  $H^0(\mathcal{O}_{\mathbb{P}^n}(\delta)) \rightarrow H^0(\mathcal{O}_Z(\delta))$  is an isomorphism. It follows that  $H^0(I_Z(\delta)) = H^1(I_Z(\delta)) = 0$ , and thus  $H^0(I_Z(d)) = 0$  for  $d < \delta$  proving (i).

Since  $\dim Z = 0$ ,  $H^i(\mathcal{O}_Z(k)) = 0$  for  $i \geq 1$ , and all  $k$ . From this it follows that  $H^i(I_Z(\delta - i + 1)) = 0$  for  $i \geq 2$ , because  $H^i(\mathcal{O}_{\mathbb{P}^n}(\delta - i + 1)) = 0$ . The vanishing for  $i = 1$  was proved above, so that  $I_Z$  is  $(\delta + 1)$ -regular and by the Castelnuovo-Mumford criterion [37, Chap. 14]  $I_Z(\delta + 1)$  is globally generated,  $H^1(I_Z(k)) = 0$  for  $k \geq \delta + 1$ , and part (iii) follows.

In order to prove (ii), consider the products  $h_I$  with  $|I| = \delta + 1$ , which are independent by Proposition 7.0.4.i, so they span a  $\binom{\delta+n}{\delta+1}$ -dimensional subspace of  $H^0(I_Z(\delta + 1))$ .

The long exact sequence in cohomology implies

$$\begin{aligned} h^0(I_Z(\delta + 1)) &= h^0(\mathcal{O}_{\mathbb{P}^n}(\delta + 1)) - h^0(\mathcal{O}_Z(\delta + 1)) + h^1(I_Z(\delta + 1)) \\ &= \binom{\delta + 1 + n}{n} - \binom{\delta + n}{n} + 0 = \binom{\delta + n}{\delta + 1} \end{aligned}$$

which concludes the proof. □

**Proposition 7.0.6** *Notations as above. Let  $L_0 \subset \{1, \dots, n + \delta\}$  have cardinality  $n$ .*

*The space of polynomials of degree  $\delta + 1$  which pass through the points  $y_L$  and are singular at  $y_{L_0}$  has a basis given by the products  $\prod_{i \in J} h_i$  for every  $J \subseteq \{1, \dots, n + \delta\}$  such that  $|J| = \delta + 1$  and  $\#(J \cap L_0) \geq 2$ . This space has the expected codimension  $\binom{\delta+n}{n} + n$ .*

*Proof* Note that if  $|J| = \delta + 1$  then  $\#(J \cap L_0) \geq 1$  and equality holds just for the  $n$  products  $h_i h_{L_0^c}$  for  $i \in L_0$ , where  $L_0^c = \{1, \dots, n + \delta\} \setminus L_0$ . Every linear combination of the  $h_J$  which has a non-zero coefficient in these  $n$  products is non-singular at  $[y_{L_0}]$ .

Hence the products which are different from these  $n$  generate the space of polynomials of degree  $\delta + 1$  which pass through the points  $[y_L]$  and are singular at  $[y_{L_0}]$ .

There are  $\binom{\delta+n}{n-1} - n$  such polynomials, which is the expected number  $\binom{\delta+n+1}{n} - (n + 1) - \left[ \binom{\delta+n}{n} - 1 \right]$ . Since the codimension is always at most the expected one, it follows that these generators give a basis. □

In the remainder of this section, we use products  $h_I$  where  $I$  is a multi-index where repetitions are allowed. Given such a multi-index  $I = \{i_1, \dots, i_{|I|}\}$ , we write  $h_I = h_1^{k_1} \dots h_{n+\delta}^{k_{n+\delta}}$ , where  $j$  appears  $k_j$  times in  $I$  and we write  $|I| = k_1 + \dots + k_{n+\delta}$ . The support  $s(I)$  of a multi-index  $I$  is the set of  $j \in \{1, \dots, n + \delta\}$  with  $k_j > 0$ .

An immediate consequence of (ii) and (iii) of Proposition 7.0.5 is:

**Corollary 7.0.7** *For  $d \geq \delta + 1$ , the vector space  $H^0(I_Z(d))$  is generated by the monomials  $h_I$  with  $|I| = d$  and  $|s(I)| \geq \delta + 1$ .*

**Proposition 7.0.8** *The space of polynomials of degree  $2\delta + 1$  which contain  $Z$  is generated by products  $h_I$  with  $|I| = 2\delta + 1$ ,  $|s(I)| \geq \delta + 1$ , and all the exponents in  $h_I$  are at most 2.*

*Proof* By Corollary 7.0.7, it just remains to prove the statement about the exponents. Let  $h_I = \prod_{i=1}^{\delta+n} h_i^{k_i}$  and let  $n_j(I) = \#\{i | k_i = j\}$  for  $j = 0, \dots, 2\delta + 1$ . Hence  $\sum_{j \geq 1} j n_j(I) = 2\delta + 1$  and  $\sum_{j \geq 1} n_j(I) \geq \delta + 1$ .

Assume that  $\gamma := \sum_{j \geq 3} n_j(I) > 0$ . It is enough to show that  $h_I = \sum c_J h_J$  where every  $J$  which appears in the sum satisfies  $|s(J)| \geq \delta + 1$ ,  $|J| = 2\delta + 1$  and  $\sum_{j \geq 3} n_j(J) < \gamma$ .

