Equations of $(1, d)$-polarized Abelian Surfaces

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16 February, 1997

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§0. Introduction.

In this paper, we study the equations of projectively embedded abelian surfaces with a polarization of type $(1, d)$. Classical results say that given an ample line bundle $\mathcal{L}$ on an abelian surface A, the line bundle $\mathcal{L}^{\otimes n}$ is very ample for $n \geq 3$, and furthermore, in case n is even and $n \geq 4$, the generators of the homogeneous ideal I_A of the embedding of A via $\mathcal{L}^{\otimes n}$ are all quadratic; a possible choice for a set of generators of I_A are the Riemann theta relations.

On the other hand, much less is known about embeddings via line bundles $\mathcal{L}$ of type $(1, d)$, that is line bundles $\mathcal{L}$ which are not powers of another line bundle on A. It is well-known that if $d \geq 5$, and A is a general abelian surface, then $\mathcal{L}$ is very ample, while $\mathcal{L}$ can never be very ample for $d < 5$. However, even if $d \geq 5$, $\mathcal{L}$ may not be very ample for special abelian surfaces. We will restrict our attention in what follows only to the general abelian surface and wish to know what form the equations take for such a projectively embedded abelian surface.

A few special cases are well-documented in the literature: $d = 4$, in which case the general surface is a singular octic in $\mathbf{P}^3$, cf. [BLvS], and $d = 5$ in which case the abelian surface is described as the zero set of a section of the Horrocks-Mumford bundle [HM], whereas its homogeneous ideal is generated by 3 (Heisenberg invariant) quintics and 15 sextics (cf. [Ma]). Also, recent work by Manolache and Schreyer [MS] and by Ranestad [Ra] provides a description of the equations and syzygies in the case $d = 7$.

In this paper, we take this question up for larger values of d, and in particular, we prove the following

* *Supported by NSF grant DMS-9400873*
** *Supported by DFG grant Po-514/1-1*

Theorem. *The homogeneous ideal of a general* $(1,d)$*-polarized abelian surface is generated by quadrics if* $d \geq 10$.

Moreover, in case the embedding considered is with level structure of canonical type, we can give a precise symmetric form for these quadrics.

Our approach is as follows: given a line bundle $\mathcal{L}$ of type $(1,d)$ on the surface A, we consider the product embedding $A \times A \subseteq \mathbf{P}^{d-1} \times \mathbf{P}^{d-1}$, with x's as coordinates on the first $\mathbf{P}^{d-1}$ factor and y's as coordinates on the second factor, and construct certain families of matrices M whose entries are bilinear in the variables $x_0, \ldots, x_{d-1}$, and $y_0, \ldots, y_{d-1}$, and which will drop rank on $A \times A$. Thus setting $(y_0 : \ldots : y_{d-1})$ to be some point in A, suitable minors of M will vanish on the surface A. Furthermore, by choosing special values for the parameter $(y_0 : \ldots : y_{d-1})$, one can obtain M's which are anti-symmetric, and hence deduce that suitable pfaffians of M will vanish on A.

These matrices prove to be quite ubiquitous: for d even, we produce a family of $\frac{d}{2} \times \frac{d}{2}$-matrices M which provide equations for elliptic normal curves of degree d, their secant varieties, and abelian surfaces of type $(1,d)$. However, in the odd case we produce $d \times d$-matrices M having similar features. Finally, we remark here that for $d = 5$ these matrices have been first introduced by R. Moore [Mo] in connection with the Horrocks-Mumford bundle on $\mathbf{P}^4$, and then later used by several other mathematicians in the same context (e.g. see [Au], [ADHPR1], [ADHPR2]). Their determinants are exactly the quintic hypersurfaces in $\mathbf{P}^4$ defined as the wedge product of two linearly independent sections of the Horrocks-Mumford bundle (cf. [Mo], [Au]).

The matrices are constructed in §2 and some of their properties are also described in the same chapter. In the remaining part of the paper, we discuss the structure of the ideal of abelian surfaces by using degeneration arguments. Thus in §3, we review the most basic facts about degenerations of abelian surfaces and elliptic curves, and in §4, we construct projectively embedded degenerations using Stanley-Reisner ideals. The deepest degenerations of abelian surfaces we make use of are described by Stanley-Reisner ideals coming from certain triangulations of the 2-torus. Combinatorics then help us to understand the ideals of these degenerations. In §5 we study basic facts and determine equations and syzygies for secant varieties of elliptic curves, while in §6 we gather all the previous information to obtain results on the ideals of general abelian surfaces.

This work was originally motivated by attempts to describe the moduli spaces $\mathcal{A}^{lev}_{(1,d)}$ of abelian surfaces with $(1,d)$-polarization and level structure of canonical type. In fact, already in this paper we obtain a great deal of information; in particular we define rational maps

$$\Theta_d : \mathcal{A}_{1,d}^{lev} \dashrightarrow X_d$$

for some suitable projective varieties X_d, and then show that for $d \geq 10$, these maps are birational onto their image.

For $d < 10$, the situation is definitely more complicated; the ideals of $(1,d)$-polarized abelian surfaces can never be generated by quadrics, and a more careful analysis is required. This will be carried out in a sequel to this paper [GP], and will include a detailed analysis of the moduli spaces $\mathcal{A}_{(1,d)}^{lev}$ for small values of d. In particular, we will obtain the following:

Theorem. *$\mathcal{A}_{(1,d)}^{lev}$ is rational for $6 \leq d \leq 10$ and $d = 12$, unirational and non-rational for $d = 11$ (being birational with the Klein cubic $\{\sum_{i\in\mathbf{Z}_5} x_i^2 x_{i+1} = 0\} \subset \mathbf{P}^4$), while $\mathcal{A}_{(1,d)}$ is unirational for $d = 14, 16, 18$ and 20.*

An important point will be that, in most of the cases covered by the theorem, pencils of abelian surfaces on Calabi-Yau 3-folds will account for the projective lines contained in these moduli spaces. The sequel to this paper will contain many more details on the geometry of these moduli spaces.

Acknowledgments: We would like to thank Alf Aure, Robert Connelly, Wolfram Decker, David Eisenbud, Kristian Ranestad, and Frank-Olaf Schreyer for many useful discussions. We are also grateful to Mike Stillman and Dave Bayer for *Macaulay* [BS], which helped us tremendously to understand the shape of the equations described in this paper.

§1. Generalities.

§1.1. Canonical Theta Functions and the Heisenberg Group.

We review here some basic facts about abelian varieties. We follow [LB].

Let A be an abelian variety of dimension g over the complex numbers, $A \cong V/\Lambda$, with V a g-dimensional complex vector space and Λ a lattice. Let $\mathcal{L}$ be an ample line bundle on A. We denote by H the first Chern class of $\mathcal{L}$; H is a *polarization* on the abelian variety. As usual, H can be thought of as a positive-definite Hermitian form, whose imaginary part, $E := \mathrm{Im}(H)$, takes integer values on Λ. $\mathcal{L}$ induces a natural map from A to its dual, $\phi_{\mathcal{L}} : A \to \hat{A}$, given by $x \mapsto t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where $t_x : A \to A$ is the morphism given by translation by $x \in A$. The kernel of $\phi_{\mathcal{L}}$, $K(\mathcal{L})$, is always of the form $K(\mathcal{L}) \cong (\mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_g\mathbf{Z})^{\oplus 2}$, where $d_1|d_2|\cdots|d_g$, and this only depends on H. The ordered g-uple $D = (d_1, \ldots, d_g)$ is then called the *type* of the polarization. We write

$\mathbf{Z}^g/D\mathbf{Z}^g$ for $\mathbf{Z}/d_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/d_g\mathbf{Z}$. On $K(\mathcal{L})$, the Weil pairing induced by E is given by

$$e^{\mathcal{L}}(x,y) = \exp(2\pi i E(x,y))$$

for $x, y \in K(\mathcal{L})$. This pairing depends only on the polarization H and thus will be denoted in the sequel mainly as e^H.

A decomposition

$$\Lambda = \Lambda_1 \oplus \Lambda_2$$

is said to be a *decomposition for* $\mathcal{L}$ if Λ_1 and Λ_2 are isotropic for E. This induces a decomposition of real vector spaces $V = V_1 \oplus V_2$. Since $K(\mathcal{L}) = \Lambda(\mathcal{L})/\Lambda$, where

$$\Lambda(\mathcal{L}) = \{v \in V | E(v, \Lambda) \subseteq \mathbf{Z}\},$$

a decomposition of Λ induces a decomposition

$$K(\mathcal{L}) = K_1(\mathcal{L}) \oplus K_2(\mathcal{L}),$$

with $K_1(\mathcal{L}) \cong K_2(\mathcal{L}) \cong \mathbf{Z}^g/D\mathbf{Z}^g$, both subgroups being isotropic with respect to the Weil pairing.

Given a decomposition, we can define a semicharacter $\chi_0 : V \to \mathbf{C}_1$ by

$$\chi_0(v) = \exp(\pi i E(v_1, v_2))$$

where $v = v_1 + v_2$ with $v_i \in V_i$. Via the Appell-Humbert theorem ([LB], Theorem 2.2.3) this determines a line bundle $\mathcal{L}_0 := \mathcal{L}(H, \chi_0)$. Then the line bundle $\mathcal{L}$ can be written as $t_c^*\mathcal{L}_0$ for some $c \in V$, unique up to translation by elements of $\Lambda(\mathcal{L})$. c is called a *characteristic* for the line bundle $\mathcal{L}$, and $\mathcal{L}_0$ is said to be a line bundle of characteristic zero with respect to the given decomposition.

Given an ample line bundle $\mathcal{L}$, a decomposition for $\mathcal{L}$, and a choice of a characteristic c for $\mathcal{L}$, there is a unique basis

$$\{\vartheta_x^c \mid x \in K_1(\mathcal{L})\}$$

of *canonical theta functions* of the space $\Gamma(X, \mathcal{L})$. (See [LB], §3.2.) We will often omit mention of c, and write $\vartheta_x^{\mathcal{L}}$ when we are dealing with several line bundles at the same time.

Given an element $x \in K(\mathcal{L})$, we have an isomorphism $t_x^*\mathcal{L} \cong \mathcal{L}$. In general x then induces a projective automorphism on $\mathbf{P}(H^0(\mathcal{L}))$ which yields a representation $K(\mathcal{L}) \to$

$PGL(H^0(\mathcal{L}))$. This representation does not lift to a linear representation of $K(\mathcal{L})$, but it does after taking a central extension of $K(\mathcal{L})$,

$$1 \to \mathbf{C}^* \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0,$$

whose Schur commutator map is the previously defined pairing $e^H = e^{\mathcal{L}}$. $\mathcal{G}(\mathcal{L})$ is called the *theta group* of $\mathcal{L}$. The theta group is isomorphic to the Heisenberg group $\mathcal{H}(D)$, which can be described as follows: as a set it is $\mathbf{C}^* \times K(D)$, where $K(D) \cong \mathbf{Z}^g/D\mathbf{Z}^g \oplus \mathbf{Z}^g/D\mathbf{Z}^g$. Let $f_1, \ldots, f_{2g}$ be the standard basis of $K(D)$, and define an alternating multiplicative form $e^D : K(D) \times K(D) \to \mathbf{C}^*$ by

$$e^D(f_\nu, f_\mu) := \begin{cases} \exp(-2\pi i/d_\nu) & \text{if } \mu = g + \nu \\ \exp(2\pi i/d_\nu) & \text{if } \nu = g + \mu \\ 1 & \text{otherwise.} \end{cases}$$

To define the group structure on $\mathcal{H}(D)$, we take for any $(\alpha, x_1, x_2), (\beta, y_1, y_2) \in \mathcal{H}(D)$

$$(\alpha, x_1, x_2)(\beta, y_1, y_2) := (\alpha\beta e^D(x_1, y_2), x_1 + y_1, x_2 + y_2).$$

An isomorphism between $\mathcal{G}(\mathcal{L})$ and $\mathcal{H}(D)$ which restricts to the identity on $\mathbf{C}^*$ is called a *theta structure* for $\mathcal{L}$. Any such isomorphism induces a symplectic isomorphism between $K(\mathcal{L})$ and $K(D)$, that is which preserves the alternating pairings e^H and e^D, respectively. Classically, a symplectic isomorphism $\bar{b} : K(\mathcal{L}) \to K(D)$ is called a *level D-structure* on (A, H), or a *level structure of canonical type*. Since we are not considering other kinds of level structures in this paper, we will refer in the sequel to these level structures simply as *level structures*.

As mentioned above, the theta group has a natural representation $\mathcal{G}(\mathcal{L}) \to GL(H^0(\mathcal{L}))$, which lifts uniquely the representation $K(\mathcal{L}) \to PGL(H^0(\mathcal{L}))$. Given a choice of theta structure, this representation is isomorphic to the Schrödinger representation of $\mathcal{H}(D)$, defined as follows. Let $W = \mathbf{C}(\mathbf{Z}^g/D\mathbf{Z}^g)$ be the vector space of complex-valued functions on the set $\mathbf{Z}^g/D\mathbf{Z}^g$. The Schrödinger representation $\rho : \mathcal{H}(D) \to GL(W)$ is given by

$$\rho(\alpha, x_1, x_2)(\gamma) = \alpha e^D(\cdot, x_2)\gamma(\cdot + x_1).$$

This representation is irreducible, and the center $\mathbf{C}^*$ acts by scalar multiplication, so it yields a projective representation of $K(D)$.

More explicitly, for a surface, the Schrödinger representation takes the following form on projective space. Let $D = (d_1, d_2)$. We can write a basis $\{\delta_\gamma \mid \gamma \in \mathbf{Z}^2/D\mathbf{Z}^2\}$ of W, where δ_γ is the delta function

$$\delta_\gamma(\gamma') = \begin{cases} 0 & \text{if } \gamma \neq \gamma'; \\ 1 & \text{if } \gamma = \gamma'. \end{cases}$$

We denote by $\mathbf{H}_{d_1,d_2}$ the subgroup of $\mathcal{H}(D)$ generated by $\sigma_1 = (1,1,0,0,0)$, $\sigma_2 = (1,0,1,0,0)$, $\tau_1 = (1,0,0,1,0)$ and $\tau_2 = (1,0,0,0,1)$, and these act on W via

$$\sigma_1(\delta_{i,j}) = \delta_{i-1,j}, \quad \sigma_2(\delta_{i,j}) = \delta_{i,j-1},$$
$$\tau_1(\delta_{i,j}) = \xi_1^{-i}\delta_{i,j}, \quad \sigma_2(\delta_{i,j}) = \xi_2^{-j}\delta_{i,j},$$

where $\xi_k := \exp(2\pi i/d_k)$.

In the case that $d_1 = 1$, both σ_1 and τ_1 are just the identity, and we shall denote by σ and τ the generators σ_2 and τ_2, and leave off the first index on the variables.

Given a decomposition for $\mathcal{L}$ inducing $K(\mathcal{L}) = K_1(\mathcal{L}) \oplus K_2(\mathcal{L})$, a basis of canonical theta functions $\{\vartheta_\gamma \mid \gamma \in K_1(\mathcal{L})\}$ for $H^0(\mathcal{L})$ yields the identification of $H^0(\mathcal{L})$ and W via $\vartheta_\gamma \mapsto \delta_\gamma$ such that the representations $\mathcal{G}(\mathcal{L}) \to GL(H^0(\mathcal{L}))$ and $\mathcal{H}(D) \to GL(V)$ coincide. Thus if we map A into $\mathbf{P}(H^0(\mathcal{L})^\vee)$ using as coordinates $x_\gamma = \vartheta_\gamma$, $\gamma \in \mathbf{Z}^g/D\mathbf{Z}^g$, the image of A will be invariant under the action of the Heisenberg group via the Schrödinger representation. In particular, if A is embedded this way in $\mathbf{P}(H^0(\mathcal{L})^\vee)$, then $H^0(\mathcal{I}_A(n))$ is also a representation of the Heisenberg group. Moreover, in case $d_1 = 1$, this is a representation of weight n (i.e., a central element $z \in \mathbf{C}^*$ acts by multiplication with z^n), and hence all its irreducible components will have dimension $\geq d_2/\gcd(d_2, n)$ (e.g. see [La], [Mu1]). These are basic facts which will be used over and over again in the sequel.

Again, if $d_1 = 1$, the action of the Heisenberg group $\mathbf{H}_d := \mathbf{H}_{1,d}$ on the coordinates of $\mathbf{P}(H^0(\mathcal{L})^\vee)$ is

$$\sigma(x_i) = x_{i-1},$$
$$\tau(x_i) = \xi^{-i}x_i.$$

If one considers $K(\mathcal{L})$ as a subgroup of the automorphism group of A via translations, then the order 2 subgroup $\langle(-1_A)\rangle$ acts on $K(\mathcal{L})$ by inner automorphisms. We may define $K^e(\mathcal{L})$ as $K(\mathcal{L}) \rtimes \langle(-1_A)\rangle$, and then define the extended theta group $\mathcal{G}(\mathcal{L})^e$ to be a central extension of $K(\mathcal{L})^e$ by $\mathbf{C}^*$. In fact $\mathcal{G}(\mathcal{L})^e := \mathcal{G}(\mathcal{L}) \rtimes \langle(-1_{\mathcal{L}})\rangle$. Similarly, one can introduce an extended Heisenberg group, defined by

$$\mathcal{H}^e(D) := \mathcal{H}(D) \rtimes \langle\iota\rangle,$$

where ι acts on $\mathcal{H}(D)$ via $\iota(\alpha, x_1, x_2) = (\alpha, -x_1, -x_2)$. An *extended* theta structure is an isomorphism between $\mathcal{G}^e(\mathcal{L})$ and $\mathcal{H}^e(D)$ inducing the identity on $\mathbf{C}^*$. Each extended theta structure restricts to a theta structure, but a theta structure does not always come from an extended theta structure. In fact, a theta structure $b : \mathcal{G}(\mathcal{L}) \to \mathcal{H}(D)$ can be extended to an extended theta structure if and only if it is a *symmetric theta structure*, that is if the diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{L}) & \xrightarrow{(-1)_{\mathcal{L}}} & \mathcal{G}(\mathcal{L}) \\ \downarrow b & & \downarrow b \\ \mathcal{H}(D) & \xrightarrow{\iota} & \mathcal{H}(D) \end{array}$$

commutes. In order for a symmetric theta structure to exist, $\mathcal{L}$ must be a symmetric line bundle, that is $(-1_A)^*\mathcal{L} \cong \mathcal{L}$. By Theorem 6.9.5 of [LB], there always exist a finite number of symmetric line bundles of a given polarization, each admitting a finite number of symmetric theta structures.

The Schrödinger representation ρ of $\mathcal{H}(D)$ extends to a representation ρ^e of $\mathcal{H}^e(D)$, with $\rho^e(\iota) \in SL^{\pm}(W) = \{M \in GL(W) | \det M = \pm 1\}$. In the case that A is a surface, ι acts on W by $\iota(\delta_{i,j}) = \delta_{-i,-j}$. We denote by $\mathbf{H}^e_{d_1,d_2}$ the subgroup of $\mathcal{H}^e(D)$ generated by $\mathbf{H}_{d_1,d_2}$ and ι. In fact, the following holds:

$$\mathbf{H}^e_{d_1,d_2} = \mathbf{H}_{d_1,d_2} \rtimes \langle\iota\rangle.$$

ι, acting as an involution on W, has two eigenspaces, with eigenvalues ± 1. We will refer to the projectivization of the positive eigenspace as $\mathbf{P}^+ \subseteq \mathbf{P}(W)$, and the negative eigenspace as $\mathbf{P}^- \subseteq \mathbf{P}(W)$.

In particular, if $D = (1, d)$, then $\mathbf{P}^+$ is given by the equations

$$\{x_i = x_{-i} \mid i \in \mathbf{Z}/d\mathbf{Z}\}$$

while $\mathbf{P}^-$ is given by the equations

$$\{x_i = -x_{-i} \mid i \in \mathbf{Z}/d\mathbf{Z}\}.$$

§1.2. Linear systems on abelian surfaces.

We recall in this paragraph general results about linear systems on abelian surfaces.

Let A be an abelian surface and let $\mathcal{L}$ be an ample line bundle on A of type $D = (d_1, d_2)$. Then Riemann-Roch gives $h^0(\mathcal{L}) = \frac{1}{2}c_1(\mathcal{L})^2 = d_1 d_2$, and so $\mathcal{L}$ defines a rational map $\psi_{\mathcal{L}} : A \dashrightarrow \mathbf{P}^{d_1 d_2 - 1}$.

