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EQUICONTINUOUS COMMUTATIVE SEMIGROUPS OF ONTO FUNCTIONS

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Equicontinuous (or regular) groups of transformations of a space onto itself have been studied extensively [1], [2], [5]. In this note we investigate equicontinuous commutative semigroups G of functions of a space X into itself. We define a product on orbit closures which makes each orbit closure a commutative semigroup. This generalizes a result of D. MONTGOMERY on equicontinuous transformation groups [5]. If X is compact Hausdorff and each $g \in G$ is onto, then each $g \in G$ is a homeomorphism and each orbit closure is a topological group. This generalizes work of P. F. DUVALL, JR. and L. S. HUSCH [1] who considered the case when X is compact metric and G is generated by a single function. Finally it is shown that if X is compact Hausdorff then the closure of G in the space of continuous maps of X into itself with the compact-open topology is a topological group and each orbit closure is the continuous homomorphic image of G.

We shall assume familiarity with [4] whose notation we shall follow. [6] contains the definitions and results from the theory of semigroups which we use. Let $(X, \mathscr{U})$ be a uniform space and let $C(X)$ be the semigroup of continuous functions of X into itself with the topology of uniform convergence on compacta. If G is a subsemigroup of $C(X)$, then G is *equicontinuous* at $x \in X$ if, for each $U \in \mathscr{U}$, there is a neighborhood V of x such that $g(V) \subseteq U[g(x)]$ for each $g \in G$. G is *equicontinuous* if it is equicontinuous at each point of X. G is *uniformly equicontinuous* if, for each $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $(g(x), g(y)) \in U$ whenever $g \in G$ and $(x, y) \in V$. If $x \in X$, let $O(x) = \overline{\{g(x) \mid g \in G\}}$. Henceforth, suppose X is Hausdorff and G is commutative.

Proposition 1. *If $x \in X$ such that G is uniformly equicontinuous on $O(x)$ and if the nets $\{g_\alpha(x), \alpha \in A\}$ and $\{g_\beta(x), \beta \in B\}$, $\{g_\alpha\}_{\alpha \in A} \cup \{g_\beta\}_{\beta \in B} \subseteq G$, are Cauchy nets, then the net $\{g_\alpha g_\beta(x), (\alpha, \beta) \in A \times B\}$ is a Cauchy net. ($A \times B$ is the product directed set [4; p. 68]).*

Proof. Suppose $U \in \mathscr{U}$ and choose $V \in \mathscr{U}$ such that $V \circ V \subseteq U$. By uniform equicontinuity there exists $W \in \mathscr{U}$ such that $(y, z) \in W$ implies $(g(y), g(z)) \in V$ for all $g \in G$. There exists α and β such that if $\alpha_1, \alpha_2 \geqq \alpha$ and $\beta_1, \beta_2 \geqq \beta$, then $(g_{\alpha_1}(x), g_{\alpha_2}(x))$ and $(g_{\beta_1}(x), g_{\beta_2}(x))$ belong to W where $\alpha, \alpha_1, \alpha_2 \in A$ and $\beta, \beta_1, \beta_2 \in B$. Note that if $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \geqq (\alpha, \beta)$, then $(g_{\alpha_1} g_{\beta_1}(x), g_{\alpha_2} g_{\beta_2}(x)) \in U$.

Definition. If $O(x)$ is complete and $y, z \in O(x)$, let $\{g_\alpha(x), \alpha \in A\}$ and $\{g_\beta(x), \beta \in B\}$ be nets which converge to y and z, respectively. Define $y \cdot z$ to be the limit of the net $\{g_\alpha g_\beta(x), (\alpha, \beta) \in A \times B\}$.