Indeed

$$\begin{aligned} 2\delta + 1 &= n_1(I) + 2n_2(I) + 3n_3(I) + \dots \geq n_1(I) + 2n_2(I) + 3\gamma \\ &\geq (\delta + 1 - n_2(I) - \gamma) + 2n_2(I) + 3\gamma \end{aligned}$$

that is,  $n_2(I) + \gamma \leq \delta - \gamma \leq \delta - 1$ . Hence there at least  $n + 1$  forms  $h_i$  which appear with exponent at most one in  $h_I$ . We can express all the remaining forms  $h_s$  as linear combinations of these  $n + 1$  forms. By expressing a form with exponent at least 3 as linear combination of these  $n + 1$  linear forms  $h_i$ , we get  $h_I = \sum c_j h_j$  where each summand has the required properties.  $\square$

To better understand the numbers in the following proposition, recall that  $\sum_{i=1}^{\delta+1} \binom{\delta+n-i}{n-1} = \binom{\delta+n}{n}$  as taking  $E = \mathbb{C}^n$ ,  $F = \mathbb{C}^1$ , one has  $S^n(E + F) = S^n E \oplus S^{n-1} E \otimes F \oplus S^{n-2} E \otimes S^2 F \oplus \dots \oplus S^n F$ .

**Proposition 7.0.9** *For  $k = 0, \dots, \delta$ , let  $I_{\delta,k,n}$  be the linear system of hypersurfaces of degree  $2\delta + 1 - k$  which contain  $Z$  and are singular on the points of  $Z$  which lie outside  $\cup_{i=1}^k H_i$ . The space  $I_{\delta,k,n}$  has the expected codimension  $(n + 1)\binom{\delta+n}{n} - n \sum_{i=1}^k \binom{\delta+n-i}{n-1}$ .*

*Proof* For  $k = \delta$  the assertion is Proposition 7.0.6 with  $L_0 = \{\delta + 1, \dots, \delta + n\}$ . We work by induction on  $n$  and by descending induction on  $k$  (for fixed  $\delta$ ). We restrict to the last hyperplane  $H_{\delta+n}$ .

Let  $V_{\delta,k+1,n} \supseteq I_{\delta,k+1,n}$  denote the set of hypersurfaces of degree  $2\delta - k$  which pass through  $Z \setminus \{H_{\delta+n} \cap (\cup_{i=1}^k H_i)\}$  and are singular on the points of  $Z$  which lie outside  $(\cup_{i=1}^k H_i) \cup \{H_{\delta+n}\}$ . We have the exact sequence

$$0 \longrightarrow V_{\delta,k+1,n} \xrightarrow{\psi} I_{\delta,k,n} \xrightarrow{\phi} I_{\delta,k,n-1}$$

where  $\phi$  is the restriction to  $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ , and  $\psi$  is the multiplication by  $h_{\delta+n}$ .

Since by induction  $I_{\delta,k+1,n}$  has the expected codimension in  $S^{2\delta-k} V^*$  it follows that also  $V_{\delta,k+1,n}$  has the expected codimension

$$n \left[ \binom{\delta + n - 1}{n} - \sum_{i=1}^k \binom{\delta + n - 1 - i}{n - 1} \right] + \binom{\delta + n}{n} - \sum_{i=1}^k \binom{\delta + n - 1 - i}{n - 2}$$

because the conditions imposed by  $V_{\delta,k+1,n}$  are a subset of the conditions imposed by  $I_{\delta,k+1,n}$  and a subset of a set of independent conditions still consists of independent conditions.

Since the codimension of  $I_{\delta,k,n-1}$  in  $S^{2\delta+1-k} \mathbb{C}^n$  is the expected one, it follows that the codimension of  $I_{\delta,k,n}$  in  $S^{2\delta+1-k} V^*$  is at least the expected one, hence equality holds.  $\square$

The following Corollary proves Lemma 7.0.3.

**Corollary 7.0.10** *Let  $Z$  be a set of  $\binom{\delta+n}{n}$  points in  $\mathbb{P}^n = \mathbb{P}V$  obtained as the intersection of  $\delta + n$  general hyperplanes  $H_1, \dots, H_{\delta+n}$  ( $\delta \geq 1$ ), and let  $h_i \in V^*$  be an equation of  $H_i$ . The linear system of hypersurfaces of degree  $2\delta + 1$  which are singular on  $Z$  has the expected codimension  $(n + 1)\binom{\delta+n}{n}$  and it is generated by the products  $h_I = \prod_{i \in I} h_i$  with  $I$  a multi-index allowing repetitions such that  $|I| = 2\delta + 1$  and the support of  $I$  is at least  $\delta + 2$  and all the exponents are at most two.*

*Proof* The statement about the codimension is the case  $k = 0$  of Proposition 7.0.9. Note that every product  $h_I$  with  $I$  such that  $|I| = 2\delta + 1$  and  $s(I) \geq \delta + 2$  is singular at any  $P \in Z$

because  $h_I$  contains at least two factors which vanish at  $P$ . Let  $f$  be a homogeneous polynomial of degree  $2\delta + 1$  which is singular on  $Z$ . By Proposition 7.0.8 we have the decomposition  $f = \sum_{|s(I)| \geq \delta+2} a_I h_I + \sum_{|s(I)| = \delta+1} b_I h_I$ , where all the exponents are at most 2. Let  $I_0$  be in the second summation with  $|s(I_0)| = \delta + 1$ . There is a unique  $h_i$  appearing in  $h_{I_0}$  with exponent 1 and  $n - 1$  hyperplanes appearing with exponent 0. Let  $P_0$  be the point where these  $1 + (n - 1) = n$  hyperplanes meet. It follows that  $h_{I_0}$  is non-singular at  $P_0$  while, for all the other summands,  $h_I$  is singular in  $P_0$ . It follows that  $b_{I_0} = 0$  and  $f = \sum_{|s(I)| \geq \delta+2} a_I h_I$  as we wanted.  $\square$