According to Lefschetz's theorem [Mu2], the line bundle $\mathcal{L}$ is very ample if $d_1 \geq 3$. Moreover, the embedding $\psi_{\mathcal{L}}$ is projectively normal in this case [Ko].

If $d_1 = 2$, then there exists an ample line bundle $\mathcal{M}$ on A with $\mathcal{L} = \mathcal{M}^2$, and moreover $|\mathcal{L}|$ is at least base point free. If $\mathcal{M}$ splits, that is $(A, \mathcal{M}) \cong (E_1 \times E_2, \mathcal{M}_1 \boxtimes \mathcal{M}_2)$, where

E_1 and E_2 are elliptic curves and $\mathcal{M}_1$ a principal polarization on E_1, then $\psi_{\mathcal{L}}$ is the composition

$$E_1 \times E_2 \stackrel{\psi_{\mathcal{M}_1^2 \times \mathcal{M}_2^2}}{\longrightarrow} \mathbf{P}^1 \times \mathbf{P}^{d_2-1} \stackrel{\text{Segre}}{\longrightarrow} \mathbf{P}^{2d_2-1}.$$

In general $\mathcal{M}$ doesn't split and one distinguishes two cases: either $d_2 = 2$ and $\psi_{\mathcal{L}} : A \to K \subset \mathbf{P}^3$ is of degree 2 on its image, the Kummer surface associated to $\mathcal{M}$, or $d_2 > 2$ and then $\psi_{\mathcal{L}}$ is an embedding [Oh], [LN]. Furthermore, in this last case the embedding is not necessarily projectively normal; see [Oh] for an explicit necessary and sufficient criterion. In all the above cases, Mumford's theory of the multiplying-sections-map shows that in case $\psi_{\mathcal{L}}$ is an embedding, the homogeneous ideal of the image is generated by quadrics when $d_1 \geq 4$ [Mu1], [Ke], or by quadrics and cubics for $d_1 = 3$ [Ke], or by quadrics, cubics and quartics for $d_1 = 2$ [Ke]. Moreover, if $2|d_1$ and $d_1 \geq 4$ then Riemann's quadratic theta relations [Mu1], [LB] describe the image $\psi_{\mathcal{L}}(A) \subset \mathbf{P}^{d_1 d_2 - 1}$ completely. Similarly, when $3|d_1$, the cubic theta relations give a complete set of equations for the image $\psi_{\mathcal{L}}(A) \subset \mathbf{P}^{d_1 d_2 - 1}$ (see [LB, Theorem 7.6.6]).

Finally, let $\mathcal{L}$ be of type $(1, d)$. Then the Decomposition theorem [LB, Theorem 4.3.1] says that $|\mathcal{L}|$ has base curves if and only if $(A, \mathcal{L}) \cong (E_1 \times E_2, \mathcal{L}_1 \boxtimes \mathcal{L}_2)$, where $\mathcal{L}_1$ is a principal polarization on E_1 and $\mathcal{L}_2$ is of type (d) on E_2. Assume from now on that $(A, \mathcal{L})$ is simple. If $d = 2$, then the linear system $|\mathcal{L}|$ has exactly 4 base points, which are 4-torsion points in case $\mathcal{L}$ is symmetric. Furthermore, in this case $\psi_{\mathcal{L}}$ is just a pencil of smooth (generically) irreducible curves of genus 3 [Ba]. If $d = 3$, then $|\mathcal{L}|$ is base point free and $\psi_{\mathcal{L}} : A \to \mathbf{P}_2$ is a 6-fold covering branched over a curve of degree 18 ([LB], Example 10.1.5). If $d = 4$, then $\psi_{\mathcal{L}} : A \to \mathbf{P}_3$ is generically birational on its image, a singular octic surface ([BLvS], [LB] §10.5), and the geometry of the situation is well understood. See [BLvS] and [LB], Proposition 10.5.7 for a complete description and for further details. Finally, if $d \geq 5$, then Reider's theorem states that $\psi_{\mathcal{L}} : A \to \mathbf{P}_{d-1}$ is an embedding if and only if there is no elliptic curve E on A with $E \cdot c_1(\mathcal{L}) = 2$. Furthermore, the embedding is even projectively normal in case d odd and $d \geq 7$, or d even and $d \geq 14$ [La].

§**1.3. Moduli.**

Let $\mathcal{A}_D$ denote the coarse moduli space of abelian varieties of dimension g with polarization of type D. $\mathcal{A}_D$ is a $g(g+1)/2$-dimensional quasi-projective variety. In case $g = 2$ and $D = (1, d)$, we'll write d instead of D in all our notation, so for instance we'll write $\mathcal{A}_d$ instead of $\mathcal{A}_{(1,d)}$. Similarly $\mathcal{A}_D^{lev}$ will denote the moduli space $\mathcal{A}_D^{lev}$ of abelian varieties with a polarization of type D and with canonical level structure.

Relating these spaces, there is a forgetful map $\pi_1 : \mathcal{A}_d^{lev} \to \mathcal{A}_d$ which is a finite dominant morphism, of degree $\sharp PSL(2, \mathbf{Z}_d) = d(d^2 - 1)/2$ when d is prime, and also a finite map induced by dividing out the level structure $\pi_2 : \mathcal{A}_d^{lev} \to \mathcal{A}_1$, which is of degree $d(d^4-1)/2$ when d is prime (see [HKW], Proposition 1.21.).

Both $\mathcal{A}_D$ and $\mathcal{A}_D^{lev}$ can be described as certain quotients of the Siegel upper half-space

$$\mathcal{H}_g = \{Z \in M_g(\mathbf{C}) \mid Z^t = Z \text{ and } \operatorname{Im}(Z) > 0\}.$$

One can think of $\mathcal{H}_g$ as parametrizing abelian varieties with period matrix (Z, D), so that $\mathcal{H}_g$ is the moduli space of type D abelian varieties with a choice of a symplectic basis for the lattice, namely the basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ given by the columns of the period matrix (Z, D). Note that such data also determines a canonical level structure, by using the basis for $K(D)$ represented by $\lambda_1/d_1, \ldots, \lambda_g/d_g, \mu_1/d_1, \ldots, \mu_g/d_g$.

There exists a universal family $\mathcal{X}_D \to \mathcal{H}_g$ with $\mathcal{X}_D := (\mathcal{H}_g \times \mathbf{C}^g)/\mathbf{Z}^{2g}$, where $\alpha \in \mathbf{Z}^{2g}$ acts on $\mathcal{H}_g \times \mathbf{C}^g$ by

$$\alpha : (Z, v) \mapsto (Z, v + (Z, D)\alpha),$$

and there is a line bundle $\mathcal{L}$ on $\mathcal{X}_D$ such that $\mathcal{L}|_{X_Z}$ is a line bundle of type D and characteristic zero with respect to the decomposition $Z\mathbf{Z}^g \oplus D\mathbf{Z}^g$, where X_Z denotes the fibre of $\mathcal{X}_D \to \mathcal{H}_g$ over $Z \in \mathcal{H}_g$. See [LB], §8.7 for details. Holomorphic sections of $\mathcal{L}$ can be defined using the classical theta functions $\vartheta\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for $c_1, c_2 \in \mathbf{R}^g$. These are holomorphic functions on $\mathcal{H}_g \times \mathbf{C}^g$. If $c_0, \ldots, c_N$ are a set of representatives for the group $D^{-1}\mathbf{Z}^g/\mathbf{Z}^g$, then the theta functions

$$\vartheta\begin{bmatrix} c_0 \\ 0 \end{bmatrix}, \ldots, \vartheta\begin{bmatrix} c_N \\ 0 \end{bmatrix}$$

descend to give sections of $\mathcal{L}$ on $\mathcal{X}_D$, such that their restrictions of X_Z form a basis for $H^0(\mathcal{L}|_{X_Z})$ for any $Z \in \mathcal{H}_g$. Thus we can use these sections to define a map $\mathcal{X}_D \to \mathbf{P}^N_{\mathcal{H}_g}$ in case $\mathcal{L}$ is generated by global sections. The image of each abelian variety X_Z under this map is Heisenberg invariant, with the elements of $\mathcal{H}(D)$ acting by translation on X_Z by the corresponding elements of $K(D)$, as can be seen from the relations

$$\vartheta\begin{bmatrix} c_\nu \\ 0 \end{bmatrix}(Z, v + e_\mu) = \exp(2\pi i (c_\nu \cdot e_\mu))\vartheta\begin{bmatrix} c_\nu \\ 0 \end{bmatrix}(Z, v),$$

where e_μ is the μ-th standard basis vector, and

$$\vartheta\begin{bmatrix} c_\nu \\ 0 \end{bmatrix}(Z, v + Ze_\mu/d_\mu) = C_\mu \vartheta\begin{bmatrix} c_\nu + e_\mu/d_\mu \\ 0 \end{bmatrix}(Z, v),$$

where C_μ is a constant which depends only on μ. The following proposition, which follows easily from the above observations, demonstrates the significance of the moduli spaces $\mathcal{A}_D^{lev}$:

Proposition 1.3.1. *Let X_Z and $X_{Z'}$ be two fibres of $\mathcal{X}_D \to \mathcal{H}_g$, and suppose $\mathcal{L}|_{X_Z}$ and $\mathcal{L}|_{X_{Z'}}$ are both very ample. Then the images of X_Z and $X_{Z'}$ coincide in $\mathbf{P}^N$ if and only if there is an isomorphism between X_Z and $X_{Z'}$ preserving their canonical level structures.*

Proof: We set $\mathcal{L}_Z := \mathcal{L}|_{X_Z}$, $\mathcal{L}_{Z'} := \mathcal{L}|_{X_{Z'}}$, and let $\phi : X_Z \hookrightarrow \mathbf{P}^N$, $\phi' : X_{Z'} \hookrightarrow \mathbf{P}^N$ be the maps induced by these bundles using bases of classical theta-functions. Now, if $\alpha : X_Z \to X_{Z'}$ is an isomorphism of polarized abelian varieties, then there exists a $c \in X_Z$ such that $\alpha^*\mathcal{L}_{Z'} = t_c^*\mathcal{L}_Z$. In particular there is a linear automorphism $\beta : \mathbf{P}^N \to \mathbf{P}^N$ such that $\beta|_{X_Z} = \alpha \circ t_{-c}$. On the other hand, α is an isomorphism of level structures, say b and b' respectively, if and only if the composition $(b')^{-1} \circ \alpha \circ b$

$$K(D) \xrightarrow{b} K(\mathcal{L}_Z) \xrightarrow{\alpha} K(\mathcal{L}_{Z'}) \xleftarrow{b'} K(D)$$

is the identity map. In other words, α is an isomorphism of level structures if and only if

$$\alpha \circ t_{b(x)} \circ \alpha^{-1} = t_{b'(x)} \quad \text{for all } x \in K(D),$$

or equivalently that

$$\beta \circ t_{b(x)} \circ \beta^{-1} = t_{b'(x)} \quad \text{for all } x \in K(D).$$

Thus thinking of $\mathbf{P}^N = \mathbf{P}(W)$, and $\beta \in PGL(W)$, we deduce that $\beta \in N(\mathcal{H}(D))/\mathbf{C}^*$, where $N(\mathcal{H}(D))$ is the normalizer of $\mathcal{H}(D)$ in $SL^{\pm}(W)$. Now by [LB], Exercise 6.14, there is an exact sequence

$$0 \to K(D) \to N(\mathcal{H}(D))/\mathbf{C}^* \to Sp(D) \to 1,$$

where $Sp(D)$ is the group of symplectic automorphisms of $(K(D), e^D)$. It follows that α is an isomorphism of level structures if and only if β induces the identity in $Sp(D)$, that is if and only if $\beta \in K(D) \subseteq PGL(W)$, which is the case if and only if $\phi'(X_{Z'}) = \beta(\phi(X_Z)) = \phi(X_Z)$. •

Thus, if $\mathcal{L}|_{X_Z}$ is very ample for general $Z \in \mathcal{H}_g$, there is an open set $U \subseteq \mathcal{A}_D^{lev}$ over which we have a family $\mathcal{X}_U \to U$, $\mathcal{X}_U \subseteq \mathbf{P}^N \times U$ of embedded abelian varieties. We will often make use of this family in the sequel. We should note, however, that in general this is not the universal family over $\mathcal{A}_D^{lev}$. Indeed, let $\psi : \mathcal{H}_g \to \mathcal{A}_D^{lev}$ be the quotient map defining $\mathcal{A}_D^{lev}$ as a quotient of $\mathcal{H}_g$, and let $U' = \psi^{-1}(U)$. Then the projective family $\mathcal{X}_U$ described above over U pulls back to the family $\mathcal{X}_D \to \mathcal{H}_g$ restricted to U'. However $\mathcal{X}_U$ is a quotient of $\mathcal{X}_D$ which may identify two abelian varieties over two different points of $\mathcal{H}_g$ via morphisms which are not homomorphisms of abelian varieties, i.e., the zero section of $\mathcal{X}_D \to \mathcal{H}_g$ is not preserved under taking this quotient. Therefore the family $\mathcal{X}_U \to U$ is in fact a twist of the universal family over U. The simplest case where this happens is for $g = 1$ and D even (see [BaH] for further details).

§2. Equations for Elliptic Curves and Abelian Surfaces.

Let A be an abelian variety of dimension g, $\mathcal{L}$ a symmetric line bundle on A of type $D = (d_1, \dots, d_g)$. Choose a decomposition for $\mathcal{L}^2$, with $K(\mathcal{L}^2) = K_1 \oplus K_2$, with $K_1 \cong K_2 \cong \mathbf{Z}^g/2D\mathbf{Z}^g$. This induces a decomposition on $2K(\mathcal{L}^2) = K(\mathcal{L}) = 2K_1 \oplus 2K_2$. There is a canonical choice of basis of $H^0(\mathcal{L}^2)$ given by $\{\vartheta_i^{\mathcal{L}^2} | i \in K_1\}$, and a canonical choice of basis of $H^0(\mathcal{L})$ given by $\{\vartheta_i^{\mathcal{L}} | i \in 2K_1\}$.

If p_1 and p_2 are the first and second projections of $A \times A$ onto A, and $\mathcal{L}$ and $\mathcal{M}$ are line bundles on A, then we write $\mathcal{L} \boxtimes \mathcal{M}$ for $p_1^*\mathcal{L} \otimes p_2^*\mathcal{M}$. Similarly, if $s \in \Gamma(\mathcal{L})$ and $t \in \Gamma(\mathcal{M})$ are sections, we write $s \boxtimes t$ for $p_1^* s \otimes p_2^* t$.

Theorem 2.1. *The* $(\#2K_1) \times (\#2K_1)$*-matrix*

$$M_{\mathcal{L}} = (\sum_{z \in Z_2} \vartheta_{i+j+z}^{\mathcal{L}^2} \boxtimes \vartheta_{i-j+z}^{\mathcal{L}^2})_{2i,2j \in 2K_1}$$

has rank at most one on $A \times A$*. This notation means that* i *(and ditto for* j*) runs through a subset* R *of* K_1 *of representatives for* K_1/Z_2*. Also* $Z_2 = A_2 \cap K_1 \cong (\mathbf{Z}/2\mathbf{Z})^g$*, where* A_2 *is the group of two-torsion points on* A*.*

Proof: Define $\alpha : A \times A \to A \times A$ by $\alpha(a_1, a_2) := (a_1 + a_2, a_1 - a_2)$. α is an isogeny with $\ker(\alpha) = \{(a, a) \mid a \in A_2\}$. By [LB], Lemma 7.1.1,

$$\alpha^*(\mathcal{L} \boxtimes \mathcal{L}) = \mathcal{L}^2 \boxtimes \mathcal{L}^2.$$

Also,

$$K(\mathcal{L}^2 \boxtimes \mathcal{L}^2) = (K_1 \times K_1) \oplus (K_2 \times K_2)$$

and

$$K(\mathcal{L} \boxtimes \mathcal{L}) = (2K_1 \times 2K_1) \oplus (2K_2 \times 2K_2),$$

and these decompositions are compatible with α in the sense that $\alpha(K_i \times K_i) \cap K(\mathcal{L} \boxtimes \mathcal{L}) = 2K_i \times 2K_i$, by [LB], Lemma 7.1.2. Thus we are in a position to apply the Isogeny theorem, [LB], Theorem 6.5.1. The Künneth isomorphism $H^0(\mathcal{L} \boxtimes \mathcal{L}) \cong H^0(\mathcal{L}) \boxtimes H^0(\mathcal{L})$ yields a basis of canonical theta-functions

$$\{\vartheta_{i'}^{\mathcal{L}} \boxtimes \vartheta_{j'}^{\mathcal{L}} \mid i', j' \in 2K_1\} = \{\vartheta_{2i}^{\mathcal{L}} \boxtimes \vartheta_{2j}^{\mathcal{L}} \mid 2i, 2j \in 2K_1\}.$$

Thus by the Isogeny theorem,

$$\alpha^*(\vartheta_{2i}^{\mathcal{L}} \boxtimes \vartheta_{2j}^{\mathcal{L}}) = \sum_{\substack{(a_1,a_2) \in \\ \alpha^{-1}(2i,2j) \cap (K_1 \times K_1)}} \vartheta_{a_1}^{\mathcal{L}^2} \boxtimes \vartheta_{a_2}^{\mathcal{L}^2}.$$

If $\alpha(a_1, a_2) = (2i, 2j)$ for $(a_1, a_2) \in K_1 \times K_1$, then we obtain

$$\alpha^*(\vartheta_{2i}^{\mathcal{L}} \boxtimes \vartheta_{2j}^{\mathcal{L}}) = \sum_{z \in Z_2} \vartheta_{a_1+z}^{\mathcal{L}^2} \boxtimes \vartheta_{a_2+z}^{\mathcal{L}^2}.$$

Since the matrix $M' = (\vartheta_{2i}^{\mathcal{L}} \boxtimes \vartheta_{2j}^{\mathcal{L}})_{2i,2j \in 2K_1}$ obviously has rank one on $A \times A$, so does $\mathcal{M}_{\mathcal{L}} = \alpha^* M'$. •

Corollary 2.2. *If $A \cong E$ is an elliptic curve, $\mathcal{L}$ a symmetric line bundle of degree $d \geq 2$ on E, and $E \subseteq \mathbf{P}^{2d-1}$ the embedding given by the sections $\vartheta_i^{\mathcal{L}^2}$, $i \in K_1(\mathcal{L}^2)$, then the 2×2 minors of the matrix*

$$M_d = (x_{i+j} y_{i-j} + x_{i+j+d} y_{i-j+d})_{2i,2j \in 2\mathbf{Z}/2d\mathbf{Z}}$$

vanish along $E \times E \subseteq \mathbf{P}^{2d-1} \times \mathbf{P}^{2d-1}$, where x_i, y_j, with $i, j \in \mathbf{Z}/2d\mathbf{Z}$, are coordinates for the first and second $\mathbf{P}^{2d-1}$, respectively.

Example 2.3. Consider an elliptic curve E with $\mathcal{L}$ a symmetric line bundle of degree 2. Embedding $E \times E$ into $\mathbf{P^3} \times \mathbf{P^3}$ via $\mathcal{L}^2 \boxtimes \mathcal{L}^2$, we find that the matrix

$$M_2 = \begin{pmatrix} x_0 y_0 + x_2 y_2 & x_1 y_3 + x_3 y_1 \\ x_1 y_1 + x_3 y_3 & x_2 y_0 + x_0 y_2 \end{pmatrix}$$

has rank one on $E \times E$. Thus, if we substitute the coordinates of a point $(y_0 : y_1 : y_2 : y_3) \in E \subset \mathbf{P^3}$, we obtain an equation for E in $x_0, \ldots, x_3$ given by

$$\det M_2 = 0.$$

Applying σ to this equation yields another equation, and these two equations cut out E:

$$\begin{aligned} (x_0 y_0 + x_2 y_2)(x_2 y_0 + x_0 y_2) - (x_1 y_1 + x_3 y_3)(x_1 y_3 + x_3 y_1) &= 0 \\ (x_3 y_0 + x_1 y_2)(x_1 y_0 + x_3 y_2) - (x_0 y_1 + x_2 y_3)(x_0 y_3 + x_2 y_1) &= 0. \end{aligned}$$

Furthermore, since $(y_0 : y_1 : y_2 : y_3) \in E$ and hence is a solution of the above equations, we deduce that the closure of the union in $\mathbf{P^3}$ of all Heisenberg invariant elliptic normal curves coincides with the surface F' whose equation is the determinant of the matrix obtained from M_2 by substituting x's for the y's:

$$F' := \{x_0 x_2 (x_0^2 + x_2^2) - x_1 x_3 (x_1^2 + x_3^2) = 0\} \subset \mathbf{P^3}.$$

It is easily seen that F' is in fact projectively equivalent with the Fermat quartic $F := \{\sum_{i=0}^3 x_i^4 = 0\} \subset \mathbf{P^3}$. Finally, as noted at the end of §1.3, we remark here that F' is not birational to the Shioda surface $S(4)$ (see [BaH] for a proof of this claim).

