EQUATIONS WITH UNBOUNDED DELAY: A SURVEY

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Technical Report No. 113
August, 1979

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INTRODUCTION

The theory of equations with delay has been considerably developed during the past 30 years. The first book dedicated to this subject has been published by A. D. Myshkis [180], in 1951, and has been subsequently translated into German (1955) and English (1966). A second Russian edition, that came out in 1972, contains a good deal of references and points out interesting features of the theory of linear equations with delayed argument, including also examples of equations with unbounded delay.

The general theory of equations with delay, as well as several basic topics in this field (stability, oscillations), have constituted successively the aim (totally or partially) of the following monographs: L. E. El'sgol'ts [68], N. N. Krasovski [137], E. Pinney [198], R. Bellman and K. Cooke [10], A. Halanay [89], L. E. El'sgol'ts [69] S. B. Norkin [190], N. M. Oguztoreli [196], V. Lakshmikantham and S. Leela [145], Yu A. Mitropolskii and D. I. Martynyuk [263], J. K. Hale [96], L. E. El'sgol'ts and S. B. Norkin [70], R. D. Driver [65].

The second edition of Hale's book [96] contains the most complete description of the subject, though it does not cover entirely several aspects discussed in some of the above quoted monographs.

It should be also mentioned that, in the past few years, several survey papers have been devoted to the theory of equations with delay: [92], [181], [182], [242], [264].

In spite of the fact that most of the above quoted sources contain results and references related to the theory of equations with unbounded delay, the coverage of this topic, in monographs and survey papers, seems to be much behind its real status in the literature. This situation might be the result of the fact that the interest in this class of functional equation has shown a dramatic increase only in the past few years. Indeed, over sixty percent of the papers included in the list of references are from the 1970's. Moreover, the main achievements, in both general theory and investigation of various special classes that occur in applications, are undoubtedly those obtained during the recent years. At least seven dissertations have been dedicated, in the past few years, to the theory of equations with unbounded delay (see [23], [56], [156], [157], [215], [216], [269]).

The aim of the authors of this survey is to provide an account of the basic results and problems of the theory of equations with unbounded delay.

Though we do not aim at a thorough presentation of the topic, including its history, we think a few remarks on the early stage of this theory may be welcome. There is no doubt, we must pay tribute to V. Volterra (see [228], [229] and [230], where further references can be found for the period up to 1930) as the founder of equations in which the unbounded delays occur. Moreover, he is to be credited for the first applications in such fields as population dynamics and mechanics

of continua (materials with memory) [228]. Of course, the development
of functional analysis at that time did not stimulate the construction
of a general theory for equations with unbounded delay (not even for
those with bounded delay). Volterra himself found it necessary to
"cut" the delay, in order to make the topic more accessible to the
investigation (see [228]). In the period 1930-1950, only a few papers
have been concerned with equations involving unbounded delays, or with
so-called "Volterra equations" (in more common terminology, these are
functional equations involving causal operators: $x(t) = V(t,x(t))$).
In particular, the right hand side V could be an integral operator of
Volterra type. The paper by N. A. Tychonoff [223], though isolated
in time, could serve as an illustration of the impact that Volterra's
ideas had on the applied research. The name of Lotka should be also
associated with applied research areas that rely on the concept of
equationswith delay, the case of the unbounded delay being implicitly
involved.

If the progress in the theory of equations with delay is hardly
noticeable in the period 1930-1950, the progress in Functional Analysis
and related areas (harmonic analysis, dynamical systems, various classes
of function spaces and operators, semigroups) has been overwhelming.
First, the theory of equations with bounded delay has taken advantage in
building up its structures. Hale's monograph [96] provides the best
illustration in this respect, but examples of the use of functional
analytic methods in investigating delay equations can be also found in
several of the above quoted monographs, and papers included in the
references.

Gradually, the theory of equations with unbounded delay has emerged as an independent branch of the modern research in the field of Mathematical Analysis, with its specific problems and many connections to the applied fields. If there is no monograph dedicated in its greatest part to the theory of equations involving unbounded delays, we can mention several books containing various applications of these equations in such fields as Continuous Mechanics [30], Population Dynamics and Ecology [57], [169], Systems Theory [59], [61], [170], [203], and Nuclear Reactor Dynamics [81].

It is interesting to point out that most of the research conducted towards the construction of a general theory of existence and stability of solutions, for equations with unbounded delay, was aimed to shape an adequate theory for materials with memory (i.e., the old problem Volterra dealt with in 1928: [228]). The group of researchers to be first credited for the promotion of this field is, undoubtedly, that with the Carnegie-Mellon University in Pittsburgh and their associates (see references: B. Coleman, B. Coleman and collab., [28]-[34]; M. J. Leitman and V. J. Mizel, [150]-[153]; R. C. MacCamy and collab., [159]-[164]).

In compiling the list of references, which shortly after its completion required a supplement, the authors have been guided by the following lines:

a) To include in the list, primarily, those papers that deal with equations involving unbounded delays, but also papers that have direct connections with the latter, or are significant in regard to certain general procedures (Liapunov's functions or functionals, semigroup approach, etc.) that might be applicable also in the case under consideration.

However, the number of papers not effectively connected to unbounded delays is relatively small.

b) Not to include papers that have been already quoted in the previously mentioned monographs, unless they are directly involved in this presentation.

c) To include papers in which the stress is on the applications, but only to the extent we need to illustrate the interconnections between the theory of equations with unbounded delay, and various applied areas. We made no attempt to give a complete list of references, as far as applications of equations with unbounded delay are concerned.

d) Papers dealing with stochastic equations involving unbounded delay, or partial differential equations of the same kind, have not been, generally, included in the list.

1. DESCRIPTION OF EQUATIONS WITH UNBOUNDED DELAY

The functional-differential equations with unbounded delay can be written as

$$(1) \qquad \dot{x}(t) = f(t, x_t),$$

where $x_t(u) = x(t+u)$, $-\infty < u \leq 0$. If x takes values in a real Banach space B, then for each fixed t, $\phi \to f(t, \phi)$ is a map from a certain function space $S = S((-\infty, 0], B)$ into B. We agree that $S((-\infty, 0], B)$ denotes a function space whose elements are defined on $(-\infty, 0]$, and take values in B. In most cases discussed in the literature dedicated to this subject, the space B is of finite dimension. However, certain papers deal with the general case when B is infinite dimensional (see, for instance, [23], [28], [209]).

The Cauchy problem related to (1) can be stated as follows: given $\phi \in S$ and $t_0 \in R$, find a function $x : (-\infty, t_0 + \delta) \to B$, $\delta > 0$, such that

$$(2) \qquad x_{t_0} = \phi,$$

and (1) is verified on $(t_0, t_0 + \delta)$.

It becomes clear from the above formulation that two basic elements are involved in the definition of an equation with unbounded delay. First, the function space from which we select the initial data ϕ, occurring in (2), and second, the operator (or, more accurately, the family of operators $f(t, \cdot)$) defined on that space and occurring in the right hand side of (1).

Unfortunately, both mathematical concepts mentioned above are not at all simple, and many peculiarities occur when dropping the assumption of boundedness for the delay.

Let us discuss first the space of initial functions, the so-called "phase-space" associated to the equation (1). Its elements are functions taking values in a Banach space, even in an Euclidean space, but their domain of definitions is a noncompact set $R = (-\infty, 0]$. One can also say that the set of definition has infinite measure, if one thinks to the ordinary Lebesgue measure. The structure of such function spaces is, generally, more intricate than the one of function spaces whose elements are defined on a compact set (see, for illustration, [38], [66], [167]). For instance, the compactness criteria are more involved in the case of function spaces with elements defined on noncompact sets. The separability property is another example of property that usually holds true in the cases of function spaces consisting of elements defined on compact sets, while it does not in most cases of function spaces whose elements are defined on noncompact sets.

Perhaps, more dramatic is the difference between the various operators acting on spaces belonging to the above mentioned classes. To illustrate only one feature, let us remind (see [96]) that the general form of linear autonomous equations with bounded delay, and continuous right hand side, is

(3) $$\dot{x}(t) = \int_{-r}^{0} [d\eta(s)]x(t+s) + f(t),$$

where $\eta(s)$ is a matrix whose entries are functions with bounded variation on $[-r, 0]$. It is assumed that $B = R^n$. If one tries to find such a form

in the case of equations with unbounded delay, then the following situation may occur. Assume that $S = S((-\infty,0],R^n)$ is the space of all continuous maps from R_- into R^n, with the topology of uniform convergence on any compact set of R. This space is not a Banach space, but its topology is a natural one. The general form of a linear continuous map from S into R^n is precisely

$$(4) \qquad L(\phi) = \int_{-r}^{0} [d\eta(s)]\phi(s),$$

with $r > 0$ depending on L, and $\eta(s)$ a matrix whose entries are real function with bounded variation on $[-r,0]$. In other words, if one looks for the general form of the linear autonomous equation, with continuous right hand side, then one finds equations like (3). The result is somewhat astonishing, because the unbounded delay is "cut" down to a bounded one. Of course, the continuity requirement on the right hand side of the equation is too strong here, and considerably reduces the class of equations with such a property.

Let us remark that similar features are present when S is the space of all locally integrable maps from R_- into R^n, with the topology given by the family of semi-norms

$$x \to \int_{-n}^{0} |x(s)|ds, \qquad n = 1, 2, 3, \ldots .$$

Therefore, a special attention should be paid when choosing the phase-space and the operators occurring in the equation, in order not to lose the most salient features of the theory of equations with

unbounded delay. Actually, the latest developments show that the phase-space should be chosen not as rich as it appears in the examples described above (see [9], [28], [31], [32], [34], [39], [41], [43], [46], [75], [93], [94], [97], [99], [104], [157], [158], [173], [209]), while the operator involved does not have to be necessarily continuous (see [249], suppl. ref.).

If one looks for the general form of the linear equation, with continuous right hand side, but not necessarily autonomous, then one finds

$$(5) \qquad \dot{x}(t) = \int_{-r(t)}^{0} [d_s \eta(t,s)]x(t+s) + f(t),$$

with convenient conditions on η (we again assume that S is the space of all continuous maps from R_- into R^n, with the topology of uniform convergence on any compact set of R_-). Since the function $r(t)$ could be unbounded, it is reasonable in such a case to consider (5) as an equation with unbounded delay. Equations like (5), and related ones, have been extensively investigated by A. D. Myshkis in his monograph [180], and by several of his followers (see reference items under names: V. B. Kolmanovskii, E. Kozakiewicz, K. M. Malov, N. P. Mironov, V. F. Subbotin, A. M. Zverkin [240]).

Nonlinear equations involving integral operators of the form indicated in the right hand side of (5), or even with the integral limits $-\infty$ and 0, have been also investigated by many authors: V. E. Benes, L. B. Bisjarina, J. Blaz, C. Corduneanu [37], A. Haimovici, A. Kamont and M. Kwapisz, M. Kisieliewicz, J. Kudrewicz, M. J. Leitman and V. J. Mizel, J. J. Levin and J. A. Nohel, R. C. MacCamy, R. K. Miller, Yu. I. Neimark,

V. R. Nosov, N. Oguztoreli, M. Picone, A. G. Ramm, Vl. Rasvan, I. W.
Sandberg, G. Seifert, V. R. Vinokurov, V. Volterra [230], K. Zima (see
list of references). Usually, the equation under investigation is
supposed to be in the integral form, for instance,

(6)
$$x(t) = f(t, x(t), \int_{-\infty}^{t} k(t,s, x(s)ds)$$

or another similar form. In most of the cases, the convolution kernel
is considered: $k(t,s,x) = k(t-s)g(t,x(s))$.

It is worthwhile to point out that the equations of the form (6),
or their counterparts involving also the derivative, have been the first
equations with unbounded delay to be investigated (see V. Volterra [230]).
An interesting feature, they come out in a very natural manner if one searches
for the limiting behavior of solutions to Volterra equations with finite
limits (see R. K. Miller [172] or C. Corduneanu [38]).

Besides functional differential equations of the form (1), and
integral equations of the form (6) or related to it, we must include in
the class of equations with unbounded delay another family of functional
equations (see N. V. Azbelev [4], N. V. Azbelev and L. F. Rakhmatulina [5],
Yu. A. Gershman and A. D. Myshkis [78], G. A. Kamenskii and A. D. Myshkis
[114], A. D. Myshkis [181]). A very recent survey paper on this topic has
been published by N. V. Azbelev and L. F. Rakhmatulina (see suppl. list
of references). These authors deal, in fact, with boundary value problems
for functional differential equations of the form $x'(t) = (Ax)(t)$,
$a \leq t \leq b$, $x = \phi$ for $t \in [a,b]$. The operator A depends on ϕ, and in
particular on the values ϕ assumes in the half-axis $t \leq a$. Keeping ϕ

fixed for $t > b$, or taking the limit case $b = +\infty$, we obtain a functional differential equation on $[a,b]$, with "initial" data on $(-\infty, a]$. Let us remark that this is not a local Cauchy problem for the equation $x'(t) = (Ax)(t)$. It is rather an existence problem "in the large," for a family of equations depending upon a functional parameter ϕ. Actually, the equation should be written as $x'(t) = (A_\phi x)(t)$, $t > a$, $x(a+) = \phi(a)$. Since ϕ belongs usually to a space with rather intricate topological structure, and the existence is required "in the large," the difficulty of the problem is considerable (even in the special case formulated above). Moreover, the difficulty is increased by the fact that the dependence of the operator (in the right hand side of the equation) upon ϕ does not seem usually to be of a simple nature.

To illustrate another feature of the equations with unbounded delay, let us consider the case

$$(7) \qquad x(t) = G(t, \int_{-\infty}^{t} k(t,s,x(s))ds), \quad t \in R.$$

If one associates to (7) the initial condition (2), i.e.,

$$(8) \qquad x(t) = \phi(t), \quad t \leq t_0, \quad t_0 \in R,$$

then (7) becomes for $t > t_0$

$$x(t) = G(t, \int_{-\infty}^{t_0} k(t,s,\phi(s))ds + \int_{t_0}^{t} k(t,s,x(s))ds),$$

or, in another form:

$$(7') \qquad\qquad x(t) = \tilde{G}(t, \int_{t_0}^{t} k(t,s,x(s))ds\,), \, t > t_0.$$

In other words, if one prescribes the values of $x(t)$ on $(-\infty, t_0]$, then $x(t)$ can be determined on $t > t_0$ by solving a Volterra equation with finite limits (or, in equivalent terms, with bounded delay). Of course, one can object that ϕ does not verify (7) for $t \leq t_0$. But this is true also in the case of equations of the form (1), with initial condition (2). It does not seem to be any ground to reject the legitimacy of the Cauchy problem for non-differential functional equations as (7). Since $\tilde{G}$ involves $\int_{-\infty}^{t_0} k(t,s,\phi(s))ds$, i.e., an operator on a function space whose elements are defined on noncompact sets, it is reasonable to regard such problems as pertaining to the theory of equations with unbounded delay.

Before concluding this section, let us mention the example of an equation of the form

$$(9) \qquad\qquad x'(t) = \lambda(t)x(t) + \mu \int_{-\infty}^{t} K(t,s)x(s)ds + f(t),$$

studied by L. P. Bisjarina[17]. It occurs, according to the author, in the nuclear-reactor kinetics. Under appropriate hypotheses, the uniqueness of solution is guaranteed by the "initial" condition $x(-\infty) = x_0$, a feature that does not seem to be very common.