### 8 Construction of $A_v^E$ via a presentation of $E$

#### 8.1 Motivating example

Set  $\dim V = n + 1$ ,  $d = 2\delta + 1$  and  $a = \lfloor \frac{n}{2} \rfloor$ . We saw in Sect. 4 that the inclusion  $S^d V \subset S_{\delta+1, 1^{n-a}} V^* \otimes S_{\delta+1, 1^a} V$  yields new modules of equations for  $\sigma_r(v_d(\mathbb{P}V))$ . The relevant vector bundle here is  $E = E_{\delta\omega_1 + \omega_{n-a+1}} = \wedge^a Q(\delta)$  where  $Q$  is the tautological quotient bundle. The equations were initially presented without reference to representation theory using (2). Note that  $\text{rank}(YF_{d,n}(w^d)) = \binom{n}{a} = \text{rank}(E)$ .

This map between larger, but more elementary, spaces has the same rank properties as the map  $S_{\delta+1, 1^{n-a}} V^* \rightarrow S_{\delta+1, 1^a} V$ , because when one decomposes the spaces in this map as  $SL(V)$ -modules, the other module maps are zero.

Observe that

$$S^\delta V^* \otimes \wedge^a V = H^0(\mathcal{O}(\delta)_{\mathbb{P}V} \otimes \wedge^a V),$$

$$\text{and } (S^\delta V \otimes \wedge^{a+1} V)^* = H^0(\mathcal{O}(\delta)_{\mathbb{P}V} \otimes \wedge^{n-a+1} V).$$

The vector bundle  $E = \wedge^a Q(\delta)$  may be recovered as the image of the map

$$L_1 = \mathcal{O}(\delta) \otimes \wedge^a V \xrightarrow{P_E} \mathcal{O}(\delta + 1) \otimes \wedge^{a+1} V = L_0.$$

#### 8.2 The general case

In the general set-up, in order to factorize  $E$ , one begins with a base line bundle  $M \rightarrow X$ , which is  $\mathcal{O}_{\mathbb{P}W}(1)$  when  $X = v_d(\mathbb{P}W) \subset \mathbb{P}S^d W = \mathbb{P}V$ , and more generally is the ample generator of the Picard group if  $X$  has Picard number one (e.g., rational homogeneous varieties  $G/P$  with  $P$  maximal).

The key is to realize  $E$  as the image of a map

$$p_E : L_1 \rightarrow L_0$$

where the  $L_i$  are direct sums of powers of  $M$  and

$$\text{Image}(p_E) = E.$$

By construction, for  $x \in \hat{X}$ ,  $\text{rank}(p_E(x)) = e$ . Let  $v \in V = H^0(X, L)^*$ . Compose the map  $H^0(L_0) \otimes H^0(L)^* \rightarrow H^0(L_0^* \otimes L)^*$  with the map on sections induced from  $p_E$ , to obtain

$$P_v^E : H^0(L_1) \rightarrow H^0(L_0^* \otimes L)^*$$

and observe the linearity  $P_{sv_1 + tv_2}^E = sP_{v_1}^E + tP_{v_2}^E$  for  $v_1, v_2 \in V, s, t \in \mathbb{C}$ .

### 8.3 Case $X = v_d(\mathbb{P}W)$

In the case  $(X, M) = (\mathbb{P}W, \mathcal{O}(1))$ , and  $L = \mathcal{O}(d)$ , this means

$$L_j = \oplus(\mathcal{O}(i))^{\oplus a_j^i}$$

for some finite set of natural numbers  $a_1^i, a_2^i$ . For the examples in this paper there is just one term in the summand, e.g., in Sect. 8.1,  $L_1 = \mathcal{O}(\delta)^{\oplus \binom{n+1}{a}}$ ,  $L_0 = \mathcal{O}(\delta + 1)^{\oplus \binom{n+1}{a+1}}$ .

When  $X = v_d(\mathbb{P}W)$ ,  $P_v^E$  may be written explicitly in bases as follows:

Let  $L_1 = \oplus_{j=1}^{m_1} \mathcal{O}(b_j)$  and  $L_0 = \oplus_{i=1}^{m_0} \mathcal{O}(a_i)$ . Let  $p_{ij} \in S^{a_i-b_j} W^*$  denote the map  $\mathcal{O}(b_j) \rightarrow \mathcal{O}(a_i)$ , given by the composition

$$\mathcal{O}(b_j) \rightarrow L_1 \xrightarrow{P^E} L_0 \rightarrow \mathcal{O}(a_i)$$

Then, for any  $\phi \in S^d W$ ,  $P_v^E(\phi)$  is obtained by taking a matrix of  $m_1 \times m_0$  blocks with the  $(i, j)$ -th block a matrix representing the catalecticant  $\phi_{b_j, d-a_i} \in S^{b_j} W^* \otimes S^{d-a_i} W^* \otimes S^d W$  contracted by  $p_{ij}$  so the new matrix is of size  $h^0(L_0^* \otimes L) \times h^0(L_1)$  with scalar entries. See Examples 1.2.1 and 8.4.4 for explicit examples.