Corollary 2.4. *If A is an abelian surface, and $\mathcal{L}$ a symmetric line bundle of type $(1,d)$, then $K_1 = K_1(\mathcal{L}^2) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2d\mathbf{Z}$, $Z_2 \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, and the matrix $\mathcal{M}_{\mathcal{L}}$*

$$(\vartheta^{\mathcal{L}^2}_{i+j} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j} + \vartheta^{\mathcal{L}^2}_{i+j+(1,0)} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j+(1,0)} + \vartheta^{\mathcal{L}^2}_{i+j+(0,d)} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j+(0,d)} + \vartheta^{\mathcal{L}^2}_{i+j+(1,d)} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j+(1,d)})_{2i,2j \in 2K_1}$$

has rank at most one on $A \times A$.

Remark 2.5. Note that for any abelian variety A and choice of $y \in A$, the 2×2 minors of $M_{\mathcal{L}}$ yield quadratic theta relations, and one might want to know if these theta relations are equivalent to the Riemann theta relations. This is indeed the case, if one also considers all translates of these relations by the Heisenberg group (as in Example 2.3, where we had to apply σ to $\det M_2$ in order to obtain defining equations for the elliptic curve E).

To see this, first recall that if $\rho \in \hat{Z}_2 = \mathrm{Hom}(Z_2, \mathbf{C}^*)$ is a character of Z_2, $y \in K_1(\mathcal{L}^4)$, then we may define

$$\vartheta^{\mathcal{L}^4}_{y,\rho} := \sum_{z \in Z_2} \rho(z) \vartheta^{\mathcal{L}^4}_{y+z}.$$

Then the Riemann theta relations (compare [LB] Theorem 7.5.2) take the form, for $y, y_1, y_2 \in K_1(\mathcal{L}^4)$, with $y \equiv y_1 \equiv y_2 \bmod 2K_1(\mathcal{L}^4)$ and $\rho \in \hat{Z}_2$,

$$\vartheta^{\mathcal{L}^4}_{y_1,\rho}(0) \sum_{z \in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_2+z} \otimes \vartheta^{\mathcal{L}^2}_{y-y_2+z} = \vartheta^{\mathcal{L}^4}_{y_2,\rho}(0) \sum_{z \in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_1+z} \otimes \vartheta^{\mathcal{L}^2}_{y-y_1+z}.$$

See also [Mu3] Chapter II, Theorem 6.1, for a slightly different version of Riemann's theta identities and [Mu3], p.223 ff, for a discussion of the relationships between the various settings.

To obtain the above theta relations using the 2×2 minors of the matrix $M_{\mathcal{L}}$, we may proceed as follows. Let $R \subseteq K_1(\mathcal{L}^2)$ be the subset in Theorem 2.1 being used to represent $K_1(\mathcal{L}^2)/Z_2$. Then it is possible to find $i, j, i', j' \in R$ such that

$$\begin{aligned} i + j &\equiv y + y_2 \bmod Z_2, \\ i' + j' &\equiv y - y_2 \bmod Z_2, \\ i + j' &\equiv y + y_1 \bmod Z_2, \\ i' + j &\equiv y - y_1 \bmod Z_2. \end{aligned}$$

Thus there exists $z_1, z_2, z_3, z_4 \in Z_2$ such that

$$\begin{aligned} i + j &= y + y_2 + z_1, \\ i' + j' &= y - y_2 + z_2, \\ i + j' &= y + y_1 + z_3, \\ i' + j &= y - y_1 + z_4 \end{aligned}$$

hold in $K_1(\mathcal{L}^2)$. Then the 2×2 minor of $M_{\mathcal{L}}$ involving rows i and i', and columns j and j' takes the form

$$
\begin{aligned}
m_{(i,j),(i',j')} := &\left(\sum_{z\in Z_2} \vartheta^{\mathcal{L}^2}_{y+y_2+z} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j+z+z_1}\right)\cdot\left(\sum_{z\in Z_2} \vartheta^{\mathcal{L}^2}_{y-y_2+z} \boxtimes \vartheta^{\mathcal{L}^2}_{i'-j'+z+z_2}\right) - \\
&- \left(\sum_{z\in Z_2} \vartheta^{\mathcal{L}^2}_{y+y_1+z} \boxtimes \vartheta^{\mathcal{L}^2}_{i-j'+z+z_3}\right)\cdot\left(\sum_{z\in Z_2} \vartheta^{\mathcal{L}^2}_{y-y_1+z} \boxtimes \vartheta^{\mathcal{L}^2}_{i'-j+z+z_4}\right).
\end{aligned}
$$

Since $M_{\mathcal{L}}$ is rank ≤ 2, $m_{(i,j),(i',j')}$ is identically zero on $A\times A$.

Suppose now $x\in K_2(\mathcal{L}^2)$. Then for $z,z'\in Z_2$, by [LB], Proposition 6.4.2, there is a constant C_x depending only on x such that

$$
\begin{aligned}
t_x^*\vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot t_x^*\vartheta^{\mathcal{L}^2}_{y-y_2+z'} &= C_x e^{\mathcal{L}^2}(y+y_2+z,x)\cdot e^{\mathcal{L}^2}(y-y_2+z',x)\cdot \vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot\vartheta^{\mathcal{L}^2}_{y-y_2+z'}\\
&= C_x e^{\mathcal{L}^2}(2y+z+z',x)\cdot \vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot\vartheta^{\mathcal{L}^2}_{y-y_2+z'}.
\end{aligned}
$$

Now

$$
\sum_{x\in K_2(\mathcal{L}^2)} e^{\mathcal{L}^2}(z+z',x) = \begin{cases} 0 & z+z'\neq 0,\\ \#K_2(\mathcal{L}^2) & z+z'=0,\end{cases}
$$

so

$$
\frac{1}{\#K_2(\mathcal{L}^2)}\sum_{x\in K_2(\mathcal{L}^2)} \frac{1}{C_x e^{\mathcal{L}^2}(2y,x)}\cdot t_x^*\vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot t_x^*\vartheta^{\mathcal{L}^2}_{y-y_2+z'} = \begin{cases} \vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot\vartheta^{\mathcal{L}^2}_{y-y_2+z} & z=z',\\ 0 & z\neq z'.\end{cases}
$$

The same holds if y_2 is replaced with y_1. Thus, if we denote translation by $(x,y)\in A\times A$ by $t_{(x,y)}$, then

$$
\begin{aligned}
n_{(i,j),(i',j')} :=& \frac{1}{\#K_2(\mathcal{L}^2)}\sum_{x\in K_2(\mathcal{L}^2)} \frac{1}{C_x e^{\mathcal{L}^2}(2y,x)}\cdot t^*_{(x,0)} m_{(i,j),(i',j')} = \\
&= \sum_{z\in Z_2} (\vartheta^{\mathcal{L}^2}_{y+y_2+z}\cdot\vartheta^{\mathcal{L}^2}_{y-y_2+z}\boxtimes \vartheta^{\mathcal{L}^2}_{i-j+z+z_1}\cdot\vartheta^{\mathcal{L}^2}_{i'-j'+z+z_2})\\
&\quad - \sum_{z\in Z_2} (\vartheta^{\mathcal{L}^2}_{y+y_1+z}\cdot\vartheta^{\mathcal{L}^2}_{y-y_1+z}\boxtimes \vartheta^{\mathcal{L}^2}_{i-j'+z+z_3}\cdot\vartheta^{\mathcal{L}^2}_{i'-j+z+z_4})
\end{aligned}
$$

Next, using the fact that if $z\in Z_2\subseteq K_1(\mathcal{L}^2)$, then $t_z^*\vartheta^{\mathcal{L}^2}_x = D_z\vartheta^{\mathcal{L}^2}_{x+z}$, where D_z is a constant depending only on z (cf. [LB], Proposition 6.4.2), we see that

$$\begin{aligned}
r_{(i,j),(i',j')} &= \sum_{z'\in Z_2} \frac{1}{D_{z'}} \rho(z') t^*_{(0,z')} n_{(i,j),(i',j')} \\
&= \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_2+z} \cdot \vartheta^{\mathcal{L}^2}_{y-y_2+z} \right) \boxtimes \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{i-j+z+z_1} \cdot \vartheta^{\mathcal{L}^2}_{i'-j'+z+z_2} \right) - \\
&\quad - \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_1+z} \cdot \vartheta^{\mathcal{L}^2}_{y-y_1+z} \right) \boxtimes \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{i-j'+z+z_3} \cdot \vartheta^{\mathcal{L}^2}_{i'-j+z+z_4} \right) \\
&= \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_2+z} \cdot \vartheta^{\mathcal{L}^2}_{y-y_2+z} \right) \boxtimes \left(\vartheta^{\mathcal{L}^4}_{y_1,\rho}(0) \cdot \vartheta^{\mathcal{L}^4}_{y',\rho} \right) - \\
&\quad - \left(\sum_{z\in Z_2} \rho(z) \vartheta^{\mathcal{L}^2}_{y+y_1+z} \cdot \vartheta^{\mathcal{L}^2}_{y-y_1+z} \right) \boxtimes \left(\vartheta^{\mathcal{L}^4}_{y_2,\rho}(0) \cdot \vartheta^{\mathcal{L}^4}_{y',\rho} \right),
\end{aligned}$$

the latter equality by the multiplication formula ([LB], Theorem 7.1.4), where y' is such that

$$y' = i - j - y_1 + z_1 = i' - j' + y_1 + z_2 = i - j' - y_2 + z_3 = i' - j + y_2 + z_4.$$

The latter equalities are a consequence of our choice for i, j, i' and j'. Finally, since $\vartheta^{\mathcal{L}^4}_{y',\rho}$ is not everywhere zero, we can divide through by this function and obtain the desired Riemann theta relation, from the fact that $r_{(i,j),(i',j')}$ is identically zero on A.

Lemma 2.6. *Let A be an abelian surface, and let $\mathcal{M}$ be a line bundle of type $(1, 2d)$ and characteristic zero with respect to some decomposition. Then there exists an abelian surface A' and a double covering $f : A' \to A$ such that $f^*\mathcal{M} \cong \mathcal{L}^2$, where $\mathcal{L}$ is a symmetric line bundle of type $(1, d)$ on A'. Furthermore, the decomposition on A for $\mathcal{M}$ induces a decomposition on A' for $f^*\mathcal{M}$ such that $f^{-1}(K_1(\mathcal{M})) = K_1(f^*\mathcal{M})$.*

Proof: An unbranched double cover of an abelian variety is determined by giving a two-torsion element τ of $\mathrm{Pic}^0(A) = \hat{A}$. We then obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & \\
 & & & \downarrow & & \downarrow & & & \\
 & & & K(f^*\mathcal{M}) & & K(\mathcal{M}) & & & \\
 & & & \downarrow & & \downarrow & & & \\
0 & \longrightarrow & G & \longrightarrow A' & \xrightarrow{f} & A & \longrightarrow & 0 & \\
 & & & \downarrow \phi_{f^*\mathcal{M}} & & \downarrow \phi_{\mathcal{M}} & & & \\
 & & 0 & \longleftarrow \hat{A}' & \xleftarrow{\hat{f}} & \hat{A} & \longleftarrow & \hat{G} & \longleftarrow 0 \\
 & & & \downarrow & & \downarrow & & & \\
 & & & 0 & & 0 & & &
\end{array}$$

where $\hat{G}$ is the subgroup of $\hat{A}$ generated by τ. This shows that

$$\begin{aligned} K(f^*\mathcal{M}) &= \phi^{-1}_{f^*\mathcal{M}}(0) \\ &= (\hat{f} \circ \phi_{\mathcal{M}} \circ f)^{-1}(0) \\ &= (\phi_{\mathcal{M}} \circ f)^{-1}(\hat{G}). \end{aligned}$$

Now, given that $\mathcal{M}$ is of type $(1, 2d)$, we have an exact sequence

$$0 \to \mathbf{Z}/2d\mathbf{Z} \oplus \mathbf{Z}/2d\mathbf{Z} \to \phi^{-1}_{\mathcal{M}}(\hat{G}) \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

This sequence is split if $\phi^{-1}_{\mathcal{M}}(\tau) \cap A_2$ is non-empty. Here A_2 denotes the set of two-torsion points of A. If the sequence splits, we must have

$$f^{-1}(\phi^{-1}_{\mathcal{M}}(\hat{G})) = (\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2d\mathbf{Z})^{\oplus 2},$$

whence $f^*\mathcal{M}$ is of type $(2, 2d)$.

Now we can write $A = V/\Lambda$ and $A' = V/\Lambda'$ for some $\Lambda' \subseteq \Lambda$. Furthermore, $\hat{A} = V/\Lambda(\mathcal{M})$. The decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ induces a decomposition $V = V_1 \oplus V_2$. Set $\Lambda_i(\mathcal{M}) = \Lambda(M) \cap V_i$.

To construct the desired A', choose a $\tau \in \frac{1}{2}\Lambda_2$, $\tau \notin \Lambda(\mathcal{M})$, representing a non-zero two-torsion element of $\hat{A}$ in $\phi_{\mathcal{M}}(A_2)$. Then $K_1(f^*\mathcal{M}) = \Lambda_1(\mathcal{M})/(\Lambda' \cap \Lambda_1(M)) = f^{-1}(K_1(\mathcal{M}))$. In addition, since $\mathcal{M}$ is of characteristic zero with respect to the decomposition, so is $f^*\mathcal{M}$. Furthermore, $A'_2 \subseteq K(f^*\mathcal{M})$. Thus by [LB], Exercise 6.12, $f^*\mathcal{M} = \mathcal{L}^2$ for some symmetric line bundle $\mathcal{L}$ on A'. •

Corollary 2.7. *Let A be an abelian surface and $\mathcal{M}$ be a line bundle of type $(1, 2d)$, $d \geq 2$, of characteristic zero with respect to some decomposition on A. Let $\psi_{\mathcal{M}} : A \to \mathbf{P}^{2d-1}$ be the map induced by the $\vartheta^{\mathcal{M}}_i$s, $i \in K_1(\mathcal{M})$. Then the $d \times d$-matrix*

$$M_d = (x_{i+j}y_{i-j} + x_{i+j+d}y_{i-j+d})_{2i,2j \in 2\mathbf{Z}/2d\mathbf{Z}}$$

has rank at most two on $(\psi \times \psi)(A \times A) \subseteq \mathbf{P}^{2d-1} \times \mathbf{P}^{2d-1}$.

Proof: Let $f : A' \to A$ be the double cover in Lemma 2.6. The decomposition on A induces a decomposition on A' for $\mathcal{L}^2$ which is compatible with f. By the Isogeny theorem, thinking of $K_1(f^*\mathcal{M}) = \mathbf{Z}/2\mathbf{Z} \oplus K_1(\mathcal{M})$,

$$f^*(\vartheta^{\mathcal{M}}_a) = \vartheta^{\mathcal{L}^2}_{(0,a)} + \vartheta^{\mathcal{L}^2}_{(1,a)},$$

for $a \in K_1(\mathcal{M})$. Thus

$$\begin{aligned}(f\times f)^*(\vartheta^{\mathcal{M}}_{i+j}\boxtimes\vartheta^{\mathcal{M}}_{i-j}+\vartheta^{\mathcal{M}}_{i+j+d}\boxtimes\vartheta^{\mathcal{M}}_{i-j+d})_{2i,2j\in 2K_1(\mathcal{M})} &=\\ =\Big((\vartheta^{\mathcal{L}^2}_{(0,i+j)}+\vartheta^{\mathcal{L}^2}_{(1,i+j)})\boxtimes(\vartheta^{\mathcal{L}^2}_{(0,i-j)}+\vartheta^{\mathcal{L}^2}_{(1,i-j)})+&\\ +(\vartheta^{\mathcal{L}^2}_{(0,i+j+d)}+\vartheta^{\mathcal{L}^2}_{(1,i+j+d)})\boxtimes(\vartheta^{\mathcal{L}^2}_{(0,i-j+d)}+\vartheta^{\mathcal{L}^2}_{(1,i-j+d)})\Big)_{2i,2j\in 2K_1(\mathcal{M})}&\\ = M_{\mathcal{L}}+(\sigma_{(1,0)}\times 1_{A'})^*(M_{\mathcal{L}}),&\end{aligned}$$

where $\sigma_{(1,0)}: A'\to A'$ is translation by $(1,0)\in K_1(\mathcal{L}^2)$, $f\times f: A'\times A'\to A\times A$ is the cartesian product of f with itself, and $\mathcal{M}_{\mathcal{L}}$ is the matrix of Corollary 2.4. Since $M_{\mathcal{L}}$ has rank at most one, so does $(\sigma_{(1,0)}\times 1_{A'})^*(M_{\mathcal{L}})$. Thus their sum has rank at most two. •

Corollary 2.8. *Let A be an abelian surface and let $\mathcal{M}$ be a line bundle of type $(1,2d+1)$, $(d\geq 2)$, of characteristic zero with respect to some decomposition on A. Let $\psi_{\mathcal{M}}: A\to \mathbf{P}^{2d}$ be the map induced by the sections $\vartheta^{\mathcal{M}}_i$, $i\in K_1(\mathcal{M})$. Then the $(2d+1)\times(2d+1)$ matrix*

$$M'_d=(x_{(d+1)(i+j)}y_{(d+1)(i-j)})_{i,j\in\mathbf{Z}/(2d+1)\mathbf{Z}}$$

has rank at most 4 on $(\psi\times\psi)(A\times A)\subseteq\mathbf{P}^{2d}\times\mathbf{P}^{2d}$.

Proof: If $f: A'\to A$ is an arbitrary double cover, then $\mathcal{L}=f^*\mathcal{M}$ must be of type $(1,4d+2)$ since $2d+1$ is odd. As in the proof of Lemma 2.6, it is possible to choose the double cover and a decomposition on A' for $f^*\mathcal{M}$ which is compatible with f, in such a way so that $f^{-1}(K_1(\mathcal{M}))=K_1(\mathcal{L})=\mathbf{Z}/(4d+2)\mathbf{Z}$. Therefore, by the Isogeny theorem,

$$f^*\vartheta^{\mathcal{M}}_x=\sum_{\substack{y\in K_1(\mathcal{L})\\ f(y)=x}}\vartheta^{\mathcal{L}}_y=\vartheta^{\mathcal{L}}_{2x}+\vartheta^{\mathcal{L}}_{2x+2d+1},$$

where $x\in K_1(\mathcal{M})\cong\mathbf{Z}/(2d+1)\mathbf{Z}$, and the indices in the last sum are considered modulo $4d+2$. Now

$$\begin{aligned}&(f\times f)^*(\vartheta^{\mathcal{M}}_{(d+1)(i+j)}\boxtimes\vartheta^{\mathcal{M}}_{(d+1)(i-j)})=\\ &=\ \vartheta^{\mathcal{L}}_{i+j}\boxtimes\vartheta^{\mathcal{L}}_{i-j}+\vartheta^{\mathcal{L}}_{i+j+2d+1}\boxtimes\vartheta^{\mathcal{L}}_{i-j+2d+1}+\vartheta^{\mathcal{L}}_{i+j+2d+1}\boxtimes\vartheta^{\mathcal{L}}_{i-j}+\vartheta^{\mathcal{L}}_{i+j}\boxtimes\vartheta^{\mathcal{L}}_{i-j+2d+1},\end{aligned}$$

since, thinking of $0\leq i,j\leq 2d$, we have $4(d+1)(i+j)=2(i+j)\bmod 4d+2$, whence $2(d+1)(i+j)=i+j$, or $i+j+2d+1\bmod 4d+2$. Thus

$$(f\times f)^*(\vartheta^{\mathcal{M}}_{(d+1)(i+j)}\boxtimes\vartheta^{\mathcal{M}}_{(d+1)(i-j)})_{i,j\in\mathbf{Z}/(2d+1)\mathbf{Z}}=M_{\mathcal{L}}+(\sigma_{2d+1}\times 1_{A'})^*M_{\mathcal{L}}.$$

Since these two matrices have rank at most two, their sum has rank at most four on $A'\times A'$, and hence also the original matrix has rank at most four on $A\times A$. •

Corollary 2.9. *Let E be an elliptic curve and let $\mathcal{M}$ be a line bundle of degree $2d+1$, $d \geq 2$, of characteristic zero with respect to some decomposition on E. Let $\psi_{\mathcal{M}} : E \to \mathbf{P}^{2d}$ be the map induced by $\vartheta_i^{\mathcal{M}}$, $i \in K_1(\mathcal{M})$. Then the $(2d+1) \times (2d+1)$-matrix*

$$M'_d = (x_{(d+1)(i+j)} y_{(d+1)(i-j)})_{i,j \in \mathbf{Z}/(2d+1)\mathbf{Z}}$$

has rank at most 2 on $(\psi \times \psi)(E \times E) \subseteq \mathbf{P}^{2d} \times \mathbf{P}^{2d}$.