Proposition 2. *The product $y \cdot z$ is well-defined*; — i.e., *$y \cdot z$ is independent of the choice of nets which converge to y and z.*

Proof. Suppose y is the limit of the nets $\{g_\alpha(x), \alpha \in A\}$ and $\{g_\gamma(x), \gamma \in \Gamma\}$ and z is the limit of the nets $\{g_\beta(x), \beta \in B\}$ and $\{g_\delta(x), \delta \in \Delta\}$. Let a and b be the limits of the nets $\{g_\alpha g_\beta(x), (\alpha, \beta) \in A \times B\}$ and $\{g_\gamma g_\delta(x), (\gamma, \delta) \in \Gamma \times \Delta\}$, respectively. Let $U \in \mathscr{U}$ and choose $V \in \mathscr{U}$ such that $V \circ V \circ V \circ V \subseteq U$. By uniform equicontinuity there exists $W \in \mathscr{U}$ such that $(a, b) \in W$ implies $(g(a), g(b)) \in V$ for all $g \in G$. Choose $\alpha, \beta, \gamma, \delta$ so that $(g_\gamma(x), g_\alpha(x)), (g_\delta(x), g_\beta(x)) \in W$ and $(b, g_\gamma g_\delta(x)), (g_\alpha g_\beta(x), a) \in V$. It is easily seen that $(b, a) \in U$ and since U is arbitrary, $a = b$.

Proposition 3. *If $y, z \in O(x)$ such that z is the limit of the net $\{z_\alpha, \alpha \in A\} \subseteq O(x)$, then the net $\{y \cdot z_\alpha, \alpha \in A\}$ converges to $y \cdot z$.*

Proof. Suppose y, z, z_α $(\alpha \in A)$ are the limit of the nets $\{g_\beta(x), \beta \in B\}$, $\{g_\gamma(x), \gamma \in \Gamma\}$ and $\{g_{\alpha,\delta}(x), \delta \in \Delta(\alpha)\}$, respectively. Let $U \in \mathscr{U}$ and choose $V \in \mathscr{U}$ such that $V \circ V \circ V \circ V \circ V \subseteq U$ and $W \in \mathscr{U}$ such that $(a, b) \in W$ implies $(g(a), g(b)) \in V$ for all $g \in G$. There exists

i) α_1 such that $\alpha \in A$ and $\alpha \geqq \alpha_1$ implies $(z, z_\alpha) \in W$,

ii) (β_1, γ_1) such that $(\beta, \gamma) \in B \times \Gamma$ and $(\beta, \gamma) \geqq (\beta_1, \gamma_1)$ implies $(y \cdot z, g_\beta g_\gamma(x)) \in V$,

iii) γ_2 such that $\gamma \in \Gamma$ and $\gamma \geqq \gamma_2$ implies $(g_\gamma(x), z) \in W$,

iv) δ_1 such that $\delta \in \Delta$ and $\delta \geqq \delta_1$ implies $(z_\alpha, g_{\alpha,\delta}(x)) \in W$,

v) (β_2, δ_2) such that $(\beta, \delta) \in B \times \Delta$ and $(\beta, \delta) \geqq (\beta_1, \delta_1)$ implies $(g_\beta g_{\alpha,\delta}(x), y \cdot z_\alpha) \in V$.

Note that δ_1 and (β_2, δ_2) depend upon α. Choose $\beta_3 \geqq \beta_1, \beta_2$; $\gamma_3 \geqq \gamma_1, \gamma_2$; $\delta_3 \geqq \delta_1, \delta_2$. Suppose $\alpha \geqq \alpha_1$ and choose $(\beta, \gamma, \delta) \geqq (\beta_3, \gamma_3, \delta_3)$. Then $(g_\beta g_{\alpha,\delta}(x), y \,.\, z_\alpha)$, $(g_\beta(z_\alpha), g_\beta g_{\alpha,\delta}(x))$, $(g_\beta(z), g_\beta(z_\alpha))$, $(g_\beta g_\gamma(x), g_\beta(z))$, $(y \,.\, z, g_\beta g_\gamma(x)) \in V$ implies $(y \,.\, z, y \cdot z_\alpha) \in U$.