### 8.4 Rank and irreducible component theorems in the presentation setting

Under mild assumptions, the following proposition shows that  $P_v^E$  can be used in place of  $A_v^E$  in the theorems of Sect. 5.

**Proposition 8.4.1** *Notations as above. Assume that  $H^0(L_1) \xrightarrow{i} H^0(E)$  and  $H^0(L_0^* \otimes L) \xrightarrow{j} H^0(E^* \otimes L)$  are surjective.*

*Then the rank of  $A_v^E$  equals the rank of  $P_v^E$ , so that the size  $(k + 1)$ -minors of  $P_v^E$  give equations for  $\sigma_k(X)$ .*

*If  $(E, L)$  is a symmetric (resp. skew-symmetric) pair, then there exists a symmetric (resp. skew-symmetric) presentation where  $L_1 \simeq L_0^* \otimes L$  and  $P_v^E$  is symmetric (resp. skew-symmetric).*

*Proof* Consider the commutative diagram

$$\begin{CD} H^0(L_1) @>P_v>> H^0(L_0^* \otimes L)^* \\ @VViV @VVj^tV \\ H^0(E) @>A_v>> H^0(E^* \otimes L)^* \end{CD}$$

The assumptions  $i$  is surjective and  $j^t$  is injective imply  $\text{rank}(A_v) = \text{rank}(P_v)$ . The symmetric and skew assertions are clear. □

With the assumptions of Proposition 8.4.1, the natural map

$$H^0(L_1) \otimes H^0(L_0^*) \rightarrow H^0(E) \otimes H^0(E^* \otimes L)$$

is surjective and the affine conormal space at  $v$  is the image of  $\ker P_v \otimes (\text{Im } P_v)^\perp \rightarrow H^0(L)$ .

The following variant of Proposition 5.1.1 is often easy to implement in practice.

**Theorem 8.4.2** *Notations as above. Let  $v \in \sigma_k(X)$ . Assume the maps  $H^0(L_1) \rightarrow H^0(E)$  and  $H^0(L_0^* \otimes L) \rightarrow H^0(E^* \otimes L)$  are surjective.*

- (i) *If the rank of  $\ker P_v \otimes (\text{Im } P_v)^\perp \rightarrow H^0(L)$  at  $v$  equals the codimension of  $\sigma_k(X)$ , then  $\sigma_k(X)$  is an irreducible component of  $\text{Rank}_k(E)$  passing through  $v$ .*

- (ii) If  $(E, L)$  is a symmetric (resp. skew-symmetric) pair, assume that the rank of  $S^2(\ker P_v) \rightarrow H^0(L)$  (resp. of  $\wedge^2(\ker P_v) \rightarrow H^0(L)$ ) at  $v$  coincides with the codimension of  $\sigma_k(X)$ .  
Then  $\sigma_k(X)$  is an irreducible component of  $\text{Rank}_k(E)$  passing through  $v$ .

Note that in the skew-symmetric case,  $\text{Rank}_k(E)$  is defined by sub-Pfaffians.

*Remark 8.4.3* In Theorem 8.4.2, the rank of the maps is always bounded above by the codimension of  $\sigma_k(X)$ , as the rank of the maps is equal to the dimension of the conormal space of  $\text{Rank}_k(E)$  at  $v$ .

The previous theorem can be implemented in concrete examples. Indeed, the pairing  $g$ , appearing in the theorem, can be described as follows: given  $f \in \ker P_v \subset H^0(L_1)$ ,  $h \in (\text{Im } P_v)^\perp \subset H^0(L_0^* \otimes L)$ , and  $\phi \in H^0(L)^*$ , one has  $g(f, h)(\phi) = h[P_\phi(f)]$ .

*Example 8.4.4* [Cubic 3-folds revisited] Let  $\dim V = 5$ , let  $E = \Omega^2(4) = \wedge^2 T^* \mathbb{P}V(4) = E_{\omega_2}$  on  $X = \mathbb{P}^4$ , the pair  $(E, \mathcal{O}(3))$  is a symmetric pair presented by

$$L_1 = \mathcal{O}(1) \otimes \wedge^3 V \xrightarrow{P} \mathcal{O}(2) \otimes \wedge^2 V = L_0.$$

Choose a basis  $x_0, \dots, x_4$  of  $V$ , so  $p$  is represented by the  $10 \times 10$  symmetric matrix called  $K_4$  in the introduction. Let  $L = \mathcal{O}(3)$ , for any  $\phi \in S^3 V^*$  the map  $A_\phi^E$  is the skew-symmetric morphism from  $H^0(\Omega^2(4))$  to its dual  $H^0(\Omega^3(5))^*$  where both have dimension 45 and  $P_\phi$  is represented by the  $50 \times 50$  block matrix where the entries  $\pm x_i$  in  $K_4$  are replaced with  $\pm(\frac{\partial \phi}{\partial x_j})_{1,1}$ , which are  $5 \times 5$  symmetric catalecticant matrices of the quadric  $\frac{\partial \phi}{\partial x_j}$ . In [40] it is shown that a matrix representing  $A_\phi^E$  is obtained from the matrix representing  $P_\phi$  by deleting five suitably chosen rows and columns. Here  $\det(A_\phi)$  is the cube of the degree 15 equation of  $\sigma_7(v_3(\mathbb{P}^4))$ .