Proof: The proof is exactly the same as the proof of Corollary 2.8. •

We remark that similar results hold for abelian varieties of dimension g and polarization of type $(1, 1, \ldots, 1, n)$, but now the above matrices will have much higher rank (depending exponentially on g) on these varieties, and it is not clear if much information is obtained.

Example 2.10. Let E be an elliptic curve endowed with a symmetric line bundle $\mathcal{L}$ of degree 7. Embedding $E \times E$ into $\mathbf{P}^6 \times \mathbf{P}^6$ via the canonical basis of $\mathcal{L} \boxtimes \mathcal{L}$, we find that the matrix M'_3 in Corollary 2.9 has rank at most 2 on $E \times E$. Therefore substituting in the matrix the coordinates of a point $(y_0 : y_1 : \ldots : y_6) \in E$, we obtain a 7×7-matrix $M'_3(y)$ with linear entries in $x_0, \ldots, x_6$ whose 3×3-minors vanish on the elliptic curve normal curve $E \subset \mathbf{P}^6$. On the other side, if we substitute for $(y_0 : y_1 : \ldots : y_6)$ the coordinates of the origin of E , which is the only point of the intersection $E \cap (\mathbf{P^2})^-$, then the matrix $M'_3(y)$ becomes skew-symmetric. Therefore, the 4×4-pfaffians of $M'_3(y)$ provide then 35 quadratic polynomials vanishing on E, which in fact generate the homogeneous ideal of E. To see this recall first that $h^0(\mathcal{I}_E(2)) = 14$ since $E \subseteq \mathbf{P}^6$ is projectively normal, and that $H^0(\mathcal{I}_E(2))$ decomposes as an $\mathbf{H}_7$-module as the direct sum of two mutually isomorphic 7-dimensional representations (of weight 2) of $\mathbf{H}_7$. Thus in order to prove our claim we need to determine the linear span of the τ-invariant pfaffians of the skew-symmetric matrix $M'_3(y)$. There are exactly four such pfaffians, namely

$$\begin{array}{ll} x_0^2 y_2 y_3 + x_1 x_6 y_3^2 - x_3 x_4 y_1^2 & x_0^2 y_1 y_3 - x_1 x_6 y_2^2 + x_2 x_5 y_1^2 \\ x_0^2 y_1 y_2 - x_3 x_4 y_2^2 + x_2 x_5 y_3^2 & x_1 x_6 y_1 y_2 - x_3 x_4 y_1 y_3 - x_2 x_5 y_2 y_3. \end{array}$$

Their matrix of coefficients in terms of the τ-invariant quadratic monomials x_0^2, $x_1 x_6$, $x_2 x_5$, $x_3 x_4$ is, up to column operations, the skew symmetric matrix

$$\begin{pmatrix} 0 & y_1 y_2 & -y_2 y_3 & -y_1 y_3 \\ -y_1 y_2 & 0 & -y_3^2 & y_2^2 \\ y_2 y_3 & y_3^2 & 0 & -y_1^2 \\ y_1 y_3 & -y_2^2 & y_1^2 & 0 \end{pmatrix}.$$

Since the entries of this matrix do not have any common zeroes, we deduce that the 4×4-pfaffians of the matrix $M_3'(y)$ span precisely a 14-dimensional subspace of $H^0(\mathcal{O}_{\mathbf{P^6}}(2))$ if and only if the point $(0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1)$ lies on the quartic plane curve K defined by the pfaffian of the coefficient matrix:

$$K := \{(y_1 : y_2 : y_3) \mid y_1^3 y_2 - y_2^3 y_3 - y_3^3 y_1 = 0\} \subset (\mathbf{P^2})^-.$$

This smooth quartic, which is in fact the unique $PSL(2, \mathbf{Z}_7)$-invariant of degree ≤ 4 in $(\mathbf{P^2})^-$ and is projectively equivalent to the Klein quartic

$$K' := \{(y_1 : y_2 : y_3) \in \mathbf{P^2} \mid y_1^3 y_2 + y_2^3 y_3 + y_3^3 y_1 = 0\},$$

is therefore the isomorphic image in $\mathbf{P}^6$ of the modular curve $X(7)$, represented here by the 0-section of $S(7)$, the Shioda surface of level 7 (compare [Kl], [Ve]). In particular this proves our claim about the homogeneous ideal of E. The equations above for an elliptic normal curve in $\mathbf{P}^6$ (up to certain linear combinations) were first found by Klein (cf. [Kl], [KlF], [Ve]). Moreover, the above setting generalizes easily to all elliptic normal curves of odd degree, giving rise to equations similar to those in [KlF], p. 245–246, [Kl2], and [Ve], Proposition 5.8. We refer also to [GP] for a detailed discussion of the geometry associated with $X(11)$.

We remark further that $Sec(E)$, the closure of the chordal variety to E, is in fact contained in the locus V defined by the 6×6-pfaffians of the matrix $M_3'(y)$, for the same choice as above of y as origin of E. Restricting to a 2-dimensional subspace (e.g. $\{x_0 = x_1 = x_3 = x_5 = 0\}$) it is easily seen that V has the expected codimension 3, whence also degree 14 by Porteous' formula. Since, on the other hand $Sec(E)$ is an irreducible 3-fold of degree 14 (compare Proposition 5.1) we deduce that the 6×6-pfaffians of the matrix $M_3'(y)$ indeed cut out $Sec(E)$.

Finally let now $(y_0 : y_1 : \ldots : y_6)$ be a general point on the elliptic normal curve $E \subset \mathbf{P}^6$. In particular, we may assume for its coordinates that $y_i \neq 0$, for all $i \in \mathbf{Z}_7$. Then the determinant of the matrix $M_3'(y)$ is a non-zero septic polynomial F (e.g. since the coefficient of x_0^7 in F is $-\prod_{i=0}^6 y_i \neq 0$) which vanishes on the variety $Sec_3(E) \subset \mathbf{P}^6$ of trisecant planes to the elliptic normal curve E. On the other hand $Sec_3(E)$ is an irreducible fivefold, hence an irreducible hypersurface in $\mathbf{P}^6$, whose equation being also $\mathbf{H}_7$-invariant must have degree divisible by 7. It follows necessarily that $Sec_3(E) = \{F = 0\}$.

In the remainder of this section, we relate the above approach to the main result of [EKS], which in turn enables us to give a more geometrical explanation for the above matrices. Let $X \subseteq \mathbf{P}^n = \mathbf{P}(V^*)$ be a non-degenerate, reduced, irreducible scheme and assume we can write $\mathcal{L} = \mathcal{O}_X(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$ for suitable line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X. Suppose also that $V_i \subset H^0(\mathcal{L}_i)$, $v_i = \dim V_i$, $i = 1, 2$, are linear series such that the image of $V_1 \otimes V_2$ through the multiplication map

$$\mu : H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \to H^0(\mathcal{L})$$

is contained in $V \subset H^0(\mathcal{L})$. Then the homogeneous ideal I_X of $X \subset \mathbf{P}(V^*)$ contains the 2×2-minors of the 1-generic $v_1 \times v_2$-matrix M with entries linear forms on V corresponding to the multiplication morphism $V_1 \otimes V_2 \to V$. As usual, 1-generic means that all generalized entries of M are non-zero. In particular, if $X \subseteq \mathbf{P}^n$ is linearly normal and D is an effective divisor on X which moves in a linear system of (affine) dimension v, and whose linear span in $\mathbf{P}^n$ has codimension w, then I_X contains the 2×2 minors of a $v \times w$-matrix $M_{|D|}$ with linear entries.

Conversely, if the homogeneous ideal of X contains the 2×2-minors of the 1-generic, $v_1 \times v_2$-matrix M of linear forms, associated to a pairing $V_1 \otimes V_2 \to V$, then we can recover line bundles and linear series as above through the following correspondence:

$$\begin{aligned}
\mathcal{L}_1 &= \mathrm{im}(M : \mathcal{O}_X^{\oplus v_1} \to \mathcal{O}_X^{\oplus v_2}(1)) \\
\mathcal{L}_2 &= \mathrm{im}(M^t : \mathcal{O}_X^{\oplus v_2} \to \mathcal{O}_X^{\oplus v_1}(1)) \\
V_1 &= \mathrm{im}(M : H^0(\mathcal{O}_X^{\oplus v_1}) \to H^0(\mathcal{L}_1)) \\
V_2 &= \mathrm{im}(M^t : H^0(\mathcal{O}_X^{\oplus v_2}) \to H^0(\mathcal{L}_2))
\end{aligned}$$

The main result of [EKS] asserts that if X is a reduced, irreducible curve of genus g and $\mathcal{L}_i$, $i = 1, 2$, are line bundles on X of degrees at least $2g+1$, nonisomorphic in case both have degree $2g + 1$ and $g > 0$, then the 2×2-minors of the matrix defined above generate the homogeneous ideal I_X of X embedded via $\mathcal{L}_1 \otimes \mathcal{L}_2$. In particular, this provides a way to write down the equations of an elliptic normal curve as the 2×2-minors of a matrix with catalecticant blocks (see [EKS], §3 (c)).

In this context, it is easy to determine to which splitting of the embedding of an elliptic normal curve do correspond the determinantal equations in Corollary 2.2. It is precisely the choice of a level structure which accounts for the nice symmetric form of the matrices involved. We'll discuss only the following case:

Let $E \subseteq \mathbf{P}^{2d-1}$ be a Heisenberg invariant normal curve of degree $2d$, with $d \geq 3$ and $d \equiv 1 \bmod 2$, and let $y = (y_0 : \ldots : y_{2d-1}) \in E \subset \mathbf{P}^{2d-1}$ be a point on the curve. Recall now from §1.1 that the generators σ and τ of $\mathbf{H}_{2d}$ act on E, via the Schrödinger representation, by translation with genuine $(2d)$-torsion elements, say ρ and ν.

Now let D be the divisor $D = \sum_{i=0}^{d-1} \tau^{2i}(y')$ of degree d on E, where $y' := \sigma^d \tau^d(y) = (y_d : -y_{d+1} : \ldots : -y_0 : y_1 \ldots) \in E$. On the other hand, since $\sum_{i=0}^{d-1} \tau^{2i}(y') \equiv d \cdot y' + \sum_{i=0}^{d-1}(2i) \cdot \nu \equiv d(d-1)\nu + d \cdot y'$ and $d \cdot y' \equiv d \cdot \sigma^d \tau^d(y) \equiv d \cdot y + d^2 \cdot \rho + d^2 \cdot \nu \equiv d \cdot y + d(\rho + \nu)$ in the group law of the elliptic curve E, we deduce that

$$\mathcal{L}_1 := \mathcal{O}_E(D) = \mathcal{O}_E(d \cdot y + \gamma),$$

where $\gamma = d \cdot (\rho + \nu)$ is a non-trivial 2-torsion point on E. Moreover, since in the group law $2kd \cdot \rho \equiv 0$ for all $k \in \mathbf{N}$, the set $\{D_k := \sigma^{2k}(D) \mid k \in \{0, \ldots, d-1\}\}$ is a basis of sections for the linear series $H^0(\mathcal{O}_E(D))$.

The linear span Π_0 of the divisor $D_0 := D$ is a d-secant $\mathbf{P}^{d-1}$ to E, whose equations are

$$\begin{aligned}\Pi_0 &= \{x_0 y_0 + x_d y_d = x_1 y_1 + x_{d+1} y_{d+1} = \ldots = x_{d-1} y_{d-1} + x_{2d-1} y_{2d-1} = 0\} \\ &= \{x_i y_i + x_{i+d} y_{i+d} = 0 \mid 2i \in 2\mathbf{Z}/2d\mathbf{Z}\}.\end{aligned}$$

Similarly, the linear span of the divisor D_k, $k \in \{0, \ldots, d-1\}$, is the d-secant $\mathbf{P}^{d-1}$ to the curve $E \subset \mathbf{P}^{2d-1}$ defined by $\Pi_k := \sigma^{2k}(\Pi_0)$. The chosen elliptic curve $E \subset \mathbf{P}^{2d-1}$ is embedded by the line bundle $\mathcal{L} := \mathcal{O}_E((2d)o_E)$, therefore we can take as a complementary line bundle $\mathcal{L}_2 := \mathcal{L} \otimes \mathcal{L}_1^{-1} = \mathcal{O}_E(\iota(D))$, where ι is the Heisenberg involution. This time, a basis of sections for $H^0(\mathcal{L}_2)$ is the set $\{\iota(D_k) \mid k \in \{0, \ldots, d-1\}\}$. The linear span of a divisor $\iota(D_k)$ is $\Pi'_k := \iota(\Pi_k)$, the d-secant $\mathbf{P}^{d-1}$ to E defined by

$$\Pi'_k = \{x_{k+i} y_{k-i} + x_{k+i+d} y_{k-i+d} = 0,\ 2i \in 2\mathbf{Z}/2d\mathbf{Z}\}.$$

Given the above chosen bases for $H^0(\mathcal{L}_i)$, $i = 1, 2$, the $d \times d$-matrix of linear forms corresponding to the multiplication map $H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \to H^0(\mathcal{L})$ has as its (i,j)-entry the equation of the hyperplane $\mathrm{span}_{\mathbf{C}}(\Pi_j, \Pi'_i)$, namely $x_{i+j} y_{i-j} + x_{i+j+d} y_{i-j+d}$. In other words, we obtain the matrix M_d in Corollary 2.2, whose 2×2-minors generate the homogeneous ideal of the elliptic curve $E \subset \mathbf{P}^{2d-1}$.

Example 2.11. Let $E \subset \mathbf{P}^5$ be a Heisenberg invariant elliptic normal curve, and let $y = (y_0 : \ldots : y_5) \in E \subset \mathbf{P}^5$ be a point on it. We've seen above that E is cut out by the 2×2-minors of the matrix

$$M_3(y) = \begin{pmatrix} x_0 y_0 + x_3 y_3 & x_1 y_5 + x_4 y_2 & x_2 y_4 + x_5 y_1 \\ x_1 y_1 + x_4 y_4 & x_2 y_0 + x_5 y_3 & x_3 y_5 + x_0 y_2 \\ x_2 y_2 + x_5 y_5 & x_3 y_1 + x_0 y_4 & x_4 y_0 + x_1 y_3 \end{pmatrix}$$

The two nets, say $\mathcal{F}_1$ and $\mathcal{F}_2$, of trisecant planes to $E \subset \mathbf{P}^5$ defined by generalized rows or columns respectively, both trace out in $\mathbf{P}^5$ the cubic hypersurface $V_y = \{\det M_3(y) = 0\}$. In particular, V_y depends only on the chosen decomposition of the line bundle $\mathcal{O}_E(6o_E)$, i.e., only on the point y, and not on our choices for bases. Analyzing further this example, we have the following results due to Veneroni and Room (see [Ro], 7.11, (ii) and 9.22)

Proposition 2.12 (Room). *$\{V_y\}_{y\in E}$ is a linear pencil. Furthermore, $Sec(E)$ is the complete intersection of any two members in this pencil. In particular, any cubic hypersurface having multiplicity two along E is a member of the above pencil, and hence is determinantal.*

Proof. Each cubic V_y is singular along E, and thus by Bézout contains any secant line to the elliptic curve. On the other hand, it is easily seen that a trisecant plane to E which meets a plane of the net $\mathcal{F}_1$ ($\mathcal{F}_2$) in a point outside $Sec(E)$ is necessarily a $\mathbf{P}^2$ in the net $\mathcal{F}_2$ ($\mathcal{F}_1$, respectively). In particular, two trisecant planes contained in different V_y's either meet on $Sec(E)$, or they are disjoint. Therefore any two different cubics V_y and $V_{y'}$ intersect properly, and so the claim follows since $\deg Sec(E) = 9$ (see Proposition 5.1). •

Remark 2.13.

i) *The matrix $M_3(y)$ is symmetric (up to row and column operations) if and only if y is a 2-torsion point of E. In each of these four cases the 2×2-minors of the matrix $M_3(y)$ cut out a Veronese surface in $\mathbf{P}^5$ containing E; the cubic hypersurface V_y is the secant variety of this Veronese surface.*

ii) *$\sigma(V_y) = V_{y+(2-\text{torsion})}$ and $\tau(V_y) = V_{y+(2-\text{torsion})}$. In particular, V_y is invariant under the action of the subgroup $\mathbf{H}' \subseteq \mathbf{H}_6$ generated by σ^2 and τ^2. The four cubic hypersurfaces V_y, $\sigma(V_y)$, $\tau(V_y)$ and $\sigma\tau(V_y)$ span only a pencil, whose base locus is $Sec(E)$.*

Proof. For part *i)* observe that $M_3(y)$ is symmetric if and only if the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide, that is exactly when y is a 2-torsion point on E. See [SR] for the assertions concerning the Veronese surfaces. Part *ii)* is easy and left to the reader. •

§3. Degenerations of abelian surfaces and elliptic curves.

We will need the following facts about degenerations of elliptic curves and abelian surfaces.

Definition. *Let $E \subseteq \mathbf{P}^{n-1}$ be a Heisenberg invariant elliptic normal curve of degree n. For a point $\tau \in E$, τ not a 2-torsion point of E, the surface*

$$S_{E,\tau} := \bigcup_{P\in E} \langle P, P+\tau\rangle$$

is called a translation scroll, where $\langle P, P+\tau\rangle$ denotes the line spanned by P and $P+\tau$.

Theorem 3.1. *Let $n \geq 5$. Let $S_{E,\sigma}$ be a translation scroll, with $\sigma \in E$ general. Then there exists a flat family $\mathcal{A} \to \Delta$, a point $0 \in \Delta$, along with a Heisenberg invariant* *embedding $\mathcal{A} \subseteq \mathbf{P}^{n-1}_\Delta$ such that $\mathcal{A}_0 \cong S_{E,\sigma}$ and $\mathcal{A}_t$ is a non-singular abelian surface for $t \in \Delta$, $t \neq 0$.*

Proof. This follows from the results of [DHS]. More precisely, let

$$\Omega_\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ 1 & 0 \\ 0 & n \end{pmatrix}$$

be the normalized period matrix for an abelian surface with a polarization of type $(1, n)$, where $\tau = \tau^t$ is a 2×2 symmetric complex matrix with $\operatorname{Im} \tau > 0$. Then [DHS], Proposition 14 and §3.4, yield a family $\mathcal{A}_V$ over $V = \operatorname{Spec} \mathbf{C}[T_1, T_2]$ such that for general $(T_1, T_2) \in V$, the fibre is a smooth abelian surface with $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$, for some τ_{11} and τ_{22}, and the fibre over $(0, T_2)$, $T_2 \neq 0$, is a translation scroll associated to an elliptic curve

$$E = \mathbf{C}/(\mathbf{Z}n + \mathbf{Z}\tau_{22})$$

for some τ_{22} depending on T_2, and each possible τ_{22} occurs. Furthermore the element in E yielding the translation scroll is given by $x = [\tau_{12}] \in E$. In this way one obtains all possible translation scrolls. In [DHS], §4, for a given neighborhood Δ of a point $(0, T_2) \in V$, a line bundle $\mathcal{L}$ is constructed on $\mathcal{A}_\Delta$, along with sections $\vartheta_0, \ldots, \vartheta_{n-1} \in H^0(\mathcal{L})$ which are invariant under the action of the Heisenberg group. It is then shown in §5 of [DHS] that these sections induce an embedding $\mathcal{A}_\Delta \hookrightarrow \mathbf{P}^{n-1}_\Delta$ for general τ_{12}, if $n \geq 5$. This gives the desired result. •

We also need similar results about degenerations of elliptic curves with level n structure.

Definition. *We call*

$$X(\Gamma_n) = \bigcup_{i \in \mathbf{Z}_n} l_{i,i+1} = \bigcup_{i \in \mathbf{Z}_n} \langle e_i, e_{i+1} \rangle,$$

the standard n-gon where $l_{i,i+1}$ is the line where all coordinates are zero except for the i-th and $(i+1)$-st in $\mathbf{P}^{n-1}$. It is a cycle of n lines.