Summing up the discussion above, we must point out that the theory of equations with unbounded delay heavily relies on the properties of function spaces, whose elements are defined on noncompact sets, and on the properties of operators acting on such spaces. As seen above,

due to the intricate structure of such spaces and operators, various oddities can occur when starting to build up a theory of equations with unbounded delay.

2. EXISTENCE, UNIQUENESS, CONTINUATION AND
CONTINUOUS DEPENDENCE OF SOLUTIONS

The basic problems mentioned in the title of this section have been discussed in most of the papers included in the list of references, under various assumptions and formulations.

For instance, the linear equations of the form (5) have been considered in A. D. Myshkis' monograph [180], and thoroughly investigated as far as existence, uniqueness, and continuous dependence are concerned. The existence has a global character and the initial data are subject to a minimum of restrictions.

Nonlinear equations, directly generalizing those considered by Myshkis, have been studied by J. Blaz [18], [19], A. Kamont and M. Kwapisz [115], M. Kisieliewicz [121]. Carathéodory type conditions have been also examined.

Most of the efforts of Pittsburgh's school, in setting up an adequate theory of existence, uniqueness and continuous dependence of solutions for equations with infinite delay, have been summarized in B. D. Coleman's paper [28]. See also [34], and for recent discussion [249]. Let us elaborate now on these contributions.

We shall consider the functional differential equation (1), under initial condition (2). But since ϕ stands in (2) for a class of (measurable) equivalent maps, it is not possible to determine the value of ϕ at any point of R_-. As suggested by M. C. Delfour and S. K. Mitter in [58], for a real meaning of such a condition, in conjunction to the functional-differential equation (1), one should prescribe the initial value of the

solution at $t = t_0$. In other words, (2) should be replaced by

$$(2') \qquad x_{t_0} = \phi, \quad x(t_0+) = x^0 \in B.$$

Of course, if ϕ belongs to a space of continuous functions at $t = 0$, it is reasonable to take $x^0 = \phi(0)$. The function space of initial functions ϕ will consist of those classes of equivalent maps (Bochner) from $(-\infty,0)$ into the Banach space B, which are strongly Bochner measurable, and such that

$$(10) \qquad \int_{-\infty}^{0} |\phi(s)|^p k(s)ds < +\infty,$$

where $p \geq 1$ is given, $|\cdot|$ denotes the norm in B, while $k : (-\infty,0) \to (0,\infty)$ is an __influence function.__ In other words, k is locally summable on $(-\infty,0)$ and for each $\sigma \geq 0$ one has

$$(11) \qquad K(\sigma) = \operatorname*{ess\,-\,sup}_{s \in (-\infty,0)} \frac{k(s-\sigma)}{k(s)} < \infty, \quad H(\sigma) = \operatorname*{ess\,-\,sup}_{s \in (-\infty,-\sigma)} \frac{k(s+\sigma)}{k(s)} < \infty.$$

If one denotes by S the product space $B \times L_k^p(R_-,B)$, i.e., $\phi \in L_k^p$ iff it satisfies (10), then a convenient norm in S is given by

$$(12) \qquad ||(x^0,\phi)||^p = |x^0|^p + \int_{-\infty}^{0} |\phi(s)|^p k(s)ds.$$

The requirements (11) on the weight function k are motivated mainly by application needs (mechanics of continua). They might be strengthened or relaxed, in accordance to the specific problem under investigation (see [28], [31], [32], [34], [254]).

If G is an open set in L_k^p, and T > 0, then assume f : [0,T] × G → B is such that

(13)
$$|f(t,\phi) - f(t,\psi)| \leq L \{\int_{-\infty}^{0} |\phi-\psi|^p kds\}^{1/p}$$

holds locally in [0,T] × G. Moreover, assume f(t,φ), that depends only on t and the equivalence class of φ, is integrable on [0,T] for each φ. Under these assumptions, the Cauchy problem (1), (2') has a unique (local) solution, defined on an interval $[t_0, t_0 + \delta]$, with $0 \leq t_0 < t_0 + \delta \leq T$. Continuous dependence of solutions, with respect to initial data, is also assured under above assumptions.

In [254], T. L. Herdman and J. A. Burns consider the equation

(14)
$$\dot{x}(t) = F(t,x(t),x_t),$$

with x, F ε R^n, under initial condition (2'). The condition (13) is now replaced by the weaker one

(15)
$$\int_{t_0}^{t} |F(s,x(s)x_t) - F(s,y(s),y_s)|ds \leq$$

$$\Gamma_r(t)\{\int_{-\infty}^{t} |x(s)-y(s)|^p k(s)ds\}^{1/p}, \quad t \in [t_0,t_0 + T],$$

where $\Gamma_r(t) \geq 0$ is a locally bounded measurable function on $[t_0,t_0+T]$, T > 0, and x, y belong to $L_k^p((-\infty,t_0+T),B)$, are continuous on $[t_0,t_0+T]$, and $||(x(t),x_t)||$, $||(y(t),y_t)|| \leq r$, t ε $[t_0,t_0+T]$ (see formula (12) for the definition of $||\cdot||$). In (15), one takes k(s) ≡ 1 for s > 0.

Local existence and uniqueness hold true, and continuous dependence follows from the same assumptions. More specifically, if the perturbed equation, associated to (14) is considered,

$$(16) \qquad \dot{x}(t) = F(t,x(t),x_t) + h(t),$$

then the (unique) solution depends continuously (in the supremum norm) on x^0, ϕ and h. The phase-space the authors deal with is the product $B \times L_k^p(R_-,B)$. Moreover, the case when F is defined only on a dense subset, in the last two arguments, is investigated. It is interesting to be pointed out that condition (15) does not necessarily imply continuity of the map $(x^0,\phi) \rightarrow F(t,x^0,\phi)$. The necessity to deal with discontinuous functions in the right hand side of the equation is not imposed, in this case, by the requirements of a theory covering the so-called systems with distributed parameters. It might occur in such a simple case like that of the equation

$$(17) \qquad \dot{x}(t) = \int_{-\infty}^{0} k(s)x(t+s)ds,$$

with integrable k(s) on R_-. For a detailed discussion, see [249], where a k(s) is indicated, with the property that the mapping $\phi \rightarrow \int_{-\infty}^{0} k(s)\phi(s)ds$ cannot be continuous on any L_k^p, p > 1, no matter how we take the influence function k(s) (see conditions (11)).

Another function space that has been used by many authors, in connection with equations with unbounded delay, can be defined as follows. Assume B is a Banach space and B^p, p > 1, represents the set of classes

of equivalent maps from $(-\infty,0]$ into B, such that they are strongly

measurable on $(-\infty,0]$ and continuous on $[-r,0]$, $r > 0$, and

$$(18) \qquad \sup_{s\epsilon[-r,0]} |\phi(s)|^p + \int_{-\infty}^{-r} |\phi(s)|^p k(s)ds < +\infty.$$

B^p spaces have been defined by B. D. Coleman and V. J. Mizel [33], and

used by J. K. Hale [94], D. Brewer [23], T. Naito [185], [186]. The

quantity in the left hand side of (18) is the norm of ϕ, at the power p,

in B^p.

   Local existence and uniqueness, for equation (1), under condition

(2'), can be obtained for a variety of phase spaces provided adequate

conditions are imposed on the right hand side $f(t,\phi)$. As remarked in

Sec. 2, the phase space must be a subspace (not necessarily closed) of

the locally convex space $L_{loc}(R_-,B)$, with topology stronger than that

of $L_{loc}(R_-,B)$. Such spaces have been thoroughly investigated in the

well known monograph of J. L. Massera and J. J. Schaffer [167]. For

further development of the theory, see [131] and [262], where the idea

of a weight function is emphasized.

   Unfortunately, the theory of operators (functionals) on such spaces

is not yet developed to the same extent. Moreover, it presents such

inconveniences as the one mentioned above, in connection to the equation

(17).

   Because of the large variety of phase spaces that could be con-

sidered in building up a theory of equations with unbounded delay, it

became desirable to approach the problem axiomatically. In other words,

to list certain axioms for the phase space and the right hand side of (1),

such that any particular space and $f(t,\phi)$ verifying these axioms,

automatically generate existence and uniqueness. Such a task has been undertaken for the first time by J. K. Hale [93]. Further contributions to this problem have been brought by Y. Hino [101]-[104], and T. Naito [185], [186].

Recently, J. K. Hale and J. Kato [99], and K. Schumacher [209], have dedicated conspicuous papers to the axiomatic approach in the phase space theory for equations of the form (1). In [99], the authors deal with the finite-dimensional systems (i.e., $x \in R^n = B$), while in [209] one builds up a theory which is valid in a Banach space. The axioms proposed in [99] and [209] are not exactly the same, though there is a strong resemblance between them. This is probably due to the tendency of shaping the new theory in accordance with the most discussed case of spaces "with memory" (see [31]-[33]). J. K. Hale and J. Kato's paper [99] deals also with stability problems. Both papers [99] and [209] came out almost simultaneously, though [99] has been submitted for publication eight months prior to the date [209] has been submitted.

Slightly modified versions of the axioms given in [99] and [209] have been recently considered by several authors. See, for instance, Y. Hino [255], T. Kaminogo [257], T. Naito [265], and K. Sawano [268]. Their contributions will be discussed in subsequent sections of the paper, due to the fact that they are aiming at such problems as stability theory, existence of almost periodic solutions, and boundary value problems.

Let us consider now the basic axioms that provide the framework for a local theory of existence, in regard to equation (1). We shall follow K. Sawano's [268] presentation of the axioms. We assume that x and f in (1) take values in the Euclidean space R^n, i.e., $B = R^n$.

Let S be a linear space whose elements are mappings from $(-\infty,0]$ into R^n, provided with a semi-norm $|\cdot|$. Two elements of S are considered identical if and only if they coincide at each point of $(-\infty,0]$. If $S_0 \subset S$ is the subspace of those ϕ that satisfy $|\phi| = 0$, then the quotient space $S^* = S/S_0$ is a normed linear space (not necessarily a Banach space).

For any $\beta \in [0,\infty)$ and $\phi \in S$, let ϕ^β denote the restriction of ϕ to the interval $(-\infty,-\beta]$. Denote further by S^β the linear space of these restricted mappings from $(-\infty,-\beta]$ into R^n, and define the semi-norm $|\cdot|_\beta$ by

$$|\psi|_\beta = \inf \{|\phi|; \, \phi \in S, \, \phi^\beta = \psi\}.$$

It is then obvious that $|\phi|_{\dot\beta} = |\phi^\beta|_\beta$, $\phi \in S$, is a semi-norm on S.

The basic axioms can now be stated as follows:

(A_1) For any $\phi \in S$, and any mapping x from $(-\infty,T]$ into R^n, $0 < T \leq \infty$, such that $x_0 = \phi$ and x is continuous on $[0,T)$, we have $x_t \in S$ for $t \in [0,T)$, and $t \to x_t$ is continuous.

(A_2) There exists a continuous function $K(\beta) > 0$, $\beta \in [0,\infty)$, such that

$$|\phi| \leq K(\beta) \sup_{t\in[-\beta,0]} ||\phi(t)|| + |\phi|_\beta,$$

for any $\phi \in S$, and $\beta \in [0,\infty)$.

In (A_2), $||\cdot||$ stands for the Euclidean norm in R^n.

Let us denote now by τ^β the linear operator from S into S^β, $\beta \in [0,\infty)$, defined by the formula

$$(\tau^\beta \phi)(\theta) = \phi(\beta+\theta), \quad \theta \in (-\infty,-\beta].$$

This formula makes sense on behalf of (A_1).

(A_3) There exists a continuous function $M(\beta) > 0$, $\beta \in [0,\infty)$, such that

$$|\tau^\beta \phi|_\beta \leq M(\beta)|\phi|, \quad \phi \in S.$$

(A_4) There exists $K_i > 0$, with the property

$$||\phi(0)|| \leq K_1|\phi|, \quad \phi \in S.$$

Axioms $(A_1) \sim (A_4)$ are formulated in terms of the elements of S. It is not difficult to see that they can be also expressed in terms of the elements of S^* (i.e., the classes of equivalence in S, according to the relation $\phi \sim \psi$ iff $|\phi-\psi| = 0$). For instance, axiom (A_4) shows that $\phi(0) = \psi(0)$ if $|\phi-\psi| = 0$, a feature stressing the fact that this axiom deals, actually, with classes of equivalent functions.

Let Ω be an open set in $R \times S$, and assume $f : \Omega \to R^n$ is a continuous map. If $(t_0,\phi) \in \Omega$, then $x:(-\infty,t_0+T) \to R^n$ with $0 < T \leq \infty$, is said to be a solution of equation (1), under initial condition (2), if $x_{t_0} = \phi$ and $x(t)$

is continuously differentiable and satisfies (1) for $t \in (t_0, t_0+T)$. One

denotes by $x(t; t_0, \phi)$ the above solution. Of course, $x(t; t_0, \phi)$ needs not

to be unique. $x_t(t_0, \phi)$ has an obvious meaning.

Axioms (A_1) and (A_2) guarantee the existence of a solution of the

functional-differential equation (1), under initial condition (2), provided

we assume the continuity of the map f. Under slightly different forms, the

existence statement appears in [99], [257], [265], and [268].

Under axioms (A_1) $\sim$ (A_4), and assuming

(19) $\qquad ||f(t,\phi) - f(t,\psi)|| \le n(t)|\phi - \psi|$ on Ω,

where n(t) is a continuous function, there exists a nonnegative continuous

function $N(t, t_0)$, $t \ge t_0$, such that

(20) $\qquad |x_t(t_0, \phi) - x_t(t_0, \psi)| \le N(t, t_0)|\phi - \psi|$,

for all $t \ge t_0$ that belong to the common interval of existence of

$x_t(t_0, \phi)$ and $x_t(t_0, \psi)$. This result is due to K. Sawano [268]. A

particular case is given in [99].

From (20) one derives easily the uniqueness (at the right) of the

solution through (t_0, ϕ). The same formula (20) shows the continuous

dependence of the solution with respect to ϕ.

If axioms (A_1) $\sim$ (A_4) hold true, and $x(t; \tau, \psi)$ exists up to $t_0 + T$ and

is unique in a neighborhood of (t_0, ϕ), then for any $\epsilon > 0$, there exists

$\delta(\epsilon) > 0$ such that

$$|x_t(\tau, \psi) - x_t(t_0, \phi)| < \epsilon, \quad t \in [\max(\tau, t_0), t_0 + T],$$

as soon as

$$|\tau - t_0|, \quad |\psi - \phi| < \delta(\epsilon), \quad (\tau,\psi) \ \epsilon \ \Omega.$$

The above result, concerning continuous dependence of solutions with respect to initial data, has been proved by K. Sawano [268]. Another version is due to Y. Hino [104]. See also [99].

Various forms are known for the continuation theorem (see [99] and [268]). A noncontinuable solution will leave any compact set $K \subset \Omega$, starting at a certain moment t_K.

Strengthening somewhat the axioms $(A_1) \sim (A_4)$, J. K. Hale and J. Kato [99] have proved the global existence of the solution (i.e., on $[t_0, \infty)$). Moreover, conditions for the precompactness of a trajectory $\{x_t; \ t \geq t_0\}$ are described. Under such conditions, the ω-limit set of a bounded solution, corresponding to completely continuous f, is nonempty, compact, connected and invariant. In other words, it meets the qualities encountered in classical dynamical systems theory.

It is also interesting to point out that the well-known Kneser's property (regarding the bunch of solutions starting at a given (t_0, ϕ)) holds true under axioms (A_1), (A_2), and continuity and boundedness of f. See T. Kaminogo [257] that uses this property in connection with boundary value problems for (1).