Note that when  $\phi = x_0^3$  is a cube, then the rank of  $A_\phi^E$  and the rank of  $P_\phi$  are both equal to 6, which is the rank of  $\Omega^2(3)$ .

### 9 Decomposition of polynomials into a sum of powers

Having a presentation for  $E$  enables one to reduce the problem of decomposing a polynomial into a sum of powers to a problem of solving a system of polynomial equations (sometimes linear) as we explain in this section. We begin with a classical example:

#### 9.1 Catalecticant

Let  $\phi \in S^d W$  be a general element of  $\hat{\sigma}_k(v_d(\mathbb{P}^n))$  so it can be written as  $\phi = \sum_{i=1}^k l_i^d$ . Let  $Z = \{l_1, \dots, l_k\}$ , let  $\delta = \lfloor \frac{d}{2} \rfloor$ , and let  $k \leq \binom{\delta+n-1}{n}$ . Take  $E = \mathcal{O}(\delta)$  so  $A_\phi^E: H^0(\mathcal{O}(\delta)) \rightarrow H^0(\mathcal{O}(d-\delta))^*$ . Then  $\ker A_\phi = H^0(I_Z(\delta))$  and  $\ker A_\phi \cap v_\delta(\mathbb{P}W) = \{l_1^\delta, \dots, l_k^\delta\}$ . Thus if one can compute the intersection, one can recover  $Z$  and the decomposition. For a further discussion of these ideas see [38].

#### 9.2 $YF_{d,n}$

For  $\phi \in S^d W$ , let  $\delta = \lfloor \frac{d-1}{2} \rfloor$ ,  $a = \lfloor \frac{n}{2} \rfloor$ ,  $k \leq \binom{\delta+n}{n}$  and take  $E = \wedge^a Q(\delta)$ , so  $A_\phi: H^0(\wedge^a Q(\delta)) \rightarrow H^0(\wedge^{n-a} Q(d-\delta))$ , then  $\ker A_\phi = H^0(I_Z \otimes \wedge^a Q(\delta))$  so that  $Z$  can be recovered as the base locus of  $\ker A_\phi$ .

### 9.3 General quintics in $S^5\mathbb{C}^3$

A general element  $\phi \in S^5\mathbb{C}^3$  is a unique sum of seven fifth powers (realized by Hilbert, Richmond and Palatini [24,42,44]), write  $\phi = x_1^5 + \dots + x_7^5$ . Here is an algorithm to compute the seven summands:

Let  $E = Q(2)$  and consider the skew-symmetric morphism  $A_\phi^E: H^0(Q(2)) \rightarrow H^0(Q(2))^*$ . (Note that  $H^0(Q(2)) = S_{3,2}V$  as an  $SL_3$ -module and that has dimension 15.) For a general  $\phi$  the kernel of  $A_\phi$  has dimension one and we let  $s \in H^0(Q(2))$  denote a section spanning it. By Proposition 5.4.1, the seven points where  $s$  vanishes correspond to the seven summands. Explicitly,  $s$  corresponds to a morphism  $f: S^2W \rightarrow W$  such that the set  $\{v \in W | f(v^2) \in \langle v \rangle\}$  consists of the seven points  $P_1, \dots, P_7$ .

One way to describe the seven points is to consider the map

$$F_\phi : \mathbb{P}W \rightarrow \mathbb{P}(\mathbb{C}^2 \otimes W)$$

$$[v] \mapsto [v \otimes e_1 + f(v^2) \otimes e_2]$$

where  $\mathbb{C}^2$  has basis  $e_1, e_2$ . Then, the seven points are the intersection  $F_\phi(\mathbb{P}W) \cap \text{Seg}(\mathbb{P}^1 \times \mathbb{P}W)$ , which is the set of two by two minors of a matrix with linear and quadratic entries. Giving  $W^*$  the basis  $x_0, x_1, x_2$ , the matrix is of the form

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ q_0 & q_1 & q_2 \end{bmatrix}$$

where the  $q_j$  are of degree two. In practice this system is easily solved with, e.g., Maple.

## 10 Grassmannians

### 10.1 Skew-flattenings

Some natural equations for  $\sigma_r(G(k, W)) \subset \mathbb{P} \wedge^k W$  are obtained from the inclusions  $\wedge^k W \subset \wedge^a W \otimes \wedge^{k-a} W$  which we will call a *skew-flattening*. Let  $W = \mathbb{C}^{n+1}$ , let  $\omega_i, 1 \leq i \leq n$  denote the fundamental weights of  $\mathfrak{sl}(W)$ , let  $E_\lambda$  denote the irreducible vector bundle induced from the  $P$ -module of highest weight  $\lambda$ . The skew-flattenings correspond to the bundle  $E_{\omega_a} \rightarrow G(k, W)$ . Note that  $\text{rank}(E_{\omega_a}) = \binom{k}{a}$ , as  $E_{\omega_a}$  is isomorphic to the  $a$ -th exterior power of the dual of the universal sub-bundle of rank  $k$ . The map  $v_{a,k-a} = A_v^E: \wedge^a W^* \rightarrow \wedge^{k-a} W$  thus has rank  $\binom{k}{a}$  (this is also easy to see directly). Since  $\dim \wedge^a W = \binom{n+1}{k}$ , one obtains equations possibly up to the  $\binom{n+1}{k} / \binom{k}{a}$ -th secant variety.