Theorem 3.2. *There is a family $\mathcal{E} \subseteq \mathbf{P}^{n-1}_\Delta$ such that $\mathcal{E}_0 = X(\Gamma_n)$ and $\mathcal{E}_t$ is a non-singular, $\mathbf{H}_n$-invariant elliptic normal curve of degree n.*

Proof: This is standard, accomplished by embedding a Tate curve (see [DR], §1) in $\mathbf{P}^{n-1}$. See [Ste], Theorem 2.3, for the precise equations of this degeneration. •

Definition. *We set $S_n \subseteq \mathbf{P}^{n-1}$ to be the closure of the union of all $\mathbf{H}_n$-invariant elliptic normal curves in $\mathbf{P}^{n-1}$.*

Note that by §1.3, S_n is an irreducible surface, and by Theorem 3.2 contains $X(\Gamma_n)$, the standard n-gon. If n is odd, the normalization of S_n is the Shioda surface of level n.

§4. Toric degenerations

In this section we'll describe a class of toric degenerations of abelian surfaces carrying a polarization of type $(1, d)$, and also certain classes of combinatorially defined "varieties" with trivial canonical sheaf. We start by recalling some definitions.

As usual, a (lattice) polytope in $\mathbf{R}^n$ is the convex hull of a finite subset of $\mathbf{Z}^n$.

A finite (integral) polyhedral complex Δ is a finite set of lattice polytopes in $\mathbf{R}^n$, such that any face of a polytope in Δ is a polytope in Δ, and such that any two of the polytopes in Δ intersect in a face of each of them. The polytopes in Δ will be called the faces of Δ. The maximal faces will be called facets. In particular, we will denote by $F(\Delta)$ the set of facets of Δ. In the special case when Δ consists of a single polytope P we will denote by ∂P the boundary complex, which is the complex formed by all proper faces of P.

As a matter of notation, the f-vector of a d-dimensional polyhedral complex Δ is the vector in $\mathbf{N}^{d+1}$ given by $f(\Delta) = (f_0, f_1, \ldots, f_d)$, where $f_k = f_k(\Delta)$ denotes the number of k-dimensional faces in Δ. Finally by the f-vector of a polytope P we mean the f-vector of its boundary complex ∂P.

Recall that if Δ is a simplicial complex on the vertex set $V = \{v_0, \ldots v_n\}$, the corresponding *face (ring) variety* $X(\Delta)$ in the sense of Stanley, Hochster and Reisner (see [Ho] for more details) is the variety defined by the ideal I_Δ in the projective space $\mathbf{P}^n = \operatorname{Proj} \mathbf{C}[x_0, \ldots x_n]$, whose coordinates correspond to the vertices of Δ, and the ideal I_Δ consists of monomials corresponding to the simplexes (faces) not contained in Δ. It is easily seen that the dimension of $X(\Delta)$ as a projective variety is the dimension of the simplicial complex Δ. The ideal I_Δ is generated by square free monomials, and in fact the above construction describes a 1:1 inclusion-reversing correspondence between simplicial complexes on the vertex set V and ideals $I \subseteq (x_0, \ldots, x_n)^2$ generated by square-free monomials.

In the special case when the topological realization $|\Delta|$ is a manifold, it follows from the work of Reisner [Re], Hochster and Roberts [HR] and Stanley [Sta] that $X(\Delta)$ is a Gorenstein scheme. Moreover, the following isomorphisms hold:

$$H^i(X(\Delta), \mathcal{O}_{X(\Delta)}(n)) = \begin{cases} H^i(\Delta, \mathbf{C}) & \text{for } n = 0 \text{ and all } i > 0; \\ 0 & \text{for all } n \neq 0 \text{ and } 0 < i < \dim \Delta, \end{cases}$$

while

$$H^1(\mathbf{P}^n, \mathcal{I}_{X(\Delta)}(n)) = \begin{cases} \widetilde{H}^0(\Delta, \mathbf{C}) & \text{if } n = 0; \\ 0 & \text{for all } n \neq 0. \end{cases}$$

In particular, $X(\Delta)$ is projectively Cohen-Macaulay if and only if Δ has no reduced cohomology below $\dim \Delta$, and $X(\Delta)$ is projectively Gorenstein if and only if $|\Delta|$ is a homology sphere. Moreover, it follows from [Ho] and [BE] that the canonical bundle $\omega_{X(\Delta)}$ is 2-torsion, and moreover that it is trivial if and only if $|\Delta|$ is an orientable manifold.

In terms of projective invariants $X(\Delta)$ has degree $\deg X(\Delta) = f_d(\Delta)$ and arithmetic sectional genus $\pi(X(\Delta)) = df_d(\Delta) - f_{d-1}(\Delta) + 1$, where $d = \dim \Delta$. Also, as expected, $\chi(\mathcal{O}_{X(\Delta)}) = \chi(\Delta) = \sum_{i=0}^{d} (-1)^i f_i(\Delta)$. In fact the Hilbert polynomial, which coincides with the Hilbert function for strictly positive values, is completely determined by the combinatorial data (see [Ho], or [BH] for more details).

A somewhat classical example of a face ring variety is the following:

Degenerations of elliptic normal curves. Let Γ_{n+1} denote the triangulation of the circle S^1 corresponding to an $(n+1)$-gon, whose vertices are labeled in a counterclockwise manner by x_i, $i \in \mathbf{Z}_{n+1} = \mathbf{Z}/(n+1)\mathbf{Z}$. Then $X(\Gamma_{n+1}) \subseteq \mathbf{P}^n$ is a projectively Gorenstein 1-dimensional scheme of degree $n+1$ and arithmetic genus 1. In case $n = 2$, $X(\Gamma_3) = \{x_0 x_1 x_2 = 0\} \subseteq \mathbf{P}^2$ is just one of the triangles in the Hesse pencil. To simplify notation in the general case we introduce the following:

Definition. *If $k, l \in \mathbf{Z}_m$, we define the distance between k and l to be*

$$d(k,l) = \min\{|\bar{k} - \bar{l}| \mid \bar{k}, \bar{l} \in \mathbf{Z} \textit{ representing } k \textit{ and } l \textit{ respectively}\}.$$

It is now easy to see that the homogeneous ideal $I_{X(\Gamma_{n+1})}$, $n \geq 3$, is generated by all quadratic monomials $x_i x_j$, where $i, j \in \mathbf{Z}_{n+1}$ with $d(i,j) \geq 2$. There are exactly $(n^2 - n - 2)/2$ such monomials. It is also not hard to see that

$$X(\Gamma_{n+1}) = \cup_{i \in \mathbf{Z}_{n+1}} l_{i,i+1} \subseteq \mathbf{P}^n,$$

where $l_{i,i+1} = \langle e_i, e_{i+1} \rangle = \{x_0 = x_1 = \ldots = x_{i-1} = x_{i+2} = \ldots = x_n = 0\}$ is the line joining the vertices e_i and e_{i+1} of the standard simplex. This fits with the notation of §3. Finally we remark that $X(\Gamma_{n+1})$ is invariant under the action of the extended Heisenberg group $\mathbf{H}^e_{n+1}$ via the Schrödinger representation.

In the sequel, we will be interested in giving a satisfactory description of the "secant varieties" of the "elliptic normal curves" $X(\Gamma_{n+1}) \subseteq \mathbf{P}^n$.

Cyclic polytopes. These polytopes were first introduced by Carathéodory and then rediscovered and studied by Gale, Motzkin and Klee among others (see [Gr] for details and historical remarks). The real algebraic curve $C_d \subseteq \mathbf{R}^d$, parametrized by

$$t \mapsto x(t) = (t, t^2, \ldots, t^d), \qquad t \in \mathbf{R},$$

is called the moment curve. The cyclic d-polytope $C(n,d)$ is defined as the convex hull of n distinct points $x_i = x(t_i)$, $t_0 < t_1 < \ldots t_{n-1}$, $n \geq d+1$, chosen on the moment curve C_d. Here are some relevant facts about cyclic polytopes:

- $C(n,d)$ is a d-dimensional simplicial polytope (any $d+1$ points on C_d being affinely independent). In particular the boundary complex $\Delta(n,d) := \partial C(n,d)$ provides a triangulation of the corresponding sphere: $|\Delta(n,d)| \cong S^{d-1}$.
- A d-uple W of points in $V_{n,d} = \{x_0, \ldots, x_{n-1}\}$, the set of vertices of $C(n,d)$, spans a facet of the cyclic polytope if and only if any two vertices in $V_{n,d} \setminus W$ are separated on the moment curve C_d by an even number of points of W. This criterion is called "Gale's evenness condition". A similar description holds for the lower dimensional faces of the cyclic polytope (see e.g. [BH], Theorem 5.2.11) and shows that $C(n,d)$ is well defined as a combinatorial equivalence class of polytopes.
- $C(n,d)$ is $[\frac{d}{2}]$-neighbourly, that is the convex hull of any $j+1$ vertices of $C(n,d)$ is a face of the polytope, for all $j \leq [\frac{d}{2}] - 1$. In particular, $C(n,d)$ is a polytope without diagonals when $d \geq 4$.
- Gale's evenness condition above implies that

$$f_{n-1}(C(n,d)) = \begin{cases} \frac{n}{n-k}\binom{n-k}{k} & \text{if } d = 2k; \\ 2\binom{n-k-1}{k} & \text{if } d = 2k+1. \end{cases}$$

In particular, for a cyclic 4-polytope:

$$f_0(C(n,4)) = n, \quad f_1(C(n,4)) = \binom{n}{2}, \quad f_2(C(n,4)) = n(n-3), \quad f_3(C(n,4)) = \frac{1}{2}n(n-3).$$

For $d = 2$, the cyclic polytopes $C(n,2)$ are combinatorially equivalent to the n-gons Γ_n introduced above. When $d = 4$, the boundary complexes of the cyclic polytopes provide triangulations of the 3-sphere, and therefore their associated face varieties will have the same numerical type as a Calabi-Yau threefold. More precisely:

Proposition 4.1. *$X(\partial C(n,4)) \subseteq \mathbf{P}^{n-1}$, $n \geq 5$, is a 3-dimensional projectively Gorenstein scheme of degree $n(n-3)/2$ and arithmetic sectional genus $n(n-3)/2+1$, with trivial Euler characteristic and trivial canonical sheaf. Moreover*
i) $X(\partial C(n,4))$ is the "secant variety" of the "elliptic normal" curve $X(\Gamma_n) \subseteq \mathbf{P}^{n-1}$.
ii) $X(\partial C(5,4)) = \{x_0x_1x_2x_3x_4 = 0\}$, while for $n \geq 6$ the homogeneous ideal $I_{X(\partial C(n,4))}$ is generated by the cubic monomials

$$x_ix_jx_k, \qquad \text{with } d(i,j) \geq 2,\ d(i,k) \geq 2 \text{ and } d(j,k) \geq 2.$$

There are $n(n^2-9n+20)/6$ such monomials.

Proof. The proof is easy and left to the reader. By "secant variety" to $X(\Gamma_n)$ we mean the Zariski closure of

$$\bigcup_{\substack{x,y \in X(\Gamma_n)\\ x \neq y}} \langle x,y \rangle,$$

where $\langle x,y\rangle$ is the line spanned by x and y. Statements (i) and (ii) are immediate consequences of Gale's evenness condition stated above. •

We remark here for later reference that the coordinate ring of $X(\partial C(n,2d))$, for $n \geq 6$ and $d \geq 2$, is an extremal graded Gorenstein algebra in the sense of [Sch] (or a compressed or extremally compressed Gorenstein algebra of type z^d in the sense of [FL]). Let A be a graded Gorenstein k-algebra (k a field) with $A_0 \cong k$, and generated by its degree one elements. Let $r = \dim_k(A_1)$ be its embedding dimension, write A as a quotient $k[x_1,\ldots,x_r]/I$ and define $d(I)$ (or $d(A)$) to be $\min\{t \mid I_t \neq 0\}$. Then the following inequality holds, see [Sch], [FL]:

$$i(A) + \dim A \geq 2d(A) - 1,$$

where $i(A)$ is the index of regularity (that is the degree from which the Hilbert polynomial and the Hilbert function of A start to agree), and $\dim A$ is the Krull dimension. The Gorenstein algebra A is called extremal whenever equality holds in the above inequality. In our case $i(I_{X(\partial C(n,2d))}) = 1$ while $d(I_{X(\partial C(n,2d))}) = d+1$. Typical other examples of extremal graded Gorenstein rings in (Krull) dimension 3 are the tangent cones of the simple elliptic surface singularities.

We discuss in the sequel the possible topological types of the face variety-like "divisors" on the 3-fold $X(\partial C(n,4))$.

Proposition 4.2. *If Δ is a triangulation of a compact, connected, orientable 2-manifold T_g which can be realized as a simplicial subcomplex of the boundary complex $\partial C(n,4)$ of* *the cyclic polytope $C(n,4)$, then $g \leq 1$. Moreover, in case Δ is a triangulation of the torus T_1, then all vertices of the cyclic polytope $C(n,4)$ are vertices of the triangulation, and the 1-skeleton of the triangulation has a Hamiltonian circuit such that every triangle in Δ has exactly one edge in common with this circuit.*

Proof. Gale's evenness condition says that the convex hull of four vertices of $C(n,4)$ is a facet of the polytope if and only if they correspond to two pairs of neighbouring points on the moment curve C_4. Similarly, three vertices of $C(n,4)$ span a 2-face of the polytope if and only if two of them are neighbours on the moment curve. For the first part of the proposition observe that since there are at most $m = f_0(\Delta) \leq n$ pairs of vertices in Δ which are adjacent on C_4 it follows that $f_2(\Delta) \leq 2m$. Now T_g is a manifold so each edge of Δ is contained in exactly two triangles of Δ. Therefore, since $f_2(\Delta) = \chi(T_g) + f_1(\Delta) - m = 2(1-g) + 3f_2(\Delta)/2 - m$, we deduce that $f_2(\Delta) = 2m - 4(1-g) \leq 2m$ which implies $g \leq 1$. We assume now that $g = 1$, and therefore that equality holds in all the above inequalities, that is $f_2(\Delta) = 2m$ and $m = n = f_0(C(n,4))$ since the edges whose spanning vertices are adjacent on the moment curve C_4 form a Hamiltonian circuit, in other words a graph with m vertices and m edges which has no proper subgraphs. But every edge of Δ is contained in exactly two triangles, and thus each of the $2n$ triangles of Δ must contain exactly one edge of the Hamiltonian cycle. •

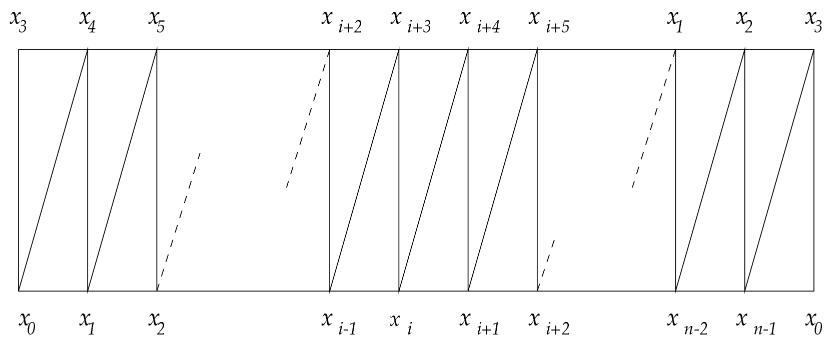
Remark 4.3. In the case $g = 1$, we can rephrase the statement of the above proposition by saying that the face variety $X(\Delta)$ is a "translation scroll" of the "elliptic normal curve" $X(\Gamma_n) \subseteq X(\partial C(n,4)) \subseteq \mathbf{P}^{n-1}$, which corresponds to the Hamiltonian cycle.

Let Δ be a triangulation of the torus T_1 and denote as usual by $f_i = f_i(\Delta)$ the number of i-dimensional faces of Δ. The triplet $f(\Delta) = (f_0, f_1, f_2)$ will be called the f-vector of the given triangulation. Each edge in the triangulation is common to exactly two triangles, so $2f_1 = 3f_2$ and hence $f_0 = f_1 - f_2 + \chi(T_1) = \frac{1}{3}f_1$. On the other hand obviously $f_1 \leq \binom{f_0}{2}$, so we deduce that $f_0 \geq 7$. Therefore a triangulation of the torus T_1 has at least 7 vertices, and moreover the above formulae show that for such a triangulation the graph of its 1-skeleton is necessarily K_7, the complete graph on seven vertices. Such a triangulation was first constructed in 1949 by Császár [Cs]. It is unique up to isomorphism and has an automorphism group of order 42. The dual graph of its 1-skeleton divides the torus in the well known 7-colourable map (see [Wh] for more details).

Inspired by this construction we describe in the proof of the following proposition a uniform series of triangulations for the torus T_1.

Proposition 4.4. *For each $n \geq 7$ there exists a $\mathbf{Z}_n \times \mathbf{Z}_2$-invariant triangulation Δ_n of the* *torus T_1, whose f-vector is $f(\Delta_n) = (n, 3n, 2n)$, and which can be realized as a subcomplex in the cyclic polytope $C(n, 4)$.*

Proof. Let $x_i = x(t_i)$, $i \in 0, 1, \ldots, n-1$, with $t_0 < t_1 < \ldots t_{n-1}$, denote the vertices of the cyclic polytope $C(n, 4)$ on the moment curve C_4 and label them in a natural way by the elements of $\mathbf{Z}_n$. Assume first that $n \geq 8$. We define then Δ_n to be the 2-dimensional simplicial complex whose faces are the triangles $(x_i\, x_{i+1}\, x_{i+4})$ and $(x_i\, x_{i+3}\, x_{i+4})$, for $i \in \mathbf{Z}_n$, their edges and their vertices. These triangles are chosen such as to form a tessellation of the usual representation of the torus as a rectangular stripe with the opposite edges identified with an appropriate Dehn twist. Namely we have the following diagram:



It is obvious now that $|\Delta_n|$ is homeomorphic to the torus T_1. The $\mathbf{Z}_n$ action on the triangulation is the one induced from the natural action on the vertices. The extra $\mathbf{Z}_2$ invariance is under the transformation $i \mapsto -i \,(\mathrm{mod}\, n)$. A similar discussion applies also in the case $n = 7$, where Δ_7 is taken to be the 2-dimensional simplicial complex whose faces are the triangles $(x_i\, x_{i+1}\, x_{i+3})$ and $(x_i\, x_{i+2}\, x_{i+3})$, for $i \in \mathbf{Z}_7$, their edges and their vertices. •

Corollary 4.5. *The face variety $X(\Delta_n) \subseteq \mathbf{P}^{n-1}$ is a locally Gorenstein, 2-dimensional projectively normal scheme of degree $2n$ and arithmetic sectional genus $n+1$, with trivial canonical sheaf and irregularity $q = 2$. In particular, $X(\Delta_n)$ has the same Hilbert polynomial as a smooth $(1, n)$-polarized abelian surface. Moreover, if $n \geq 13$, then the homogeneous ideal $I_{X(\Delta_n)}$ is generated by the quadratic monomials*

$$x_i x_{i+2}, \quad \text{and} \quad x_i x_{i+5}, \ldots, x_i x_{i+[\frac{d}{2}]}, \quad i \in \mathbf{Z}_n,$$

where $[\ldots]$ denotes the integral part. If $n = 11$ or 12, then $I_{X(\Delta_n)}$ is generated by the previous quadrics and the additional n cubics

$$x_{i-4} x_i x_{i+4}, \quad i \in \mathbf{Z}_n,$$

while if $n = 10$, $I_{X(\Delta_n)}$ is generated by the above quadrics and the additional 10 cubics

$$x_{i-3}x_ix_{i+3}, \quad i \in \mathbf{Z}_{10}.$$

Proof. In order to prove the statement about $I_{X(\Delta_n)}$, $n \geq 13$, it is enough to check that for each $(i,j,k) \in \mathbf{Z}_n^3$, with $i \neq j$, $i \neq k$ and $j \neq k$, either $(x_i\, x_j\, x_k)$ is a face of Δ_n, or one of the edges $(x_i\, x_j)$, $(x_i\, x_k)$, $(x_j\, x_k)$ is not an edge in Δ_n. This is clear if either of $d(i,j)$, $d(i,k)$, and $d(j,k)$ is two or at least 5. On the other hand, if $n \geq 13$ and all of $d(i,j)$, $d(i,k)$, and $d(j,k)$ are at most 4, then it is easily seen that one of them is necessarily 2, if $(x_i\, x_j\, x_k)$ is not a face in Δ_n, while for $10 \leq n \leq 12$, the only non-faces $(x_i\, x_j\, x_k)$ with edges in Δ_n are the ones listed. •

We now define

$$Q_i^\lambda := \{x_ix_{i+2} + \lambda x_{i-1}x_{i+3} = x_j = 0, \text{ for } j \in \mathbf{Z}_n \setminus \{i, i+2, i-1, i+3\}\},$$

and then set

$$X_n^\lambda := \cup_{i \in \mathbf{Z}_n} Q_i^\lambda, \quad \text{ for } \lambda \neq 0.$$

Let W_n be the one dimensional simplicial complex obtained by taking the one-skeleton of Δ_n and removing the diagonals $(x_i\, x_{i+4})$. We consider here W_n as a simplicial subcomplex of the boundary complex of the cyclic polytope $\partial C(n,4)$. Note that the base locus B of the family X_n^λ is in fact

$$B := \bigcap_{\lambda \in k^*} X_n^\lambda = X(W_n).$$

This follows immediately from the fact that the ideal $I_{X(W_n)}$ is generated by the monomials

$$\{x_ix_j \mid d(i,j) = 2 \text{ or } 4, \text{ or } d(i,j) \geq 5\}.$$

In particular, since the Hamiltonian cycle Γ_n is a subcomplex of W_n, we get $X(\Gamma_n) \subseteq X(W_n)$. The following was inspired and is closely related to [ES], Proposition 4.8:

Theorem 4.6.

a) *The ideal $I(X_n^\lambda)$ is generated by the quadrics*

$$\{x_ix_{i+2} + \lambda x_{i-1}x_{i+3} \mid i \in \mathbf{Z}_n\} \cup \{x_ix_j \mid d(i,j) \geq 5\}$$

for $\lambda \neq 0$.

b) *If $n = 10$, then $x_{i-3}x_ix_{i+3} \in I(X_n^\lambda)$ for all $i \in \mathbf{Z}_{10}$, $\lambda \neq 0$, while if $n = 11$ or 12, then $x_{i-4}x_ix_{i+4} \in I(X_n^\lambda)$ for all $i \in \mathbf{Z}_n$.*

c) The family $\{X_n^\lambda \mid \lambda \in k^*\} \subseteq k^* \times \mathbf{P}^{n-1}$ extends uniquely to a flat family $\{X_n^\lambda \mid \lambda \in \mathbf{A}^1\} \subseteq \mathbf{A}^1 \times \mathbf{P}^{n-1}$ over $\mathbf{A}^1$, and $X_n^0 = X(\Delta_n)$. In particular, X_n^λ has the same Hilbert polynomial as a smooth $(1,n)$-polarized abelian surface.