K. Schumacher [209] built up his theory starting with a system of axioms that allows to cover the so-called Caratheodory conditions. The phase space is not necessarily a normed space, but is always Hausdorff separable. Several particular function spaces, including the "histories"

or "memory spaces" are carefully described in connection to the proposed axioms. Interesting remarks are also made in regard to the early stage of development of this theory, with references to B. D. Coleman and V. J. Mizel [31]-[33], J. K. Hale and C. Imaz [98], J. L. Gonzales [80], J. K. Hale [93], Y. Hino [102]-[104], G. Ladas and V. Lakshmikantham [142].

Existence, uniqueness and continuous dependence for functional-differential equations with unbounded delay have been discussed by R. Driver [62], [63], [64] in the early 60's. One finds also basic references to Soviet literature on the subject. In particular, the role of differential inequalities is emphasized in obtaining estimates for the solutions. For more details, see V. Lakshmikantham [142]-[144] and the monograph [145].

Functional differential inequalities have been investigated and applied in finding estimates for the solutions of functional-differential equations by many authors (see [89], [146], [178]-[180]). The case of inequalities with unbounded delay has been considered by K. Zima [240], and E. Kozakiewicz [134], [136], [259].

Problems related to the existence of a maximal solution, and inequalities under monotonicity assumptions, are discussed by J. Blaz and K. Zima [22].

Initial value problems for equations of the form

$$
(21) \qquad \dot{x}(t) = f(t, x[h(t)]) \qquad t \in [a,b],
$$

or

$$
(21') \qquad \dot{x}(t) = f(t, x[h(t)], \dot{x}[g(t)]) \qquad t \in [a,b),
$$

where t - h(t), t - g(t) $\geq$ 0 are considered by N. V. Azbelev and L. F. Rakhmatulina in [5] and [244]. Further references are provided in [244]. More general equations, of the form

$$(22) \qquad \dot{x}(t) = f(t,Tx, S\dot{x}),$$

with T and S some operators conveniently chosen, are also discussed in [244]. Under appropriate hypotheses, the Cauchy problem related to such equations can be reduced to the solution of an integral equation of Volterra type.

The interest for Volterra operators and equations, including the case of unbounded delay, has been growing steadily in the past two decades. The existence, uniqueness and convergence of successive approximations for vector equations of the form

$$(23) \qquad y(x) = y_0(x) + \int_{-\infty}^{x} f(x,t,y(t))dt$$

have been investigated by A. Erdelyi in [71]. M. Picone [197] considered similar problems in the linear case, investigating also the dependence of solution with respect to initial data (see formula (8)). J. A. Nohel [189] alluded to the significance of Volterra equations (involving general operators), and L. W. Neustadt [188] gives a theory for such operators and equations, with applications to control problems.

The equations with Volterra operators have been also investigated by C. V. Coffman and J. J. Schaffer [26], [27]. They consider the linear case and develop a theory with very weak restrictions on the initial data.

Another approach to the theory of initial value problems for equations with hereditary structure has been given by G. S. Jones [110]. He represents the functional-differential equation in the form

$$(24) \qquad \dot{x}(t) = f(t,x(\alpha_t(x(t)))), \quad x \in R^n$$

$\alpha_t(\xi) \subset (-\infty,t]$ being defined as follows: $\alpha : R \times R^n \to \Omega = $ the set of all closed subsets of R, and $\alpha_t(\xi) = \alpha(t,\xi)$, with $t \in x_t(\xi)$.

The existence and uniqueness for functional-differential equations, with state dependent delay, have been also examined by E. Winston [236] - [238]. The case of an unbounded delay is not excluded.

The uniqueness problem for functional-differential equations with unbounded delay has been considered in [208] and [219].

Equations of the form

$$(25) \qquad x'(t) = \int_0^\infty f[t,x(t-s)]d_s r(t,s) + g(t),$$

or

$$(26) \qquad \dot{x}(t) = f(t,G(t,x)),$$

with G(t,x) an operator subject to adequate conditions, have been investigated by A. Bielecki and M. Maksym [15], J. Blaz [19], Z. Kamont and M. Kwapisz [115], M. Kisielewicz [121], [122], [124], and B. Rzepecki [206]. Continuous dependence with respect to parameters is also investigated. One derives existence and uniqueness theorems in various function

spaces, with rather general assumptions on the data. It would be useful to compare such results with those mentioned above, in the axiomatic framework.

A. D. Myshkis and Z. B. Tsalyuk [183] investigate the continuability of solutions. See also [181] for examples of equations with unbounded delay, such that some solutions are noncontinuable.

V. F. Subbotin [220] establishes Kneser's type results for certain classes of equations with unbounded delay. See also T. Kaminogo [257].

B. I. Ananev [3] gives existence results for differential inclusions of the form

$$(27) \qquad \dot{x}(t) \; \varepsilon \; R(t, x_t(\cdot)),$$

where $x_t(\xi) = x(t+\xi)$, $-h(t) \leq \xi \leq 0$, with $h(t) \geq 0$ a continuous function.

For equations with inclusion and unbounded delay, M. G. Crandall, S. O. Londen and J. A. Nohel [97] have recently obtained (global) existence results. Their paper emphasizes the fact that the function spaces whose elements are defined on a half-axis, and operators acting on such spaces, constitute the basic tools in the investigation of equations with unbounded delay.

V. Lakshmikantham and collab. [147] dealt with delay equations on closed subsets of a Banach space, assuming the delay is bounded. It would be interesting to approach the same problems (existence, uniqueness) for equations with unbounded delay.

In [226], I. Ya. Viner investigates the linear equation

$$(28 \qquad \dot{x}(t) = x(t^{-1}), \quad t > 0.$$

Some solutions of this equation can be found differentiating both sides of (28), then eliminating $\dot{x}$ from the two equations: $t^2\ddot{x}(t) + x(t) = 0$. For equation (28), that can be also written as

(28')
$$\dot{x}(t) = x(t-h(t)), \quad h(t) = t - t^{-1},$$

one can consider the Cauchy problem corresponding to the initial data

(29)
$$x(t) = \phi(t), \quad t \, \varepsilon \, (0,t_0], \; t_0 \geq 1.$$

If we assume $\phi(t)$ to be continuous, the space of initial functions consists, for each $t_0 \geq 1$, of all continuous maps from $(0,t_0]$ into R. Therefore, it is convenient to endow it with the topology of uniform convergence on each interval $[\varepsilon,t_0]$, $\varepsilon > 0$. There is no difference between this space and the analogous space that corresponds to the half-axis $(-\infty,t_0]$. Finally, let us remark that $h(t) \to \infty$ as $t \to \infty$.

From (27), one sees that for the construction of the solution satisfying (29), one needs only the values of $x(t)$ (or $\phi(t)$) on $(0,t_0^{-1}]$. We can change $\phi(t)$ on $[t_0^{-1},t_0)$ as we like (of course, preserving continuity), without any change in the solution $x(t)$ for $t > t_0$. If there is no reason to wonder about such situations, the backward uniqueness being an improper concept for equations with delay, one can however remark the extremely weak connection between initial data and the resulting solution. It appears more reasonable, perhaps, to associate with each $t_0 \geq 1$ a certain function space, in our case, the space C_{t_0} of continuous maps from $(0,t_0^{-1}]$ into R, and consider the scale of spaces

(t_0, C_{t_0}), $t_0 \geq 1$, as a framework for the discussion of the Cauchy problem. A recent contribution on these lines has been brought recently by A. L. Bukheim [247] who considers Volterra equations in the abstract setting. V. A. Jakubovich [109] has used such structures in dealing with absolute stability in abstract systems.

3. STABILITY PROBLEMS

The stability problems for equations with unbounded delay have attracted the interest of researchers long before any attempt has been made to set up a general theory of these equations. For instance, Yu. I. Neimark [187] investigated stability problems for equations of the form

$$(30) \quad u(t) = \int_{-\infty}^{t} G(t,s) \, [f(s,u(s)) + w(s)] \, ds \, ,$$

in which $G(t,s)$ is an operator-valued function, with regard to the validity of the linearization procedure. Various examples of equations with unbounded delay, whose stability can be investigated with specific methods, can be found in the monographs [180], [89], [81] .

A first systematic approach to this topic can be found in R.D. Driver's paper [62], in which he deals with equations of the form

$$(31) \quad y'(t) = \mathcal{F}(t,y(\cdot)), \quad t > t_0 \, ,$$

where $\mathcal{F}(t,y(\cdot))$ denotes a Volterra operator (or functional). Of course, the values of $y(t)$ are prescribed for all $t \leq t_0$. After developing a theory of existence, uniqueness, and continuation of solutions, the second Liapunov method is applied to investigate stability for (31) . More precisely, the author gives a rather general theorem of stability for the zero solution of (31), using comparison method. He assumes the existence of a functional $V(t,\phi(\cdot))$, satisfying usual conditions required for a candidate Liapunov function, and a differential inequality of the form

$$(32) \quad V'(t,\phi) \leq \omega(t,V(t,\phi)) \, .$$

The comparison equation $y' = \omega(t,y)$ plays a central role in the investigation of stability properties for (31). Let us point out that several particular systems and Liapunov functionals are discussed in [62].

A basic inequality following from (32) is obtained by V. Lakshmikantham in [142], and related topics are discussed by the same author in [143]. In [144], V. Lakshmikantham investigates stability properties for systems with unbounded delay, in the form considered by G.S. Jones [110] (see formula (24) above).

In [128], V.B. Kolmanovskii and R.Z. Hasminskii deal with the L^2-stability of the integro-differential equation

$$(33) \quad \dot{x}(t) = \int_0^\infty x(t-s)d\, K_0(s) \quad , \, t > 0 \, ,$$

under initial conditions of the form $x(t) = \phi(t)$, $t \le 0$.

Liapunov's method is applied in [125] to equations slightly more general than (33) , in which $K_0(s)$ is replaced by $K_0(t,s)$. These ideas are then developed in [126] and [129]. In [129], neutral equations of the form

$$(34) \quad x'(t) = -\int_0^\infty x(t-s)d\, K_0(s) + \int_0^\infty x'(t-s)d\, K_1(s) + b(t,x_t)$$

are discussed in regard to the stability of solutions. In [127], more emphasis is placed on the use of frequency techniques in investigating stability for equations of the form (34) .

Equations of the form

$$(35) \quad x'(t) = \int_0^\infty x(t-s)dr(t,s)$$

are investigated by V. Ya. Grebennikov [83], using rather elementary considerations.

B.D. Coleman and V.J. Mizel [33] use, in fact, Liapunov function technique. They call the functionals involved in the formulation of the stability results _energy functions_, a term inspired by the mechanical interpretation of the system with unbounded delay. Phase spaces consisting of functions integrable with respect to a certain measure (or weight), as those described in Section 2, are used in order to investigate stability problems for rather general equations. In [159], R.C. MacCamy is investigating exponential asymptotic stability for autonomous systems of the form (1), using also phase space approach as in [33].

J.K. Hale [93] approaches the theory of equations with unbounded delay, including some stability aspects from the standpoint of dynamical systems theory. This paper contains the first attempt made in the literature to build up an axiomatic phase-space theory, and hints developments that have been achieved later by the author and his followers: [99], [119], [103], [105], [265]. Some of these developments will be discussed in more detail in a subsequent paragraph.

In [94] and [95], J.K. Hale deals with linear equations such as

$$(36) \quad \dot{x}(t) = \sum_{j=0}^{N} A_j \, x(t-t_j) + \int_{-\infty}^{0} B(s) \, x(t+s) \, ds \, ,$$

using Kuratowski's measure of noncompactness and Sadovskii's fixed point theorem. The characteristic equation associated to (36) is involved in obtaining estimates for the solutions, and, in particular, stability conditions. These papers generalize several results obtained by means of different techniques, among them those due to V. Barbu and S. Grossman [8].

A stability theory for equations with unbounded delay of the form

$$(37) \quad \dot{x}(t) = \sum_{j=0}^{\infty} A_j \, x(t-t_j) + \int_{0}^{\infty} B(t-s) \, x(s) \, ds \, , \quad t > 0 \, ,$$

under initial conditions

(38) $x(t) = h(t)$, $t < 0$, $x(0+) = x^0$,

has been built up by C. Corduneanu [39], [41], [42], C. Corduneanu and N. Luca [46], N. Luca [157], [261] .

It is assumed that x, $x^0 \in R^n$, A_j are n by n matrices with real entries, $B: [0,\infty) \to \mathcal{L}(R^n, R^n)$, and $h:(-\infty,0) \to R^n$.

The following basic assumptions are made on the system (37) :

(39) $t_j \geq 0$, $j=0,1,2,\ldots;$ $\sum_{j=0}^{\infty} \|A_j\| < \infty$, $\|B\| \in L(R_+, R)$.

Assumptions (39) guarantee the existence and uniqueness of solution for (37), provided $h \in L^p(R_-, R^n)$, $1 \leq p < \infty$, [39], [46] . If one denotes by $X(t)$ the map from R into $\mathcal{L}(R^n, R^n)$, such that

(40) $\dot{X}(t) = \sum_{j=0}^{\infty} A_j X(t-t_j) + \int_0^t B(t-s) X(s) \, ds$, $t > 0$,

(41) $X(t) = 0$, $t < 0$, $X(0+) = I$,

then $\|X(t)\| \in L(R_+, R)$ if and only if the stability condition holds true:

(42) $\det [s \, I - \mathcal{A}(s)] \neq 0$, $\mathrm{Re} s \geq 0$,

where

(43) $\mathcal{A}(s) = \sum_{j=0}^{\infty} A_j e^{-t_j s} + \int_0^{\infty} B(t) e^{-ts} \, dt$, $\mathrm{Re} s \geq 0$.

Since (40) implies $\|\dot{X}(t)\| \in L(R_+, R)$ for $\|X(t)\| \in L(R_+, R)$, one sees that condition (42) provides asymptotic stability:

(44) $\lim \|X(t)\| = 0$ as $t \to \infty$.

Actually, (42) is equivalent to the condition $\|X(t)\| \in L(R_+,R)$, and, as shown in [41], it is equivalent to $\|X(t)\| \in L^p(R_+,R)$ for any p , $1 \le p < \infty$.

The basic tool in obtaining the above result is the theory of matrix-valued function algebras whose elements are of the form (43), under assumptions (39). The theory of operators occuring in the right hand side of (37), as well as the theory of function algebras generated by them can be found in some basic references as [59], [79], [235] . In [59] and [235], several interesting applications of such topics in system engineering are emphasized. Y.V. Venkatesh [234] also uses these techniques in investigating stability and instability for various classes of systems encountered in the applications (without particular regard to equations with unbounded delay).

The above results, concerning the behavior of the fundamental matrix $X(t)$ defined by (40) and (41), pose the following problem: Is the asymptotic stability, implied by (39) and (42), of exponential type? In other words, is it possible to find positive constants K and α , such that $\|X(t)\| \le K \exp(-\alpha t)$, for $t \ge 0$? We are entitled to rise such a problem, mainly on the ground that the right hand side of (36) is an operator of the time-invariant type.