The case  $k = 2$  is well known, and  $I(\sigma_r(G(2, W)))$  is generated in degree  $r + 1$  by sub-Pfaffians of size  $2r + 2$ .

Let  $E \in G(k, W) = \mathbb{G}(\mathbb{P}^{k-1}, \mathbb{P}W)$ . We slightly abuse notation and also write  $E \subset W$  for the corresponding linear subspace. Then

$$\hat{N}_{E^*}^* G(k, W) = \wedge^2(E^\perp) \wedge (\wedge^{k-2} W^*) \subset \wedge^k W^*. \tag{18}$$

Now let  $[E_1 + E_2] \in \sigma_2(G(k, W))$ . By Terracini's lemma  $\hat{N}_{[E_1+E_2]}^* G(k, W) = \hat{N}_{E_1}^* G(k, W) \cap \hat{N}_{E_2}^* G(k, W)$ . Let  $U_{12} = E_1^\perp \cap E_2^\perp$ , and  $U_j = E_j^\perp$ . We have

$$\hat{N}_{[E_1+E_2]}^* G(k, W) = (\wedge^2 U_{12}) \wedge (\wedge^{k-2} W^*) + U_{12} \wedge U_1 \wedge U_2 \wedge (\wedge^{k-3} W^*)$$

$$+ (\wedge^2 U_1) \wedge (\wedge^2 U_2) \wedge (\wedge^{k-4} W^*). \tag{19}$$



To see this, note that each term must contain a  $\wedge^2(E_1^\perp)$  and a  $\wedge^2(E_2^\perp)$ . The notation is designed so that  $\wedge^2(E_j^\perp)$  will appear if the  $j$  index occurs at least twice in the  $U$ 's. Similarly below, one takes all combinations of  $U$ 's such that each of 1, 2, 3 appears twice in the expression.

With similar notation,

$$\begin{aligned}
 N_{[E_1+E_2+E_3]}^*G(k, W) = & (\wedge^2U_{123}) \wedge (\wedge^{k-2}W^*) \\
 & + [U_{123} \wedge U_{12} \wedge U_3 + U_{123} \wedge U_{13} \wedge U_2 \\
 & + U_{123} \wedge U_{23} \wedge U_1] \wedge (\wedge^{k-3}W^*) \\
 & + U_{12} \wedge U_{13} \wedge U_{23} \wedge (\wedge^{k-3}W^*) \\
 & + U_{123} \wedge U_1 \wedge U_2 \wedge U_3 \wedge (\wedge^{k-4}W^*) \\
 & + [(\wedge^2U_{12}) \wedge (\wedge^2U_3) + (\wedge^2U_{13}) \wedge (\wedge^2U_2) \\
 & + (\wedge^2U_{23}) \wedge (\wedge^2U_1)] \wedge (\wedge^{k-4}W^*) \\
 & + (\wedge^2U_1) \wedge (\wedge^2U_2) \wedge (\wedge^2U_3) \wedge (\wedge^{k-6}W^*). \tag{20}
 \end{aligned}$$

We emphasize that with four or more subspaces, analogous formulas are much more difficult, because four subspaces have moduli, see [19].

We now determine to what extent the zero sets of the skew-flattenings provide local equations for secant varieties of Grassmannians.

The first skew-flattening  $\wedge^k W \subset W \otimes \wedge^{k-1} W$  gives rise to the *subspace varieties*: let

$$\text{Sub}_p(\wedge^k W) = \{[z] \in \mathbb{P}(\wedge^k W) \mid \exists W' \subset W, \dim W' = p, z \in \wedge^k W'\}$$

it admits a description via a Kempf-Weyman ‘‘collapsing’’ of the vector bundle  $\wedge^k S_p \rightarrow G(p, W)$ , where  $S_p$  is the tautological rank  $p$  subspace bundle, i.e., the variety is the projectivization of the image of the total space of  $\wedge^k S_p$  in  $\wedge^k W$ . From this description one sees that  $\text{Sub}_p(\wedge^k W)$  is of dimension  $p(n + 1 - p) + \binom{p}{k}$  and its affine conormal space at a smooth point  $z \in \wedge^k W' \subset \wedge^k W$  is

$$\hat{N}_z^* \text{Sub}_p(\wedge^k W) = \wedge^2(W'^\perp) \otimes \wedge^{k-2} W^*$$

As a set,  $\text{Sub}_p(\wedge^k W)$  is the zero locus of  $\wedge^{p+1} W^* \otimes \wedge^{p+1} (\wedge^{k-1} W^*)$ , the size  $(p + 1)$ -minors of the skew-flattening. The ideal of  $\text{Sub}_p(\wedge^k W)$  is simply the set of all modules  $S_\pi W^* \subset \text{Sym}(W^*)$  such that  $\ell(\pi) > p$ . Its generators are known only in very few cases, see [49, §7.3].

Question

In what cases do the minors of the skew-flattenings generate the ideal of  $\text{Sub}_p(\wedge^k W)$ ?