Proposition 4. *If $y, z \in O(x)$ are the limits of the nets $\{y_\sigma, \sigma \in \Sigma\}$ and $\{z_\alpha, \alpha \in A\}$, respectively, which are contained in $O(x)$ and if $U \in \mathscr{U}$, then there exists $(\sigma_1, \alpha_1) \in \in \Sigma \times A$ such that if $(\sigma, \alpha) \in \Sigma \times A$ and $(\sigma, \alpha) \geqq (\sigma_1, \alpha_1)$, then $(y_\sigma \cdot z_\alpha, y \cdot z_\alpha) \in U$.*

Proof. In addition to the nets used in the previous proof, let y be the limit of the net $\{g_{\sigma,\tau}(x), \tau \in T(\sigma)\} \subseteq O(x)$. Choose $V \in \mathcal{U}$ such that $V \circ V \circ V \circ V \circ V \subseteq U$ and $W \in \mathcal{U}$ such that $(a, b) \in W$ implies $(g(a), g(b)) \in V$ for all $g \in G$. There exists

i) σ_1 such that $\sigma \in \Sigma$ and $\sigma \geqq \sigma_1$ implies $(y, y_\sigma) \in W$,

ii) (β_1, δ_1) such that $(\beta, \delta) \in B \times \Delta$ and $(\beta, \delta) \geqq (\beta_1, \delta_1)$ implies $(y \cdot z_\alpha, g_\beta g_{\alpha,\delta}(x)) \in \in V$,

iii) β_2 such that $\beta \in B$ and $\beta \geqq \beta_2$ implies $(g_\beta(x), y) \in W$,

iv) τ_1 such that $\tau \in T$ and $\tau \geqq \tau_1$ implies $(y_\sigma, g_{\sigma,\tau}(x)) \in W$,

v) (τ_2, δ_2) such that $(\tau, \delta) \in T \times \Delta$ and $(\tau, \delta) \geqq (\tau_2, \delta_2)$ implies $(g_{\sigma,\tau} g_{\alpha,\delta}(x), y_\sigma \cdot z_\alpha) \in V$.

Choose $\beta_3 \geqq \beta_1, \beta_2$; $\delta_3 \geqq \delta_1, \delta_2$; $\tau_3 \geqq \tau_1, \tau_2$. Let $\alpha_1 \in A$ and suppose $(\sigma, \alpha) \in \in \Sigma \times A$ such that $(\sigma, \alpha) \geqq (\sigma_1, \alpha_1)$. Choose $(\beta, \delta, \tau) \geqq (\beta_3, \delta_3, \tau_3)$ (note that $\beta_3, \delta_3, \tau_3$ depend on (σ, α)); then $(g_{\sigma,\tau} g_{\alpha,\delta}(x), y_\sigma \cdot z_\alpha)$, $(g_{\alpha,\delta}(y_\sigma), g_{\sigma,\tau} g_{\alpha,\delta}(x))$, $(g_{\alpha,\delta}(y)\, g_{\alpha,\delta}(y_\sigma))$, $(g_{\alpha,\delta} g_\beta(x), g_{\alpha,\delta}(y))$, $(y \cdot z_\alpha, g_\beta g_{\alpha,\delta}(x)) \in V$ implies $(y_\sigma \cdot z_\alpha, y \cdot z_\alpha) \in U$.

From Propositions 3 and 4 we get the follow theorem.

Theorem 5. *Let $(X, \mathcal{U})$ be a Hausdorff uniform space and let G be a commutative subsemigroup of $C(X)$. If $x \in X$ such that $O(x)$ is complete and G is uniformly equicontinuous on $O(x)$, then $O(x)$ is a commutative topological semigroup.*

Definition. If $g \in G$, let $O(x; g) = \{g^i(x) \mid i$ is a positive integer$\}$ and $K(x; g) = = \bigcap_{i=0}^{\infty} O(g^i(x); g)$. We omit the proof of the following.

Proposition 6. *If $z \in O(x; g)$, then either $z = g^i(x)$ for some positive integer i or $z \in K(x; g)$. $z \in K(x; g)$ if and only if there exists a strictly monotone increasing sequence of positive integers $\{i_n\}_{n=1}^{\infty}$ such that $z = \lim_{n \to +\infty} g^{i_n}(x)$.*

Theorem 7. *Let $(X, \mathcal{U})$, G and $x \in X$ be as in Theorem 5. If $K(x; g)$ is nonempty for some $g \in G$, then $K(x; g)$ is an ideal in $O(x; g)$. If $O(x; g)$ is compact, then $K(x; g)$ is a minimal ideal in $O(x; g)$ and is a topological group.*

Proof. The first part is a consequence of Proposition 6 and the second part follows from [6; p. 109].