The basic properties of the fundamental matrix $X(t)$, defined by (40) and (41), can be exploited to further stability theory for functional-differential equations of the form (37), or as recently shown by N. Luca [261], for equations of the form

(45) $\dot{y}(t) = \sum_{j=0}^{\infty} A_j \, y(t-t_j) + \int_{-\infty}^{t} B(t-s) \, y(s) \, ds$,

under initial conditions $y(t) = h(t)$ for $t < 0$, and $y(0+) = y^0 \in R^n$. The definitions of stability, uniform stability etc. are formulated according to the classical pattern. For instance, the uniform stability of either (37) or (45) is

equivalent to the boundedness of $X(t)$, and the boundedness of all solutions of (45), with arbitrary $h \in L(R_-, R^n)$, $y^0 \in R^n$. Also, the *uniform stability of the zero solution of (37)* implies $\det [s I - \mathcal{A}(s)] \neq 0$ for $\text{Re} s > 0$ *(naturally, a condition weaker than (42))*. Another set of conditions equivalent to the uniform stability of the zero solution of (37) or (45) is: the functions $X(t)$,

$$(46) \qquad \phi(t) = \int_0^\infty \left\| \int_0^t X(t-s) \ B(s+u) \ ds \right\| \ du \ ,$$

and

$$(47) \qquad \psi(t) \ = \ \text{ess} - \sup_{s \in R_-} \|K(t,s)\| \ ,$$

with

$$(48) \qquad K(t,s) \ = \ \sum_{j=0}^\infty \chi_j(s) \ X(t-t_j-s) \ A_j$$

are defined and bounded on R_+ . By $\chi_j(t)$ one denotes the characteristic function of the interval $[-t_j, 0]$.

Any equation of the form

$$(49) \qquad \dot{x}(t) \ = \ \sum_{j=0}^\infty A_j x(t-t_j) \ + \int_0^t B(t-s) \ x(s) \ ds \ + \ f(t;x) \ ,$$

with $f(t;x) = (fx)(t)$ an operator adequately defined, and the usual initial conditions $x(t) = h(t)$ for $t < 0$, $x(0+) = x^0 \in R^n$, can be transformed by means of the variation of constants formula [41], [46], into an integral equation of the form

$$(50) \qquad x(t) \ = \ X(t)x^0 + (Yh)(t) \ + \int_0^t X(t-s)f(s;x) \ ds \ ,$$

where

$$(51) \qquad (Yh)(t) = \sum_{j=0}^{\infty} \int_{-t_j}^{0} X(t-t_j-s) \, A_j h(s) \, ds \, , \qquad t \in R_+ \, .$$

The convergence properties of the series occuring in (51) depend on the nature of h .

Adequate results are obtained in [41], [46] for $h \in L^p(R_-, R^n)$, $1 \le p < \infty$.

Stability results for nonlinear systems of the form (49) have been obtained,

mainly using frequency domain techniques, in a series of papers: [39], [40], [42],

[44], [45], [157], [261]. A problem rised in [40] has been solved by V. Rasvan [204].

Another reference regarding the systems with infinite delay of the form (37),

and their nonhomogeneous counterparts, is D.-P.K. Hsing [108]. The semigroup theory

is used in order to find estimates for solutions.

In almost all cases mentioned above, the stability is usually meant in the

norm of the R^n . In other words, the quantity $\|X(t)\|$ is estimated, and relations

such as (44) have the usual interpretation we are acquainted with, from the theory

of ordinary differential equations. If instead of the Euclidean norm $\|\cdot\|$, in

estimating the solution, one uses function-space norms (for instance, the norm

of the phase-space involved in the description of the system), some difficulties

arise as pointed out from the very beginning of the theory by R.D. Driver [62].

Let us illustrate the above statement. For instance, when the phase space

is a memory space, corresponding to a weight function k(s) the p-th power of the

norm is given either by formula (12), or (18). Assume we deal with (12). Then,

assymptotic stability should mean

$$(52) \quad \|x(t)\|^p + \int_{-\infty}^{0} k(s) \, \|x(t+s)\|^p \, ds \; \to \; 0 \quad as \quad t \to \infty \, ,$$

a condition that does not necessarily hold, even in the case $x(t) \to 0$, as $t \to \infty$.

Similar illustrations are provided by J. Kato [119] . A more detailed

discussion of this problem, and relationships between stabilities in the

sense of the phase space norm or the Euclidean norm, can be found in [99].
These authors, treat stability problems in the framework of an axiomatic phase
space theory. While this approach tends to become preponderant in the investigation
of equations with unbounded delay of the form (1) (see [118], [119], [102], [105],
[268]), it should be pointed out, however, that not all the available results seem
to be obtainable in this framework. The main difficulty seems to be in writing
most particular equations in the form (1), with f continuous in the second
argument, from the phase space S , into the space R^n .

Let us consider now the system (1), under initial condition (2), and assume
we deal with a phase space S satisfying axioms $(A_1) - (A_4)$, formulated in
Section 2. Therefore, in condition (2) $\phi \in S$. We shall not repeat here the
definitions of various types of stability, they being very much alike to those
well known from the stability theory for ordinary differential equations or
equations with finite delay [89], [96].

The following result is due to J. Kato [119]: Assume condition (19) holds
true, and there exists a continuous real valued function $V(t,\phi)$, defined
on $R_+ \times S_r$, $S_r = \{\phi; \phi \in S , |\phi| \le r\}$, such that

(53) $a(\| \phi(0)\| \|) \le V(t,\phi)$,

(54) $V(t,\phi) \le b(t,|\phi|)$,

(55) $V'(t,\phi) \le c(t,V(t,\phi))$,

where

(56) $V'(t,\phi) = \limsup_{h \to 0+} \dfrac{V(t+h,x_{t+h}) - V(t,\phi)}{h}$, $x_t = \phi$,

with $a(r)$, $b(t,r)$, $c(t,r)$ non-negative, continuous, non-decreasing in r , $r \geq 0$, and $a(r) > 0$ for $r > 0$, $b(t,0) \equiv 0$. Then the zero solution of (1) is stable in R^n . Moreover, it is asymptotically stable in R^n if

$$(57) \qquad \int_t^{t+T} c(s,r)\, ds \to \infty \qquad \text{as} \quad T \to \infty \ ,$$

for any $r \geq 0$ and $t \in R_+$. Furthermore, it is uniformly asymptotically stable in R^n if (57) holds true, uniformly with respect to $t \in R_+$, and $b(t,|\phi|) \leq b(|\phi|)$.

The following converse theorem for uniform asymptotic stability is also given in [119]: Assume condition (19) holds true with $n(t) = \text{const.} > 0$, and the zero solution of (1) is uniformly asymptotically stable in S . Then, there exists a real-valued continuous function $V(t,\phi)$, defined on $R_+ \times S_r$ $(r > 0)$, such that

$$(58) \qquad a(|\phi|) \leq V(t,\phi) \leq b(|\phi|) \ ,$$

$$(59) \qquad V'(t,\phi) \leq -c\, V(t,\phi) \ ,$$

$$(60) \qquad |V(t,\phi) - V(t,\psi)| \leq L|\phi - \psi| \quad ,$$

with $a(r)$ and $b(r)$ positive definite, and c,L some positive constants.

Since stability in S implies stability in R^n [99] under certain additional axioms for S , the above theorem gives only a partial answer to the problem of characterizing stability theorems in terms of Liapunov functionals. Nevertheless, the converse theorem allows the study of some perturbed systems, as shown in [119]. Applications regard such systems as

$$(61) \qquad \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \int_{-\infty}^0 A(s)\, x(t+s)\, ds \ ,$$

with h > 0 , and

(62)
$$\int_{-\infty}^{0} \|A(s)\| \, e^{-\gamma s} ds < \infty \, , \quad \text{for a} \quad \gamma > 0 \, .$$

The result states exponential asymptotic stability in the space $S = C_\gamma$, i.e.

(63)
$$C_\gamma = \{x; R_- \to R^n \, , \text{ continuous, } \lim_{t \to -\infty} e^{\gamma t} x(t) \text{ exists}\}$$

In a slightly different form, the existence of $V(t,\phi)$ satisfying (58) - (60) is proven by K. Sawano [268].

Further results in [119] regard the so-called Razumikhin type stability theorems. In the case of finite delay, a recent paper by V. Lakshmikantham and S. Leela [146] develop such techniques, starting from equations written in the form (14), and the associated differential-difference inequality

(64)
$$V'(t,\phi(0),\phi) \le \omega(t, V(t,\phi(0),\phi), V_t) \, .$$

An extension of the results in [146] to the case of equations with unbounded delay, in the framework of the axiomatic phase space theory, would certainly be welcome.

K. Sawano [268] investigates the linear system of the form (1), i.e. $f:R_+ \times S \to R^n$ is continuous and linear in the second argument, with regard to the exponential asymptotic stability. If S is such that $(A_1) - (A_4)$ hold true, and the zero solution of the linear system (1) is uniformly asymptotically stable in S , then it is exponentially asymptotically stable in the large. In other words, one can find positive constants M and c , such that

$$|x_t(t_0,\phi)| \le M|\phi| \, e^{-c(t-t_0)} \, , \quad t \ge t_0 \, ,$$

for any $\phi \in S$. This result has the classical flavor of the similar ones from the theory of ordinary differential equations or the theory of equations with finite delay [89]. As mentioned above in this Section, the asymptotic stability in S has some drawbacks (see formula (52)). Nevertheless, it would be interesting to see whether S can be chosen conveniently for the system (37) or (49), such that the theorem of K. Sawano stated above provide an answer to the question rised in respect to the behavior of the fundamental matrix $X(t)$ defined by (40) and (41).

J.K. Hale and J. Kato [99], under supplementary axioms for the space S , investigate also the limit set of bounded solutions, and their relationship to stability. See also [84], [210], [211], [162], [163].

Though most methods that provided satisfactory results in case of ordinary differential equations, or for equations with bounded delay, have been tested in the case of equations with unbounded delay (for instance, Liapunov functions, frequency domain techniques, abstract dynamical systems, semi-group theory), we point out that other powerful methods, such as LaSalle's invariance principle [148], [149], have not yet been used in approaching stability problems for this class of equations (excepting [116]).

Several papers have been devoted to the stability theory for integral equations with unbounded delay. The term stability may have the classical sense, i.e., the solution must be small (or tend to zero) if the data that determine it are small enough, or it may mean a certain generalized kind of continous dependence, within a function space endowed with a convenient norm.

In a recent survey paper [272], Z.B. Tsalyuk discusses most of the achievements in the field of Volterra equations, including stability problems.

The study of stability for integral equations with unbounded delay, envisaged as a specific kind of asymptotic behavior, goes back to the early stage of this theory. More recent contributions, emphasizing relatively new methods, can be found in [13], [207], [89], [172], [38].

Let us consider, with M.J. Leitman and V.J. Mizel [153], the integral equation of Volterra type

$$(65) \qquad x(t) + \int_{-\infty}^{t} g(s,x(s)) \, a(t-s) \, ds = f(t), \quad t \in R \; ,$$

where $a:R_+ \to R_+$ is nonincreasing and integrable. The nonlinearity $g:R \times R \to R$ satisfies usual Carathéodory conditions, and is such that $ug(s,u) \geq 0$ for every u, and almost all s. Further conditions are assumed, such as the monotonicity in u, and a Lipschitz condiiton at infinity, in the second argument. Then $x(t) - y(t) \to 0$ as $t \to \infty$, where $y(t)$ is the solution of the Volterra equation with finite delay,

$$(66) \qquad y(t) + \int_{0}^{t} g(s,y(s)) \, a(t-s) \, ds = f(t) \; , \quad t \in R_+ \; ,$$

provided $a(t)$ does not vanish almost everywhere and at least one function $x(t)$ or $y(t)$ be bounded on R_+.

The above result provides a very convenient tool to investigate stability for equations of the form (65), because for equation (66) we know a good deal of stability conditions. See, for instance, the monographs [38], [172], and references [160], [174], [269]. Further results are derived in [153].

Another recent contribution to the theory of integral equations with finite (and infinite) delay is due to M.G. Crandall, S.O. Londen and J.A. Nohel [47]. They investigate problems of the form

$$(67) \qquad u'(t) + Bu(t) + \int_{-\infty}^{t} a(t-s) \, Au(s) \, ds \ni f(t) \; , \quad t > 0 \; ,$$

with $u(t) = h(t)$ on $t \leq 0$. The solution is sought in spaces like $W_{loc}^{1,1}(R_+,H)$, where H is a Hilbert space. The (nonlinear) operators occuring in (67) are chosen to be the subdifferentials of some convex funtions. Some of their results

are applicable to equations of the form

$$(68) \qquad u_t(t,x) - \Delta u(t,x) + \int_{-\infty}^{t} a(t-s) \, g(u(s,x)) \, ds = f(t,x) \, ,$$

occuring in elasticity theory. Stability results for (67) are also derived from the existence theorems proven in [47].

In [161], R.C. MacCamy and V. Mizel deal with equation (65), with $f \in W^{1,q}(R_+, V)$, where $V \subset H \subset V'$, and V is a Banach space with dual V' . H is a Hilbert space, and the solution is constructed in the space $L^p(R_+, V) \cap W^{1,q}(R_+, V')$, with $p^{-1} + q^{-1} = 1$. More precisely, the solution and its translations must belong to that space.

For other stability results regarding integral equations see [154], [164], [200].

A generalized concept of integral equation is investigated in [107]. Stability results for such equations would be welcome.

An interesting point of view in investigating linear functional differential euqations, including equations with unbounded delay and stability of their solutions, can be found in E.W. Kamen [112]. His theory is predominantly algebraic, and is applicable to systems in which the right hand side is a generalized convolution product. This allows to treat the equations as equations over a ring, thus making use of rather sophisticated algebraic results.

Certain stability results for periodic or almost periodic solutions of equations with unbounded delay will be discussed in Section 4.

In Section 6, we will include stability results for various classes of equations with unbounded delay that occur in applications.

equation

$$(71) \qquad x(t) + \int_{-\infty}^{t} k(t-s) \, f(s,x(s)) \, ds = h(t) \; .$$

In [14], further references are given to the work of these authors, regarding periodic or almost periodic solutions for integral equations.

C. Corduneanu [37] and A.G. Ramm [202] use various functional-analytic methods in finding existence conditions for periodic or almost periodic solutions to the equation

$$(72) \qquad x(t) + \int_{-\infty}^{\infty} k(t-s) \, f(s;x) \, ds = h(t) \quad , \; t \in R \; ,$$

where $f(t;x) = (fx)(t)$ is a nonlinear operator acting on the space of periodic (almost periodic) functions. Accordingly, $h(t)$ is assumed periodic or almost periodic. Let us point out that (72) reduces to (71) when $k(t) \equiv 0$ for $t < 0$, and $(fx)(t) = f(t,x(t))$.

J. Kudrewicz [138] also investigates existence of periodic solutions for equations of the form (71), in the autonomous case $f(t,x) \equiv f(x)$, assuming $h(t)$ periodic.

Small periodic solutions are sought by J. Kudrewicz and M. Odyniec [139] to the equation

$$(73) \qquad x(t,\mu) = \int_{0}^{\infty} k(\tau,\mu) \, f[x(t-\tau,\mu),\mu] \, d\tau \; ,$$

with f analytic, and under the assumption that (73) has for each μ a unique constant solution $c = c(\mu)$. Frequency domain methods are largely applied.