The most relevant cases for the study of secant varieties of Grassmannians are when  $p = rk$ , as  $\sigma_r(G(k, W)) \subset \text{Sub}_{rk}(\wedge^k W)$ . Note that in general  $\sigma_r(G(k, W))$  is a much smaller subvariety than  $\text{Sub}_{rk}(\wedge^k W)$ .

Now we compute  $\ker \otimes \text{Image}^\perp$ :

For  $E \in G(k, W)$ , write  $E_{p,k-p} : \wedge^p W^* \rightarrow \wedge^{k-p} W$  for the flattening. Then

$$\ker(E_{2,k-2}) = \wedge^2(E^\perp) \tag{21}$$

$$\text{Image}(E_{2,k-2})^\perp = (\wedge^{k-2} E)^\perp = E^\perp \wedge (\wedge^{k-3} W^*) \tag{22}$$

and similarly

$$\ker([E_1 + E_2]_{2,k-2}) = U_{12} \wedge W^* + U_1 \wedge U_2 \tag{23}$$

$$\text{Image}([E_1 + E_2]_{2,k-2})^\perp = U_{12} \wedge (\wedge^{k-3} W^*) + U_1 \wedge U_2 \wedge (\wedge^{k-4} W^*) \tag{24}$$

And thus

$$\begin{aligned} &\ker([E_1 + E_2]_{2,k-2}) \wedge \text{Image}([E_1 + E_2]_{2,k-2})^\perp \\ &= \wedge^2 U_{12} \wedge (\wedge^{k-2} W^*) + U_{12} \wedge U_1 \wedge U_2 \wedge (\wedge^{k-3} W^*) + \wedge^2 U_1 \wedge \wedge^2 U_2 \wedge (\wedge^{k-4} W^*) \end{aligned}$$

which agrees with (19). (Note that if we had tried the  $(1, k - 1)$ -flattening, we would have missed the third term in (19).)

Similarly

$$\ker([E_1 + E_2 + E_3]_{2,k-2}) = U_{123} \wedge W^* + U_{12} \wedge U_3 + U_{13} \wedge U_2 + U_{23} \wedge U_1 \tag{25}$$

$$\begin{aligned} \text{Image}([E_1 + E_2 + E_3]_{2,k-2})^\perp &= U_{123} \wedge (\wedge^{k-3} W^*) \\ &\quad + [U_{12} \wedge U_3 + U_{13} \wedge U_2 + U_{23} \wedge U_1] \wedge (\wedge^{k-4} W^*) \\ &\quad + U_1 \wedge U_2 \wedge U_3 \wedge (\wedge^{k-5} W^*) \end{aligned} \tag{26}$$

and  $\ker \otimes \text{Image}^\perp$  does not contain last term in (20) if it is non-zero. However, when we consider the  $(3, k - 3)$  flattening, we will recover the full conormal space. In general, we obtain:

**Theorem 10.1.1** *The variety  $\sigma_3(G(k, W)) \subset \mathbb{P} \wedge^k W$  is an irreducible component of the zero set of the size  $3k + 1$  minors of the skew-flattening  $\wedge^s W \otimes \wedge^{k-s} W$ ,  $s \leq k - s$ ,  $k \leq \dim W - k$ , as long as  $s \geq 3$ .*

This theorem does not extend to  $\sigma_4(G(k, W))$  because there it is possible to have, e.g., sums of vectors in pairwise intersections of spaces that add to a vector in a triple intersection, which does not occur for the third secant variety.

*Remark 10.1.2* When  $\dim W$  is small, the range of Theorem 10.1.1 can be extended. For example, when  $\dim W = 8$ ,  $\sigma_4(G(4, 8))$  is an irreducible component of the size 9 minors of  $\wedge^2 \mathbb{C}^8 \otimes \wedge^6 \mathbb{C}^8$ .

We expect the situation to be similar to that of Veronese varieties, where for small secant varieties, skew-flattenings provide enough equations to cut out the variety, then for a larger range of  $r$  the secant variety is an irreducible component of the variety of skew-flattenings, and then for larger  $r$ , more equations will be needed.

### 10.2 Skew-inheritance

If  $\dim W > m$ , and  $\pi$  is a partition with at most  $m$  parts, we say the module  $S_\pi W$  is *inherited* from the module  $S_\pi \mathbb{C}^m$ . The following is a straightforward variant of [30, Prop. 4.4]:

**Proposition 10.2.1** *Equations (set-theoretic, scheme-theoretic or ideal theoretic) for  $\sigma_r(G(k, W))$  for  $\dim W > kr$  are given by*

- *the modules inherited from the ideal of  $\sigma_r(G(k, kr))$ ,*
- *and the modules generating the ideal of  $\text{Sub}_{kr}(\wedge^k W)$ .*

### 11 Homogeneous varieties $G/P$

Let  $X = G/P \subset \mathbb{P}V_\lambda$  be a homogeneously embedded homogeneous variety, where  $V_\lambda$  denotes the irreducible  $G$ -module of highest weight  $\lambda$ . In particular, when  $\lambda = \omega_i$  is fundamental,  $P$  is the parabolic obtained by deleting the root spaces corresponding to the negative roots  $\alpha = \sum m^j \alpha_j$  such that  $m^i \neq 0$ . Let  $G_0$  be the Levi-factor of  $P$ , if  $W_\mu$  is an irreducible  $\mathfrak{g}_0$ -module given by its highest weight  $\mu$ , we consider it as a  $\mathfrak{p}$ -module by letting the nilpotent part of  $\mathfrak{p}$  act trivially. We let  $E_\mu \rightarrow G/P$  denote the corresponding irreducible homogeneous vector bundle. If  $\mu$  is also  $\mathfrak{g}$ -dominant, then  $H^0(E_\mu)^* = V_\mu$ .