Proof: Let J be the ideal generated by

$$\{x_i x_{i+2} + \lambda x_{i-1} x_{i+3} \mid i \in \mathbf{Z}_n\} \cup \{x_i x_j \mid d(i,j) \geq 5\}.$$

It is easy to see that $J \subseteq I(X_n^\lambda) = \bigcap_{i \in \mathbf{Z}_n} I(Q_i^\lambda)$. For the purposes of this proof, we define the width of a monomial $x^I \in k[x_0, \ldots, x_{n-1}]$, where I is a multindex, as

$$w(x^I) := \max\{d(i,j) \mid x_i | x^I \text{ and } x_j | x^I\}.$$

Suppose $f \in I(X_n^\lambda)$. Since $I(X_n^\lambda) \subseteq I_{X(W_n)}$, it is clear that f can have no terms of width 0 or 1, and no terms of width three of the form $x_i^d x_{i+3}^e$. If f has a term of width two, say $x_i^{d_1} x_{i+1}^{d_2} x_{i+2}^{d_3}$, $d_1 d_3 \neq 0$, then by using the relation $x_i x_{i+2} = -\lambda x_{i-1} x_{i+3}$, we can replace this term, modulo J, with a term of width four or more. If f has a term of width three, say $x_i^{d_1} x_{i+1}^{d_2} x_{i+2}^{d_3} x_{i+3}^{d_4}$, $d_1 d_4 \neq 0$, then either d_2 or d_3 is non-zero by the above discussion, and likewise we can replace this term with a term of width at least 4. Since J contains all monomials of width at least 5, we thus find that f must be congruent modulo J to a polynomial with terms of width exactly 4.

If $n \geq 13$, then any width four term is of the form $x_i^{d_1} x_{i+1}^{d_2} x_{i+2}^{d_3} x_{i+3}^{d_4} x_{i+4}^{d_5}$, and if $d_3 \neq 0$, we can again replace this term with a term of width at least five, which thus is in J. In case $d_3 = 0$, then restricting f to the $\mathbf{P^3} := \{x_j = 0 \mid j \in \mathbf{Z}_n \setminus \{i, i+1, i+3, i+4\}\}$, f must vanish on the quadric $x_{i+1} x_{i+3} + \lambda x_i x_{i+4}$, and thus must be a multiple of this quadric. Therefore we find that f is congruent modulo J to a polynomial containing no terms of width less than or equal to four and hence f is zero. Thus $I(X_n^\lambda) = J$ in this case.

If $n = 10, 11$ or 12, the following minimal width four terms could also appear:

$$\begin{aligned} &x_{i-3} x_i x_{i+3} \quad \text{if } n = 10; \\ &x_{i-4} x_i x_{i+4} \quad \text{if } n = 11, 12. \end{aligned}$$

Note that these are exactly the cubics appearing in Corollary 4.5. But then

$$\lambda^2 x_{i-3} x_i x_{i+3} = x_{i+3} \cdot (x_{i+1} x_{i+6}) - x_{i+6} \cdot (x_{i+1} x_{i+3} + \lambda x_i x_{i+4}) + \lambda x_i \cdot (x_{i+4} x_{i+6} + \lambda x_{i+3} x_{i-3}),$$

so $x_{i-3} x_i x_{i+3} \in J$. Similarly, $x_{i-4} x_i x_{i+4} \in J$ for $i \in \mathbf{Z}_n$, $n = 11, 12$, via

$$\lambda x_{i-4} x_i x_{i+4} = x_{i+4} \cdot (x_{i-3} x_{i-1} + \lambda x_{i-4} x_i) - x_{i-3} \cdot (x_{i-1} x_{i+4}).$$

This completes the proof of a) and b). To prove c), consider the family $\mathcal{X}_n$ in $\mathbf{A}^1 \times \mathbf{P}^n$ defined by the equations

$$\{x_i x_{i+2} + \lambda x_{i-1} x_{i+3} \mid i \in \mathbf{Z}_n\} \cup \{x_i x_j \mid d(i,j) \geq 5\}$$

along with the additional cubics listed in part b) if $10 \leq n \leq 12$. Then the fibre of this family over $\lambda \neq 0$ is X_n^λ, and by Corollary 4.5, the fibre over $0 \in \mathbf{A}^1$ is $X(\Delta_n)$. Thus $\mathcal{X}_n$ is pure dimension three, and every component of $\mathcal{X}_n$ dominates $\mathbf{A}^1$. Therefore, by [Ha] §III Proposition 9.7, $\mathcal{X}_n \to \mathbf{A}^1$ is flat. •

Remark 4.7. The smooth part $X(\Gamma_n)_{sm}$ of $X(\Gamma_n)$ has a natural group structure, isomorphic to $\mathbf{G}_m \times \mathbf{Z}/n\mathbf{Z}$. As such, it is still possible to define a translation scroll for $X(\Gamma_n)$ by taking a given $\tau \in X(\Gamma_n)_{sm}$, and letting $S_{X(\Gamma_n),\tau}$ be the Zariski closure of

$$\bigcup_{x \in X(\Gamma_n)_{sm}} \langle x, x+\tau \rangle.$$

Then it is not difficult to see that the X_n^λ defined above are translation scrolls for $X(\Gamma_n)$, and that the pencil X_n^λ traces out $Sec(X(\Gamma_n)) = X(\partial C(n,4))$, with appropriate definitions for $\lambda = \infty$.

§5. Secant varieties of elliptic normal curves.

Let $E \subseteq \mathbf{P}^{n-1}$ be an elliptic normal curve of degree n with a level n structure. This is equivalent to saying that the embedding of E is $\mathbf{H}_n \rtimes \langle \iota \rangle$-equivariant under the Schrödinger representation, where $\mathbf{H}_n$ acts on E via translation with n-torsion points while ι acts as the natural involution given by negation. We'll denote by $Sec(E)$ the closure of the chordal variety to E. We have

Proposition 5.1. *Let $E \subseteq \mathbf{P}^{n-1}$ be an elliptic normal curve of degree n. Then*

(i) *$Sec(E)$ is an irreducible threefold of degree $n(n-3)/2$.*

(ii) *$Sec(E)$ is non-singular outside of E, and is singular along E with multiplicity $n-2$.*

(iii) *A natural desingularization of $Sec(E)$ is given by $\pi : \mathbf{P}^1_{S^2E} \to Sec(E)$, where*

$$\mathbf{P}^1_{S^2E} = \{(p, \{e_1, e_2\}) \mid e_1, e_2 \in E, p \in \operatorname{span}_{\mathbf{C}}(e_1, e_2)\} \subseteq \mathbf{P}^{n-1} \times S^2E,$$

and where π is the projection on the first factor. $\mathbf{P}^1_{S^2E}$ has the structure of a $\mathbf{P}^1$-bundle over the elliptic scroll S^2E.

(iv) $\omega_{Sec(E)} \cong \mathcal{O}_{Sec(E)}$, *while* $h^1(\mathcal{O}_{Sec(E)}) = h^2(\mathcal{O}_{Sec(E)}) = 0$.

Proof: *i)* $Sec(E)$ is clearly an irreducible threefold. To compute its degree, take a general $L \cong \mathbf{P}^{n-4} \subseteq \mathbf{P}^{n-1}$ and project E to $\mathbf{P}^2$ from L. The number of nodes of the projection of E will be exactly $\frac{1}{2}(n-1)(n-2) - 1 = n(n-3)/2$, and this is precisely the number of secants of E which L meets. This is also the degree of $Sec(E)$.

ii) (See [ADHPR1], [Seg] in the case $n = 5$.) To see that the multiplicity of $Sec(E)$ along E is $n-2$, take a linear space $L \cong \mathbf{P}^{n-4}$, $L \subseteq \mathbf{P}^{n-1}$, which contains a point p of E, so that L is neither tangent to E nor contains a secant of E. Choosing L generally, we can assume that L meets $Sec(E)$ transversally at a finite number of smooth points outside of p. Now, two secants or tangents of E cannot meet at a point outside of E. If they do, then there is a plane M containing both, and the linear system of hyperplanes containing M yields a linear system on E residual to $M \cap E$ which is of dimension $n-4$ and degree $\leq n-4$, a contradiction for an elliptic curve. Thus there is a unique secant or tangent line to E through each point of $L \cap Sec(E)$ outside of p. Projecting E from L yields a curve in $\mathbf{P}^2$ of degree $n-1$, which must then have $\frac{1}{2}(n-2)(n-3) - 1$ nodes. Thus L intersects $Sec(E)$ in $\frac{1}{2}(n-2)(n-3) - 1$ points outside of p, and so p has multiplicity

$$\deg Sec(E) - \left[\frac{1}{2}(n-2)(n-3) - 1\right] = n - 2.$$

We will show that $Sec(E)$ is non-singular outside of E after showing *iii)*.

iii) It is clear that $\mathbf{P}^1_{S^2E}$ is a $\mathbf{P}^1$-bundle over S^2E via the second projection $f : \mathbf{P}^1_{S^2E} \to S^2E$, and thus is non-singular. Therefore π is a desingularization of $Sec(E)$.

To show that $Sec(E)$ is singular only along E, it is enough to show that π is an isomorphism outside of $\pi^{-1}(E)$. Since no two bisecants of E pass through the same point of $\mathbf{P}^{n-1}$ outside of E, $\pi : \mathbf{P}^1_{S^2E} \backslash \pi^{-1}(E) \to Sec(E) \backslash E$ is one-to-one, thus to check that π is an isomorphism, we need to consider the maps on tangent spaces. Let $x_1, x_2 \in E$ be distinct points, and let U_1, U_2 be disjoint small open neighborhoods of x_1, x_2, respectively, with local coordinates u_1 and u_2. These yield a neighborhood of $\mathbf{P}^1_{S^2E}$ of the form $U_1 \times U_2 \times \mathbf{A}^1$, with coordinates (u_1, u_2, t). Let $\mathbf{A}^{n-1} \subseteq \mathbf{P}^{n-1}$ be an affine subspace containing U_1 and U_2, and let $\phi_1 : U_1 \to \mathbf{A}^{n-1}$, $\phi_2 : U_2 \to \mathbf{A}^{n-1}$ be the embeddings of U_1 and U_2, respectively, in $\mathbf{A}^{n-1}$. Then we may identify the restriction of π with the map $\psi : U_1 \times U_2 \times \mathbf{A}^1 \to \mathbf{A}^{n-1}$ defined by

$$\psi(u_1, u_2, t) = t\phi_1(u_1) + (1-t)\phi_2(u_2).$$

For fixed u_1 and u_2, this yields the secant line joining $\phi_1(u_1)$ and $\phi_2(u_2)$. Now for a given point $(u_1, u_2, t) \in U_1 \times U_2 \times \mathbf{A}^1$, the image $\psi_*(T)$ of the tangent space $T =$

$T_{U_1\times U_2\times \mathbf{A}^1,(u_1,u_2,t)}$ at that point is spanned by the rows of the jacobian matrix

$$\begin{pmatrix} \partial\psi/\partial u_1(u_1,u_2,t) \\ \partial\psi/\partial u_2(u_1,u_2,t) \\ \partial\psi/\partial t(u_1,u_2,t) \end{pmatrix} = \begin{pmatrix} t\phi_1'(u_1) \\ (1-t)\phi_2'(u_2) \\ \phi_1(u_1)-\phi_2(u_2) \end{pmatrix}.$$

Therefore, if $t \neq 0,1$ (values where $Sec(E)$ is singular), then $\psi_*(T)$ contains the tangent vectors to E at u_1 and u_2, and hence $\psi_*(T)$ is tangent to E at u_1 and u_2. If $\psi_*(T)$ had dimension less than 3, then this would yield a plane which intersects E at least four times counted with multiplicities, but again this is not possible. Thus $\psi_* : T \to T_{\mathbf{P}^{n-1},\psi(u_1,u_2,t)}$ is injective, and thus $Sec(E)$ is nonsingular away from E at points on secant lines which are not tangent lines.

To deal with points on tangent lines, we need suitable coordinates on S^2E. Let $x \in E$, and let U be an open neighborhood of x with coordinate u, with $u = 0$ at x for convenience. Let $\phi : U \to \mathbf{A}^{n-1}$ be the embedding of U as before. Consider now $\psi : U \times U \times \mathbf{A}^1 \to \mathbf{A}^{n-1}$ defined by

$$\psi(u,v,t) = \begin{cases} \frac{1}{2}(\phi(u)+\phi(v)) + t\left(\frac{\phi(v)-\phi(u)}{v-u}\right) & v \neq u; \\ \phi(u) + t\phi'(u) & v = u. \end{cases}$$

ψ is clearly holomorphic and its image is contained in the secant scroll. We can expand ϕ in a Taylor series,

$$\phi(u) = \phi(0) + \phi'(0)u + \frac{\phi''(0)}{2}u^2 + \cdots,$$

and then write ψ as

$$\psi(u,v,t) = \phi(0) + \frac{1}{2}\left(\phi'(0)(u+v) + \frac{\phi''(0)}{2}(u^2+v^2) + \frac{\phi^{(3)}(0)}{6}(u^3+v^3) + \cdots\right) +$$
$$+\, t\left(\phi'(0) + \frac{\phi''(0)}{2}(u+v) + \frac{\phi^{(3)}(0)}{6}(u^2+uv+v^2) + \cdots\right).$$

Now, on S^2U, we can use the symmetric coordinates $s_1 = u+v$ and $s_2 = uv$. Since ψ is symmetric with respect to interchanging u and v, ψ descends to a function on $S^2U \times \mathbf{A}^1$, which we may write as

$$\psi(s_1,s_2,t) = \phi(0) + \frac{1}{2}\left(\phi'(0)s_1 + \frac{\phi''(0)}{2}(s_1^2-2s_2) + \frac{\phi^{(3)}}{6}(s_1^3-3s_1s_2) + \cdots\right) +$$
$$+\, t\left(\phi'(0) + \frac{\phi''(0)}{2}s_1 + \frac{\phi^{(3)}(0)}{6}(s_1^2-s_2) + \cdots\right).$$

As before, if T is the tangent space of $S^2U \times \mathbf{A}^1$ at $(0,0,t)$, $t \neq 0$, we can write $\psi_*(T)$ as the span of the rows of the jacobian of ψ:

$$\begin{pmatrix} \partial\psi/\partial s_1(s_1,s_2,t) \\ \partial\psi/\partial s_2(s_1,s_2,t) \\ \partial\psi/\partial t(s_1,s_2,t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\phi'(0) + t\frac{\phi''(0)}{2} \\ -\frac{\phi''(0)}{2} - t\frac{\phi^{(3)}(0)}{6} \\ \phi'(0) \end{pmatrix}.$$

So given that $t \neq 0$, $\psi_*(T)$ is spanned by $\phi'(0)$, $\phi''(0)$ and $\phi^{(3)}(0)$. Thus $\psi_*(T)$ intersects E at $x \in U$ with multiplicity four, and hence, as before, must be three dimensional. Thus we have shown that $Sec(E)$ is smooth at all points in $Sec(E) \setminus E$.

iv) There is a natural inclusion $E \times E \subseteq \mathbf{P}^1_{S^2E}$, given by $(e_1, e_2) \mapsto (e_1, \{e_1, e_2\})$. Thus the map $\pi : \mathbf{P}^1_{S^2E} \to \mathbf{P}^{n-1}$ contracts $E \times E \subseteq \mathbf{P}^1_{S^2E}$ to $E \subseteq \mathbf{P}^{n-1}$, via projection onto the first coordinate. If $\widetilde{X}$ is the normalization of $Sec(E)$, then $\widetilde{X}$ is isomorphic precisely to the threefold obtained by contracting $E \times E$ inside of $\mathbf{P}^1_{S^2E}$ in this manner. Let $p : \mathbf{P}^1_{S^2E} \to \widetilde{X}$ be the contraction. $\widetilde{X}$ then has a curve of simple elliptic singularities, so $\widetilde{X}$ is Gorenstein and $K_{\mathbf{P}^1_{S^2E}} \sim p^* K_{\widetilde{X}} - E \times E$. If $E \times E \sim -K_{\mathbf{P}^1_{S^2E}} + D$ for some divisor D, adjunction tells us that

$$\begin{aligned} 0 &= K_{E \times E} \\ &= (E \times E) \cdot (-K_{\mathbf{P}^1_{S^2E}} + D + K_{\mathbf{P}^1_{S^2E}}) \\ &= (E \times E) \cdot D. \end{aligned}$$

It is easy to see that the restriction map $\mathrm{Pic}(\mathbf{P}^1_{S^2E}) \to \mathrm{Pic}(E \times E)$ is injective. Thus $D \sim 0$ and $E \times E$ is an anticanonical divisor in the linear system $|-K_{\mathbf{P}^1_{S^2E}}|$. Thus

$$K_{\mathbf{P}^1_{S^2E}} \sim p^* K_{\widetilde{X}} + K_{\mathbf{P}^1_{S^2E}},$$

so $\mathcal{K}_{\widetilde{X}} \sim 0$. Finally using intersection theory in $\mathbf{P}^1_{S^2E}$, one can see that $\widetilde{X}$ is singular with multiplicity $n-2$ along E, and these singularities are locally of the form

$$\text{(simple elliptic surface singularity of multiplicity } n-2) \times \text{curve.}$$

Thus the Zariski tangent space at each singular point of $\tilde{X}$ has dimension $n-1$, and also the Zariski tangent space of each singular point of $Sec(E)$ is of dimension $n-1$ (since such a tangent space contains the cone over E with vertex a point on E), and hence $\widetilde{X} \cong Sec(E)$. Therefore $\omega_{Sec(E)} \cong \mathcal{O}_{Sec(E)}$, as desired.