M.J. Leitman and V.J. Mizel [153] deal with the equation (71), with $f(t,x)$ periodic in t : $f(t+T,x) = f(t,x)$. Moreover, $h(t)$ is assumed to have the same period $T > 0$. If the kernel $k(t)$ satisfies certain conditions

(see conditions on a(t) in Section 3), and denote $k_T(t) = \sum\limits_{j=0}^{\infty} k(t+jT)$, then the periodic solution x(t) of (71) will satisfy

$$(74) \qquad x(t) + \int_0^T \overset{\sim}{k_T}(t-s) \, f(s,x(s)) \, ds \ = \ h(t) \, , \quad t \in [0,T] \, ,$$

where $\overset{\sim}{k_T}(t)$ is the periodic extension of $k_T(t)$ from [0,T) , to [-T,T) . In other words, the problem of finding periodic solutions for equations with unbounded delay of the form (71) is reduced to a similar problem for the (Fredholm) integral equation (74).

B.D. Coleman and G.H. Renninger [35], [36] investigate the existence of periodic solutions to certain integral equations with infinite delay. These equations are of the form

$$y(t) = m(a(t) - \int_0^\infty e^{-S} g(y(t-\tau-s)) \, ds) \, ,$$

where $m(u) = \frac{1}{2}(u + |u|)$. Such equations are motivated by the description of neural interactions. The authors find conditions under which a unique periodic solution exists, and also discuss related problems.

V.R. Nosov [192] - [194] studies the periodicity of solutions to the class of integro-differential equations of the form

$$(75) \qquad \int_R d_s \, r(t,s) \, [\dot{x}(t+s)] \ = \ \int_R d_s p(t,s) \, [x(t+s)] \ + \ f(t) \, ,$$

in which f(t) is ω-periodic (not necessarily continuous), while r(t,s), p(t,s) possess periodicity property in the first argument, and have bounded variation with respect to the second argument. The author remarks that, under adequate conditions, the operator

$$(76) \qquad (Tz)(t) = \int_R d_s r(t,s) \, [z(t+s)]$$

carries the space C_ω of continuous functions, with period ω , into itself. Therefore, it can be represented as

$$(77) \qquad (Tz)(t) = \int_0^\omega d_s R(t,s) \, [z(s)] \, ,$$

where $R(t,s)$ can be constructed starting from $r(t,s)$, in a manner similar to the one used above, when constructing $\tilde{k}_T(t)$ by means of $k(t)$. Of course, $R(t,s)$ in (77) is periodic in t , of period ω . Using representation (77) for the operator T defined by (76), the problem of searching periodic solutions for (75) reduces to the corresponding problem for $Tz = f$. More precisely, the Fredholm alternative is simultaneously true for (75) and $Tz = f$ (and their adjoint equations). Of course, particularizing conveniently $r(t,s)$ and $p(t,s)$ in (75), one obtains equations with unbounded delay. In particular,

$$(78) \qquad \dot{x}(t) = \int_{-\infty}^0 d_s p(t,s) \, [x(t+s)] \, + \, f(t)$$

constitutes a model for linear equations with unbounded delay and periodic coefficients. It should be noticed that the powerful functional-analytic methods used by the author provide an adequate treatment of equation (75), in which the argument presents both advanced and delayed deviations.

J.M. Cushing [49] – [51] dealt with the existence problem of periodic solutions for certain equations with unbounded delay that occur in Population Dynamics. Rather general results are obtained for systems of the form

$$(79) \qquad \dot{x}(t) = \Lambda \int_{-\infty}^0 d \, H(s) \, [x(t+s)] \, + \, (gx)(t) \, ,$$

where $x \in R^n$, $\Lambda = \text{diag} \, (\lambda_1, \lambda_2, \ldots \lambda_n)$, and $H = (h_{ij})_{n \times n}$ is of bounded variation on $(-\infty, 0]$. g stands for a nonlinear operator, assumed to

be of higher order in x, near x = 0 . More exactly, the author provides
conditions that guarnatee the existence of nontrivial periodic solutions
for (79), for certain values of Λ . The homogeneous linear equation
attached to (79) must possess at least one nontrivial periodic solution,
corresponding to some Λ . As one can see, the author obtains a bifurcation
result. The characteristic equation associated with the linear part of (79)
is involved in formulating the exact requirements on the equation (79).

H.C. Simpson [269], [270] has undertaken the study of various problems
regarding the periodic solutions of some classes of equations with unbounded
delay, such as

$$(80) \qquad \dot{x}(t) = f(x(t), \int_0^\infty G(s,\lambda) \, x(t-s) \, ds, \lambda) \quad,$$

and their linearized version

$$(81) \qquad \dot{x}(t) = A(t) \, x(t) + \int_0^\infty B(t,s) \, x(t-s) \, ds + c(t) \quad.$$

Besides existence and bifurcation, he pays attention to the stability of periodic
solutions, as well as to the investigation of various equations that arise in
Pupulation Dynamics. Among other topics, a diffusional model described by
the equation

$$(82) \qquad V_t = \Delta V + f(\int_0^\infty k(s) \, V(x,t-s) \, ds) \quad,$$

where V = V(x,t) , and Δ represents the Laplacian with respect to x-variables,
is analysed.

For linear equations of the form (81), the author constructs a Floquet
theory, taking C_γ as phase-space (see Section 3). The representation of
solutions as finite linear combinations of the form

$$(83) \qquad \sum_{\mu} P_\mu(t)\, e^{B_\mu t}\, b_\mu$$

where $P_\mu(t)$ are ω-periodic matrices, B_μ are constant matrices and b_μ some constant vectors, is proven for the homogeneous equation associated to (81). Furthermore, the Fredholm alternative is obtained for the equation (81), its adjoint equation

$$(84) \qquad \dot{y}(t) = -y(t)\, A(t) - \int_{-\infty}^{0} y(t-s)\, B(t-s,-s)\, ds + c^*(t) \ ,$$

and their homogeneous counterparts. In (84), y and c^* are row-vectors. Since (81) is a special case of (78) and (75), it would be interesting to compare the conditions under which the Fredholm alternative is valid in each case. Formally, the result for (81) could be derived from the general one, valid for (75).

The stability analysis conducted by H.C. Simpson in the papers quoted above relies on the representation of the solution of the homogeneous equation in the form (83). An interesting result regards the orbital stability of a periodic solution $p(t)$ of the "autonomous" system

$$(85) \qquad \dot{z}(t) = f(z(t),(k*z)(t)) \ ,$$

where $*$ indicates the convolution product. The linearized system of (85), about $p(t)$, will be of the form

$$(86) \qquad \dot{v}(t) = A(t)\, v(t) + \int_{0}^{\infty} k(t,s)\, v(t-s)\, ds \ ,$$

and the characteristic values associated with (86) are the eigenvalues of the matrices B_μ occuring in the representation of solutions according to Floquet

theory (see (83)). If all characteristic values of (86) have negative real part, except for a simple eigenvalue that is zero, then p(t) is asymptotically orbitally stable.

H.W. Stech [216] is also investigating linear periodic systems with infinite delay, using the phase space B^p , in which the norm is given by formula (18). He builds up a Floquet theory, finds the adjoint system for a periodic system, discusses the stability and the Fredholm alternative. The linear system considered in [216] is represented in the form

$$(87) \qquad \dot{x}(t) = L(t,x_t) + h(t) ,$$

with $L(t+\omega,\phi) = L(t,\phi)$, $t \in R$, $\phi \in B^p$, $h(t+\omega) = h(t)$, $t \in R$. It is assumed that L is continuous from $R \times B^p$ into R^n , while h is locally integrable. Representation for solutions in the form (83) is obtained. Since L can be represented in integral form, a comparison of this theory with the one provided by V.R. Nosov [194], in regard to the Fredholm alternative, should be considered. In [216], the author gives a detailed discussion of the behavior of solutions of an autonomous system $\dot{x}(t) = f(x_t)$ in the neighborhood of an orbit. $f(\phi)$ is supposed differentiable in Fréchet sense, so that a linearized system can be attached. Like the result of H.C. Simpson, H.W. Stech obtains his result on orbital stability. A comparison of their results would be interesting. There are somewhat different features.

In his thesis, P.F. Lima [156] is developing a theory of equations with unbounded delay, taking as phase space a somewhat more general space than that resulting from the norm (12). Namely, he considers the norm

$$(88) \qquad |\phi|_S^p = \|\phi(0)\|^p + \int_{-\infty}^{0} \|\phi(s)\|^p \, d\mu(s) ,$$

where μ is a measure satisfying appropriate conditons. Main attention is

paid to the bifurcation problem for periodic systems.

In the framework of an axiomatic theory for the phase space, several authors dealt with the almost periodicity of solutions of functional differential equations of the form (1). For instance, J.K. Hale and J. Kato [99] give various results concerning the limit set of a bounded solution, and the relationships between stabilities, the case of periodic or almost periodic systems being covered. As an illustration, if the phase space S satisfies certain additional axioms, and f is periodic in t , then asymptotic stability for the zero solution of (1) implies uniform asymptotic stability. Further oscillation results concerning equations with unbounded delay have been obtained by Y. Hino [102], [105], [106], [255], and K. Sawano [268].

In [105], Y. Hino generalizes to the case of equations with infinite delay a classical result of L. Amerio for ordinary differential equations. Roughly speaking, the almost periodicity of a bounded solution of (1), with $f(t,\phi)$ almost periodic in t , uniformly with respect to $\phi \in B \subset S$ for any bounded B , is the consequence of a <u>separation</u> property for (1), and for any other system in the hull of (1) : H(f) . The class of systems H(f) is defined as being the closure of the set of translations $\{f(t+h,\phi); h \in R\}$, with respect to the uniform convergence on R in the first argument, uniformly for $\phi \in B \subset S$, with B an arbitrary bounded set. Let $D = R \times B$, with fixed bounded $B \subset S$, be a set containing the graph of a solution of the system (1): $(t,x_t) \in D$, $t \in R$. Then x is said <u>separated</u> in D if it is either the only solution of (1) lying in D , or, in case another solution y satisfies also $(t,y_t) \in R \times B$, one has $\inf |x_t - y_t|_S > 0$, $t \in R$. The separation requirement regards not only f , but any $g \in H(f)$.

In [106], Y. Hino deals also with almost periodicity of solutions of linear systems of the form (87), taking $\mathscr{C}_\gamma$ or B^p as a phase space. The following generalization of Favard's theorem is obtained: Assume $\dot{x}(t) = A(t,x_t)$ is a

linear almost periodic system, with $A(t,\phi)$ continuous from $R \times \mathscr{C}_\gamma (R \times B^p)$ into R^n , such that any bounded nonidentically zero solution of $\dot{x}(t) = B(t,x_t)$, with arbitrary $B \in H(A) =$ the hull of A , satisfies $\inf |x_t|_S > 0$, $t \in R$ ($S = \mathscr{C}_\gamma$ or $S = B^p$) . Then for each almost periodic $f:R \to R^n$, the system $\dot{x}(t) = A(t,x_t) + f(t)$ has an almost periodic solution, whenever it has a bounded solution on R_+ .

Relationships between stability and almost periodicity of solutions, for equations with unbounded delay of the form (1), are investigated in [102] by Y. Hino. The phase space S is supposed to satisfy some additional axioms, among them being the separability. As a consequence of the developments in this paper, one shows that for linear almost periodic systems of the form $\dot{x}(t) = A(t,x_t) + f(t)$, the boundedness of a solution implies existence of an almost periodic solution, provided the zero solution of the homogeneous system is uniformly stable (in $\mathscr{C}_\gamma$ or B^p).

A result of a different nature is given in [268]. First, the phase space S is assumed to satisfy the follwoing extra axioms: (A_5) S is separable ; (A_6) $M(\beta) \to 0$ as $\beta \to \infty$, with $M(\beta)$ the function occuring in axiom (A_3); (A_7) If $\{\phi_k\} \subset S$. is uniformly bounded, and converges uniformly on any compact to a function ϕ , then $\phi \in S$, and $|\phi_k - \phi|_S \to 0$ as $k \to \infty$.

Let (1) be an almost periodic system, and assume there exists a real valued functional $V(t,\phi,\psi)$ defined on $R_+ \times S_r \times S_r$, $S_r = \{\phi; \phi \in S , |\phi| \leq r\}$, such that

    (i) $a(|\phi-\psi|) \leq V(t,\phi,\psi) \leq b(|\phi-\psi|)$,

with $a(\lambda)$ and $b(\lambda)$ positive definite functions ;

    (ii)   there exists $L > 0$, such that

$$|V(t,\phi,\psi) - V(t,\tilde{\phi},\tilde{\psi})| \leq L \{|\phi-\tilde{\phi}| + |\psi-\tilde{\psi}|\} ,$$

    (iii)   there exists $c > 0$, such that

$$V'(t,\phi,\psi) \leq -c\, V(t,\phi,\psi) ,$$

where V' denotes the derivative of V with respect to the system

$$\dot{x} = f(t, x_t) \ , \quad \dot{y} = f(t, y_t) \ .$$

Then, if (1) has a bounded solution on a half-axis $[t_0, \infty)$, there exists a solution of (1) that is bounded in the norm of S by the same constant, and is uniformly asymptotically stable. In particular, when $f(t, \phi)$ is periodic in t of period ω , the uniformly asymptotically stable solution is periodic of period ω . Furthermore, the construction of $V(t, \phi, \psi)$ is given in the linear case, using the converse theorem on uniform asymptotic stability (see Section 3) .

An interesting result concerning periodic solutions for autonomous systems $\dot{x}(t) = f(x_t)$, with f analytic on the space of continuous bounded maps from $(-\infty, 0]$ into R^n , is given by R.D. Nussbaum [195]. It is shown, under conditions that we do not list here, that a periodic solution of such a system can be analytically extended to a neighborhood of the real axis (in the complex plane). Therefore, the periodic solution is analytic.

Further results regarding oscillations in nonlinear systems with unbounded delay, including bifurcation aspects, can be found in [82], [120].

Finally, oscillating solutions for equations with infinite delay are investigated in [191], [214], without concern for their periodicity or almost periodicity.

Also, related to this section are the papers [16] and [25].

5. FURTHER TOPICS

This Section is devoted to the discussion of various results and methods pertaining to the theory of equations with infinite delay, that did not naturally find place into the preceding Sections. In particular, we shall survey the literature related to the theory of linear equations, the semigroup approach to the construction of solutions, boundedness and related behavior, and boundary value problems.

A.D. Myshkis [179] finds asymptotic estimates for linear integro-differential equations of the form

$$(89) \qquad y'(t) = \int_0^\infty [d_s \, r(t,s)] \, y(t-s) \, ,$$

under appropriate conditions for $r(t,s)$. Inequalities with delay are used in his approach.

E. Kozakiewicz [132], [133], [135] deals also with equation (89) , improving results due to Myshkis and generalizing the equation to the case when Perron's integral is involved.

I. Györi [85] provides conditions under which (89) has solutions satisfying

$$\lim_{t \to \infty} y(t) \exp \{- \int_A^t R(s) \, ds\} = C \, ,$$

where $R(t)$ is defined by means of $r(t,s)$.

Z.B. Tsalyuk [222] studies the behavior at infinity of solutions of the equation

$$(90) \qquad \dot{x}(t) = \sum_{j=1}^m A_j(t) \, x(g_j(t)) \, , \quad t \in R_+ \, ,$$

determined by initial conditions of the form $x(t) = \phi(t)$, $t \le 0$.

H. Grabmüller [252] investigates the behavior at infinity of solutions of the integro-differential equation

$$(91) \quad y'(t) + c \, A \, y(t) + \int_0^\infty h(t-s) \, A \, y(s) \, ds + \varepsilon \int_0^\infty k(t-s) \, y'(s) \, ds + \phi(t) = 0 \, ,$$

in a reflexive Banach space E. $A:D(A) \to E$ is a closed linear operator with dense domain, and $-A$ generates a semigroup on E. Though (91) is not itself an equation with unbounded delay, it has common features with the "adjoint" of such an equation.