#### 11.1 Example: The usual flattenings for triple Segre products

Let  $A, B, C$  be vector spaces and use  $\omega, \eta, \psi$  respectively for the fundamental weights of the corresponding  $SL$ -modules. Then take  $E$  with highest weight  $(\omega_1, \eta_1, 0)$ , so  $E^* \otimes L$  has highest weight  $(0, 0, \psi_1)$ . Note each of these bundles has rank one, as is the rank of  $A_v$  when  $v \in \hat{\text{Seg}}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$  as the theory predicts. We get the usual flattenings as in [28] and [18]. The equations found for the unbalanced cases in [9, Thm. 2.4] are of this form.

#### 11.2 Example: Strassen’s equations

Let  $\dim A = 3$ . Taking  $E$  with highest weight  $(\omega_2, \eta_1, 0)$  so  $E^* \otimes L$  has highest weight  $(\omega_2, 0, \psi_1)$ , yields Strassen’s equations (see [29,39]). Each of these bundles has rank two, as is the rank of  $A_v$  when  $v \in \hat{\text{Seg}}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ .

#### 11.3 Inheritance from usual Grassmannians

Let  $\omega$  be a fundamental weight for  $\mathfrak{g}$ , and consider  $G/P \subset \mathbb{P}V_\omega$ . Say the Dynkin diagram of  $\mathfrak{g}$  is such that  $\omega$  is  $s$ -nodes from an end node with associated fundamental representation  $V_\eta$ . Then  $V_\omega \subseteq \wedge^s V_\eta$  and  $G/P \subset G(s, V_\eta)$ , see, e.g., [30]. Thus  $\sigma_r(G/P) \subset \sigma_r(G(s, V_\eta))$ , so any equations for the latter give equations for  $\sigma_r(G/P)$ . There may be several such ways to consider  $\omega$ , each will give (usually different) modules of equations.

For example, in the case  $D_n/P_{n-2}$ ,  $V_{\omega_{n-2}}$  is a submodule of  $\wedge^2 V_{\omega_n}, \wedge^2 V_{\omega_{n-1}}$  and  $\wedge^{n-2} V_{\omega_1}$ .

In the case of the  $E$  series we have inclusions

$$\begin{aligned} V_{\omega_3} &\subset \wedge^2 V_{\omega_1} \\ V_{\omega_4} &\subset \wedge^2 V_{\omega_2} \\ V_{\omega_{n-1}} &\subset \wedge^2 V_{\omega_n} \end{aligned}$$

Here, in each case the vector bundle  $E$  has rank two, so we obtain minimal degree equations for the secant varieties of  $E_n/P_3, E_n/P_4, E_n/P_{n-1}$ . The dimension of  $H^0(E)$  for  $(\epsilon_6, \epsilon_7, \epsilon_8)$  is respectively: (27, 133, 3875) for  $\omega_3$ , (78, 912, 147250) for  $\omega_4$ , and (27, 56, 248) for  $\omega_{n-1}$ .

#### 11.4 Other cases

From the paragraph above, it is clear the essential cases are the fundamental  $G/P \subset \mathbb{P}V_\omega$  where  $\omega$  appears at the end of a Dynkin diagram. The primary difficulty in finding equations is locating vector bundles  $E$  such that  $E$  and  $E^* \otimes L$  both have sections. If  $E$  corresponds to a node on the interior of a Dynkin diagram, then  $E^*$  will have a  $-2$  over the node of  $\omega$

and  $E^* \otimes L$  a minus one on the node and thus no sections. However if  $E$  corresponds to a different end of the diagram, one obtains non-trivial equations.

*Example 11.4.1*  $E_n/P_{\alpha_1} \subset \mathbb{P}V_{\omega_1}$  for  $n = 6, 7, 8$ . Taking  $E = E_{\omega_n}$ , then  $E^* \otimes L = E$  and  $H^0(E_{\omega_n}) = V_{\omega_n}$ . The rank of  $E$  is  $2(n-1)$ , and the fiber is the standard representation of  $SO(2n-2)$ .  $\dim V_{\omega_n}$  is respectively 27, 56, 248, so one gets equations for  $\sigma_r(E_n/P_{\alpha_1})$  for  $r$  respectively up to 2, 4, 17. Since the matrix is square, it is possible these equations have lower degrees than one naively expects. Especially since in the case of  $E_6$ , the secant variety is known to be a cubic hypersurface.

*Example 11.4.2*  $D_n/P_n \subset \mathbb{P}V_{\omega_n}$ . Here we may take  $E = E_{\omega_1}$  and then  $E^* \otimes L = E_{\omega_{n-1}}$ . The fiber of  $E$  is the standard representation of  $A_{n-1}$ ; in particular, it is of dimension  $n$ ,  $H^0(E) = V_{\omega_1}$  which is of dimension  $2n$  and  $H^0(E^* \otimes L) = V_{\omega_{n-1}}$ . Thus these give (high degree) equations for the spinor varieties  $D_n/P_n$  but no equations for their secant varieties. Some equations for  $\sigma_2(D_n/P_n)$  are known, see [33, 35]

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