To conclude, by Serre duality it is enough to show that $H^1(\mathcal{O}_{Sec(E)}) = 0$. Now $H^1(\mathcal{O}_{Sec(E)})$ is the Lie algebra of $\mathrm{Pic}^0(Sec(E))$, which is reduced since we are in characteristic zero. Thus it is enough to show that $\mathrm{Pic}(Sec(E))$ is a discrete group, since then $H^1(\mathcal{O}_{Sec(E)}) = 0$. To show this, we use the map

$$\pi^* : \mathrm{Pic}(Sec(E)) \to \mathrm{Pic}(\mathbf{P}^1_{S^2E}),$$

which is injective, and whose image is contained in the subgroup

$$\mathcal{P} = \{\mathcal{L} \in \mathrm{Pic}(\mathbf{P}^1_{S^2E}) \mid \mathcal{L}|_{E \times E} = \rho_1^* \mathcal{M} \text{ for some } \mathcal{M} \in \mathrm{Pic}(E)\},$$

where $\rho_1 : E \times E \to E$ denotes the first projection. So it is enough to check that $\mathcal{P}$ is discrete. We will do this by showing that if $D, D' \in \mathcal{P}$ are two divisors which are algebraically equivalent on $\mathbf{P}^1_{S^2E}$, then $D \sim D'$. Indeed $D - D' \in \mathrm{Pic}^0(\mathbf{P}^1_{S^2E})$, and any element of $\mathrm{Pic}^0(\mathbf{P}^1_{S^2E})$ can be written as a multiple of $f^*(C_0 - C_0')$, where $f : \mathbf{P}^1_{S^2E} \to S^2E$ is the second projection, and $C_0, C_0' \subseteq S^2E$ are images of fibres of ρ_1 in S^2E. But

$$f^*(C_0 - C_0')|_{E\times E} = E \times \{p_1\} - E \times \{p_2\} + \{p_1\} \times E - \{p_2\} \times E,$$

and this is not trivial on the fibers of the first projection $\rho_1 : E \times E \to E$ unless $p_1 = p_2$, in which case $D \sim D'$. Thus if $D - D' \in \mathcal{P}$, then $D \sim D'$; and therefore $\mathrm{Pic}(Sec(E))$ is discrete, and $H^1(\mathcal{O}_{Sec(E)}) = 0$. $\bullet$

We now give the equations for these secant varieties. In this direction we mention also Ravi's results [Rav] which in the special case of an elliptic normal curve $E \subset \mathbf{P}^{n-1}$ say that if M is the $m \times (n-m)$-matrix with linear entries corresponding to a splitting of the embedding as a tensor product of two line bundles of degrees m and $n-m$, respectively, with $4 \le m \le n-4$, then the 3×3-minors of M set-theoretically define $Sec(E)$.

Theorem 5.2. *Let $E \subseteq \mathbf{P}^{2d-1}$ be a Heisenberg invariant normal curve of degree $2d$, $d \ge 3$. If $(y_0 : \ldots : y_{2d-1}) \in E$ is a general point, then the 3×3 minors of the matrix*

$$M_d = (x_{i+j}y_{i-j} + x_{i+j+d}y_{i-j+d})_{i,j\in 2\mathbf{Z}/2d\mathbf{Z}}$$

vanish on $Sec(E)$. In particular, if $d = 3$, then $\det M_d$ and $\sigma(\det M_d)$ cut out $Sec(E)$, and if $d \ge 4$, then for general E, the ideal of 3×3 minors of M_d cuts out $Sec(E)$.

Proof: By Corollary 2.2, the 2×2 minors of M_d vanish on E. It is easy to see that the variety given by the determinant of a 3×3 matrix of forms is singular along the vanishing locus of the 2×2 minors. Thus each 3×3 minor of M_d determines a cubic hypersurface which is singular along E. Any secant line to E intersects this cubic hypersurface in at least four points, counted with multiplicity, and so is contained in this hypersurface. Thus each 3×3 minor of M_d vanishes along $Sec(E)$.

Now if $d = 3$, it is easy to check that $\det M_3$ and $\sigma(\det M_3)$ are independent cubics, and hence $\det M_3 = \sigma(\det M_3) = 0$ determines a three dimensional complete intersection of degree 9. Since $Sec(E)$ is also degree 9, these two varieties must coincide (see Proposition 2.12 and Remark 2.13, *ii*)).

Define now $\mathcal{S} \subseteq \mathbf{P}^{2d-1} \times S_{2d} \subseteq \mathbf{P}^{2d-1} \times \mathbf{P}^{2d-1}$, with x coordinates on the first $\mathbf{P}^{2d-1}$ and y coordinates on the second $\mathbf{P}^{2d-1}$, as the variety defined by the 3×3 minors of the matrix M_d. (See §3 for the definition of S_{2d}). If $E \subset \mathbf{P}^{2d-1}$ is a Heisenberg invariant elliptic curve and $y \in E$, then $y \in S_{2d}$ and we have already observed that the fibre $\mathcal{S}_t$ of the second projection $\mathcal{S} \to S_{2d}$ contains $Sec(E)$. On the other hand, if there is a point $y \in S_{2d}$ such that $\mathcal{S}_y$ is Cohen-Macaulay and of degree and dimension the same as the secant variety of an elliptic normal curve of degree $2d$, then for the general $y \in S_{2d}$, $\mathcal{S}_y$ must coincide with the secant variety $Sec(E)$, where $y \in E$. Thus in order to prove the theorem, we need only to exhibit one such point.

If $d > 3$, consider $(y_0 : \ldots : y_{2d-1}) := (1 : 1 : 0 : \ldots : 0)$. Then for this choice of the parameter

$$M_d = \begin{pmatrix} x_0 & 0 & 0 & \cdots & 0 & x_{2d-1} \\ x_1 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & x_3 & x_4 & \cdots & 0 & 0 \\ & \cdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & x_{2d-3} & x_{2d-2} \end{pmatrix}.$$

This matrix has the property that any of its 3×3 minors is either zero or consists of one monomial. Furthermore, if $d(i,j) = 1$, then x_i and x_j appear in the same row or column, so no 3×3 minor can be of the form $x_i x_j x_k$ with $d(i,j) \leq 1$. Conversely, if $x_i x_j x_k$ is a monomial such that $d(i,j) \geq 2$, $d(i,k) \geq 2$, $d(j,k) \geq 2$, then x_i, x_j and x_k appear in distinct rows and columns in the above matrix, and thus there is a 3×3 minor of M_d of the form $x_i x_j x_k$. Therefore the ideal I_d generated by the 3×3 minors of M_d is precisely $I_{X(\partial C(2d,4))}$, by Proposition 4.1. In particular, I_d cuts out a projectively Gorenstein scheme of dimension 3 and degree $\frac{1}{2}(2d)(2d-3)$. This is exactly what we had to show. •

In the case of the secant variety of a degree 5 elliptic normal curve, $Sec(E)$ is a well known quintic hypersurface in $\mathbf{P^4}$, and its geometry has been extensively studied in the literature beginning with [Seg]. We refer to [ADHPR1] and [ADHPR2] for details and for further references to the literature. For an explicit form of its equation see [Hu], p. 109, or [ADHPR1], [ADHPR2]. We give in the following a slightly different version of this quintic equation (compare [ADHPR2], Proposition 4.12, *i*).

Theorem 5.3. *Let $y = (y_0 : y_1 : y_2 : y_3 : y_4) \in E \subset \mathbf{P^4}$ be a general point. Then*

$$Sec(E) = \{\det(x_{3(i+j)} y_{3(i-j)})_{i,j \in \mathbf{Z}_5} = 0\}.$$

Proof. By Corollary 2.9, the 3×3 minors of the matrix $M_2' = (x_{3(i+j)} y_{3(i-j)})_{i,j \in \mathbf{Z}/5\mathbf{Z}}$ vanish along E, and thus the quintic $Q = \det(x_{3(i+j)} y_{3(i-j)})$ is singular with multiplicity at least 3 along E. In particular, $\{Q = 0\}$ contains every secant line to E. Since on the other hand, $Sec(E)$ is a hypersurface of degree 5, we deduce that $Sec(E) = \{Q = 0\}$. •

Theorem 5.4. *Let $E \subseteq \mathbf{P}^{2d}$ be a Heisenberg invariant elliptic normal curve of degree $2d+1$, $d \geq 3$. Then the intersection $E \cap (\mathbf{P}^{d-1})^-$ is non-empty, and for a point $y = (y_0 : \ldots : y_{2d}) \in E \cap (\mathbf{P}^{d-1})^-$ the matrix*

$$M'_d = (x_{(d+1)(i+j)}y_{(d+1)(i-j)})_{i,j \in \mathbf{Z}/(2d+1)\mathbf{Z}}$$

is skew-symmetric, and the ideal I_{2d+1} generated by the 6×6-pfaffians of M'_d is the ideal of $Sec(E)$, for the general curve E.

Proof. By [LB], Proposition 4.7.5, $E \cap (\mathbf{P}^{d-1})^-$ is non-empty. Now a point $y = (y_0 : \cdots : y_{2d})$ in $(\mathbf{P}^{d-1})^-$ satisfies $y_i = -y_{-i}$, for all $i \in \mathbf{Z}_d$, whence

$$\begin{aligned}
(M'_d)_{ij} &= x_{(d+1)(i+j)}y_{(d+1)(i-j)} \\
&= -x_{(d+1)(i+j)}y_{(d+1)(j-i)} \\
&= -(M'_d)_{ji}.
\end{aligned}$$

Therefore M'_d is skew-symmetric, and by Corollary 2.9, the 4×4-pfaffians of M'_d vanish along the curve E. In particular, the 6×6-pfaffians of M'_d are cubic hypersurfaces which are singular along E, and hence contain $Sec(E)$. By taking $y = (y_i)$, whose coordinates are

$$y_i = \begin{cases} 1 & \text{if } i = d; \\ -1 & \text{if } i = d+1; \\ 0 & \text{otherwise,} \end{cases}$$

and which is a point lying on the standard $(2d+1)$-gon $X(\Gamma_{2d+1})$, we obtain

$$M'_d = \begin{pmatrix}
0 & x_{d+1} & 0 & 0 & \cdots & 0 & 0 & -x_d \\
-x_{d+1} & 0 & x_{d+2} & 0 & \cdots & 0 & 0 & 0 \\
0 & -x_{d+2} & 0 & x_{d+3} & \cdots & 0 & 0 & 0 \\
 & \vdots & & & & & \vdots & \\
0 & 0 & 0 & 0 & \cdots & -x_{d-2} & 0 & x_{d-1} \\
x_d & 0 & 0 & 0 & \cdots & 0 & -x_{d-1} & 0
\end{pmatrix}$$

It is now easily seen that the 6×6-pfaffians of the above matrix consist of all monomials $x_i x_j x_k$ with $d(i,j) \geq 2$, $d(i,k) \geq 2$, and $d(j,k) \geq 2$. Indeed, to obtain the monomial $x_i x_j x_k$ with this restriction, we have to take the minor whose rows (and columns) are indexed by the set

$$\{i+d, i+d+1, j+d, j+d+1, k+d, k+d+1\} \subseteq \{0, \ldots, 2d\},$$

where all indices are computed mod $2d+1$. This yields the minor (except for the case that $d \in \{i,j,k\}$, when a slightly different configuration results, which may be handled similarly):

$$\begin{pmatrix} 0 & x_i & 0 & 0 & 0 & 0 \\ -x_i & 0 & \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & x_j & 0 & 0 \\ 0 & 0 & -x_j & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta & 0 & x_k \\ 0 & 0 & 0 & 0 & -x_k & 0 \end{pmatrix}$$

where the values α and β are irrelevant. The pfaffian of this matrix is $x_i x_j x_k$. Furthermore, a minor which does not contain three adjacent pairs of rows, must contain a zero row and hence its pfaffian is identically zero.

Therefore, by Proposition 4.1, we deduce that $I_{2d+1} = I_{X(\partial C(2d+1,4))}$. Moreover, using the same argument as in the even case, this again implies that for the general elliptic normal curve E and $y \in E \cap (\mathbf{P}^{d-1})^-$, the homogeneous ideal I_{2d+1} cuts out $Sec(E)$. •

Finally the results in §4 allow us also to say something about minimal free resolutions. Part *a*) below is a rather well-known fact (see e.g. [E], Exercise A2.22):

Theorem 5.5. *Let $E \subseteq \mathbf{P}^n$ be a Heisenberg invariant elliptic normal curve. Then*

a) *For $n \geq 4$ the homogeneous ideal I_E has a minimal free resolution of type*

$$0 \leftarrow I_E \leftarrow R(-2)^{b_1} \leftarrow R(-3)^{b_2} \leftarrow \cdots \leftarrow R(-n+1)^{b_{n-2}} \leftarrow R(-n-1) \leftarrow 0,$$

where

$$b_i = i\binom{n}{i+1} - \binom{n-1}{i-1},$$

for all $1 \leq i \leq n-2$, and where R is the homogeneous coordinate ring of $\mathbf{P}^n$.

b) *$Sec(E)$ is projectively Gorenstein and its homogeneous ideal $I_{Sec(E)}$ has a minimal free resolution of type*

$$0 \leftarrow I_{Sec(E)} \leftarrow R(-3)^{b_1} \leftarrow R(-4)^{b_2} \leftarrow \cdots \leftarrow R(-n+2)^{b_{n-4}} \leftarrow R(-n-1) \leftarrow 0,$$

if E is general and $n \geq 6$, where

$$b_i = \binom{n-1}{i+2}\binom{i+1}{2} + \binom{n-1}{i}\binom{n-2-i}{2} - \binom{n-3}{i}\binom{n-2}{2},$$

for all $1 \leq i \leq n-4$.

Proof. To prove *b*) it is enough to check the claims in a special case, for instance the secant variety of $X(\Gamma_{n+1})$ which in turn (as seen in §4) may be identified with the face variety $X(\partial C(n+1,4))$. As mentioned in §4, its homogeneous coordinate ring is an extremal compressed Gorenstein algebra, and thus our claims follow from [Sch], Theorem B (see also [FL]), which essentially asserts that the Betti numbers of such extremal rings are uniquely determined by their Hilbert function. Part *a*) may also be regarded as a corollary of this result since homogeneous rings of elliptic curves are obviously extremal.●

§6. Equations defining abelian surfaces and the structure of $\mathcal{A}_n^{lev}$.

The goal in this section is to understand the structure of the ideal of a general abelian surface embedded via a polarization of type $(1,n)$, for $n \geq 10$. In this case, the homogeneous ideal will be generated by quadrics which are easy to write down.

n even: We write $n = 2d$. In this case, the projectivization of the negative eigenspace of ι is $\mathbf{P}^- \cong \mathbf{P}^{d-2}$.

Lemma 6.1.

a) *Let A be an abelian surface with a line bundle $\mathcal{L}$ of type $(1,2d)$, $d \geq 2$, $\mathcal{L}$ being of characteristic zero with respect to a given decomposition, and let $\phi : A \to \mathbf{P}^{2d-1}$ be the morphism induced by a basis of canonical theta functions. Then $\phi(A) \cap \mathbf{P}^-$ consists of four distinct points.*

b) *Let $S_{E,\tau}$ be a Heisenberg invariant translation scroll in $\mathbf{P}^{2d-1}$, with E an elliptic curve in $\mathbf{P}^{2d-1}$ of degree $2d$. Then $S_{E,\tau} \cap \mathbf{P}^-$ consists also of four distinct points.*

Proof: *a*) Let $A = V/\Lambda$, and let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be a symplectic basis for Λ compatible with a given decomposition, with $E(\lambda_1, \mu_1) = 1$, $E(\lambda_2, \mu_2) = 2d$, following the notation of §1. Then for a 2-torsion point $v \in A$, represented by $\lambda/2$, $\lambda \in \Lambda$, we define

$$q_{\mathcal{L}}(v) := \exp(\pi i E(\lambda', \mu')),$$

where $\lambda = \lambda' + \mu'$ is the decomposition of λ with respect to the chosen decomposition of Λ. It is then easy to see from explicit calculations that

$$\#\{v \in A_2 \mid q_{\mathcal{L}}(v) = +1\} = 12,$$
$$\#\{v \in A_2 \mid q_{\mathcal{L}}(v) = -1\} = 4.$$

By [LB], Proposition 4.7.2, for $v \in A_2$, if D is a symmetric divisor, then

$$(-1)^{mult_v(D) - mult_0(D)} = q_L(v).$$

So if D is an even symmetric divisor, that is $mult_0(D)$ is even, we see that

$$\#\{v \in A_2 \mid mult_v(D) \text{ even}\} = 12,$$
$$\#\{v \in A_2 \mid mult_v(D) \text{ odd}\} = 4.$$

The claim then follows, since $0 \in A$ is mapped to $\mathbf{P}^+$.

b) Let $E \subseteq \mathbf{P}^{2d-1}$ be a Heisenberg invariant elliptic normal curve. Let $\langle x, y\rangle$ be a secant line of E joining the points $x, y \in E$. If $\langle x, y\rangle$ intersects $\mathbf{P}^-$, then $\langle x, y\rangle$ and $\iota\langle x, y\rangle$ span at most a $\mathbf{P}^2$. But a $\mathbf{P}^2$ cannot intersect E in four points, so either x or y is fixed by ι or the whole line $\langle x, y\rangle$ is fixed by ι. In the former case, this tells us that either x or y is in $(\mathbf{P}^d)^+$, since $E \cap \mathbf{P}^- = \emptyset$ by [LB], Corollary 4.7.6. Thus two points of $\langle x, y\rangle$ are fixed by ι, and so the line $\langle x, y\rangle$ is fixed by ι anyway. Hence $\langle x, y\rangle \cap \mathbf{P}^- \neq \emptyset$ if and only if $x = -y$ in the group law on E. Now a translation scroll $S_{E,\tau}$ is defined as

$$\bigcup_{x \in E} \langle x, x+\tau\rangle,$$

so $x + \tau = -x$ if and only if $2x = -\tau$, and there are precisely four such possible values of x. •

Both σ^d and τ^d act on $\mathbf{P}^-$, and this defines a $\mathbf{Z}_2 \times \mathbf{Z}_2$ action on $\mathbf{P}^-$. Thus, by Lemma 6.1, if A is a Heisenberg invariant abelian surface or translation scroll in $\mathbf{P}^{2d-1}$, then $A \cap \mathbf{P}^-$ consists of a $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbit on $\mathbf{P}^-$.

Using Proposition 1.3.1, we can identify a point of $\mathcal{A}_{2d}^{lev}$ with a Heisenberg invariant abelian surface A contained in $\mathbf{P}^{2d-1}$, if the restriction of the universal line bundle is very ample on A. Based on this, we give the following definition:

Definition. *For $d \geq 2$, we may define a rational map*

$$\Theta_{2d} : \mathcal{A}_{2d}^{lev} \dashrightarrow \mathbf{P}^-/\mathbf{Z}_2 \times \mathbf{Z}_2$$

by mapping A to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbit $A \cap \mathbf{P}^-$.

Theorem 6.2.

a) *For a general $\mathbf{H}_{2d}$-invariant, $(1, 2d)$-polarized abelian surface $A \subseteq \mathbf{P}^{2d-1}$, and a point $y \in A \cap \mathbf{P}^-$, the 4×4-pfaffians of the matrices*

$$\begin{cases} M_5(x,y), M_5(x,\sigma^5(y)), M_5(x,\tau^5(y)) & \text{if } d = 5; \\ M_d(x,y), M_d(x,\sigma^d(y)) & \text{if } d \geq 7,\ d \text{ odd}; \\ M_6(x,y), M_6(\sigma(x),y), M_6(\tau(x),y) & \text{if } d = 6; \\ M_d(x,y), M_d(\sigma(x),y) & \text{if } d \geq 8,\ d \text{ even} \end{cases}$$

generate the homogeneous ideal of A.

b) *For $d \geq 5$, Θ_{2d} is birational onto its image.*

Proof: Let $\pi : \mathbf{P}^- \to \mathbf{P}^-/\mathbf{Z}_2 \times \mathbf{Z}_2$ be the quotient map, and let $Z_{2d} := \pi^{-1}(\operatorname{im}\Theta_{2d})$. Let also $S_{E,\tau}$ be a translation scroll. By Theorem 3.1, there exists a flat family $\mathcal{A} \to \Delta$,

$\mathcal{A} \subseteq \mathbf{P}^{2d-1}_{\Delta}$, of Heisenberg invariant surfaces with $\mathcal{A}_0 = S_{E,\tau}$, and $\mathcal{A}_t$ a non-singular $(1, 2d)$-polarized abelian surface for the general $t \in \Delta$. Then it is clear that $S_{E,\tau} \cap \mathbf{P}^- \subseteq \overline{Z_{2d}}$. We deduce that for any elliptic normal curve, $Sec(E) \cap \mathbf{P}^- \subseteq \overline{Z_{2d}}$, and so the same is true if instead of a non-singular elliptic curve, we take E to be the "standard $2d$-gon" and $Sec(E)$ to be its "secant variety". Thus, in particular, the point

$$p_0 := (0 : 1 : 1 : 0 : \cdots : 0 : -1 : -1)$$

lies in $\overline{Z_{2d}}$. Consider now the family

$$\mathcal{A} \subseteq \mathbf{P}^{2d-1} \times \overline{Z_{2d}}$$

over $\overline{Z_{2d}}$ defined by the 4×4-pfaffians of the matrices $M_d(x, y)$, etc (where the choice is made depending on d as in the statement of the theorem). Here, the x's are coordinates on the first $\mathbf{P}^{2d-1}$ while the y's are coordinates for $\overline{Z_{2d}} \subseteq \mathbf{P}^{2d-1}$. By Corollary 2.7, $\mathcal{A}_y$ contains all abelian surfaces represented by the points of $\Theta_{2d}^{-1}(\pi(y))$, for $y \in Z_{2d}$.