The general theory of linear functional differential equations with unbounded delay, i.e., when the equation is represented in the form (1), with f linear in the second argument, and the phase space is chosen among the those described in Section 2 or defined axiomatically, has been recently considered by several authors: [94], [185], [186], [24], [216], [217], [265], [266] .

In [94], J.K. Hale deals mainly with the space B^1 as a phase space, using semigroup theory in order to obtain exponential estaimates for the solutions, in the autonomous case. He points out the fact that the phase space B^1 could be replaced by other spaces, still following the same approach.

T. Naito [185], [186] investigates the linear autonomous equation $\dot{x}(t) = f(x_t)$, the phase space being the space B^p (see formula (18) for the definition of the norm), with complex-valued elements. It is assumed that $k(t)$ in (18) is integrable on $(-\infty,0)$, positive, and non-decreasing. The relation $T(t)\phi = x_t(\phi)$, $\phi \in B^p$, $t > 0$, defines a semigroup on B^p , whose infinitesimal generator A is determined as follows: $(A\phi)(u) = \dot{\phi}(u)$, a.e. on $(-\infty,0)$, $(A\phi)(0) = f(\phi)$, for all $\phi \in B^p$ that are absolutely continuous on any compact interval of $(-\infty,0]$, and for which $\dot{\phi} \in B^p$. This result constitutes the extension of a classical result due to J.K. Hale [94; Ch.7] for the case of autonomous linear equations with finite delay. See also [140]. The spectral properties of A are then

thoroughly investigated in [185]. As a final application of the general theory, one obtains in the absence of the "characteristic values" with zero real parts a decomposition of the phase space B^p , say $B^p = U \oplus V$, with U finite dimensional, and such that $T(t)$ can be extended to the whole real axis on U . Moreover, the following estimates are valid:

$$|T(t)\phi| \leq K e^{\alpha t} |\phi| \ , \quad t \leq 0 \ , \quad \phi \in U \ ,$$
$$|T(t)\phi| \leq K e^{-\alpha t} |\phi| \ , \quad t \geq 0 \ , \quad \phi \in V \ ,$$

with K and α two positive constants.

In [186], T. Naito defines the adjoint system of $\dot{x}(t) = f(x_t)$, using extensively semigroup theory. The spaces B^p and $\mathscr{C}_\gamma$ are taken as phase spaces.

H.W. Stech [217] also deals with the adjoint theory for autonomous linear equations with unbounded delay, relying on J.K. Hale's [94] and T. Naito's results [185], but making use, in a more systematic manner, of the concept of duality. His phase space is the space B^1 , with $k:(-\infty,0] \to (0,\infty)$ continuous, nondecreasing, integrable on $(-\infty,0]$, and such that $k(u+v) \leq k(u)k(v)$, $u,v \leq 0$. The norm is (see 18))

$$(92) \qquad |\phi| = \sup_{u \in [-r,0]} \|\phi(u)\| + \int_{-\infty}^{-r} k(u) \|\phi(u)\| \, du \ .$$

The dual space $(B^1)^*$ consists of all $\psi:(-\infty,0] \to R^n$, such that the restriction of ψ to $(-\infty,-r)$ belongs to $L^\infty ((-\infty,-r), R^n)$, while the restriction to $[-r,0]$ is of bounded variation, left continuous on $[-r,0)$, and $\psi(0) = 0$. The duality pairing between $\phi \in B^1$ and $\psi \in (B^1)^*$ is given by

$$(93) \qquad < \psi,\phi > \ = \int_{-\infty}^{-r} \psi(u) \ \phi(u) \ k(u) \ du \ + \int_{-r}^{0} [d\psi(u)] \ \phi(u) \ ,$$

with $\psi\phi$ and $[d\psi]\,\phi$ standing for scalar products in the Euclidean space.

The linear equation $\dot{x}(t) = f(x_t)$, with $f:B^1 \to R^n$ continuous, can be always represented as

$$(94) \quad f(\phi) = \int_{-\infty}^{-r} k(s)\, \eta(s)\, \phi(s)\, ds + \int_{-r}^{0} [d\eta(s)]\, \phi(s) ,$$

with $\eta : (-\infty,0] \to \mathcal{L}(R^n,R^n)$ a matrix whose columns are in $(B^1)^*$.

If A is the infinitesimal generator of the semigroup attached to $\dot{x}(t) = f(x_t)$, then its adjoint A^* can be determined from $< \psi,A\phi > = < A^*\psi,\phi >$. The expression for A^* is given, and its domain $D(A^*) \subset (B^1)^*$ is determined. Therefore, the adjoint system can be directly investigated. The role of the characteristic equation $\det[\lambda I - f(e^{\lambda \cdot}I)] = 0$ is also emphasized, and the spectral properties of A are discussed.

In their paper [24], J.A. Burns and T.L. Herdman also investigate the problem of the adjoint system. The phase space is now $L^p(R_-,R^n) \times R^n$, and the right hand side of the linear equation is not necessarily a map from the phase space into R^n . As already remarked in this survey, this is a very restrictive assumption that excludes the treatment of such simple equations as (17), with integrable kernel. The authors assume, actually, that the map $(t,\phi) \to f(\phi)(t)$, $f(\phi) \in S_1$, where S_1 stands for a function space, satisfies conditions similar to the ones imposed on equation (14) (see (15)) .

The case when the phase space is defined axiomatically, and the right hand side of (1) is linear in the second argument, has been recently considered by T. Naito [265], [266] . Since the phase space is not a specific one, we do not know how to represent (by an integral) the linear functional occurring in the system, and therefore, a theory that is independent of such a representation must be built up. The author carries out this task, giving several interesting results. For instance, if A is the infinitesimal generator of the semigroup

$T(t)$ attached to the linear equation $\dot{x}(t) = f(x_t)$, with f continuous from the phase space S into C^n (some axioms are added to $(A_1) - (A_4)$) , then the point spectrum $P_\sigma(A)$ is the set of those λ , for which there exists $b \in C^n$, $b \neq 0$, such that $e^{\lambda \cdot} b \in S$, and $\lambda b - f(e^{\lambda \cdot} b) = 0$. The spectral radius $r_\sigma(T(t))$ is also estimated. It is shown that, for $\text{Re}\lambda$ sufficiently large, $e^{\lambda \cdot} b \in S$. An interesting result regarding the adjoint of A is that $D(A^*)$ does not depend on the particular choice of the right hand side (provided all axioms and conditions are fulfilled) . Another topic discussed in [265] is the validity of the variation of constants formula, for the equation $\dot{x}(t) = f(x_t) + h(t)$. If $X(t)$ denotes the fundamental matrix (see (40) and (41), for equation (37)) associated with $\dot{x}(t) = f(x_t)$, then the solution of the nonhomogeneous system with $x(t) = \phi(t)$, $t \leq 0$, is given by

$$x(t) = \phi(0) + \int_0^t X(t-s)\, h(s)\, ds + \int_0^t X(t-s)\, f(T_0(s)\phi)\, ds ,$$

where $T_0(t)\phi$ is the operator semigroup corresponding to the linear equation $\dot{x}(t) = 0$.

The paper [266] is dedicated to the definition of the fundamental matrix $X(t)$, as the inverse Laplace transform of the matrix $\Delta(\lambda)^{-1}$, with $\Delta(\lambda) = \lambda I - f(e^{\lambda \cdot} I)$. Related problems are discussed

Various problems pertaining to the variation of constants formula for equations with infinite delay are discussed in A. Halanay [89], H.T. Banks [6], C. Corduneanu [39], [41] – [43], C. Corduneanu and N. Luca [46]. Also, most papers dealing with semigroup theory contain representation formulas for the solutions of nonhomogeneous equations. A very recent reference is M.C. Delfour [248].

The approach in [248] is based on semigroup theory, the initial function space being $M^p = R^n \times L^p$, with the norm given by (12), for $k(t) \equiv 1$. A subspace of M^p , $1 \leq p < \infty$, is the Sobolev space $W^{1,p} = W^{1,p}(R_-, R^n)$, and it will play a significant role in discussing linear equations. Consider the linear system

(95) $\dot{x}(t) = \int_{-\infty}^{0} [A_1(s) \, x(t+s) + A_2(s) \, \dot{x}(t+s)] \, ds + f(t)$,

under initial condition

(96) $x(t) = \phi(t)$, $t < 0$, $x(0+) = x^0$.

A basic assumption on the kernels is

(97) $A_1, A_2 \in L^q(R_-, \mathcal{L}(R^n, R^n))$,

where p in the definition of M^p , and q in (97) , are conjugate indices: $p^{-1} + q^{-1} = 1$. Formally, (95) and (96) lead to

(98) $x(t) = x^0 + \int_{-\infty}^{0} \left[A_1(s) \begin{cases} x(u+s) & , u+s \geq 0 \\ \phi(u+s) & , u+s < 0 \end{cases} \right. +$

$\left. A_2(s) \begin{cases} x(t+s) - \phi(s) & , t+s \geq 0 \\ \phi(t+s) - \phi(s) & , t+s < 0 \end{cases} \right] ds + \int_{0}^{t} f(s) \, ds$.

While the integral in the right hand side of (95) represents the general linear functional continuous on $W^{1,p}$, the right hand side of (98) has a meaning for any couple $(x^0, \phi) \in M^p$. The following basic results are obtained in [248]:

(i) For any $(x^0, \phi) \in M^p$, and $f \in L_{loc}(R_+, R^n)$, the equation (98) has unique continuous solution $x: R_+ \to R^n$. There exists $c(T) > 0$, for all $T > 0$, such that

(99) $\|x(t)\| \leq c(T) [|(x^0, \phi)|_{M^p} + \int_{0}^{T} \|f(s)\| \, ds]$, $t \in [0, T]$.

(ii) For any $\phi \in W^{1,p}$ and continuous $f: R_+ \to R^n$, the equation (95)

has a unique continuously differentiable solution $x:R_+ \to R^n$, the same as the solution of (98) for $x^0 = \phi(0)$.

(iii) When $f \equiv 0$, the relation

(100) $S(t)\phi = (x(t), x_t)$, $t \geq 0$,

defines a strongly continuous operator semigroup on M^p , of class $\mathcal{C}_0$, whose infinitesimal generator A is described by the formula

(101) $D(A) = \{(\phi(0),\phi) ; \phi \in W^{1,p} , A_\phi = (L\phi,\dot{\phi})\}$,

where $L\phi$ is given by

(102) $L\phi = \int_{-\infty}^{0} [A_1(s) \phi(s) + A_2(s) \dot{\phi}(s)] ds$.

(iv) For all $(x^0,\phi) \in M^p$ and $f \in L_{loc}(R_+,R^n)$, one has for $t \geq 0$,

(103) $(x(t),x_t) = S(t) (x^0,\phi) + \int_{0}^{t} S(t-u)(f(u),0) du$.

The results (i) - (iv) generalizes most known results regarding the linear case. The following remark [248] will give a better idea on this matter. Since the linear map L , given by (103) on $W^{1,p}$, cannot be continuously extended to L^p , that serves as a phase space, there results that the theory above is actually concerned with unbounded operators on L^p , with dense domain. This is one of the most promising features in further developing the theory of linear equations with unbounded delay, such that simple equations like (17) be covered satisfactorily.

Let us remark that Delfour's theory does not cover the case $q = 1$. Using function algebra techniques, this case is thoroughly investigated in [39], [41], [42], [46], [158], [259], where $L\phi$ looks somewhat different, but any M^p , $1 \leq p < \infty$, is admissible for building up a theory that allows investigation of stability, other behavior,the use of variation of constants formula etc. In the case of operators that can be represented by

$$Lx = Ax(t) + \int_{-\infty}^{t} B(t-s)\, x(s)\, ds \ , \quad \text{R.K. Miller [173] dealt with the situation}$$

when $B(t)$ is integrable $(q=1)$, using M^p spaces , and semigroup theory.

We also mention the fact that (103), and the variation of constants formula due to T. Naito [185], look very much alike, though they have been obtained under conditions that are independent. The spaces M^p are covered by Naito's theory, while the continuity of L is not required in Delfour's theory.

A pioneering paper in regard to the use of M^p (or L^p) spaces as initial spaces, and with the right hand side of the equation non necessarily a continuous operator, is due to J.G. Borisovic̆ and A.S. Turbabin [246]. They consider only the case of finite delays.

R. Datko [53] deals with equations of the form

$$(104) \qquad \dot{x}(t) = A(t)\, x(t) + \sum_{j=1}^{\infty} H_j(t-\omega_j)\, x(t-\omega_j) \ ,$$

with x taking values in a Banach/Hilbert space E . For each $t \in R_+$, $A(t):E \to E$ is a linear operator, not necessarily bounded, while $H_j(t):E \to E$ is bounded. One assumes that $0 < \omega_1 < \omega_2 < ... < \omega_m < ...$. Existence, uniqueness, and variation of constants formula are obtained for the Cauchy problem attached to (104) . In the autonomous case, a C_0 semigroup is constructed and the solution of (104) is obtained. When E is a Hilbert space, the stability properties are also investigated. The author also deals with related neutral equations. As applications, some partial differential equations with delay are

investigated.

For linear integral equations of the form

$$x(t) + \int_R x(t-s) \, dA(s) = h(t) \ , \quad t \in R \ ,$$

J.J. Levin and D.F. Shea [155] undertook a systematic study of the asymptotic behavior of bounded solutions. A key element in these studies is the characteristic equation

$$\mathscr{A}(\lambda) = 1 + \int_R e^{i\lambda t} \, dA(t) = 0 \ .$$

A brief account is given in [38]. Solutions with limit at infinity, with asymptotic almost periodic behavior, and other behavior, can occur. Their ideas are developed in [212], [234] and [256].

Further references, concerning the use of operational calculus in finding the solution of a linear equation with infinite delay, are [112] and [205].

The nonlinear semigroup theory has become one of the most widely used tool, in the past decade. See V. Barbu [8] for basic results, and numerous applications to various classes of functional equations, including equations of Volterra type and finite delay. In the case of nonlinear equations with finite delay, the semigroup theory has been used by G.F. Webb [232], [273] H. Flaschka and M.J. Leitman [77], A.T. Plant [199], J. Dyson and R. Villella Bressan [250], [251] .

In the past few years, the nonlinear semigroup theory has been used in connection to the theory of equations with unbounded delay by D.W. Brewer [23], J. Dyson and R. Villella Bressan [249] - [251], W.E. Fitzgibbon [73] - [76], A.T. Plant [200], and G.F. Webb [233], [273]. Further references are contained in the above quoted papers.

D.W. Brewer [23] develops a semigroup theory for autonomous nonlinear equations in a Banach space E

(105) $\dot{x}(t) = f(x_t)$, $t > 0$,

(106) $x_0 = \phi \in B^1$,

where B^1 is defined as usual , and $k(u)$ involved in formula (92) is positive, and nondecreasing on $(-\infty, -r]$. The operator semigroup acting on $B^1(E)$ is defined by means of the relations

(107) $D(A) = \{\phi; \phi, \dot{\phi} \in B^1 , \dot{\phi}(0) = F(\phi)\}$, $A\phi = -\dot{\phi}$.

If f in (105) satisfies a global Lipschitz condition on B^1

(108) $\|f(\phi) - f(\psi)\| \leq M |\phi - \psi|_{B^1}$,

then $-A$ generates a nonlinear quasicontraction strongly continuous semigroup on B^1 , say $S(t)$, such that $S(t)\phi = x_t$ for $t \geq 0$, and $\phi \in B^1$. For $\omega \geq M + k(-r)$, one has for $t \geq 0$, and $\phi, \psi \in B^1$:

(109) $|S(t)\phi - S(t)\psi| \leq e^{\omega t} |\phi - \psi|$.