Proposition 8. *If $z \in O(x)$ and $g \in G$, then $g(z) = g(x) \cdot z$.*

Proof. Let $\{g_\alpha(x), \alpha \in A\}$ be a net which converges to z. Then $g(z)$ is the limit of the net $\{g g_\alpha(x), \alpha \in A\}$ and the proposition follows from the definition of multiplication in $O(x)$.

Theorem 9. *Let S be a compact Hausdorff space and let G be a commutative equicontinuous subsemigroup of $C(X)$ such that each $g \in G$ is onto. Then each $g \in G$ is a homeomorphism and $x \in O(x; g) = K(x; g)$.*

Proof. Having developed the necessary machinery above, the proof of this theorem can be gotten by mimicing the proof of Theorem 33 of [1]. Since [1] has not yet appeared, we sketch a proof for completeness.

If $y \in K(x; g)$, then $K(y; g) \subseteq O(g; g) \subseteq K(x; g)$ and it is easily seen that $K(y; g)$ (with the multiplication from $O(x)$ is an ideal in $O(x; g)$. By Theorem 7, $K(y; g) = K(x; g)$. It follows that $K(w; g) \cap K(z; g) \neq \emptyset$ if and only if $K(w; g) = K(z; g)$, $w, z \in X$. By using the group structure of $K(x; g)$ and Propositions 6 and 8, one sees that $g \mid K(x; g)$ is a homeomorphism of $K(x; g)$ onto itself. To finish the proof it suffices to show that $X = \bigcap_{x \in X} K(x; g)$.

This is shown by noting that, for each i, $O(g^i(x); g)$ is an upper semicontinuous compact set-valued function $X \to 2^X$ [3]. Since, for each i, $X = \bigcap_{x \in X} O(g^i(x); g)$ and $K(x; g) = \bigcap_{i=0}^{\infty} O(g^i(x); g)$, it follows from a slight modification of arguments in [3] that $X = \bigcap_{x \in X} K(x; g)$.

Definition. By [6; p. 18], $O(x; g)$ is contained in a unique maximal subgroup $M(x; g)$ of $O(x)$. If $g, h \in G$, then $x \in M(x; g) \cap M(x; h)$ and, hence, by [6; p. 18], $M(x; g) = M(x; h)$. Let $M(x) = M(x; g)$.

Theorem 10. *Let X be a compact Hausdorff space and let G be a commutative equicontinuous subsemigroup of $C(X)$ such that each of $g \in G$ is onto. Then* a) *for each $x \in X$, $O(x)$ is a topological group,* b) *if $O(x) \cap O(y) \neq \emptyset$, then $O(x) = O(y)$ and* c) *the closure of G in $C(X)$ is an equicontinuous compact topological group and the mapping $\lambda : \bar{G} \to O(x)$ defined by $\lambda(g) = g(x)$ is a continuous epimorphism.*

Proof. a) Let $g \in G$ and $x \in X$. Since $g(x) \in O(x; g)$, $g(x) \in M(x)$. Suppose $z \in M(x)$; by Proposition 8, $g(z) = g(x) \cdot z$ and hence $g(z) \in M(x)$. Since $g(M(x)) \subseteq M(x)$, it follows that $O(x) \subseteq M(x)$ and hence $O(x) = M(x)$ is a topological group.

b) Suppose $z \in O(x)$ and $g \in G$. Since $g(z) = g(x) \cdot z$ $g(x) \in O(z)$. Therefore $O(x; g) \subseteq O(z)$; since $x \in O(x; g)$, $O(x) = O(z)$.

c) By [4; p. 240], the closure of G in $C(X)$ with respect to the topology of pointwise convergence is uniformly equicontinuous; hence the closure of G, $\bar{G}$, with respect to the topology of uniform convergence on compacta is also uniformly equicontinuous. Note that each element of $\bar{G}$ is an onto map and hence by Theorem 9 is a homeomorphism. By Ascoli's Theorem [4; p. 233], $\bar{G}$ is compact and by Theorem 1.1.15 of [6], $\bar{G}$ is a topological group. We leave to the reader the verification of the second part.

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