Now some easy combinatorics show that the 4×4-pfaffians of the matrices in the statement of the Theorem generate the ideal $I_{X^1_{2d}}$, using the description of this ideal given in Theorem 4.6. We prove this in the general case for $d \geq 7$, and leave it to the reader to check the cases $d = 5$ and $d = 6$. We first consider certain pfaffians of $M_d(x, y_0)$. If i is odd, $0 \leq i < 2d$, then the 4×4-submatrix consisting of the rows and columns indexed by

$$\frac{i-1}{2}, \frac{i-1}{2} + 1, \frac{i-1}{2} + 2, \frac{i-1}{2} + 3,$$

modulo d, is the matrix

$$\begin{pmatrix} 0 & -x_i & -x_{i+1} & 0 \\ x_i & 0 & -x_{i+2} & -x_{i+3} \\ x_{i+1} & x_{i+2} & 0 & -x_{i+4} \\ 0 & x_{i+3} & x_{i+4} & 0 \end{pmatrix}$$

whose pfaffian is easily seen to be

$$x_i x_{i+4} - x_{i+1} x_{i+3}.$$

If i and j are both odd, and $d(i, j) \geq 6$, then the submatrix consisting of the rows and columns

$$\frac{i-1}{2}, \frac{i+1}{2}, \frac{j-1}{2}, \frac{j+1}{2}$$

is now

$$\begin{pmatrix} 0 & -x_i & 0 & -\alpha \\ x_i & 0 & -\beta & 0 \\ 0 & \beta & 0 & -x_j \\ \alpha & 0 & x_j & 0 \end{pmatrix}$$

where α and β depend on the precise choice of i and j, but at least one of them is zero. The pfaffian of this matrix is $x_i x_j$. Finally, if i is odd and j even, with $d(i,j) \geq 5$, then the submatrix with rows

$$\frac{i-1}{2}, \frac{i+1}{2}, \frac{j}{2}-1, \frac{j}{2}+1$$

is

$$\begin{pmatrix} 0 & -x_i & -\alpha & -\beta \\ x_i & 0 & -\gamma & -\delta \\ \alpha & \gamma & 0 & -x_j \\ \beta & \delta & x_j & 0 \end{pmatrix}$$

with $\alpha, \ldots, \delta$ depending on the precise choice of i and j, but with $\alpha\delta = \beta\gamma = 0$; in particular the pfaffian of this matrix is $x_i x_j$. In this way, we have obtained the following sets of equations

$$\{x_{i+1}x_{i+3} - x_i x_{i+4} \mid i \text{ odd}\} \cup \{x_i x_j \mid d(i,j) \geq 5,\ d(i,j) \text{ odd}\} \cup$$

$$\{x_i x_j \mid d(i,j) \geq 6,\ d(i,j) \text{ even},\ i,\ j \text{ odd}\}.$$

To obtain the remaining equations, if d is even, we apply σ to these equations. If d is odd, we apply a similar procedure to the matrix $M_d(x, \sigma^d(p_0))$.

Recall now from Theorem 4.6 that X_{2d}^1 is a Cohen-Macaulay surface of degree $4d$. Since for a general $y \in Z_{2d}$, $\mathcal{A}_y$ is either of dimension > 2, or is a surface of degree at least $4d$, we see that for general y, $\dim \mathcal{A}_y = 2$ and $\deg \mathcal{A}_y = 4d$. Since furthermore $\mathcal{A}_{p_0} = X_{2d}^1$ has the same Hilbert polynomial as a $(1, 2d)$-polarized abelian surface, there must exist an open neighborhood $U \subseteq \overline{Z_{2d}}$ of p_0 such that $\mathcal{A}_U \subseteq \mathbf{P}_U^{2d-1}$ is flat over U, and each smooth fibre of $\mathcal{A}_U \to U$ is an abelian surface. In particular, since the ideal of $\mathcal{A}_{p_0}$ is generated by the pfaffians in question, the same is true for the ideal of $\mathcal{A}_y$ for general $y \in U$. Thus *a)* follows, and *b)* is now clear since $\Theta_{2d} : \Theta_{2d}^{-1}(\pi(U \cap Z_{2d})) \to \pi(U \cap Z_{2d})$ is an isomorphism. •

n odd: Let $n = 2d+1$. The projectivization of the negative eigenspace of the involution ι acting on $\mathbf{P}^{2d}$ is now $\mathbf{P}^- \cong \mathbf{P}^{d-1}$. The group $\mathbf{H}_{2d+1}$ acts on $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$, and one sees readily that $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ splits up into $d+1$ mutually isomorphic irreducible representations of $\mathbf{H}_{2d+1}$. Let R_d be the $(d+1) \times (2d+1)$ matrix given by

$$(R_d)_{ij} = x_{j+i} x_{j-i}, \quad 0 \leq i \leq d,\ 0 \leq j \leq 2d.$$

The rows of this matrix each span an irreducible subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ and yield the decomposition of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ into $d+1$ mutually isomorphic representations.

Let $D_i \subseteq \mathbf{P}^-$ be the locus in $\mathbf{P}^-$ where R_d has rank $\leq 2i$. Note that on $\mathbf{P}^-$, $x_0 = 0$ and $x_i = -x_{-i}$ for $i \neq 0$, so using coordinates $x_1, \ldots, x_d$ on $\mathbf{P}^-$, R_n restricted to $\mathbf{P}^-$ has the property that for $j \neq 0$, the j-th column and the $(2d+1-j)$-th column coincide, since

$$\begin{aligned}(R_d)_{i,2d+1-j} &= x_{(2d+1-j)+i}x_{(2d+1-j)-i} \\ &= (-x_{j-i})(-x_{j+i}) \\ &= x_{j-i}x_{j+i} \\ &= (R_d)_{ij}\end{aligned}$$

on $\mathbf{P}^-$. Also, the leftmost $(d+1) \times (d+1)$ block of R_d when restricted to $\mathbf{P}^-$ is anti-symmetric, since on $\mathbf{P}^-$:

$$\begin{aligned}(R_d)_{ij} &= x_{j+i}x_{j-i} \\ &= x_{j+i}(-x_{i-j}) \\ &= -(R_d)_{ji}.\end{aligned}$$

We denote by T_d the restriction of this $(d+1) \times (d+1)$ block to $\mathbf{P}^-$. D_i is then the locus where T_d is rank $\leq 2i$.

Lemma 6.3. *For a general $\mathbf{H}_{2d+1}$-invariant abelian surface $A \subseteq \mathbf{P}^{2d}$, $d \geq 3$, we have $A \cap \mathbf{P}^- \subseteq D_2$ and $A \cap \mathbf{P}^- \not\subseteq D_1$.*

Proof: Note that R_d, up to transpose and permutations of rows and columns, is a submatrix of the matrix $M_d'(x, x)$ of Corollary 2.8, which is rank at most 4 on A. Thus $A \cap \mathbf{P}^- \subseteq D_2$. We need then to show that for general A, $A \cap \mathbf{P}^- \not\subseteq D_1$. Let $S_{E,\tau}$ be a translation scroll. By Theorem 3.1, we can find a flat family $\mathcal{A} \to \Delta$ in $\mathbf{P}^{2d}_{\Delta}$ with $\mathcal{A}_0 = S_{E,\tau}$, and such that $\mathcal{A}_t$ a non-singular abelian surface for general $t \in \Delta$. Thus it is enough to show that $S_{E,\tau} \cap \mathbf{P}^- \not\subseteq D_1$ for general $S_{E,\tau}$, or equivalently, it is enough to show that $Sec(E) \cap \mathbf{P}^- \not\subseteq D_1$ for the general elliptic normal curve $E \subset \mathbf{P}^{2d}$. By Theorem 3.2, it is then enough to check that $Sec(E) \cap \mathbf{P}^- \not\subseteq D_1$ for $E = X(\Gamma_{2d+1})$, the standard $(2d+1)$-gon. This is clear since the point

$$p_0 = (0 : 1 : 1 : 0 : \cdots : 0 : -1 : -1)$$

lies in $Sec(E) \cap \mathbf{P}^- = X(\partial C(2d+1, 4)) \cap \mathbf{P}^-$, while the matrix

$$T_d(z_0) = \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \vdots & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

has rank 4. •

We may define now a morphism

$$s : D_2 \backslash D_1 \to Gr(d-3, d+1)$$

by taking $x \in D_2 \backslash D_1$ to $\ker(R_d(x)^t)$. We think of the points of $Gr(d-3, d+1)$ as parametrizing $(d-3)(2d+1)$-dimensional sub-$\mathbf{H}^e_{2d+1}$-representations of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$: if N is the $(d+1) \times (d-3)$-matrix whose columns span a $(d-3)$-dimensional subspace of $\mathbf{C}^{d+1}$, then the entries of $R_d^t \cdot N$ span the corresponding $(d-3)(2d+1)$-dimensional subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$. In other words, the map s takes a point $x \in D_2 \backslash D_1$ to the largest subrepresentation of quadrics in $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ vanishing at x.

Lemma 6.4. *For a general $\mathbf{H}_{2d+1}$-invariant abelian surface $A \subseteq \mathbf{P}^{2d}$, $d \geq 3$, the set $s(A \cap \mathbf{P}^- \cap (D_2 \backslash D_1))$ consists of exactly one point $p \in Gr(d-3, d+1)$. If V_p is the corresponding $(d-3)$-dimensional subspace of $\mathbf{C}^{d+1}$, and N_p is a matrix whose columns span V_p, then the space of quadrics spanned by the entries of $R_d^t \cdot N_p$ is $H^0(\mathcal{I}_A(2))$, and each column of $R_d^t \cdot N_p$ spans a sub-$\mathbf{H}_{2d+1}$-representation of $H^0(\mathcal{I}_A(2))$.*

Proof: By Riemann-Roch,

$$\begin{aligned} h^0(\mathcal{I}_A(2)) &\geq h^0(\mathcal{O}_{\mathbf{P}^{2d}}) - h^0(\mathcal{O}_A(2)) \\ &= \binom{2d+2}{2} - 4(2d+1) \\ &= (d-3)(2d+1). \end{aligned}$$

If $x \in A \cap \mathbf{P}^-$, $x \in D_2 \backslash D_1$, and $p := s(x)$, then the entries of $R_d^t \cdot N_p$ span the largest sub-$\mathbf{H}_{2d+1}$-representation of quadrics in $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ vanishing at x. Call this subrepresentation I_2. Now $\dim I_2 = (d-3)(2d+1)$. Since $H^0(\mathcal{I}_A(2))$ is also a sub-$\mathbf{H}_{2d+1}$-representation of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ consisting of quadrics vanishing at x, we get $H^0(\mathcal{I}_A(2)) \subseteq I_2$. From the dimension estimate above (or since the representations have weight 2, cf. [La]), we must then have $H^0(\mathcal{I}_A(2)) = I_2$. This is true for each $x \in A \cap \mathbf{P}^-$, $x \in D_2 \backslash D_1$, so we see that $s(x)$ is the point corresponding to the subrepresentation $H^0(\mathcal{I}_A(2))$ of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$. In particular $s(A \cap \mathbf{P}^- \cap (D_2 \backslash D_1))$ consists of one point. •

Definition. *We define a rational map*

$$\Theta_{2d+1} : \mathcal{A}^{lev}_{2d+1} \dashrightarrow Gr(d-3, d+1)$$

by taking an abelian surface A to $s(A \cap \mathbf{P}^- \cap (D_2 \backslash D_1))$, or equivalently, to the point of $Gr(d-3, d+1)$ corresponding to the subrepresentation $H^0(\mathcal{I}_A(2))$ of $H^0(\mathcal{O}_A(2))$.

Theorem 6.5.

a) *The homogeneous ideal of a general* $\mathbf{H}_{2d+1}$*-invariant abelian surface* $A \subseteq \mathbf{P}^{2d}$ *of type* $(1, 2d+1)$, $d \geq 5$, *is generated by quadrics.*

b) Θ_{2d+1} *is birational onto its image.*

Proof: Let $Z := s^{-1}(\operatorname{im}\Theta_{2d+1}) \subseteq D_2 \backslash D_1$, and let $\overline{Z}$ be the closure of Z in $D_2 \backslash D_1$. Let $\mathcal{A} \subseteq \mathbf{P}^{2d}_{\overline{Z}}$ be the family defined by the condition that the ideal of $\mathcal{A}_z$, $z \in \overline{Z}$, is the subrepresentation of $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ determined by $s(z)$, that is the subrepresentation of quadrics in $H^0(\mathcal{O}_{\mathbf{P}^{2d}}(2))$ vanishing at z. By Lemma 6.4, if $z \in s^{-1}(\Theta_{2d+1}(A))$, then $\mathcal{A}_z$ contains the surface A. Now exactly the same argument as in the proof of Lemma 6.3 shows that the point

$$z_0 = (0 : 1 : 1 : 0 : \ldots : 0 : -1 : -1)$$

is in $\overline{Z}$. Moreover, $\ker(R_d(z_0)^t) = \ker(T_d(z_0)^t)$, which in turn is spanned by the vectors $(0, 1, -1, 0, \ldots)$ and $e_4, \ldots, e_{d-1}$, where $e_1, \ldots, e_{d+1}$ denote the standard basis of $\mathbf{C}^{d+1}$. Hence $\mathcal{A}_{z_0}$ is defined by the ideal generated by the set of quadrics

$$\{x_{i+1}x_{i+3} - x_i x_{i+4} \mid i \in \mathbf{Z}/(2d+1)\mathbf{Z}\} \cup \{x_i x_j \mid d(i,j) \geq 5\}.$$

By Theorem 4.6, the scheme $\mathcal{A}_{z_0}$ has the same Hilbert polynomial as a non-singular $(1, 2d+1)$-polarized abelian surface. So an argument as in the proof of Theorem 6.2 shows that there exists an open set $U \subseteq \overline{Z}$ such that $\mathcal{A}_U \to U$ is flat, and such that each smooth fibre is an abelian surface; the claims in *a)* and *b)* then follow. •

Example 6.6. The results in Theorems 6.2 and 6.5 *a)* do not hold for all $(1, d)$-polarized abelian surfaces. More precisely, as the following example shows, cubics may also be needed to generate the homogeneous ideal (compare also [ADHPR1]):

Let $E := \{x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0\} \subseteq \mathbf{P^2} = \mathbf{P}(V)$, $\lambda \notin \{\infty, -3, -3\epsilon_3, -3\epsilon_3^2\}$, where ϵ_3 is a third root of unity, be a smooth cubic in the Hesse pencil (i.e., an elliptic normal curve in $\mathbf{P^2}$ embedded with canonical level structure). We also choose as origin for the group law on E an inflection point, say $o_E = (0 : 1 : -1)$. Fix $d \in \mathbf{Z}$, with $d \geq 3$ and $d \cong 2 \bmod 3$, and let now $\{y_0, y_1, \ldots, y_{3d-1}\}$ be a basis of canonical theta functions of $H^0(\mathcal{O}_E(3do_E))$. Recall that the Heisenberg group $\mathbf{H}_{3d}$ acts on this basis via the Schrödinger representation, namely

$$\sigma(y_i) = y_{i-1}, \quad \tau(y_i) = \xi^{-i} y_i, \quad i \in \mathbf{Z}_{3d},\ \xi := \exp(2\pi i/3d).$$

Since $[\sigma^d, \tau^d] = \xi^{-d^2} \cdot \mathrm{id}$, we may identify $\mathbf{H}_3 \subseteq SL(V)$ in its Schrödinger representation with the subgroup of $\mathbf{H}_{3d}$ generated by σ^d and τ^d, where we take as third root of unity $\epsilon_3 := \xi^{d^2}$.

Let now $E' := \{x_0^3 + x_1^3 + x_2^3 + \lambda' x_0x_1x_2 = 0\} \subseteq \mathbf{P}(V^*)$, $\lambda' \notin \{\infty, -3, -3\epsilon_3, -3\epsilon_3^2\}$, with origin $o_{E'} = (0 : 1 : -1) \in \mathbf{P}(V^*)$, and consider the diagonal action of $\mathbf{H}_3$ on the product $E \times E' \subseteq \mathbf{P}(V) \times \mathbf{P}(V^*)$, where the action on the first factor is the one induced from $\mathbf{H}_{3d}$. It is easy to see that the center of $\mathbf{H}_3$ acts trivially on the line bundle $\mathcal{M} := \mathcal{O}_E(3do_E) \boxtimes \mathcal{O}_{E'}(3o_{E'})$, hence $\mathcal{M}$ descends to a line bundle $\mathcal{L} := \mathcal{M}/\mathbf{Z}_3 \times \mathbf{Z}_3$ over the quotient abelian surface A defined via the 9-fold unramified cover

$$\pi : E \times E' \to E \times E'/\mathbf{Z}_3 \times \mathbf{Z}_3 =: A.$$

Since $\mathcal{M}$ is of type $(3, 3d)$, and $\gcd(d, 3) = 1$, we deduce that $\mathcal{L}$ defines on A a polarization of type $(1, d)$. By using Reider's criterion it is also easily seen that $\mathcal{L}$ defines an embedding to $\mathbf{P}^{d-1}$, whenever $d \geq 5$. A basis of sections for $W = H^0(\mathcal{L})$ is defined by the following invariant sections of $\mathcal{M}$:

$$s_i = y_{3i} \boxtimes x_0 + y_{3i+d} \boxtimes x_1 + y_{3i+2d} \boxtimes x_2, \qquad i \in \mathbf{Z}_d,$$

where the indices of the $y's$ are taken modulo $3d$. The subgroup of $\mathbf{H}_{3d}$ generated by σ^3 and τ^3 acts naturally on $\mathcal{M} = \mathcal{O}_E(3do_E) \boxtimes \mathcal{O}_{E'}(3o_{E'})$, where the action on the second factor is the trivial one, and since it commutes with the above diagonal action of $\mathbf{H}_3$ we deduce that this action descends to an action on the line bundle $\mathcal{L}$, whence in particular on its sections. Moreover, since $[\sigma^3, \tau^3] = \xi^{-9} \cdot \mathrm{id}$, we may identify this subgroup with the Heisenberg group $\mathbf{H}_d \subseteq SL^{\pm}(W)$. It is also readily checked that $\mathbf{H}_d$ acts via the Schrödinger representation on the chosen basis $\{s_i \mid i \in \mathbf{Z}_d\}$ of $H^0(\mathcal{L})$. In other words, we've picked this way a canonical level structure on $(A, \mathcal{L})$.

Let now F and F' denote the images on A of the elliptic curves E and E', respectively. Since $\deg \mathcal{L}_{|F'} = 3$, all curves in the pencil $|F'|$ are embedded via $\mathcal{L}$ in $\mathbf{P}^{d-1}$ as plane cubic curves, and hence any quadric hypersurface containing the abelian surface $\psi_{\mathcal{L}}(A)$ must also contain the threefold traced out by the planes spanned by the plane cubics in the pencil $|F'|$. In particular, we deduce that the homogeneous ideal of $\psi_{\mathcal{L}}(A)$ cannot be generated only by quadrics. It is also easy to check in this case that for general choices of E and E', the homogeneous ideal of $\psi_{\mathcal{L}}(A)$ is generated by quadrics and cubics when $d \geq 7$.

The above results and rather extensive checks in examples have lead us to formulate the following

Conjectures.

a) The homogeneous ideal of an embedded $(1, d)$-polarized abelian surface is generated by quadrics and cubics, for $d \geq 9$.

b) The general $(1,d)$-polarized abelian surface, for $d \geq 10$, has property $N_{[\frac{d}{2}]-4}$.

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