Applications are given, among them being the Volterra integro-differential equation

(110) $\dot{u}(t) = C u(t) - \int_{-\infty}^{t} a(t-s) g(u(s)) ds$,

with initial condition (106) .

In [251], J. Dyson and R. Villella Bressan deal with nonlinear equations in a Hilbert space, of the form

$$(111) \qquad \dot{x}(t) = f(t, x(t)) + g(t, x_t) , \quad t > 0 ,$$

and initial condition (106). The phase space is $E \times B^1$, with $r = 0$, and $k(u)$ monotone increasing and Lebesque integrable on $(-\infty, 0]$. Since (111) is not an autonomous system, the semigroup definition is somewhat more involved. It is assumed that f is m-accretive, and g is Lipschitzian with $\beta(t)$ instead of a constant. Then

$$(112) \qquad D(A(t)) = \{(h, \phi) \in E \times B^1 , \ \phi \text{ absolutely continuous on bounded}$$
$$\text{sets}, \ \dot{\phi} \in B^1 , \ h = \phi(0) \in D(f(t, \cdot))\} .$$

$$(113) \qquad A(t) (h, \phi) = \{-f(t, h) - g(t, \phi) , \ -\dot{\phi} \} ,$$

define the nonlinear semigroup attached to (111). It is assumed that $D(f(t, \cdot))$ is dense in E. The set $\cup_t D(A(t))$ is characterized. The transition operator $U(t, s)$ is defined by means of the linear equation $\dot{\phi} = -A(t)\phi$. Estimates for the solutions of (111) are found.

W.E. Fitzgibbon [73] – [76] uses the nonlinear semigroups in investigating various properties of functional-integral equations of the form

$$(114) \qquad x(\phi)(t) = W(t, s) \ \phi(0) + \int_s^t W(t, u) \ F(u, x_u(\phi)) \ du ,$$

that arise when equations with unbounded delay of the form (111) are reduced to integral equations. $W(t, s)$ stands for the linear evolution operator (transition operator). The author deals with various phase spaces, including the space of uniformly continuous maps from $(-\infty, 0]$ into the space E (a Banach space), and the space B^1 used above. Existence, stability, and other behavior is investigated in [73] – [76]. Applications are given to some partial differential

equations involving infinite delay.

A.T. Plant [199] applies semigroup method to autonomous equations of the form (111), under accretivity and Lipschitz conditions. He also deals with inclusions associated to (111): $\dot{x}(t) \in f(x(t)) + g(x_t)$. His space consists of functions ϕ , such that $e^{-\lambda u} \phi(u)$ is bounded on $(-\infty, 0]$, $\lambda > 0$.

In his papers [233], [273], G.F. Webb treats, by the method of nonlinear semigroup theory, equations of the form (111). Also, he shows how to extend this method to nonlinear Volterra equations of the form

$$(115) \qquad x(t) = y(t) + \int_0^t G(t-s, x(s)) \, ds , \quad t \geq 0 .$$

For equations of the form (111), he considers phase space of the B^p-type. Then, (111) is regarded as an abstract ordinary differential equation in the phase space (let us mention that the basic space, to which x belongs is a Hilbert space). Of course, finding this equation or the corresponding semigroup are equivalent problems. Under appropriate conditions, (115) can be differentiated, and subsequently reduced to an equation of the form (111). An alternate method is given in [273], where (115) is shown to be equivalent to

$$(116) \qquad x(t) = h - G(\phi) + G(x_t) , \quad t \geq 0 ,$$

under initial conditions

$$(117) \qquad x_0 = \phi \in B^p , \quad x(0) = h ,$$

$h \in E$ = the underlying Banach space, with

$$(118) \qquad G(\phi) = \int_{-\infty}^0 g(-s, \phi(s)) \, ds , \quad \phi \in B^p .$$

Of course, g is subject to convenient conditions. One obtains a G satisfying

a convenient Lipschitz condition. The connection between y(t) and h can

be easily found comparing (115) and (116) . Since (116) is not a differential-

functional equation, this constitutes a good illustration of the statement made

in the Section 1, regarding the fact that Cauchy's problem (initial value problem)

makes sense for other kinds of evaluation equations than differential ones.

We shall survey now further contributions to the theory of equations with

unbounded delay, that deal with various topics and emphasize a variety of methods

and results.

Y. Hino [101], using B^p phase spaces considers stability problems by means of

Liapunov functions, and almost periodicity.

G.F. Webb [231] finds existence/behavior results, using accretive operator

theory. He considers equations of the form $y'(t) = F(t,y(t), y(\omega(t)))$, with

usual initial conditions in spaces of functions that are bounded with respect

to a weight function, and continuous or uniformly continuous.

J. Błaz [18] finds boundedness results for equations of the form (25).

D. Fargue [72] gives conditions under which systems of the form

$$(119) \qquad \dot{x}(t) = f(t,x(t)) + \int_{-\infty}^{t} k(t,s,x(s))\ ds$$

can be reduced to ordinary differential equations or to partial differential

equations.

V.R. Vinokurov [227] investigates boundedness of solutions for certain

equations of the form (119).

T.G. Hallam [100] studies boundedness of solutions for systems of the

form $\dot{x}(t) = A(t)\ x(t) + f(t;x)$, where $f(t;x)$ stands for an operator that

could be chosen, in particular, to be an operator with unbounded delay.

M. Kisielewicz [123] deals with category theorems, and shows that for equations

equations with delay, as they are formulated by Soviet mathematicians (see [243] and [244] for recent contributions, the last reference being a survey on the topic).

N.V. Azbelev [243] deals with boundary value problems of the form $\dot{x} = Fx$ on $[a,b]$, $\ell x = 0$, where $F:D \rightarrow L$ is a map from the space D of absolutely continuous functions on $[a,b]$, into the space L of Lebesque integrable functions (with values in R^n). ℓ is a linear continuous map from D, into R^n. To this scheme, one can reduce problems formulated as follows: find $x:[a,b] \rightarrow R^n$, such that $\dot{x}(t) = f(t,x(t), x(h(t)))$, $t \in (a,b)$, $x(t) = \phi(t)$, $t \in [a,b]$. The function $h(t)$ could represent a delay for some values of t, i.e., $h(t) \leq t$, but it might not satisfy the above inequality for all values of $t \in [a,b]$. Moreover, neutral functional-differential equations of the form

$$\dot{x}(t) = f(t,x[h(t,x)], \dot{x}[g(t,x)]) , \quad t \in [a,b] ,$$

with boundary value conditions $x(u) = \phi(u)$, $\dot{x}(u) = \psi(u)$, $u \in [a,b]$, can be also treated within the functional scheme described in [243], [244]. This scheme leads to a Hammerstein integral equation of the form

$$z(t) = \int_a^b \Gamma(t,s) \, f(s;z) \, ds , \quad t \in [a,b],$$

provided certain specific conditions are verified. An interesting result [244] is due to L.F. Rakhmatulina : for each functional ℓ on D, there exists a linear invertible operator $W:L \rightarrow R(W) = \{x; x \in D , \ell x = 0\}$, such that the product of the differentiation operator by W is a Fredholm operator on L. This result, and similar ones, allow to reduce rather general boundary value problems to nonlinear integral equations of Hammerstein type (not necessarily symmetric). The same procedure is developed in [4].

like (25), the nonuniqueness may occur only in a class that constitutes a set of Baire's first category (in a convenient topology for the right hand sides).

R.K. Miller's paper [174], though not dedicated to equations with unbounded delay, contains interesting features and invites to extend the study to such equations.

G.S. Jordan [111], generalizing some results of A.S. Lodge, J.B. McLeod, and J.A. Nohel [260], deals with the nonlinear integral equation with infinite delay

$$(120) \qquad -\mu\, y'(t) \;=\; \int_{-\infty}^{t} b(t,s)\, F(y(t),y(s))\, ds\ ,\quad t > 0\ ,$$

under initial condition $y(t) = g(t)$, $t \in (-\infty,0]$. The parameter $\mu > 0$ is assumed "small", $b(t,s)$, $t \geq s$, is integrable on $(-\infty,t]$ with respect to s , and satisfies various restrictions that will not be stated here. The nonlinearity $F: R_+ \times R_+ \to R$ is of class C^1 , and $F(x,x) = 0$ for $x > 0$, $F_x(x,y) > 0$, and $F_y(x,y) < 0$. Finally, g is continuous and nondecreasing on $(-\infty,0]$, and satisfies $g(-\infty) = 1$, $g(0) > 1$. On behalf of these hypotheses, the equation (120) has a unique solution $\phi(t,\mu)$, defined for $t \in R_+$ and small μ , such that $\dot\phi(t,\mu) < 0$, and $1 < \phi(t,\mu) \leq g(0)$, $t \in R_+$. Estimates are found for $\phi'(t,\mu)$, and the limit equation $(\mu = 0)$ is shown to possess a unique solution $\phi_0(t)$ on R_+ , $\phi(t,\mu) - \phi_0(t)$ being estimated.

In [47], the authors obtain for the integro-differential relation (67), besides stability, various results regarding the asymptotic behavior of solutions (for instance, the boundedness on R_+) .

Many results concerning various types of asymptotic behavior have been obtained in connection to equations generated in applied fields. They will be surveyed in Section 6.

In the Section 1, we made a few comments on boundary value problems for

J. Eisenfeld and V. Lakshmikantham [67] investigate boundary problems
of the form $x''(t) + f(t,x(t), x(h(t))) = 0$, $t \in [a,b]$, with $x(t) = \alpha(t)$,
$t \leq a$, and $x(t) = \beta(t)$, $t \geq b$. They also reduce the problem to a nonlinear
integral equation on $[a,b]$.

J. Błaz [20] deals with the following boundary value problem: find $x(t)$
on $[0,a]$, such that

$$x'(t) = \lambda \int_0^\infty f(t,x(t-s)) \, d_s \, r(t,s) , \quad t \in (a,b) ,$$

and $x(t) = \phi(t)$, $t \leq 0$, $x(a) = \eta \in R^n$. Of course, λ is a parameter to
be determined in solving the problem.

Recently, T. Kaminogo [257] dealt with boundary value problems for equations
of the form (1) , in the framework of the axiomatic theory for the phase space.
More specifically, he considers the scalar second order equation $x''(t) = f(t,x_t,x'(t))$,
with boundary value conditions $x_\sigma = \phi \in S$, $x(T) = A$, $T > \sigma$. If one denotes
$x'(t) = y(t)$, then the second order equation becomes a system of the form (1)
for col (x,y). His approach relies upon the method of subsolutions and supersolutions,
i.e., he assumes the existence of two functions $\alpha(t)$ and $\beta(t)$, such that
$\alpha''(t) \geq f(t,\alpha_t,\alpha'(t))$, and $\beta'(t) \leq f(t,\beta_t,\beta'(t))$, $t \in (a,b)$. The solution
will satisfy $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [a,b]$. Supplementary conditions are
imposed on f, α, and β , that will not be reproduced here. The analysis of the
problem involves previous investigation of certain basic properties for equations
of the form (1), including Kneser's type results.

In concluding this Section, we must remark that the theory of boundary
value problems for equations with unbounded delay is at its very beginning.

6. APPLICATIONS

Several results and topics mentioned in the preceeding sections of this survey have been motivated by various applications of equations with unbounded delay. As stated in the introduction, the fields in which the equations with unbounded delay have already found interesting applications are Mechanics of Continua, Population Dynamics and Ecology, Systems Theory (mainly engineering and control systems), and Nuclear Reactor Dynamics. We shall briefly review some contributions to the above listed areas of applied research, trying to avoid repetition of those topics that have been already discussed in the preceeding Sections.

The original approach of Volterra to the theory of hereditary phenomena in Continuum Mechanics [228] leads to equations with unbounded delay, and opened the way for a great deal of contemporary research work. In [29], B. D. Coleman and H. Dill investigate stability problems in the theory of imcompressible materials with memory, using the energy functional to provide adequate conditions of stability (see also [33]). For instance, the study of inflation of a circular tube leads to the system of equations

(121)
$$\begin{cases} \dot{\beta}(t) = v(t) \ , \\ \dot{v}(t) = kv^2(t) + h(\beta_t) + C \ , \end{cases}$$

with $k > 0$, and C real. The functional $h(\phi)$ is given on a history space, and is assumed to be continuous. Of course, the initial condtions for (121) must be of the form $v(0) = v_o$, $\beta_0 = \phi$ (i.e., $\beta(u) = \phi(u)$,

$u \leq 0$). The energy functional is explicity constructed in terms of the data in the problem. The study of inflation of a spherical shell leads to a system with unbounded delay similar to (121).

Further applications of equations with unbounded delay in Continum Mechanics regard the study of wave propagation in dissipative materials with memory, and can be found in the book [30], representing a collection of papers published in Archive for Rational Mechanics and Analysis by B. D. Coleman, M. E. Gurtin, I. Herrera R., and C. Truesdell.

M. J. Leitman and V. J. Mizel [150] - [152] paid much attention to the study of hereditary phenomena, and emphasized the role of integral equations in investigating them.

C. M. Dafermos [52] reduces the problem of asymptotic stability for viscoelastic materials to the investigation of stability properties for the equation

$$(122) \qquad \frac{d}{dt}\left(\rho \frac{du}{dt}\right) + C\, u(t) + \int_{-\infty}^{t} G(t-s)\, u(s)ds = 0 \quad ,$$

under initial condition $u(s) = v(s)$, $s \in (-\infty, 0]$. In (122), u takes the values in a real separable Hilbert space H, ρ is a bounded self-adjoint operator on H, while C and $G(t)$ stand for (unbounded) self-adjoint operators in H, such that $D(C) \subset D(G(t))$, $t \in R_+$. Roughly speaking, the asymptotic stability of the zero solution of equation (122) is a consequence of the following property: each eigen-solution w_n of the problem $Cw - \lambda \rho w = 0$ is such that, one can find $\xi_n \in R_+$ with $\dot{G}(\xi_n)w_n \neq 0$. In other words, the stability condition

relies on the spectral properties of the operators C and ρ . The applications to viscoelasticity involve, of course, some integro-differential equations with partial derivatives.

S. Adali [1] considers equations that generalize (122), namely

$$(123) \qquad \frac{d}{dt}\left(\rho(t)\frac{du}{dt}\right) + C(t)u(t) + \int_{-\infty}^{t} G(t-s,t)\,u(s)ds = f(t) \ ,$$

with initial condition $u(\tau+s) = v(\tau)$, $\tau \in (-\infty,0]$. He obtains existence and uniqueness conditions for the solution of (123), and gives criteria of asymptotic stability for the zero solution of the homogeneous equation attached to (123).

Somewhat different types of equations with unbounded delay appear in the paper [55] by P. L. Davis (see also [54]). By means of adequate transformations, P. L. Davis reduces such equations to

$$(124) \qquad u_t = k\Delta u + mu + \int_{-\infty}^{t} \left\{a(t-s)\,u(s) + b(t-s)\Delta u\right\} ds \ ,$$

where $u = u(t,x)$ denotes the deviation of temperature with respect to a standard distribution, Δ stands for the Laplace operator in the space variable x , and $a(t)$, $b(t)$ mean operator valued functions acting on convenient function spaces. The applications concern heat conduction in materials with memory. In [160], R. C. MacCamy deals with similar problems.

M. Slemrod [213], [271], uses various classes of equations with un-bounded delay in studying problems related to fluid mechanics. The velocity

vector satisfies an equation of the form

$$(125) \qquad \dot{v}(t) = -\nabla p + \Delta \int_0^\infty G(s,\cdot)\, v(t-s)\,ds \quad ,$$

with initial condition $v = v_0 \in$ memory space. The Laplace operator
is also meant in the space variables.

N. Distefano [61], dedicates two chapters of this book to the
description of hereditory processes in Continuum Mechanics, with special
emphasis on viscoelastic materials. The role of Volterra is illustrated,
and the significance of Volterra equations with infinite lower limit is
also pointed out.

In Population Dynamics, the work started by Volterra [229] has been
considerably developed during the past few years. The paper [168] and
the monographs [169], [51], contain a good deal of comments and results
concerning the role of delays in population models. In [51], J. M.
Cushing provides a broad survey of the literature pertaining to the area
of delay models in population dynamics, and includes some basic results,
as those mentioned in Section 4. The main emphasis is placed on stability
of such models, and the occurence of oscillations. Various special
models are examined, such as Volterria's predator-prey model with delays.
It is shown, among other things, that the presence of delaying terms in
an ecological system can stabilize an otherwise instable equilibrium.
See also [48] - [50] for contributions that have been covered in [51],
especially in regard to the presence of oscillations in population models
with delay.

In [86] and [87], A. Haimovici investigated the existence and uniqueness problem for equations with unbounded delay motivated by Volterra's population theory.

R. K. Miller [171] develops interesting topics related to Volterra's population equation, providing various results that overreach the initial framework. For instance, if $x(t)$ is a bounded solution (on R_+) of the equation

$$(126) \qquad \dot{x}(t) = F(x(t)) + \int_0^t A(t-s)\, G(x(s))ds \quad ,$$

where $A(t)$ is integrable on R_+ , then the limit set $\Omega(x(t))$ is an invariant set for the equation with infinite lower limit attached to (126):

$$(127) \qquad \dot{y}(t) = F(y(t)) + \int_{-\infty}^t A(t-s)\, G(y(s))ds \quad .$$

In the special case of the Volterra population equation

$$(128) \qquad \frac{\dot{N}(t)}{N(t)} = a - N(t) - \int_{-\infty}^t f(t-s)\, N(s)ds \quad ,$$

with $a > 0$, $b > 0$, $f(t)$ continuous and integrable on R_+ , and such that $f(t) \not\equiv 0$, there exists a unique solution on R_+ that reduces to $g(t)$ on the negative half-axis, and satisfying

$$(129) \qquad \lim_{t \to \infty} N(t) = N^* = a[\, b + \int_0^\infty f(s)ds\,]^{-1} \quad ,$$

provided $b > \int_0^\infty |f(s)| ds$.

H. W. Stech [218] is concerned with the effect of delays on the stability of equilibrium in a population model, considering the integro-differential equation

$$(130) \qquad \dot{N}(t) = \alpha \, N(t)[1 - N_0^{-1} \int_{-\infty}^0 N(t-s) d\eta(s)] \quad .$$

A. Wörz-Busekros [239] investigates global stability properties in population models described by Volterra integro-differential equations with unbounded delay. A method proposed by D. Fargue [72] is used in order to reduce the equations to ordinary differential ones.

H. C. Simpson [269] dedicates most of his thesis to the investigation of various problems generated by the Volterra's equations in population dynamics (see equations (85) above).

The applications of equations with unbounded delay in Systems Theory and Control Engineering are very numerous, and several monographs have been already dedicated to such topics.

I. W. Sandberg and V. E. Beneš [207] are investigating Volterra integral equations (in which the lower limit of the integral is $-\infty$), showing their significance in the theory of some dynamical systems.

Following V. M. Popov [201], I. Barbălat and A. Halanay [7] give necessary and sufficient conditions for the hyperstability of certain linear systems whose description involves operators of the form

$$(131) \qquad (Ax)(t) = \sum_{j=0}^{\infty} A_j \, x(t-t_j) + \int_{-\infty}^0 B(s)x(t-s)ds \quad ,$$

with $\{t_j\}$, A_j , and B satisfying conditions (39).

The monographs by J. C. Willems [235] , C. A. Desoer and M.
Vidyasagar [59], Vl. Rasvan [203], A. N. Michael and R. K. Miller [170],
and M. Vidyasagar [225] provide a good deal of results regarding
various stability properties of systems (in most cases, feedback sys-
tems) whose description involves operators of the form (131), or
particular cases of such operators, as well as some non linearities.

The papers [39], [40], [42], [44], [45] by C. Corduneanu, and
[46] by C. Corduneanu and N. Luca, contain several stability results
for feedback systems described by the equations

$$(132) \qquad \begin{cases} \dot{x}(t) = (Ax)(t) + (B\phi(\sigma))(t) \quad , \\ \sigma(t) = (Cx)(t) \quad , \end{cases}$$

where A , B and C stand for operators of the form (131). Vl. Rasvan
[204] has found the transfer function for (132), and has given the most
general stability result, by means of frequency domain techniques.

M. Podowski [267] illustrates the use of functional analytic methods,
with main emphasis on stability of nonlinear systems. Operators and
equations with unbounded delay are investigated, including some integro-
differential equations of the form

$$(133) \qquad \dot{x}(t) = Px(t) + \int_{-\infty}^{t} Q(t-s)x(s)ds + f(t) \quad , \quad t > 0 \quad ,$$

as well as nonlinear equations obtained by perturbing (133).

M. N. Oğuztöreli [196] considers systems of the form (36), in connection to various problems in control theory.

M. C. Delfour [57] deals with equations with unbounded delay of the form (36), being mainly concerned with the optimal control theory in case of a quadratic cost functional.

V. Barbu [245] applies some general results regarding convex control problems to the case of systems

$$(134) \quad \begin{cases} \dot{x}(t) = Ax(t) + \int_{-\infty}^{t} B(t-s)x(s)ds + C\,u(t) \;, \quad t \geq 0 \;, \\ x(0+) = x^0 \;, \quad x(s) = \phi(s) \;, \quad s < 0 \;, \end{cases}$$

with a cost functional of the form $\int_{0}^{\infty} L(x(t),u(t))dt$, where L is lower-semicontinuous and convex. He uses results from [9].

Let us conclude the comments regarding the use of equation with unbounded delay in Control Theory, by sending the reader to L. W. Neustadt [188], where he sketches a theory of optimization in a class of processes described by Volterra operators (the case of unbounded delay being, apparently, admissible).

The literature on Nuclear Reactor Dynamics is considerable, and the use of equations with unbounded delay appears in most textbooks and monographs dedicated to this field. For instance, V. D. Gorjatchenko [81] dedicates several sections of the monograph to the discussion of some basic problems for functional differential equations with unbounded delay, and gives various applications of such equations (in most cases, integro-differential equations) to the dynamics of nuclear reactors. See also [2]

where the use of integral operators and attached equations play a signifi-
ant role. Both books [2] and [81] contain lists of references that in-
clude many cases of equations with unbounded delay occuring in reactor
dynamics.

F. Kappel [116], F. Kappel and F. DiPasquantonio [60], [117], con-
tinuing research topics conducted by J. J. Levin and J. A. Nohel [154],
V. M. Popov, E. Gyftopoulos, Z. Akcasu and P. Akhtar, dealt with equations
of the form

(135)
$$\begin{cases} \dot{p}(t) = \sum_{i=1}^{M} \Lambda^{-1}\beta_i[p(t)-c_i(t)] - \Lambda^{-1}P[1+p(t)] \int_{-\infty}^{t} k(t-s)p(s)ds \quad , \\ \dot{c}_i(t) = \lambda_i[p(t)-c_i(t)] \quad , \quad i = 1,2,\dots,M \quad , \end{cases}$$

that characterize the dynamics of a nuclear reactor, under adequate
hypotheses. Their method relies mainly on Liapunov's technique. It is
worth to be pointed out that they generalize previous work done in the
field, starting with the so-called Welton-type criterion for stability
of nuclear reactors.

A. Halanay [90], and A. Halanay and Vl. Rasvan [91], further gen-
eralize the stability results by considering delays in the state variables
of the system. They make extensive use of the frequency domain techniques,
and obtain a general criterion that leads, in particular, to the criteria
earlier known.

The approach in [91], combined with the asymptotic stability result
in [41] (see Section 3 above), conducted C. Corduneanu [42] to an even
more general stability criterion. Let us dwell on this criterion.

The system describing the dynamics of the nuclear reactor is

$$
(136) \quad
\begin{cases}
\dot{x}(t) = (Ax)(t) + (b\rho)(t) \quad , \\[2mm]
\dot{\rho}(t) = -\displaystyle\sum_{k=1}^{M} \beta_k \Lambda^{-1}[\rho(t) - \eta_k(t)] - P\Lambda^{-1}[1+\rho(t)]\nu(t) \quad , \\[2mm]
\dot{\eta}_k(t) = \lambda_k[\rho(t) - \eta_k(t)] \quad , \quad k = 1,2,\ldots,M \quad , \\[2mm]
\nu(t) = (c^*x)(t) + (\alpha\rho)(t) \quad ,
\end{cases}
$$

with A , b , c^* and α standing for certain difference-integral operators, of the form:

$$
(137) \quad (Ax)(t) = A_o x(t) + \sum_{j=1}^{\infty} A_j x(t-t_j) + \int_0^t B(t-s)x(s)ds \quad ,
$$

$$
(138) \quad (b\xi)(t) = b_o \xi(t) + \sum_{j=1}^{\infty} b_j \xi(t-t_j) + \int_0^t \beta(t-s)\xi(s)ds \quad ,
$$

$$
(139) \quad (c^*x)(t) = c_o^* x(t) + \sum_{j=1}^{\infty} c_j^* x(t-t_j) + \int_0^t d^*(t-s)x(s)ds \quad ,
$$

$$
(140) \quad (\alpha\rho)(t) = \alpha_o \rho(t) + \sum_{j=1}^{\infty} \alpha_j \rho(t-t_j) + \int_{-\infty}^t \gamma(t-s)\rho(s)ds \quad ,
$$

where $t_j > 0$, $j = 1,2,\ldots$, and conditions

$$
(141) \quad \{\|A_j\|\} \ , \ \{\|b_j\|\} \ , \ \{\|c_j\|\} \ , \ \{|\alpha_j|\} \in \ell^1
$$

$$
(142) \quad \|B(t)\| \ , \ \|\beta(t)\| \ , \ \|d(t)\| \ , \ |\gamma(t)| \in L^1(R_+,R) \quad ,
$$

hold true. The asterisk indicates the transpose of the vector (c_j is suppose to be a column n-vector, for every j). The initial conditions that determine a unique solution are

$$(143) \qquad x(t) = h(t) \ , \ \rho(t) = \lambda(t) \ , \ t < 0 \ ,$$

$$(144) \qquad x(0+) = x^o \in R^n \ , \ \rho(0+) = \rho_o \in R \ , \ \eta_k(0) = \eta_k^o \ , \ k = 1,2,\ldots,M \ .$$

The constants Λ , β_k and λ_k are assumed positive.

The following linearized system, depending upon a real parameter h , is attached to (136):

$$(145) \qquad \begin{cases} \dot{y}(t) = (Ay)(t) + (b\xi)(t) \ , \\[2mm] \dot{\xi}(t) = -\displaystyle\sum_{k=1}^{M} \beta_k \Lambda^{-1}[\xi(t)-\zeta_k(t)] - P\Lambda^{-1}h[(c^*y)(t)+(\alpha\xi)(t)] \ . \\[2mm] \dot{\zeta}_k(t) = \lambda_k[\xi(t)-\zeta_k(t)] \ , \ k = 1,2,\ldots,M \ . \end{cases}$$

A basic assumption for the sequel is the existence of two numbers h_1 , h_2 , such that $0 < h_1 < 1 < h_2$, for which (145) is asymptotically stable (i.e., when $h = h_1$ or $h = h_2$) .

Before stating further conditions needed in the stability criterion, let us define the transfer functions occuring in the formation of this criterion. The symbol $\mathscr{A}(s)$ of the operator A has been defined by (43). Then let $\tilde{b}(s)$, $\tilde{c}^*(s)$ be the corresponding symbols attached to b(t) and $c^*(t)$. Denote

(146) $\tilde{k}(s) = \tilde{c}*(s)[sI - \mathscr{A}(s)]^{-1}\tilde{b}(s)$,

(147) $\gamma_1(s) = R(s)[1 + P\Lambda^{-1}h_1 R(s)\gamma_0(s)]^{-1}$,

(148) $\gamma_2(s) = P\Lambda^{-1}h_1\gamma_0(s)\gamma_1(s)$,

where $\gamma_0(s) = \tilde{k}(s) + \tilde{\alpha}(s)$, and $[sR(s)]^{-1} = 1 + \Lambda^{-1}\sum\limits_{k=1}^{M}\beta_k(s+\lambda_k)^{-1}$.

Finally, let

(149) $H(s) = \delta_0[(h_2-h_1)^{-1} + h_1^{-1}\gamma_2(s)] + \delta_1\gamma_1(s)$

$$+ \delta_2 P\Lambda^{-1}(h_2-h_1)|\gamma_1(s)|^2\gamma_0(s)$$

be such that

(150) $\text{Re } H(i\omega) > 0$ for $\omega \in R$,

when $\delta_0 > 0$, and besides (150), assume

(151) $\delta_1\Lambda^{-1}\sum\limits_{k=1}^{M}\beta_k + [\delta_1 h_1 + \delta_2(h_2-h_1)](\alpha_0 - \sum\limits_{j=1}^{\infty}|\alpha_j|) > 0$

holds true when $\delta_2 = 0$. The numbers δ_0 , δ_1 , δ_2 in (149) are
supposed non-negative, $\delta_2 \geq \delta_1$, and $\delta_1 + \delta_2 > 0$.

Under the above assumption on (136), each solution $x(t)$, $\rho(t)$,
$y_k(t)$, $k = 1,2,\ldots,M$ of that system, for which

(152) $\qquad \|h(t)\|$, $|\lambda(t)| \in L^1(R_-,R) \cap L^2(R_-,R)$,

is defined on the positive half-axis, and satisfies

(153) $\qquad \lim_{t \to \infty} (\|x(t)\| + |\rho(t)| + \sum_{k=1}^{M} |\eta_k(t)|) = 0$,

provided certain initial constraints are satisfied.

In order to describe these initial constraints (actually, they determine the zone of attraction of the zero solution), let us consider the function of a real variable

$$\phi(u) = u - \ln(1+u) - \frac{1}{2h_2} u^2 \quad ,$$

and choose $u_0 \in (0,1-h_1)$, $\bar{u}_0 \in (0,\sqrt{h_2}-1)$, such that

$$\phi(u) \leq \phi(\sqrt{h_2}-1)\min\{1,\beta_k\lambda_k^{-1}\Lambda^{-1} ; \quad k = 1,2, \dots ,M\} \quad ,$$

for $u \in [-u_0,\bar{u}_0]$. Then, we must choose $\rho(0)$ and $\eta_k(0)$, $k = 1,2, \dots M$, in the interval $(-u_0,\bar{u}_0)$.

For concluding this Section, let us point out that the equations with unbounded delay have found applications in other research fields. For instance, in R. Bellman and G. M. Wing [11], one can find such equations in connection to some pseudo-transport problems. B. D. Coleman and G. H. Renninger [35],[36], encountered equations with unbounded delay in the study of neural interactions (see Section 4 of this survey).

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